Commun. Math. Phys. 151, 467–479 (1993)

Communications in
Mathematical
Physics

Holonomy Groups and W-Symmetries

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Received January 2, 1992

Abstract. Irreducible sigma models, i.e. those for which the partition function does not factorise, are defined on Riemannian spaces with irreducible holonomy groups. These special geometries are characterised by the existence of covariantly constant forms which in turn give rise to symmetries of the supersymmetric sigma model actions. The Poisson bracket algebra of the corresponding currents is a W-algebra. Extended supersymmetries arise as special cases.

1. Introduction

It has been known for many years that the geometry of the target space of two dimensional supersymmetric sigma models is restricted when there are further supersymmetries; in particular, $N = 2$ supersymmetry requires that the target space be a Kähler manifold [1], and $N = 4$ supersymmetry requires that it be a hyperkähler manifold [2]. More exotic geometries arise in heterotic sigma models with torsion and in one-dimensional models [3, 4, 9]. More recently it has been realised that sigma models can admit further symmetries which are non-linear in the derivatives of the sigma model field. The prototype of this type of symmetry is the non-linear realisation of supersymmetry using free fermions [5]; further instances have been given in the context of supersymmetric particle mechanics [6, 7] and in $N = 2$ two-dimensional models, where it has been realised that it is not necessary to impose the vanishing of the Nijenhuis tensor [9, 10]. In [8] a preliminary investigation into non-linear symmetries of other two-dimensional supersymmetric sigma models was presented. A related type of symmetry occurs in bosonic sigma models, the so-called W-symmetry [12, 13].

In this article we combine the issues of the geometry of the target spaces and the non-linear symmetries of two dimensional supersymmetric sigma models. In the case of $N = 2$ and $N = 4$ supersymmetries, for example, the additional structures on the (Riemannian) target spaces reduce the holonomy groups from $O(n)$ to $U\left(\frac{n}{2}\right)$ and $Sp\left(\frac{n}{4}\right)$ respectively, where $n = \dim M$, M being the target space. We shall investigate the symmetries associated with other holonomy groups, restricting our study to manifolds which are not locally symmetric spaces and which have irreducible holonomy groups. Irreducibility here means that the n-dimensional representation of $O(n)$ remains an irreducible representation of the holonomy group $G \subset O(n)$. In the case that the connection is the Levi–Civita connection, the irreducibility of the holonomy implies that M is an irreducible Riemannian manifold (if $\pi_1(M) = 0$) [14], i.e. M is not a product $M_1 \times M_2 \times \cdots$ such that the metric can be written as a direct sum with each component depending only on the co-ordinates of the corresponding factor of the target manifold. In field-theoretic terms, sigma models on metrically reducible spaces factorise into sigma models on factor spaces in the sense that the partition function factorises. However, interesting symmetries can arise on reducible manifolds in which the factors transform into each other; an example of this behaviour occurs in the case of W-symmetry where the target spaces are reducible (for non-locally symmetric spaces).

The irreducible holomony groups associated with Levi–Civita connections on Riemannian manifolds have been classified by Berger [15]. The possible holonomy groups that can arise are $SO(n)$, $U\left(\frac{n}{2}\right)$, $SU\left(\frac{n}{2}\right)$, $Sp\left(\frac{n}{4}\right)$ and $Sp(1)\cdot Sp\left(\frac{n}{4}\right) = Sp(1) \times_{Z_2} Sp\left(\frac{n}{4}\right)$ together with the exceptional cases, G_2 ($n = 7$) and Spin(7) ($n = 8$). In each case there is an associated covariantly constant (with respect to the Levi–Civita connection) totally antisymmetric tensor, and it is this fact which implies the existence of an associated symmetry of the corresponding supersymmetric sigma model [11, 8]. We call such Riemannian geometries special. This classification is not strictly applicable to models with torsion for which the corresponding analysis has not been done. Nevertheless, irreducible holonomy is a useful restriction to impose and the covariantly constant tensors are the same as in the torsion-free case. In many cases of interest we shall in any case set the torsion to zero. Indeed, for both the exceptional cases, G_2 and Spin(7), it turns out that the torsion must vanish. The Riemannian (i.e. torsion-free) case is the most interesting one from the point of view of the algebraic structure of the non-linear symmetries under consideration, since in this case, as we shall show, the corresponding currents, together with the (super) energy-momentum tensor, generate super W-algebras via Poisson Brackets. These algebras are extensions of the (classical) superconformal algebra by additional currents which are, in general, of higher spin. It is of interest to note that the field theory models which provide realisations of classical W-algebras presented here are highly non-trivial field theories. This fact makes the analysis of the corresponding quantum algebras more complicated and we shall not pursue this topic in this paper.

