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Partially $U(1)$ Compactified Scalar Massless Field on the Compact Riemann Surface and the Bosonic String Amplitudes

Yu. M. Zinoviev

Steklov Mathematical Institute, Vavilov St. 42, Moscow 117966, GSP-1, Russia

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Abstract. The theory of the partially $U(1)$ compactified scalar massless field on the compact Riemann surface with Nambu-Goto action is defined. The partition function is determined completely by a choice of the finite-dimensional approximations. The correlation functions are the only correctly defined objects of the theory. The averages of the correlation function asymptotic values provide the amplitudes. For the compact Riemann surfaces of any genus the usual bosonic string amplitudes are the special cases of the above amplitudes.

1. Introduction

Let M be a compact orientable surface of genus g endowed with the Riemannian metric $g_{ij}(x)$, $i, j = 1, 2$. In the bosonic string theory the Nambu-Goto action for the scalar massless fields $X^\mu(x)$, $\mu = 1, \ldots, D$ on the surface M is given by

$$S(X^\mu) = -1/2\alpha^2 \int_M d_2x(\det g_{ij}(x))^{1/2} \sum_{i,j=1}^{2} \sum_{\mu=1}^{D} g^{ij}(x) \frac{\partial X^\mu}{\partial x^i} \frac{\partial X^\mu}{\partial x^j}, \tag{1}$$

where $g^{ij}(x)$ is the inverse matrix for the metric matrix $g_{ij}(x)$. It was shown [1, 2] that in the partition function

$$Z = \sum_{g=0}^{\infty} \int Dg_{ij}(x) DX^\mu(x) \exp[S(X^\mu)] \tag{2}$$

for the space dimension $D = 26$ the integration over the metrics $g_{ij}(x)$ is reduced to the integration over the complex structure parameters of the Riemann surfaces M. The bosonic string amplitudes are the special correlation functions defined in the following way [3, Vol. 1, Sect. 1.4.2]:

$$\begin{gathered} 1/Z \sum_{g=0}^{\infty} \int Dg_{ij}(x) DX^\mu(x) \left(\prod_{l=1}^{N} V_l(k_l) \right) \exp[S(X^\mu)], \\ V(k) = \int_M d^2x(\det g_{ij}(x))^{1/2} v(x) \exp[i(k, X(x))], \end{gathered} \tag{3}$$

where $(k, X(x)) = \sum_{\mu=1}^{D} k^\mu X^\mu(x)$ and a vector k is a D-dimensional momentum. For $g=0,1$ the amplitudes (3) are known [3]. For the fixed compact Riemann surface of higher genus the correlation functions (3) for the fields $X^\mu(x)$ compactified on a torus and for the vertex operators with $v(x)=1$ are computed in [4].

The action (1) is invariant under the shift $X^\mu \to X^\mu + a^\mu$ for a constant vector a^μ and therefore, the integral (2) with respect to $X^\mu(x)$ diverges. The finite part of this integral [1, 2] is not uniquely defined. In order to calculate the integral (2) let us consider the fields $X^\mu(x)$ taking values in the circle of radius R or in the quotient group $\mathbf{R}/2\pi\mathbf{Z}$, where $\mathbf{R}$ is the group of real numbers and $\mathbf{Z}$ is the group of integers. Hence we consider the functions $X^\mu(x)$ and $X^\mu(x) + 2\pi R n(x)$ as equivalent. The integer value function $n(x)$ is smooth if it is constant. Thus to consider the smooth functions $X^\mu(x)$ with the same action (1) it is necessary for the field $X^\mu(x)$ to belong the quotient group $C^\infty(M)/2\pi R\mathbf{Z}$, where $C^\infty(M)$ is the space of the smooth functions on the Riemann surface M and $2\pi R\mathbf{Z}$ is the group of the constant $2\pi R\mathbf{Z}$-valued functions on the Riemann surface M. Therefore, the field $X^\mu(x)$ takes values in the quotient group $\mathbf{R}/2\pi R\mathbf{Z}$ at an arbitrary but fixed point of the Riemann surface M. Such field is called of the partially $U(1)$ compactified. Let us compute the auxiliary integral for the integrals (2) and (3)

$$\int_{(C^\infty(M)/2\pi R\mathbf{Z})^{\times D}} \exp\left[i \sum_{\mu=1}^{D} (Y^\mu, X^\mu) + S(X^\mu) \right] DX^\mu(x), \tag{4}$$

where the inner product of the functions on the Riemann surface M

$$(\phi, \psi) = \int_M \phi(x)\psi(x)(\det g_{ij}(x))^{1/2} d^2x \tag{5}$$

and for every $\mu = 1, \ldots, D$ the function $Y^\mu(x)$ satisfies the condition

$$(Y^\mu, 1) \in R^{-1}\mathbf{Z}. \tag{6}$$

Here 1 is the function equal to 1 everywhere on M. The condition (6) provides the invariance of the integrand (4) under the shifts $X^\mu \to X^\mu + 2\pi R n^\mu$, $n^\mu \in \mathbf{Z}$. In other words the condition (6) provides $\exp\left[i \sum_{\mu=1}^{D} (Y^\mu, X^\mu) \right]$ to be a character of the quotient group $(C^\infty(M)/2\pi R\mathbf{Z})^{\times D}$.

For an arbitrary lattice gauge theory with an abelian compact gauge group the non-gauge invariant correlation functions are identically zero [5]. The definitions of the partition function and the gauge invariant correlation functions allow the generalizations to the lattice gauge theories with non-compact abelian gauge group [5]. The simplest non-compact abelian group is the group of real numbers $\mathbf{R}$. By using de Rham idea [6] it is possible to transfer the definitions of the lattice $\mathbf{R}$ gauge theory to the definitions of the partition function and the correlation functions of the $\mathbf{R}$ gauge theory on the Riemann manifold [7]. In particular, the correlation functions of the scalar massless field theory on a Riemann surface with the action (1) were calculated [7]. The partition function of this theory is meaningless since it depends on a choice of the finite-dimensional approximations.

In the next section it will be proved that for the partially $U(1)$ compactified scalar massless field theory (1), (4), (5), (6) on the Riemann surface the non-gauge invariant correlation functions with $(Y^\mu, 1) \neq 0$ are identically zero. The gauge invariant correlation functions with $(Y^\mu, 1) = 0$ and the partition function coincide with the correlation functions and the partition function of the **R**-gauge scalar massless field theory on a Riemann surface [7]. Hence the partition function or the integral (4) with $Y^\mu=0$ is completely determined by a choice of the finite-dimensional approximations of the integral (4). The special choice of the finite-dimensional approximations gives the result of [1, 2]. Therefore, the statistical theory (1), (2), (3) with $U(1)$ compactified zero mode is meaningless. The only correctly defined object is the correlation function of the theory (1), (4), (5), (6) for the fixed Riemann surface M. This correlation function for the vector function $Y^\mu(x)$ satisfying the conditions $(Y^\mu(x), 1)=0$ is given by

$$\exp\left[-\alpha^2/2 \int_{M^{\times 2}} (Y(x), Y(y))G(x,y)(\det g_{ij}(x))^{1/2}(\det g_{ij}(y))^{1/2}d^2x d^2y\right], \qquad (7)$$

where the inner product $(Y(x), Y(y))=\sum_{\mu=1}^{D} Y^\mu(x)Y^\mu(x)Y^\mu(y)$ and $G(x,y)$ is the Green's function for the Laplace-Beltrami operator on the Riemann surface M. The amplitude (3) corresponds with the vector function $Y^\mu(x) =(\det g_{ij})^{-1/2}\sum_{l=1}^{N} k_l^\mu\delta(x,x_l)$. The substitution of this vector function into the expression (7) gives the diverging integral. Usually the finite part of (7) inserted into the integral (3) provides the amplitude. Because of conditions $(Y^\mu, 1)=0$, or $\sum_{l=1}^{N} k_l^\mu=0$ in our case, it is possible to replace the Green's function $G(x,y)$ in (7) by a function $G(x,y)+f(x)+g(y)$, where the functions $f(x)$ and $g(y)$ are arbitrary. Thus the finite part of the correlation function (7) is not connected in general with the geometry of the Riemann surface M. For example, the simplest amplitude corresponding the Riemann sphere $\mathbf{CP}^1$ is usually computed by using the Green's function for the Laplace-Beltrami operator on the complex plane **C**. Our definition of the amplitude is similar to the integral (3) but it has a simple geometrical interpretation. Under the assumptions $\sum_{l=1}^{N} k_l^\mu=0$, $\mu=1,\ldots,D$, $(k_i,k_i)=m_i^2,\ldots,$ $i=1,\ldots,N$ this definition provides the following N-point amplitude:

$$\int_{M^{\times N}} \left(\prod_{l=1}^{N} v_l(x_l)(\det g_{ij}(x_l))^{1/2}d^2x_l\right)\exp\left[-\alpha^2\sum_{i<j}(k_i,k_j)G(x_i,x_j)\right]. \qquad (8)$$