In Sect. 2 we discuss the general form of symmetries generated by covariantly constant antisymmetric tensors. At the classical level these symmetries are of semi-local (superconformal) type, i.e. the parameters depend on some, but not all, of the coordinates of superspace, and in general generate an infinite number of symmetries of this type. However, there are examples of finite dimensional semi-local symmetry algebras, for example on manifolds with $G = SO(n)$. In some cases it is possible to get a finite-dimensional Lie algebra by restricting the parameters to be constant; an example of this type is given by Calabi–Yau manifolds $\left(G = SU\left(\frac{n}{2}\right)\right)$. When the torsion vanishes, as we remarked above, we obtain finite-dimensional W-algebras, i.e. W-algebras generated by a finite number of currents. For some purposes it is useful to regard the invariant antisymmetric tensors associated with the special geometries as vector-valued forms, and we include in this section a brief review of the way such vector-valued forms give rise to derivations of the algebra of differential forms on the target space [16, 17]. In Sect. 3, we introduce Poisson Brackets and compute them for the currents of the type we are interested in. In Sect. 4 we study the various cases listed above, and in Sect. 5 we make some concluding remarks.

2. General Formalism

Let Σ denote the $(1,0)$ (or $N=1$) superspace extensions of two-dimensional Minkowski space, with real light-cone co-ordinates $(y^{\ddagger}, y^{=}, \theta^{+})$ (resp. $(y^{\ddagger}, y^{=}, \theta^{+}, \theta^{-})$). The supercovariant derivatives D_{+} (D_{+}, D_{-}) obey

$$D_{+}^{2} = i\,\partial_{\ddagger} \tag{2.1}$$

and

$$D_{+}^{2} = i\,\partial_{\ddagger}\,; \quad D_{-}^{2} = i\,\partial_{=}; \quad \{D_{+}, D_{-}\} = 0 \tag{2.2}$$

respectively. Let (M, g) be a Riemannian target space (metric g) equipped if necessary with a closed three-form $H = 3db$, where b is a locally defined two form. Local co-ordinates on M will be denoted x^{i}, $i = 1, \dots, n$, and the sigma model superfield by X^{i}. The $(1,0)$ action is

$$S = \int d^{2}y d\theta^{+}(g_{ij} + b_{ij}) D_{+}X^{i}\,\partial_{=}X^{j}\,, \tag{2.3}$$

and the $(1,1)$ action is

$$S = \int d^{2}y d^{2}\theta(g_{ij} + b_{ij}) D_{+}X^{i} D_{-}X^{j}\,. \tag{2.4}$$

Let ω_{L} be an $(l+1)$-form on M

$$\omega_{L} = L_{i_{1}\dots i_{l+1}}\, dx^{i_{1}} \wedge \dots \wedge dx^{i_{l+1}}\,. \tag{2.5}$$

We introduce a vector-valued l-form, L^{i}, and a Lie$(O(n))$ valued $(l-1)$-form $\mathscr{L}^{i}_{j}$ by defining

$$L^{i} = L^{i}_{L}\, dx^{L}\,, \tag{2.6}$$

$$\mathscr{L}^{i}_{j} = L^{i}_{jL_{2}}\, dx^{L_{2}}\,, \tag{2.7}$$

where

$$dx^{L} := dx^{i_{1}} \wedge \dots \wedge dx^{i_{l}} \tag{2.8}$$

$$dx^{L_{2}} := dx^{i_{2}} \wedge \dots \wedge dx^{i_{l}}\,. \tag{2.9}$$

If ω_{L} is covariantly constant, i.e. if

$$\nabla^{(+)}_{j} L_{i_{1}\dots i_{l+1}} = 0 \tag{2.10}$$

with

$$\Gamma^{(\pm)i}{}_{jk} = \Gamma^{i}_{jk} \pm \frac{1}{2} H^{i}{}_{jk}\,, \tag{2.11}$$

then the transformation

$$\delta_L X^i = a_{-l}\, L^i_L\, D_+ X^L\,, \tag{2.12}$$

where

$$D_+ X^L := D_+ X^{i_1} \ldots D_+ X^{i_l} \tag{2.13}$$

is a symmetry of the $(1, 0)$ action if the parameter a_{-l} satisfies $\partial_= a_{-l} = 0$ and a symmetry of the $(1, 1)$ action if $D_- a_{-l} = 0$. The notation for the parameter indicates that it has Lorentz weight $-l/2$ and is thus Grassmann even or odd according to whether l is an even or odd integer. The $(1, 1)$ action is also invariant under

$$\delta_L X^i = a_{+l}\, L^i_L\, D_- X^L \tag{2.14}$$

if

$$\nabla^{(-)}_j L_{i_1 \ldots i_{l+1}} = 0 \tag{2.15}$$

and $D_+ a_{+l} = 0$.

The above symmetry transformations are associated with derivations of the algebra of forms, Ω on M. Let Ω_p denote the space of p-forms, so that $\Omega = \bigoplus_{p=0}^{n} \Omega_p$, and Ω^1_l the space of vector-valued l-forms. We recall that a derivation D of degree r satisfies the following properties:

a) $D(a\omega + b\rho) = aD\omega + bD\rho$; $\quad a, b \in \mathbf{R}$ $\quad$ Linearity

c) $D\Omega_p \subset \Omega_{p+r}$; $\quad$ Degree r

d) $D(\omega \wedge \rho) = D\omega \wedge \rho + (-1)^{pr}\omega \wedge D\rho$, $\quad \omega \in \Omega_p$; $\quad$ Leibniz property . (2.16)

The commutator of two derivations D_r and D_s of degrees r and s is defined by

$$[D_r, D_s] := D_r D_s - (-1)^{rs} D_s D_r \tag{2.17}$$

and the Jacobi identity

$$[D_r, [D_s, D_t]] + (-1)^{t(r+s)}[D_t, [D_r, D_s]] + (-1)^{r(s+t)}[D_s, [D_t, D_r]] = 0 \tag{2.18}$$

holds for any three derivations. Thus the space of derivations is a $\mathbf{Z}$-graded super Lie algebra.

There are two types of derivation, both of which are defined by vector-valued forms. If v is a vector field, i.e. a vector-valued 0-form, then the interior product of v with a p-form, denoted $\iota_v \omega$ is a derivation given by

$$\iota_v \omega = p\, v^i \omega_{i P_2} dx^{P_2}\,. \tag{2.19}$$

Since d is also a derivation we can generate another one from its commutator with ι_v,

$$\iota_v d + d\iota_v = d_v\,. \tag{2.20}$$

This is just the Lie derivative, normally denoted as $\mathscr{L}_v$. A similar construction can be carried out for a general vector-valued form. If $L \in \Omega_l^1$ and $\omega \in \Omega_p$ we define their interior product $\iota_L \omega$ by

$$\iota_L \omega := p\, \omega_{iP_2} L^i_L\, dx^L \wedge dx^{P_2} \,. \tag{2.21}$$

Another notation for this construct is $\omega \bar{\wedge} L$ [17]; we shall use both. It is easy to check that ι_L is a derivation. Taking the commutator of ι_L with d we get a new derivation d_L which generalises the Lie derivative,

$$\iota_L d + (-1)^l d\, \iota_L = d_L \,. \tag{2.22}$$

It has the property that it commutes with d, $d_L d = (-1)^l d d_L$, and is determined by its action on Ω_0,

$$d_L f = df \bar{\wedge} L \,. \tag{2.23}$$

On a p-form ω,

$$d_L \omega = d\omega \bar{\wedge} L + (-1)^l d(\omega \bar{\wedge} L) \,. \tag{2.24}$$

The Nijenhuis tensor (concomitant) $[L, M]$ of two vector-valued forms L and M of degrees l and m is defined by

$$[d_L, d_M] = d_{[L,M]} \,. \tag{2.25}$$

The Nijenhuis tensor can be worked out by observing that

$$d_L x^i = L^i \tag{2.26}$$

so that

$$\begin{aligned} [d_L, d_M] x^i &= [L, M]^i \\ &= dL^i \bar{\wedge} M + (-1)^l d(L^i \bar{\wedge} M) \\ &\quad - (-1)^{lm} (dM^i \bar{\wedge} L + (-1)^m d(M^i \bar{\wedge} L)) \,, \end{aligned} \tag{2.27}$$

where L^i is regarded as an l-form for each value of i. In more detail,

$$\begin{aligned} [L, M]^i &= [L, M]^i_{LM} dx^L \wedge dx^M \\ &= (L^j_L \partial_j M^i_M - M^j_M \partial_j L_L - l L^i_{jL_2} \partial_{l_1} M^j_M + m M^i_{jM_2} \partial_{m_1} L^j_L)\, dx^{LM} \,. \end{aligned} \tag{2.28}$$

It is straightforward to verify that $[I, I]$ is the usual Nijenhuis tensor, $N(I)$, for the case $L = M = I$, an almost complex structure. Hence the integrability condition for an almost complex structure to be complex, $N(I) = 0$, is equivalent to the condition $d_I^2 = 0$.