Now the space dimension $D=26$ is not preferred and the masses m_i are arbitrary. The last property is physically natural since we investigate the scattering amplitude in the Euclidean space and the particle masses usually are fixed in the Minkowski space. The integral (8) is convergent because of the smoothing functions $v_l(x_l)$. By means of the regularization procedure in the integral (8) we obtain the generalized function on the space $M^{\times N}\times T_g$, where the point of the Teichmüller space T_g corresponds with the complex structure of the Riemann surface M. It is possible to prove the modular invariance of the amplitude (8). For the analytic regularization in the parameters (k_i,k_j) the amplitude (8) has the pole singularities similar to those of the Veneziano amplitude.

If we choose the special coupling constant α^2, the masses m_i and the smoothing functions $v_l(x_l)$ in the amplitude (8) for the genus $g=0$ we obtain the N-point amplitude for the closed bosonic strings of genus zero [3, Vol. 1, formula (1.4.13)]. Another choice gives us the Koba-Nielsen amplitude for the open bosonic strings of genus zero [3, Vol. 1, formula (1.5.11)]. If we integrate the amplitude (8) for the genus $g=1$ with the special measure with respect to the parameter of the complex structure of the torus and if we choose the special coupling constant α^2, the masses m_i and the smoothing functions $v_l(x_l)$ we obtain the N-point amplitude for the closed bosonic strings of genus 1 [3, Vol. 2, formula (8.2.17)]. Another choice of the measure, the coupling constant, the masses and the smoothing functions provides the N-point amplitude for the open bosonic strings of genus 1 [3, Vol. 2, formula (8.1.55)]. For higher genus $g>1$ the substitution of the special coupling constant, the masses and the smoothing functions in the amplitude (8) gives us the expression similar to the amplitude obtained in [4] for infinite radius of the compactification torus.

In the next section we study the partition function, the correlation functions and the amplitudes of the partially $U(1)$ compactified scalar massless field theory on the compact Riemann surface. The third, fourth, and fifth sections are devoted to study the Green's functions for the Laplace-Beltrami operators and the amplitudes for the compact Riemann surface of genus: $g=0$, $g=1$, and $g>1$, respectively.

2. Correlation Functions and Amplitudes

Let M be a compact smooth connected orientable surface endowed with a complete Riemannian metric $g_{ij}(x)$, $i,j=1,2$. An element of the quotient group $(C^\infty(M)/2\pi R\mathbf{Z})^{\times D}$ is called a partially $U(1)$ compactified field on the surface M. For these fields we introduce the differential operator $dX^\mu(x)=\sum_{i=1}^{2}\frac{\partial}{\partial x^i}X^\mu(x)dx^i$, where x^1,x^2 are the local coordinates on M and the differential 1-form $dX^\mu(x)$ depends only on the equivalence class $[X^\mu(x)]\in(C^\infty(M)/2\pi R\mathbf{Z})^{\times D}$. The Nambu-Goto action (1) may be written in the following form:

$$S(X^\mu)=-1/2\alpha^2\sum_{\mu=1}^{D}(dX^\mu,dX^\mu), \tag{9}$$

where the inner product of the differential 1-forms $\alpha=\sum_{i=1}^{2}\alpha_i dx^i$ and $\beta=\sum_{i=1}^{2}\beta_i dx^i$ is defined by

$$(\alpha,\beta)=\int_M\sum_{i,j=1}^{2}g^{ij}(x)\alpha_i(x)\beta_j(x)(\det g_{ij}(x))^{1/2}d_2x. \tag{10}$$

Here the matrix $\{g^{ij}(x)\}$ is the inverse for the metric $\{g_{ij}(x)\}$ on the surface M.

Definition 2.1. *For the partially $U(1)$ compactified scalar massless field theory with the action* (9) *a correlation function for a vector function $Y^\mu(x)\in C^\infty(M)$, $\mu=1,\dots,D$ satisfying the condition* (6) *is defined in the following way*

$$W(Y^\mu)=\lim_{L_n\to(C^\infty(M)/2\pi R\mathbf{Z})^{\times D}}1/Z_n \times\int_{L_n}\exp\left[\sum_{\mu=1}^{D}[i(Y^\mu,X^\mu)-1/2\alpha^2(dX^\mu,dX^\mu)]\right]d[X^\mu], \tag{11}$$

where L_n is a n generator subgroup of the quotient group $(C^\infty(M)/2\pi R\mathbf{Z})^{\times D}$ and

$d[X^\mu]$ is any Haar measure on the group L_n. The normalizing multiplier is given by

$$Z_n = \int_{L_n} \exp\left[-1/2\alpha^2 \sum_{\mu=1}^{D} (dX^\mu, dX^\mu)\right] d[X^\mu]. \tag{12}$$

The partition function of the partially $U(1)$ compactified scalar massless field theory with the action (9) is defined as a limit

$$Z = \lim_{(C^\infty(M)/2\pi R\mathbf{Z})^{\times D}} Z_n. \tag{13}$$

Let d^* be the adjoint operator of the differential operator d with respect to the inner products (5), (10). The operator d^* sends a differential 1-form into a function on the surface M. The operator $\Delta = d^*d$ is called the Laplace-Beltrami operator on the functions on the surface M. A function ϕ is said to be harmonic if $\Delta\phi = 0$. The Hodge theorem [6, Sect. 31, Corollaire 4] implies that on the compact smooth connected orientable surface endowed with a Riemannian metric any harmonic function is constant. We use H to denote the orthogonal projector on the one-dimensional space of the harmonic functions in the Hilbert space of the functions on the surface M with the inner product (5). The operator G on this space is called a Green operator for the Laplace-Beltrami operator on the functions on the surface M if it satisfies the relations

$$\Delta G = G\Delta = I - H, \qquad GH = HG = 0. \tag{14}$$

Proposition 2.1. *For the partially $U(1)$ compactified scalar field theory with the action (9) the partition function (13) is determined completely by a choice of the subgroups $L_n \in (C^\infty(M)/2\pi R\mathbf{Z})^{\times D}$ and the Haar measures on them. The correlation function (11) is independent of a choice of the subgroups L_n and the Haar measures on them. If for some $\mu = 1, \dots, D$ the relation $(Y^\mu, 1) \neq 0$ holds then the correlation function*

$$W(Y^\mu) = 0. \tag{15}$$

If for all $\mu = 1, \dots, D$ the relations $(Y^\mu, 1) = 0$ hold then the correlation function

$$W(Y^\mu) = \exp\left[-\alpha^2/2 \sum_{\mu=1}^{D} (Y^\mu, GY^\mu)\right], \tag{16}$$

where G is the Green operator for the Laplace-Beltrami operator on surface M.
Proof. Let $n = (n_1, \dots, n_D) \in \mathbf{Z}^D$ and L_n be a subgroup of quotient group $(C^\infty/2\pi R\mathbf{Z})^{\times D}$ generated by the linear independent for every $\mu = 1, \dots, D$ functions $X_0^\mu(x) \equiv 1$, $X_j^\mu(x)$, $j = 1, \dots, n_\mu - 1$. An arbitrary element of the subgroup L_n has a form

$$\left[r_\mu + \sum_{j=1}^{n_\mu - 1} t_{j\mu} X_j^\mu\right], \tag{17}$$

where $0 \leqq r_\mu < 2\pi R$, $t_{j\mu} \in \mathbf{R}$, $j = 1, \dots, n_\mu - 1$, $\mu = 1, \dots, D$. The addition of the two elements of the form (17) is defined by the usual addition of the corresponding real numbers $t_{j\mu}$ and by the addition of the corresponding numbers r_μ as the elements of the quotient group $\mathbf{R}/2\pi R\mathbf{Z}$, namely $r_\mu \uplus r'_\mu = r_\mu + r'_\mu$ if $r_\mu + r'_\mu < 2\pi R$ and $r_\mu \uplus r'_\mu$