The other commutators are

$$[\iota_L, d_M] = d_{M \bar{\wedge} L} + (-1)^m \iota_{[L,M]} \tag{2.29}$$

and

$$[\iota_L, \iota_M] = \iota_{M \bar{\wedge} L} + (-1)^{l+m+lm} \iota_{L \bar{\wedge} M} \,, \tag{2.30}$$

where

$$(M \bar{\wedge} L)^i := m\, M^i_{jM_2} L^j_L\, dx^L \wedge dx^{M_2} \,. \tag{2.31}$$

From the Jacobi identity one can derive a number of identities for the tensors which arise in the commutators, for example,

$$[L, [M, N]] + (-1)^{n(l+m)} [N, [L, M]] + (-1)^{l(m+n)} [M, [N, L]] = 0 \tag{2.32}$$

and

$$[L \bar{\wedge} M, N] + (-1)^{(m+1)l}[L, N \bar{\wedge} M] - [L, N] \bar{\wedge} M$$
$$= (-1)^{n(l+1)} L \bar{\wedge} [M, N] + (-1)^{l+1} N \bar{\wedge} [M, L] . \tag{2.33}$$

We can now compute the commutator of two transformations of the type (2.12). It is

$$[\delta_L, \delta_M] X^i = \delta_{LM}^{(1)} X^i + \delta_{LM}^{(2)} + \delta_{LM}^{(3)} , \tag{2.34}$$

where

$$\delta_{LM}^{(1)} = a_{-m} a_{-l} [L, M]^i_{LM} D_+ X^L D_+ X^M ,$$
$$\delta_{LM}^{(2)} = a_{-m} D_+ a_{-l} (M \bar{\wedge} L)^i_{LM_2} D_+ X^L D_+ X^{M_2}$$
$$- a_{-l} D_+ a_{-m} (L \bar{\wedge} M)^i_{ML_2} D_+ X^M D_+ X^{L_2}$$

and

$$\delta_{LM}^{(3)} = i l m (-1)^l a_{-m} a_{-l} (\mathscr{L}^i_j \wedge \mathscr{M}^j_k$$
$$+ (-1)^{(l+1)(m+1)} \mathscr{M}^i_j \wedge \mathscr{L}^j_k)_{L_2 M_2} \partial_{\ddagger} X^k D_+ X^{L_2} D_+ X^{M_2} . \tag{2.35}$$

In general the three terms on the right-hand side of (2.34) are not symmetries by themselves, so that a much larger and more complicated algebra of transformations will be generated.

In the case that the torsion vanishes it is straightforward to show that $[L, M]$ also vanishes, given that L and M are covariantly constant. For (1, 1) models it is straightforward to show that the left and right transformations (2.12) and (2.14) commute up to the equations of motion.

3. Poisson Brackets

Let $\{j_A\}$ be the currents of a set of symmetries of a two-dimensional field theory. The Poisson Bracket algebra

$$\{j_A, j_B\}_{PB} = P_{AB}(\{j_A\}) \tag{3.1}$$

of these currents forms a W algebra provided that P_{AB} is a polynomial in the currents $\{j_A\}$ and their derivatives.

The currents of the symmetries (2.12) of the action (2.3) (or (2.4)) are

$$j_L = \frac{1}{l+1} L_{i_1 \dots i_{l+1}} D_+ X^{i_1} \dots D_+ X^{i_{l+1}} . \tag{3.2}$$

These currents are conserved, $D_- j_L = 0$ (or $\partial_= j_L = 0$), subject to the equations of motion of the action (2.3) (or (2.4)). The form ω_L (2.5) satisfies Eq. (2.10).

To get a complete set of currents, it is necessary to include the (super) energy-momentum tensor T, given by

$$T = g_{ij} D_+ X^i \partial_{\ddagger} X^j . \tag{3.3}$$

T generates left-handed supersymmetry transformations and translations.

In the rest of this section we shall assume that the torsion vanishes, $H = 0$, and we shall also focus only on left-handed currents having the form (3.2) or (3.3); any dependence on the right-handed co-ordinates $(y^{=}, \theta^{-})$ will be suppressed.

To calculate the Poisson brackets of the currents j_L (3.2), we introduce the Poisson bracket

$$\{D_+X^i(z_1), D_+X^j(z_2)\} = g^{ij}\nabla_{+1}\delta(z_1, z_2)\,, \tag{3.4}$$

where $z = (y^{\ddagger}, \theta^+)$. This Poisson bracket is constructed from light-cone considerations where the co-ordinate $y^{=}$ of the flat superspace is taken as "time."

Next we define the "smeared" currents $j_L(a_l)$ by

$$j_L(a_l) = \int dy^{\ddagger}\, d\theta^+\; a_{-l}j_L\,, \tag{3.5}$$

where a_l is a function of z with Grassmannian parity $(-1)^l$. The Poisson bracket of two currents of the form (3.5) is

$$\{j_L(a_{-l}), j_M(a_{-m})\}_{PB}$$
$$= (-1)^{lm+m+1}\left(\frac{l+m}{l+1}\, j_{L\bar{\wedge} M}\,(D_+a_{-l}a_{-m}) + \bar{j}_{L,M}(a_{-l}a_{-m})\right), \tag{3.6}$$

where

$$j_{L\bar{\wedge} M}\,(D_+a_{-l}a_{-m}) = \frac{(-1)^{lm}}{l+1}\int dy^{\ddagger}\, d\theta^+\, D_+a_{-l}a_{-m}(\omega_L \bar{\wedge} M)_{LM}\, D_+X^{LM} \tag{3.7}$$

and

$$\bar{j}_{L,M}(a_{-l}a_{-m}) = -\frac{(-1)^m}{l}\int dy^{\ddagger}\, d\theta^+\, i\, a_{-l}a_{-m}(L\bar{\wedge} M)_{jL_2M}\,\partial_{\ddagger}X^jD_+X^{L_2M}\,. \tag{3.8}$$

In the examples we shall see that $\bar{j}_{L,M}$ can be written as a product of the original currents, the energy-momentum tensor T and their derivatives.

The Poisson bracket of T with j_L is

$$\{T(a_{=}), j_L(a_{-l})\}_{PB} = (l+1)j_L(\partial_{\ddagger}a_{=}a_{-l} + 2a_{=}\partial_{\ddagger}a_{-l}) + \frac{1}{i}j_L(D_+(D_+a_{=}a_{-l}))\,. \tag{3.9}$$

This formula reflects the fact that j_L has Lorentz weight $\frac{1}{2}(l+1)$.

4. Applications

4.1. $SO(n)$. The simplest case to analyse is $SO(n)$. The corresponding invariant tensor is the ε-tensor, $\varepsilon_{i_1\ldots i_n}$. The symmetry transformation is

$$\delta X^i = a_{1-n}\,\varepsilon^i{}_{j_1\ldots j_{n-1}}D_+X^{j_1}\ldots D_+X^{j_{n-1}}\,. \tag{4.1}$$

This is a bosonic symmetry for n odd, and it is easy to see that the commutator of two such transformations is zero. Comparing with (2.34), we observe that the first and third terms in the right-hand side vanish automatically when $L = M$ and the symmetry is bosonic. The second term is trivially zero for $n \geqq 5$, and for $n = 3$ can been seen to be zero by a short explicit computation. When n is even (3.1) defines a fermionic symmetry which is also nilpotent, except in the case $n = 2$. The first and second terms on the right-hand side of (2.34) vanish trivially unless $n = 2$ or 4. In the case $n = 4$, the properties of the ε-tensor imply that both terms are again zero. In the case $n = 2$, the ε-tensor defines an almost complex structure on M which is integrable (the torsion vanishes identically); hence, the first term in (2.34) is zero, the second is a first supersymmetry transformation and the third is a translation.

Thus, for $n \geqq 3$, sigma models with $SO(n)$ holonomy can be characterised by the existence of an Abelian (super)conformal symmetry. In the case $n = 2$, this becomes $N = 2$ (or (2, 0) supersymmetry).

4.2. $U\left(\frac{n}{2}\right) = U(m)$. In this case the antisymmetric tensor is derived from an almost complex structure $I^i{}_j$, $I^2 = -1$. Models with $U(m)$ holonomy have been extensively studied in the literature, including the case where I is not complex, i.e. $N(I) \neq 0$ [9–11]. The commutator of two transformations defined by I is

$$[\delta_I, \delta'_I]X^i = a'_{-1}a_{-1}N^i_{j_1 j_2}D_+X^{j_1}D_+X^{j_2} + D_+(a'_{-1}a_{-1})D_+X^i + 2i(a'_{-1}a_{-1})\partial_{\ddagger}X^i\,. \tag{4.2}$$

The second and third terms correspond to first supersymmetry transformation and translations, while the Nijenhuis tensor term defines a new symmetry of the type (2.12). Since the second and third terms are symmetries by themselves, so is the first term and this implies that N_{ijk} must be totally antisymmetric and covariantly constant.