$=r_\mu+r'_\mu-2\pi R$ otherwise. Let us introduce the Haar measure on the group L_n,

$$d\left[r_\mu+\sum_{j=1}^{n_\mu-1} t_{j\mu}X_j^\mu(x)\right]=\prod_{\mu=1}^{D}\left[(2\pi R)^{-1}dr_\mu\left(\prod_{j=1}^{n_\mu-1} dt_{j\mu}\right)\right]. \tag{18}$$

Two Haar measures on the group L_n differ from each other by a constant multiplier only. The substitution of the expressions (17) and (18) into the integral (12) gives a Gauss integral. As a consequence we obtain

$$Z_n=\prod_{\mu=1}^{D}\left[(2\pi\alpha^2)^{(n_\mu-1)/2}(\det\{(dX_i^\mu,dX_j^\mu)\})^{-1/2}\right]. \tag{19}$$

The differential 1-forms dX_j^μ, $j=1,\dots,n_\mu-1$ are linearly independent. In fact, otherwise there exist the real numbers such that $\sum_{j=1}^{n_\mu-1}\lambda_j dX_j^\mu=0$. The kernel of the differential operator d coincides with the space of the constant functions on the surface M. Thus there exists a real number λ_0 such that $\sum_{j=0}^{n_\mu-1}\lambda_j X_j^\mu(x)=0$ which contradicts the assumption that the functions $X_0^\mu(x),\dots,X_{n_\mu-1}(x)$ are linearly independent. Therefore, the expression (19) is not zero. We may choose the functions $X_0^\mu(x),\dots,X_{n_\mu^\mu-1}(x)$ in such a way that the expression (19) takes any given nonzero value. For example, let us assume that $\alpha^2=(2\pi)^{-1}$, $(X_0^\mu,X_j^\mu)=0$ for $j=1,\dots,n_\mu-1$ and $\det\{(X_i^\mu,X_j^\mu)\}_{i,j=0,\dots,n_\mu-1}=1$. Then $\det\{(X_i^\mu,X_j^\mu)\}_{i,j=1,\dots,n_\mu-1}=(X_0^\mu,X_0^\mu)^{-1}$ and the definition of the Laplace-Beltrami operator implies

$$Z_n=\prod_{\mu=1}^{D}\left(\frac{\det\{(X_i^\mu,\Delta X_j^\mu)\}_{i,j=1,\dots,n_\mu-1}}{\det\{(X_i^\mu,X_j^\mu)\}_{i,j=1,\dots,n_\mu-1}}\right)^{-1/2}(X_0^\mu,X_0^\mu)^{1/2}. \tag{20}$$

The constant function X_0^μ is a zero mode of the Laplace-Beltrami operator on the functions. Hence the limit (13) of the expression (20) coincides with the result [1, 2].

The substitution of (17), (18) into the right-hand side of the definition (11) yields

$$\begin{aligned}1/Z_n\prod_{\mu=1}^{D}(2\pi R)^{-1}\int_0^{2\pi R}dr_\mu\int_{\mathbf{R}^{n_\mu-1}}d^{n_\mu-1}t\exp[ir_\mu(Y^\mu,1)]\\ \times\exp\left[i\sum_{j=1}^{n_\mu-1}t_{j\mu}(Y^\mu,X_j^\mu)-1/2\alpha^2\sum_{j,k=1}^{n_\mu-1}t_{j\mu}t_{k\mu}(dX_j^\mu,dX_k^\mu)\right].\end{aligned} \tag{21}$$

If one of the numbers $(Y^\mu,1)\in(R)^{-1}\mathbf{Z}$ is not equal to zero the integral (21) equals zero and the relation (15) is proved.

Let $(Y^\mu,1)=0$ for all $\mu=1,\dots,D$. Then for all $\mu=1,\dots,D$ the first equation (14) implies that

$$Y^\mu=d^*dGY^\mu. \tag{22}$$

Substituting the equalities (22) and $(Y^\mu,1)=0$ into the integral (21) and computing integrals with respect to the variables r_μ we have

$$1/Z_n\prod_{\mu=1}^{D}\left[\int_{\mathbf{R}^{n_\mu-1}}d^{n_\mu-1}t\exp\left[i\sum_{j=1}^{n_\mu-1}t_{j\mu}(dGY^\mu,dX_j^\mu)-1/2\alpha^2\sum_{j,k=1}^{n_\mu-1}t_{j\mu}t_{k\mu}(dX_j^\mu,dX_k^\mu)\right]\right].$$

By using this integral in the definition (11) we obtain the definition of a correlation function from the paper [7]. Due to [7] the correlation function is independent of a choice of the subgroups L_n and the Haar measures on them. It is equal to (16) [7].

Thus for the partially $U(1)$ compactified scalar massless field theory with the Nambu-Goto action (9) the correlation functions (16) are the unique correctly defined objects. The amplitude must be constructed from the correlation functions (16). The correlation function (16) has a simple geometrical meaning. Let $D^p(T^*M)$ denote the space of the smooth differential p-forms on the surface M. If a function $Y(x)$ on the surface M satisfies the equation $(Y, 1)=0$ the relation (22) implies that $Y(x)=d^*\omega(x)$, where $\omega(x)$ is a differential 1-form on the surface M and d^* is the adjoint operator of the differential operator d with respect to the inner products (5), (10). Let us establish the following relation:

$$(Y, GY)=\inf_{\substack{\omega\in D^1(T^*M)\\ d^*\omega=Y}}(\omega,\omega). \tag{23}$$

Hence the left-hand side of the relation (23) is the minimal "length" of a differential 1-form whose "boundary" coincides with the function $Y(x)$ on the surface M. To prove the relation (23) we introduce the differential operator $d: D^1(T^*M) \to D^2(T^*M)$ in the following way $d\left(\sum_j \alpha_j(x)dx^j\right)=\left(\frac{\partial\alpha_2}{\partial x^1}-\frac{\partial\alpha_1}{\partial x^2}\right)dx^1\wedge dx^2$. Let us define on the space $D^2(T^*M)$ the inner product

$$(\alpha, \beta=\int_M \alpha_{12}(x)\beta_{12}(x)(\det g_{ij}(x))^{-1/2}d^2x. \tag{24}$$

Let d^* be an operator on the space $D^2(T^*M)$ which is adjoint of the differential operator d with respect to the inner products (10), (24). The operator $\Delta\equiv d^*d+dd^*$ is called the Laplace-Beltrami operator on the differential 1-forms on the surface M. The differential 1-form ω is said to be harmonic if it satisfies the equation $\Delta\omega=0$. Due to the Hodge theorem [6, Sect. 31, Corollaire 4] for a compact smooth connected orientable surface provided with a Riemannian metric the dimension of the space of the harmonic 1-forms coincides with the number of the generators of the homology group $H_1(M, \mathbf{R})$. Any harmonic 1-form belongs to the space $D^1(T^*M)$ [6, Sect. 29, Corollaire 1]. Let H denote the orthogonal with respect to the inner product (10) projector on the space of the harmonic 1-forms. Let G be a linear operator on the Hilbert space of the differential 1-forms endowed with the inner product (10) and G satisfy the equations of the form (14). G is called a Green operator for the Laplace-Beltrami operator on the differential 1-forms on the surface M. The definitions of the operators Δ and G imply the decomposition of the differential 1-form

$$\omega=dd^*G\omega+d^*dG\omega+H\omega. \tag{25}$$

By [6, Sect. 26, Théorème 20] any harmonic 1-form α on the surface M is closed and co-closed, i.e. $d\alpha=0$ and $d^*\alpha=0$. Since $d^2=0$ and $(d^*)^2=0$ all terms in the decomposition (25) are orthogonal to each other. The operators G and d^* are commuting [6, Sects. 31, 33]. Now the decomposition (25) implies the equality (23).