One can now investigate the algebra generated by δ_I, i.e. compute $[\delta_I, \delta_N]$, etc. Referring again to Eq. (2.34), the third term on the right-hand side can be shown to vanish by virtue of the identity

$$I^i{}_l N^l{}_{jk} + N^i{}_{jl} I^l{}_k = 0\,. \tag{4.3}$$

The second term gives a contribution

$$[\delta_I, \delta_N]X^i = -(a_{-1}D_+a_{-2} + 2a_{-2}D_+a_{-1})\hat{N}^i_{jk}D_+X^jD_+X^k + \cdots, \tag{4.4}$$

where $\hat{N} = I \barwedge N$. $\hat{N}_{ijk}$ is again totally antisymmetric and covariantly constant so we have a new symmetry of type (2.12).

Finally the first term gives rise to a transformation involving the Nijenhuis concomitant of I and N, $[I, N]$. This is the Slebodzinski tensor introduced in reference [18]; however, it has been pointed out that this tensor is identically zero [19]. This can been seen very easily from the Jacobi identity (2.32), since $[I, N] = [I, [I, I]]$.

We can continue to compute commutators (or Poisson brackets), but it seems that this not a finitely-generated W-algebra [11]. However, if the transformations are restricted to be rigid, then the $\hat{N}$ symmetry will not be generated starting from δ_I and the algebra generated by δ_I and δ_N closes. This is therefore a finite-dimensional rigid symmetry algebra.

In the case of zero Wess–Zumino term $H = 0$, this case reduces to $N = 2$ superconformal symmetry.

4.3. $SU\left(\frac{n}{2}\right) = SU(m)$. In the case of $SU(m)$ we have, in addition to the almost complex structure I, an m-form ω_L which is the sum of an $(m, 0)$ and a $(0, m)$ form, $\omega_L = \varepsilon + \bar{\varepsilon}$. In a unitary basis $\varepsilon_{a_1 \ldots a_m}$ is the usual ε-tensor in m-dimensions. We also have another m-form $\omega_{\hat{L}} = \frac{1}{i}(\varepsilon - \bar{\varepsilon})$. We shall suppose that I is a complex structure. There are thus three transformations to consider, δ_I, δ_L and $\delta_{\hat{L}}$, where

$$\omega_L = L_{i_1 \ldots i_m} dx^{i_1} \wedge \cdots \wedge dx^{i_m}\,, \tag{4.5}$$

$l = (m-1)$ in the notation of Sect. 2, and $\hat{L}^i{}_L = I^i{}_j L^j{}_L$. The algebra generated by δ_I closes as $N = 0$. In the commutator of δ_I with δ_L one finds that the terms involving $I^i{}_j \mathscr{L}^j{}_k + \mathscr{L}^i{}_j I^j{}_k$ and $[I, L]$ are zero using the fact that $\omega_L = \varepsilon + \bar{\varepsilon}$. Thus we are left with

$$[\delta_I, \delta_L] X^i = -((m-1) a_{1-m} D_+ a_{-1} + a_{-1} D_+ a_{1-m}) \hat{L}^i_L D_+ X^L\,. \tag{4.6}$$

In the commutator of δ_I and $\delta_{\hat{L}}$ the third term vanishes because $\hat{L}$ is the sum of an $(m, 0)$-form and a $(0, m)$-form. The first term can be shown to be zero by using the Jacobi identity (2.33) and the fact that the Nijenhuis tensor $[I, L]$ vanishes. Finally

$$\begin{aligned}[]
&[\delta_I, \delta_{\hat{L}}] X^i \\
&\quad = ((m-1) a_{1-m} D_+ a_{-1} + a_{-1} D_+ a_{1-m}) L^i{}_{j_1 \ldots j_{m-1}} D_+ X^{j_1} \cdots D_+ X^{j_{m-1}}
\end{aligned} \tag{4.7}$$

closes to a δ_L transformation. The commutator of two δ_L transformations yields

$$\begin{aligned}[]
[\delta_L, \delta'_L] X^i = {} & a_{1-m} a'_{1-m} [L, L]^i{}_{j_1 \ldots j_{2m-2}} D_+ X^{j_1} \ldots D_+ X^{j_{2m-2}} \\
& + (a'_{1-m} D_+ a_{1-m} - a_{1-m} D_+ a'_{1-m})\, (L \bar{\wedge} L)^i{}_{j_1 \ldots j_{2m-3}} \\
& \cdot D_+ X^{j_1} \ldots D_+ X^{j_{2m-3}} \\
& + i(m-1)^2 a_{1-m} a'_{1-m} (1 - (-1)^{m-1}) \\
& \cdot (\mathscr{L}^i{}_l \mathscr{L}^l{}_k)_{j_1 \ldots j_{2m-4}} \partial_{\mp} X^k D_+ X^{j_1} \cdots D_+ X^{j_{2m-4}}\,.
\end{aligned} \tag{4.8}$$

For m odd the first and last terms vanish, but the second term does not vanish for any m. In general, therefore, the algebra generated by I and L is very complicated and leads to an infinite number of (super)conformal symmetries. We can get a finite-dimensional W-algebra by taking the Wess–Zumino H term to vanish. In this case we recover the W-algebra presented in ref. [8]. If in addition we assume that the parameters of the δ_I and δ_L are rigid and m is an odd number, $[\delta_L, \delta_L] X^i = 0$ and the $\hat{L}$ transformations are not generated as the parameters are restricted to be constant. This subset of cases includes six- (real) dimensional Calabi–Yau spaces. The Poisson bracket algebra of the currents of the symmetries of sigma models with target manifold M with $SU(m)$ holonomy and without

Wess–Zumino term closes as a W-algebra. Indeed

$$\{j_I(a_{-1}), j_I(a'_{-1})\}_{PB} = -iT(a_{-1}a'_{-1}),$$

$$\{j_I(a_{-1}), j_L(a_{-l})\}_{PB} = -j_{\hat{L}}\left(\frac{1}{l+1}D_+ a_{-1}a_{-l} - D_+(a_{-1}a_{-l})\right);$$

$$\{j_I(a_{-1}), j_{\hat{L}}(a_{-l})\}_{PB} = -j_L\left(\frac{1}{l+1}D_+ a_{-1}a_{-l} - D_+(a_{-1}a_{-l})\right).$$

For l odd

$$\begin{aligned}\{j_L(a_{-l}), j_L(a'_{-l})\}_{PB} &= -il\cdot l!\,T(a_{-l}a'_{-l}j_I{}^{l-1}),\\ \{j_{\hat{L}}(a_{-l}), j_{\hat{L}}(a'_{-l})\}_{PB} &= -il\cdot l!\,T(a_{-l}a'_{-l}j_I{}^{l-1}),\\ \{j_{\hat{L}}(\hat{a}_{-l}), j_L(a_{-l})\}_{PB} &= l!\,j_I(\{2D_+\hat{a}_{-l}a_{-l} + D_+(\hat{a}_{-l}a_{-l})\}j_I{}^{l-1}),\end{aligned} \tag{4.9}$$

and for l even

$$\begin{aligned}\{j_L(a_{-l}), j_L(a'_{-l})\}_{PB} &= l!\,j_I(\{2D_+a_{-l}a'_{-l} - D_+(a_{-l}a'_{-l})\}j_I{}^{l-1}),\\ \{j_{\hat{L}}(a_{-l}), j_{\hat{L}}(a'_{-l})\}_{PB} &= l!\,j_I(\{2D_+a_{-l}a'_{-l} - D_+(a_{-l}a'_{-l})\}j_I{}^{l-1}),\\ \{j_{\hat{L}}(\hat{a}_{-l}), j_L(a_{-l})\}_{PB} &= -il\,l!\,T(\hat{a}_{-l}a_{-l}j_I{}^{l-1}).\end{aligned} \tag{4.10}$$

4.4. $Sp\left(\frac{n}{4}\right) = Sp(m)$; $Sp(1)\cdot Sp(m)$. If the holonomy group can be reduced to $Sp(m)$ there are almost complex structures I_r, $r = 1, \ldots 3$, which satisfy the algebra of imaginary unit quaternions,

$$I_r I_s = -\delta_{rs} + \varepsilon_{rst} I_t. \tag{4.11}$$

The corresponding covariantly constant forms are obtained by lowering an index with the metric, which is hermitian with respect to all three complex structures. These structures can be used to define three additional supersymmetries, in the usual way,

$$\delta_r X^i = a^r_{-1} I_r{}^i{}_j D_+ X^j. \tag{4.12}$$

The commutator of the algebra closes, except for the terms involving the Nijenhuis tensors $[I_r, I_s]$. These generate new symmetries as in the $N = 2$ case discussed above.

In the case of zero Wess–Zumino term, the algebra of currents of the above transformations is the $N = 4$ superconformal algebra.

In the case $Sp(1)\cdot Sp(m)$ the three complex structures are not globally defined on the target space. The symmetry transformations (4.12) may be defined only in the case of local supersymmetry [20, 10]. However, there is a covariantly constant four-form ω_L given by

$$\omega_L = \sum_{r=1}^{3} \omega_r \wedge \omega_r, \tag{4.13}$$

where ω_r is the two form corresponding to I_r. This can be used to define a transformation of the type (2.12). The Poisson algebra (3.6) of the current j_L of the corresponding symmetry is

$$\{j_L(a_{-3}), j_L(a'_{-3})\}_{PB} = \frac{i}{4} j_L(a_{-3}a'_{-3}T) . \tag{4.14}$$