Let $h(x, y)$ be a smooth function of the variables $x, y\in M$. We assume that for every $y\in M$, $(h(\bullet, y), 1)=1$. In order to construct the amplitude we choose a set of the momentum vectors k_j^μ, $j=1,\dots,N$, $\mu=1,\dots,D$, satisfying the conditions

$$\sum_{j=1}^N k_j^\mu=0, \quad \mu=1,\dots,D. \tag{26}$$

Instead of the function $v_1(x_1)\dots v_N(x_N)$ in the integral (3) we introduce a smooth distribution function $f(x_1,\dots,x_N)$ which vanishes with all its derivatives when $x_i=x_j$.

Definition 2.2. *The scattering amplitude of N tachyons with the masses $m_1,\dots,m_N$ and the distribution function $f(x_1,\dots,x_N)$ on the surface M is the following limit:*

$$\begin{aligned}&\int_{M^{\times N}} A_N(k|x)f(x)\prod_{l=1}^{N}(\det g_{ij}(x_l))^{1/2}d^2x_l\\ &= \lim_{h(x,y)\to(\det g_{ij}(x))^{-1/2}\delta(x,y)} \int_{M^{\times N}}\left(\prod_{l=1}^{N}(\det g_{ij}(x_l))^{1/2}d^2x_l\right)\\ &\quad\times f(x)\exp\left[\alpha^2/2\sum_{j=1}^{N}m_j^2(h(\bullet,x_j),Gh(\bullet,x_j))\right]W\left(\sum_{j=1}^{N}k_j^\mu h(\bullet,x_j)\right),\end{aligned}\tag{27}$$

where $W\left(\sum_{j=1}^{N}k_j^\mu h(\bullet,x_j)\right)$ is a correlation function (16) *of the partially $U(1)$ compactified scalar massless field theory on the surface M and G is the Green operator for the Laplace-Beltrami operator on the functions on the surface M.*

The last two equations (14) imply that $(h(\bullet,x_j), Gh(\bullet,x_j))=((h(\bullet,x_j)-(1,1)^{-1}), G(h(\bullet,x_j)-(1,1)^{-1}))$. Now the relation (23) provides a geometrical meaning of the complementary multiplier which distinguishes the expression (27) from the integral (3) for the fixed surface M.

We assume that the Green operator for the Laplace-Beltrami operator on the functions on the surface M is an integral operator

$$G\phi(x)=\int_M G(x,y)\phi(y)(\det g_{ij}(y))^{1/2}d^2y.\tag{28}$$

The kernel $G(x,y)$ of the Green operator is called the Green's function.

Proposition 2.2. *Let for a compact smooth connected orientable surface M endowed with a Riemannian metric a Green's function $G(x,y)$ be a continuous function on $M\times M$ except for the diagonal points where it has the singularity $-1/2\pi\log|x-y|$. Then the scattering amplitude* (27) *of N tachyons with the masses $m_1,\dots,m_N$ and the distribution function $f(x_1,\dots,x_N)$ on the surface M equals*

$$\begin{cases}0 & \text{if}\quad (k_j,k_j)>m_j^2, \quad j=1,\dots,N,\\ \infty & \text{if}\quad (k_j,k_j)<m_j^2, \quad j=1,\dots,N.\end{cases}\tag{29}$$

If

$$(k_j,k_j)=m_j^2,\qquad j=1,\dots,N,\tag{30}$$

then the scattering amplitude (27) *equals*

$$\int_{M^{\times N}} f(x)\left(\prod_{l=1}^{N}(\det g_{ij}(x_l))^{1/2}d^2x_l\right)\prod_{1\leq i<j\leq N}(\exp[-4\pi G(x_i,x_j)])^{\alpha^2(k_i,k_j)/4\pi}.\tag{31}$$

Proof. The distribution function $f(x_1,\dots,x_N)$ vanishes with all its derivatives when $x_i=x_j$. Now taking into account the expressions (16), (27) and the explicit form of the singularities of the Green's function $G(x,y)$ it is easy to prove the equalities (29), (31).

Remark 2.1. The amplitude (31) has the same form for all masses $m_1, \ldots, m_N$. Therefore, it is possible to consider the expression (31) as a scattering amplitude for N particles in the Euclidean space. The analytic continuation in the momentum variables k_j^μ of the expression (31) gives a scattering amplitude for the N particle in the Minkowski space.

Remark 2.2. The amplitude $A_N(k_1, \ldots, k_N | x_1, \ldots, x_N)$ in (31) has the singularities of the form $|x_i - x_j|^\lambda$, where x_i, x_j are the two-dimensional vectors. In the spherical coordinates these singularities have the form x_+^λ, where x is the norm of the vector $x_i - x_j$. When the variables are rightly chosen in the integral (31) allows the continuation for the functions $f(x_1, \ldots, x_N)$ which do not vanish when $x_i = x_j$. One of these continuations is the analytic continuation in the variables (k_i, k_j). Due to [8, Chap. 1, Sect. 3.2] the generalized function x_+^λ is a holomorphic function of the variable λ except for the points $\lambda = -k$, $k = 1, 2, \ldots$, where it has the simple poles with the residues $(-1)^k((k-1)!)^{-1}\delta^{(k-1)}(x)$. Applying this result it is easy to compute [8, Chap. 1, Sect. 3.8] the poles and the residues of the beta function $B(\lambda, \mu)$ and consequently of the Veneziano amplitude. It seems reasonable that the amplitude (31) has in the variables (k_i, k_j) similar singularities.

Remark 2.3. The amplitude (31) is constructed from the functions $\exp[-4\pi G(x, y)]$. These functions have simple geometrical meaning [9]. Let us introduce the complex structure on the surface M. With the local coordinates (x^1, x^2) and with the Riemannian metric the following function

$$\mu(z) = \mu(x^1 + ix^2) = (g_{11} - g_{22} + 2ig_{12})(g_{11} + g_{22} + 2(g_{11}g_{22} - g_{12}^2)^{1/2})^{-1} \quad (32)$$

is related. The function f is said to be holomorphic if it satisfies the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}. \quad (33)$$

By [10, Chap. 1, Theorem 4.3] Eqs. (32) and (33) define the structure of a Riemann surface on the smooth connected orientable surface M endowed with the Riemannian metric. A Riemann surface is a one complex dimensional connected complex analytic manifold M with a maximal set of charts $\{U_\alpha, z_\alpha\}_{\alpha \in A}$ on M. The set $\{U_\alpha\}_{\alpha \in A}$ constitutes an open cover of M and a map $z_\alpha : U_\alpha \to \mathbf{C}$ is a homeomorphism onto an open subset of the complex plane $\mathbf{C}$ such that the transition functions $z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$ are holomorphic whenever $U_\alpha \cap U_\beta \neq \emptyset$. Let $\pi : E \to M$ be a linear holomorphic fibre bundle over the Riemann surface M [11, Chap. 1, Definitions 2.1, 2.2]. Let a function h define a hermitian metric on the fibre bundle $\pi : E \to M$ [11, Chap. 3, Definition 1.1]. The hermitian metric induces a canonical holomorphic connection D compatible with this metric [11, Chap. 3, Theorem 2.1]. The curvature form $\theta(D)$ of this connection is equal to $-\frac{\partial^2}{\partial z \partial \bar{z}} \log h dz \wedge d\bar{z}$ [11, Chap. 3, Theorem 2.1, Proposition 2.2]. The first Chern form for the linear fibre bundle $\pi : E \to M$ endowed with a connection D is the 2-form $c_1(E, D) = (i/2\pi)\theta(D) = (1/2\pi i)\frac{\partial^2}{\partial z \partial \bar{z}} \log h dz \wedge d\bar{z}$ [11, Chap. 3, Definition 3.4]. By the relations (32) and (33) in the holomorphic coordinates the Riemannian metric has the following form: $g_{xy} = 0$, $g_{xx} = g_{yy} = \varrho > 0$. Now using the inner products (5), (10) it is easy to calculate explicitly the Laplace-Beltrami operator on the functions: $\Delta = -4\varrho^{-1}\frac{\partial^2}{\partial z \partial \bar{z}}$. The equations (14) are equivalent to the following equations for the Green's function $G(z,\omega)$ [see (28)]:

$$\begin{aligned}
&-4\frac{\partial^2}{\partial z\partial\bar{z}}G(z,w)=\delta(z,w)-(\det g_{ij}(z))^{1/2}(\mathrm{vol}(M))^{-1},\\
&-4\frac{\partial^2}{\partial w\partial\bar{w}}G(z,w)=\delta(z,w)-(\det g_{ij}(w))^{1/2}(\mathrm{vol}(M))^{-1},\\
&\int_M G(z,w)(\det g_{ij}(z))^{1/2}(i/2)dz\wedge d\bar{z}=0,\\
&\int_M G(z,w)(\det g_{ij}(w))^{1/2}(i/2)dw\wedge d\bar{w}=0,
\end{aligned}\tag{34}$$

where

$$\mathrm{vol}(M)=\int_M(\det g_{ij}(z))^{1/2}(i/2)dz\wedge d\bar{z}. \tag{35}$$

Choose the right-hand side of the first equation (34) multiplied by the 2-form $(1/2i)dz\wedge d\bar{z}$ as the first Chern form. Comparing this first Chern form with the first Chern form $c_1(E,D)$ we obtain that the function $\exp[-4\pi G(z,w)]$ defines a hermitian metric on the linear holomorphic fibre bundle over M satisfying the third equation (34) and having a zero of order two at the point $z=w$. [We assume that the Green's function $G(z,w)$ has the singularity $-1/2\pi\log|z-w|$.] Therefore, the function $\exp[-4\pi G(z,w)]$ defines a hermitian metric on the sheaf $\mathcal{O}(w)$. The sheaf $\mathcal{O}(w)$ is determined by its local sections. If the open set U contains the point $w\in M$ the local sections $\mathcal{O}(w)(U)$ are the functions analytic on U except for a possible pole of first order at w; otherwise the local sections $\mathcal{O}(w)(U)$ are the functions analytic on U.

The topological model of a compact connected orientable surface M is a two-dimensional sphere or a polygon whose sides are identified according to $A_1B_1A_1^{-1}B_1^{-1}\dots A_gB_gA_g^{-1}B_g^{-1}$, $g=1,2,\dots$ [12, p. 17]. In the former case we say that the genus of M is zero and in the latter case we say that the genus is g. The sides of the polygon give a basis for the homology group $H_1(M,\mathbf{Z})$ [12, p. 18]. Due to [13, Sect. 0.4] the intersection number of two cycles on the Riemann surface is defined in the following way. If two cycles A and B intersect transversally at the point P the local intersection number $(A\bullet B)_P$ is equal to $+1$ if the tangent vectors for A and B provide the basis for the tangent space at $P\in M$. In the case of inverse orientation $(A\bullet B)_P=-1$. If the intersection of the cycles A and B is not transversal at the point P the local intersection number $(A\bullet B)_P=0$. The intersection number $A\bullet B$ is the sum of the local intersection numbers. In the case of a compact Riemann surface of genus g the cycles $A_1,\dots,A_g$, $B_1,\dots,B_g$ have the following intersection numbers [12, p. 54]:

$$A_i\bullet A_j=B_i\bullet B_j=0,\qquad A_i\bullet B_j=\delta_{ij}. \tag{36}$$

In the forthcoming sections we study the Green's functions and the amplitudes (31) for the compact Riemann surfaces of arbitrary genus.

3. Riemann Sphere

Every compact simply connected Riemann surface M is conformally equivalent to the Riemann sphere $\mathbf{CP}^1 \equiv \mathbf{C} \cup \infty$ [12, Theorem 4.4.1]. It is homeomorphic to the unit sphere $S^2 \subset \mathbf{R}^3$. The Euclidean metric on $\mathbf{R}^3$ and the stereographic projection $S^2 \setminus \{(0,0,1)\}$ onto $\mathbf{C}$ induces the Riemannian metric on $\mathbf{C}$

$$ds^2 = (\pi(1+|z|^2)^2)^{-1}(dx^2+dy^2). \tag{37}$$

It is invariant under the substitution $z \to z^{-1}$ and, consequently, induces the Riemannian metric on $\mathbf{CP}^1$. The coefficient in (37) is chosen in such a way that $\mathrm{vol}(\mathbf{CP}^1)=1$ [see (35)].

Proposition 3.1. *The function*

$$\begin{aligned} G(z,w) = & -1/2\pi \log|z-w| \\ & + i/(2\pi)^2 \int_{\mathbf{CP}^1} (\log|z-z_1| + \log|w-z_1|) \frac{dz_1 \wedge d\bar{z}_1}{(1+|z_1|^2)^2} \\ & + 1/(2\pi)^3 \int_{(\mathbf{CP}^1)^{\times 2}} (\log|z_1-w_1|) \frac{dz_1 \wedge d\bar{z}_1 \wedge dw_1 \wedge d\bar{w}_1}{(1+|z_1|^2)^2(1+|w_1|^2)^2} \end{aligned} \tag{38}$$

satisfies Eqs. (34) *on* $\mathbf{CP}^1$ *endowed with the Riemannian metric* (37). *The first term in the right-hand side of the equality* (38) *determines the singularity of the function* $G(z,w)$.

Proof. By using the substitution $z \to z^{-1}$ it is easy to show that

$$\int_{\mathbf{CP}^1} (\log|z|)(1+|z|^2)^{-2} dz \wedge d\bar{z} = 0. \tag{39}$$

Applying this relation it is possible to prove the following equalities

$$G(z^{-1}, w^{-1}) = G(z,w), \tag{40}$$

$$\begin{aligned} G(z^{-1},w) = & -1/2\pi \log|1-zw| \\ & + i/(2\pi)^2 \int_{\mathbf{CP}^1} (\log|1-zz_1| - \log|1-wz_1|) \frac{dz_1 \wedge d\bar{z}_1}{(1+|z_1|^2)^2} \\ & + 1/(2\pi)^3 \int_{(\mathbf{CP}^1)^{\times 2}} (\log|1-z_1w_1|) \frac{dz_1 \wedge d\bar{z}_1 \wedge dw_1 \wedge d\bar{w}_1}{(1+|z_1|^2)^2(1+|w_1|^2)^2}. \end{aligned} \tag{41}$$

The relations (40), (41) imply that the function $G(z,w)$ is defined on $\mathbf{CP}^1 \times \mathbf{CP}^1$.
Two first equalities (34) for the $\mathbf{CP}^1$ follow from the equality

$$4\frac{\partial^2}{\partial z \partial \bar{z}} \log|z-w| = 2\pi\delta(z-w)\delta(\bar{z}-\bar{w}). \tag{42}$$

The last two equalities (34) are verified immediately. The singularity of the function $G(z,w)$ is defined by the first term in the right-hand side of the equality (38) since the subsequent terms are continuous functions of the local coordinates z, w.