4.5. G_2 and Spin(7). These two cases are closely related. We begin with G_2. It is the subgroup of $SO(7)$ which leaves the antisymmetric three-index tensor defined by the structure constants of the imaginary unit octonions invariant. If e^a is a basis of orthonormal frames on M the corresponding three-form, φ, is

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{356} - e^{347} , \tag{4.15}$$

where

$$e^{abc} = e^a \wedge e^b \wedge e^c . \tag{4.16}$$

We also write

$$\varphi = \omega_L = L_{ijk}\, dx^i \wedge dx^j \wedge dx^k . \tag{4.17}$$

We observe that the covariant constancy of L_{ijk} with respect to the connection $\Gamma^{(+)}$ implies that the torsion H must vanish. The equation of covariant constancy can be written in the form

$$\nabla_i L_{jkl} - \frac{1}{2} H^m{}_{i[j} L_{kl]m} = 0 , \tag{4.18}$$

where ∇ is the Levi–Civita connection. Using (4.15) one observes that

$$\nabla_i L_{jkl} = 0 \tag{4.19}$$

and

$$H^m{}_{i[j} L_{kl]m} = 0 \tag{4.20}$$

are valid separately. Finally one can show that (4.20) implies the vanishing of H_{ijk}.

A second invariant tensor can be defined as the dual of ω_L,

$$*\omega_L = M_{ijkl}\, dx^i \wedge \cdots \wedge dx^l . \tag{4.21}$$

Therefore we have an algebra generated by δ_L and δ_M. The commutator of two L-transformations gives an M-transformation, the parameter of which vanishes in the rigid case,

$$[\delta_L, \delta'_L] X^i = -2(a'_{-2} D_+ a_{-2} - a_{-2} D_+ a'_{-2}) M^i_{jkl} D_+ X^j \ldots D_+ X^l . \tag{4.22}$$

Thus, if we take the parameters to be constant there is an Abelian symmetry algebra generated by L alone.

In general, the Poisson bracket algebra of these symmetries closes as a W algebra. Indeed,

$$\begin{aligned}
\{j_L(a_{-2}), j_L(a'_{-2})\}_{PB} &= -2 j_M(2 D_+ a_{-2} a'_{-2} - D_+(a_{-2} a'_{-2})) , \\
\{j_M(a_{-3}), j_L(a_{-2})\}_{PB} &= 27\, i\, j_L(a_{-3}\, a_{-2}\, T) , \\
\{j_M(a_{-3}), j_L(a'_{-3})\}_{PB} &= \frac{9i}{4} j_M(a_{-3}\, a'_{-3}\, T) - 9 j_L(a_{-3}\, a'_{-3}\, D_+ j_L) .
\end{aligned} \tag{4.23}$$

Finally, we turn to Spin(7). The target manifold in this case has dimension 8 and the invariant tensor is a self-dual 4-form Φ which can be constructed from φ. Let e^0, e^a, $a = 1, \ldots, 7$, be an orthonormal basis, then

$$\Phi = e^0 \wedge \varphi + {}^*\varphi \; ;$$
$$\Phi = \omega_L = L_{ijkl} \, dx^i \wedge \cdots \wedge dx^l \; . \tag{4.24}$$

It is straightforward to verify that the torsion H vanishes in the Spin(7) case as it does in the G_2 case. The Poisson bracket algebra of two transformations generated by ω_L closes as a W algebra; it is

$$\{j_L(a_{-3}), j_L(a'_{-3})\}_{PB} = \frac{9i}{4} j_L(a_{-3} \, a'_{-3} \, T) \; . \tag{4.25}$$

5. Concluding Remarks

In this paper we have seen that two-dimensional supersymmetric sigma models on Riemannian target spaces with special geometries have associated symmetries and that, classically, the algebraic structure of these symmetries is of W-type, i.e. higher spin extensions of the superconformal algebra. It would clearly be of interest to analyse these symmetries at the quantum level, but, as we remarked in the introduction, this is non-trivial in view of the non-linearities involved. If one makes the assumption that symmetries of this type are preserved quantum mechanically, then they would seem, in certain cases, to imply strong constraints on the renormalisation of the models concerned. For example, $N = 1$ sigma models on Calabi–Yau target spaces have additional symmetries of this type as we have seen, and these, if preserved, would imply, in conjunction with the Calabi–Yau theorem, the perturbative finiteness of such models. Since this would contradict explicit calculations [22] (except for $n = 4$), the conclusion seems to be that these symmetries are in general anomalous quantum mechanically. We have carried out a preliminary calculation for the case $n = 6$ which lends support to this conjecture, but a complete analysis remains to be done.

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Communicated by N.Yu. Reshetikhin