Thus the Green's function (38) satisfies all conditions of the Proposition 2.2 and it is possible to define the scattering amplitude (31) for $\mathbf{CP}^1$. To compare the obtained amplitude with the usual one we need to choose the parameters and a distribution function $f(x_1, \ldots, x_N)$ such that three terms except for the first in the right-hand side of Eq. (38) are compensated. The first term in Eq. (38) is the Green's function for the Laplace-Bertrami operator on the functions on the complex plane **C**.

We assume $\alpha^2=\pi$ and introduce a function

$$\Phi_{m^2}(z)=(\pi(1+|z|^2)^2)^{-1}\exp\left[im^2/4\pi \int_{\mathbf{CP}^1}(\log|z-w|)\frac{dw\wedge d\bar{w}}{(1+|w|^2)^2}\right]. \tag{43}$$

It follows from the relations (26), (30), (37), (38), and (43) that the expression for the scattering amplitude (31) may be rewritten as

$$C\int_{(\mathbf{CP}^1)^{\times N}}\left(\prod_{j=1}^{N}\Phi_{m^2}(z_j)(i/2)dz_j\wedge d\bar{z}_j\right)f(z)\prod_{1\leq i<j\leq N}|z_i-z_j|^{(k_i,k_j)/2}. \tag{44}$$

It is easy to verify that

$$\Phi_{m^2}(z^{-1})=|z|^{4-m^2/2}\Phi_{m^2}(z). \tag{45}$$

Hence the function $\Phi_{m^2}(z)$ on the Riemann sphere $\mathbf{CP}^1$ is smooth only for $m^2=0,4,8$. The relation (45) implies that $\Phi_0(z)\approx|z|^{-4}$ and $\Phi_4(z)\approx|z|^{-2}$ as $z\to\infty$. Now the Proposition 3.1 shows that for $m_j^2=0,4$ the last multiplier in the integrand (44) has the non-integrable singularity at the point $z_j=\infty$. Thus the unique possibility to remove the functions $\Phi_{m^2}(z_j)$ in (44) by means of a choice of the distribution function $f(z_1,\dots,z_N)$ is to fix the tachyon masses $m_1=\dots=m_N=8$. However, if now the function $f(z_1,\dots,z_N)\prod_{j=1}^{N}\Phi_8(z_j)$ is taken to be constant the integral (44) diverges since the integrand (44) is invariant under the linear fractional transformations of $\mathbf{CP}^1$ with the complex coefficients. The linear fractional transformation which leaves fixed three distinct points of the Riemann sphere $\mathbf{CP}^1$ is the identity transformation. Let us fix the points $z_1=0$, $z_2=1$, $z_3=\infty$ and take a function

$$f(z)=\delta(z_1)\delta(\bar{z}_1)\delta(z_2-1)\delta(\bar{z}_2-1)\delta(z_3^{-1})\delta(\bar{z}_3^{-1})\left(\prod_{j=4}^{N}\Phi_8(z_j)\right)^{-1} \tag{46}$$

in the integral (44). Then we obtain the scattering amplitude for the closed bosonic strings [3, Vol. 1, formula (1.4.13)].

To obtain the scattering amplitude for the open bosonic strings we need to restrict the integration over the Riemann sphere in (31) to the integration over the real axis. A simple restriction to the real axis gives the generalized functions of the type $|x_i-x_j|^\lambda$. Due to [8, Chap. 1, Sect. 3.3] the generalized functions $|x|^\lambda$ and x_+^λ have the different pole singularities. Therefore, taking into the amplitude (31) for $\mathbf{CP}^1$ the coupling constant $\alpha^2=2\pi$, the tachyon masses $m_1^2=\dots=m_N^2=4$ and the distribution function

$$\begin{aligned}f(z)=&\,\delta(z_1)\delta(\bar{z}_1)\delta(z_2-1)\delta(\bar{z}_2-1)\delta(z_3^{-1}\delta(\bar{z}_3^{-1})\left(\prod_{j=4}^{N}\Phi_8(z_j)\right)^{-1}\\&\times\left(\prod_{j=4}^{N}\delta(y_j)\right)\theta(x_4)\left(\prod_{j=4}^{N-1}\theta(x_{j+1}-x_j)\right)\theta(1-x_N),\end{aligned}$$

where $z_j=x_j+iy_j$, we obtain the N-point Koba-Nielsen generalization for the Veneziano amplitude [3, Vol. 1, formula (1.5.11)].

4. Torus

Every smooth compact connected orientable surface M of genus 1 is homeomorphic to the parallelogram whose sides are identified according to $ABA^{-1}B^{-1}$ [12, p. 17]. In particular, all vertices of the parallelogram are identified. Using the common vertex as a base point for the fundamental group, one shows that $\pi_1(M)$ is generated by the closed loops A and B subject to the single relation $ABA^{-1}B^{-1}=1$. Hence $\pi_1(M)$ is the free abelian group isomorphic to $\mathbf{Z}\oplus\mathbf{Z}$. In view of [12, Theorem 4.6.1] a Riemann surface M is conformally equivalent to a torus $\mathbf{T}(1,\tau)\equiv\mathbf{C}/\Gamma(1,\tau)$, where the group $\Gamma(1,\tau)$ is generated by the shifts $z\to z+1$ and $z\to z+\tau$, $\operatorname{Im}\tau>0$. Choose on the torus $\mathbf{T}(1,\tau)$ the Riemannian metric

$$ds^2=(\operatorname{Im}\tau)^{-1}(dx^2+dy^2) \tag{47}$$

and the corresponding canonical 2-form

$$\varrho_{\mathbf{T}(1,\tau)}(z)=(\operatorname{Im}\tau)^{-1}(i/2)dz\wedge d\bar{z}. \tag{48}$$

It is quite obvious that $\operatorname{vol}(\mathbf{T}(1,\tau))=1$ [see (35)]. The smooth functions on the torus $\mathbf{T}(1,\tau)$ are equivalent to the smooth functions on the complex plane $\mathbf{C}$ which are invariant under the transformations from the group $\Gamma(1,\tau)$. Let $D^0(T^*\mathbf{T}(1,\tau))$ denote the space of the smooth functions on the torus $\mathbf{T}(1,\tau)$. The spaces $D^p(T^*\mathbf{T}(1,\tau))$, $p=1,2$, of the smooth differential p-forms are defined in a similar way. Let us introduce the mapping $\alpha_\tau:\mathbf{T}(1,i)\to\mathbf{T}(1,\tau)$ and its inverse mapping

$$\begin{aligned}\alpha_\tau(x+iy)&=x+\tau y,\\ \alpha_\tau^{-1}(x+iy)&=(\operatorname{Im}\tau)^{-1}(\operatorname{Im}(\tau(x-iy))+iy).\end{aligned} \tag{49}$$

Since the functions from $D^0(T^*\mathbf{T}(1,\tau))$ are invariant under the shifts $z\to z+1$ and $z\to z+\tau$ there exists the Fourier expansion

$$\phi(z)=\sum_{m\in(\mathbf{Z})^2}\exp[-2\pi i\operatorname{Re}[(m_1-im_2)\alpha_\tau^{-1}(z)]]\tilde{\phi}(m). \tag{50}$$

The differential operator d on the complex plane $\mathbf{C}$ induces the differential operator d on the space $D^p(T^*\mathbf{T}(1,\tau))$. It follows from the definitions (5), (10), and (47) that the Laplace-Beltrami operator on the functions on the torus $\mathbf{T}(1,\tau)$ has the form $\Delta=-4(\operatorname{Im}\tau)\frac{\partial^2}{\partial z\partial\bar{z}}$. Hence the Green operator G satisfying Eqs. (14) acts on the function (50) in the following way:

$$G\phi(z)=\sum_{\substack{m\in(\mathbf{Z})^2\\ m_1^2+m_2^2\neq 0}}\exp[-2\pi i\operatorname{Re}[(m_1-im_2)\alpha_\tau^{-1}(z)]]\frac{(\operatorname{Im}\tau)\tilde{\phi}(m)}{(2\pi)^2|m_1\tau-m_2|^2}. \tag{51}$$

Proposition 4.1. *The Green's function for the Laplace-Beltrami operator on the functions on the torus* $\mathbf{T}(1,\tau)$ *is equal to* $G(z,w)=G_0(\alpha_\tau^{-1}(z),\alpha_\tau^{-1}(w);\tau)$. *For every variable* $z,w\in\mathbf{T}(1,i)$ *the function*

$$G_0(z,w;\tau)=\sum_{\substack{m\in\mathbf{Z}^2\\ m_1^2+m_2^2\neq 0}}\cos[2\pi\operatorname{Re}[(m_1-im_2)(z-w)]]\frac{\operatorname{Im}\tau}{(2\pi)^2|m_1\tau-m_2|^2} \tag{52}$$

belongs to Hilbert space of the functions on the torus $\mathbf{T}(1,i)$ *endowed with the inner product* (5) *corresponding to the Riemannian metric* (47). *For any matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}) \tag{53}$$

the function $G_0(z,w;\tau)$ *satisfies the relation*

$$G_0\left(x+iy,u+iv;\frac{a\tau+b}{c\tau+d}\right)=G_0(by+dx+i(ay+cx),bv+du+i(av+cu);\tau)\,. \tag{54}$$

Proof. It follows from the definition (28) and the equality (51) that the Green's function for the Laplace-Beltrami operator on the functions on a torus $\mathbf{T}(1,\tau)$ is $G_0(\alpha_\tau^{-1}(z),\alpha_\tau^{-1}(w);\tau)$, where the function $G_0(z,w;\tau)$ is given by Eq. (52). The definitions (5), (47), and (52) imply that for every $w\in\mathbf{T}(1,\tau)$,

$$(G_0(\bullet,w;\tau),G_0(\bullet,w;\tau))=\sum_{\substack{m\in(\mathbf{Z})^2\\ m_1^2+m_2^2\neq 0}}\frac{(\operatorname{Im}\tau)^2}{(2\pi)^4|m_1\tau-m_2|^4}\,. \tag{55}$$

The series in the right-hand side of this equality is absolutely convergent. The analogous equality holds for another argument $z\in\mathbf{T}(1,i)$. To prove the relation (54) we note that

$$\left|m_1\frac{a\tau+b}{c\tau+d}-m_2\right|^{-2}\operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right)=|(am_1-cm_2)\tau-(-bm_1+dm_2)|^{-2}\operatorname{Im}\tau\,,$$

$$\begin{pmatrix} dm_2-bm_1 \\ -cm_2+am_1 \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} m_2 \\ m_1 \end{pmatrix}.$$

Now the replacement of the summation variables in the right-hand side of Eq. (52) yields the relation (54).

In order to study the singularities of the Green's function we introduce the theta functions

$$\theta(z,\tau)=\sum_{n\in\mathbf{Z}}\exp[\pi in^2\tau+2\pi inz]\,, \tag{56}$$

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(z,\tau)=\exp[i\pi\tau/4+i\pi z+i\pi/2]\theta(z+(1+\tau)/2,\tau)\,, \tag{57}$$

where $z\in\mathbf{C}$ and $\tau\in H_1$. We denote the upper half plane by H_1. The function (56) is called the Riemann's theta and the function (57) is called the first order theta function with integer characteristic $1,1$.

Proposition 4.2. *The function*

$$f(z)=\left|\theta\begin{bmatrix}1\\1\end{bmatrix}(z,\tau)\right|\exp[-\pi(\operatorname{Im}\tau)^{-1}(\operatorname{Im}z)_2] \tag{58}$$

is invariant under the transformations from the group $\Gamma(1,\tau)$ *and induces a function on the torus* $\mathbf{T}(1,\tau)$. *The function*

$$\begin{aligned}G(z,w)={}&-1/2\pi\log f(z-w)\\&+1/2\pi\int_{\mathbf{T}(1,\tau)}\varrho_{\mathbf{T}(1,\tau)}(z_1)(\log f(z-z_1)+\log f(z_1-w))\\&-1/2\pi\int_{\mathbf{T}(1,\tau)^{\times 2}}\varrho_{\mathbf{T}(1,\tau)}(z_1)\wedge\varrho_{\mathbf{T}(1,\tau)}(w_1)\log f(z_1-w_1)\end{aligned} \tag{59}$$

satisfies Eqs. (34) *for the torus* $\mathbf{T}(1,\tau)$ *endowed with the Riemannian metric* (47). *The function* $G(z,w)$ *is continuous everywhere on* $\mathbf{T}(1,\tau)\times\mathbf{T}(1,\tau)$ *except for the diagonal points where it has the singularity* $-1/2\pi\log|z-w|$.

Proof. In view of [12, Chap. 6, formula (1.4.6)] the function (58) is invariant under the transformations from the group $\Gamma(1,\tau)$. Hence it induces a function on the torus $\mathbf{T}(1,\tau)$. The function $(\operatorname{Im} z)$ is locally harmonic on the torus $\mathbf{T}(1,\tau)$. The theta function (57) is locally holomorphic on the torus $\mathbf{T}(1,\tau)$ and it vanishes at the point $z=0$ [12, Proposition 6.1.5]. It follows from the Riemann theorem [14, Chap. 2, Theorem 3.1] that the point $z=0$ is the first order zero and the function $\theta\begin{bmatrix}1\\1\end{bmatrix}(z,\tau)$ has no other zeros. Now the equality (42) implies that the function (59) satisfies the first and the second equations (34) for the torus $\mathbf{T}(1,\tau)$ endowed with the Riemannian metric (47). The function (59) is continuous on $\mathbf{T}(1,\tau)\times\mathbf{T}(1,\tau)$ except for the zeros of the function $\theta\begin{bmatrix}1\\1\end{bmatrix}(z-w,\tau)$, i.e. the diagonal points, where it has the singularity $-1/2\pi\log|z-w|$. The last two equations (34) for the torus $\mathbf{T}(1,\tau)$ are verified immediately.

Since every harmonic function on the torus is constant, the solution of Eqs. (34) is unique. Hence the function (59) coincides with the function $G(z,w)$ defined in Proposition 4.1. Proposition 4.2 shows that the conditions of Proposition 2.2 are satisfied. Take in the scattering amplitude (31) for the torus $\mathbf{T}(1,\tau)$ the distribution function $f(\alpha_\tau^{-1}(z_1),\dots,\alpha_\tau^{-1}(z_N);\tau)$, where the function $f(z_1,\dots,z_N;\tau)$ is smooth and it has a compact support in the variable τ. We use the change (49) of the variables in the integral (31) and integrate it over the upper half plane H_1 with the measure $(\operatorname{Im}\tau)^{-2}(i/2)d\tau\wedge d\bar{\tau}$. Then we have

$$\int_{H_1}(\operatorname{Im}\tau)^{-2}(i/2)d\tau\wedge d\bar{\tau}\int_{\mathbf{T}(1,\tau)^{\times N}}\left(\prod_{j=1}^{N}(i/2)dz_j\wedge d\bar{z}_j\right)A_N(k|z,\tau)f(z;\tau),\tag{60}$$

$$A_N(k|z,\tau)=\exp\left[-\alpha^2\sum_{1\leqq i<i\leqq N}(k_i,k_j)G_0(z_i,z_j;\tau)\right].\tag{61}$$

The relation (54) implies that for every matrix (53)

$$\begin{aligned}&A_N\left(k|x_1+iy_1,\dots,x_N+iy_N;\frac{a\tau+b}{c\tau+d}\right)\\&=A_N(k|by_1+dx_1+i(ay_1+cx_1),\dots,by_N+dx_N+i(ay_N+cx_N);\tau).\end{aligned}\tag{62}$$

The analytic regularization of the amplitude (61) in the parameters (k_i,k_j) gives the amplitude satisfying also the relation (62).

The geometric interpretation of the relation (62) is very simple. Due to [12, Sect. 4.7.3] two tori $\mathbf{T}(1,\tau_1)$ and $\mathbf{T}(1,\tau_2)$ are conformally equivalent if and only if $\tau_2=M(g)(\tau_1)=\dfrac{a\tau_1+b}{c\tau_1+d}$, where g is the matrix (53). The group of all such linear fractional transformations of the upper half plane H_1 is isomorphic to the modular group $SL(2,\mathbf{Z})/\{\pm I\}$. On the other hand, every matrix (53) defines the homeomorphism $x+iy\to by+dx+i(ay+cx)$ of the torus $\mathbf{T}(1,i)$ onto itself. The equality (62) shows that the amplitudes for the conformally equivalent tori are related to each other by means of the corresponding homeomorphism of the torus $\mathbf{T}(1,i)$.

The domain F in the upper half plane H_1 is said to be fundamental for the modular group $SL(2,\mathbf{Z})/\{\pm I\}$ if for every orbit of this group at least one element lies into the closure of domain F and two elements of the orbit belong to the closure of F only if they belong to the boundary of F. It follows from [15, Chapitre 7, Théorème 1] that the domain $F=\{z\in H_1: |z|>1, |\mathrm{Re}\,z|<1/2\}$ in the upper half plane H_1 is fundamental for the modular group $SL(2,\mathbf{Z})/\{\pm I\}$. In the integral (60) with the analytically regularized amplitude (61) we take a distribution function $f(\tau)$ which does not depend on the variables $z_1,\ldots,z_N$. Since the measure $(\mathrm{Im}\,\tau)^{-2}(i/2)d\tau\wedge d\bar{\tau}$ is invariant under the transformations from the modular group the relation (62) implies that the integration over the upper half plane in (60) is reduced to the integration over the fundamental domain F and the distribution function $f(\tau)$ is replaced by the function

$$M[f](\tau)=1/2\sum_{g\in SL(2,\mathbf{Z})} f(M(g)(\tau)). \tag{63}$$

In order to compare our amplitude (60), (61) with the usual amplitude we introduce a notion of a modular form. The mapping $z\to q(z)=\exp[2\pi iz]$ defines a holomorphic mapping of the upper half plane H_1 onto the punctured complex plane $\mathbf{C}^*=\mathbf{C}\backslash\{0\}$. Let us denote $H_1/\{M(T)\}$ the quotient space where the group $\{M(T)\}$ is generated by the shift $M(T)(z)=z+1$. The mapping q induces the analytical isomorphism between $H_1/\{M(T)\}$ and the punctured complex plane $\mathbf{C}^*$. Therefore, the meromorphic function $f(z)$ invariant under the shift $M(T)$ induces the meromorphic function $f_\infty(q)$ on the punctured complex plane $\mathbf{C}^*$. The function $f_\infty(q)$ is meromorphic at the point 0, if for some integer n the function $q^n f_\infty(q)$ is bounded in some neighbourhood of the point 0. The minimal such integer n is called an order of the function $f(z)$ at infinity. It is denoted by $v_\infty(f)$. A holomorphic on the upper half plane function $f(z)$ is called a modular form of the weight k if a function $f(z)$ is meromorphic at infinity, i.e. $v_\infty(f)<\infty$, and for every matrix (53) the following relation satisfies

$$f\left(\frac{az+b}{cz+d}\right)=(cz+d)^k f(z). \tag{64}$$

A modular form $f(z)$ is said to be parabolic if $v_\infty(f)<0$. As an example of a parabolic modular form we consider the function $\theta_1(z|\tau)=-\theta\begin{bmatrix}1\\1\end{bmatrix}(z,\tau)$. In view of [14, Chap. 1, Proposition 14.1]

$$\begin{aligned}\theta_1(z|\tau)=2(q(\tau))^{1/8}\sin\pi z&\prod_{n=1}^{\infty}(1-q(\tau)^n)\\ \times\prod_{n=1}^{\infty}&(1-2(q(\tau))^n\cos 2\pi z+q(\tau)^{2n}).\end{aligned} \tag{65}$$

The function (65) is called the Jacobi theta function [3, Vol. 2, formulae (8.A.2), (8.A.6), (8.A.7)]. It follows from the equality (65) that the function $\theta_1'(0|\tau)=\frac{\partial}{\partial z}\theta_1(z|\tau)|_{z=0}$ has the form

$$\theta_1'(0|\tau)=2\pi(q(\tau))^{1/8}\prod_{n=1}^{\infty}(1-q(\tau)^n)^3. \tag{66}$$

Hence the order at infinity of the function $(\theta_1'(0|\tau))^8$ equals -1. By [3, Vol. 2, formula (8.A.25)] the function $(\theta_1'(0|\tau))^8$ satisfies Eq. (64) for $k=12$, i.e. it is a parabolic modular form of the weight 12. The Green's function (59) is not changed if we replace the function (58) by the function

$$\chi(z|\tau)=2\pi\exp[-\pi(\mathrm{Im}\,\tau)^{-1}(\mathrm{Im}\,z)^2]|\theta_1(z|\tau)|\,|\theta_1'(0|\tau)|^{-1}. \tag{67}$$

Now if we take in the amplitude (31) for the torus $\mathbf{T}(1,\tau)$ the coupling constant $\alpha^2=\pi$, the tachyon masses $(k_j,k_j)=4$, $j=1,\dots,N$ and the distribution function

$$(\mathrm{Im}\,z_N)^N\exp\left[-2\sum_{j=1}^{N}\int_{\mathbf{T}(1,\tau)}\varrho_{\mathbf{T}(1,\tau)}(w)\log\chi(z_j-w|\tau)\right]\delta(z_N,\tau), \tag{68}$$

and if we integrate the obtained expression over the fundamental domain F with the differential 2-form

$$\mu(\tau)=(\mathrm{Im}\,\tau/2)^{-13}|(\theta_1'(0|\tau))^8/2\pi|^{-2}(i/2)d\tau\wedge d\bar{\tau}, \tag{69}$$

then omitting the constant multiplier we have

$$\int_F\mu(\tau)\int_{\mathbf{T}(1,\tau)^{\times(N-1)}}\left(\prod_{j=1}^{N-1}(i/2)dz_j\wedge d\bar{z}_j\right)\left[\prod_{1\leqq l<j\leqq N}(\chi(z_l-z_j|\tau))^{(k_l,k_j)/2}\right]_{z_N=\tau}. \tag{70}$$

The expression (70) coincides with the N-point scattering amplitude for the closed bosonic strings corresponding to the tori [3, Vol. 2, formula (8.2.17)]. Note that due to (66) the coefficient of the differential 2-form (69) increases exponentially as $\tau\to\infty$.

To obtain the amplitude for the open bosonic strings corresponding to the tori we reduce the integration over the torus $\mathbf{T}(1,\tau)$ in (31) to the integration over the *unique real side* $[0,1]$ *of the parallelogram related with the torus* $\mathbf{T}(1,\tau)$. We choose now in the expression (31) the coupling constant $\alpha^2=2\pi$, the tachyon masses $(k_j,k_j)=2,\dots,$ $j=1,\dots,N$, and the distribution function

$$f(x_1,\dots,x_N)=\exp\left[-2\sum_{j=1}^{N}\int_{\mathbf{T}(1,\tau)}\varrho_{\mathbf{T}(1,\tau)}(w)\log\chi(x_j-w|\tau)\right]$$
$$\times\,\theta(x_1)\left(\prod_{l=1}^{N-1}\theta(x_{l+1}-x_l)\right)\delta(x_N-1).$$

Then the integration of the obtained expression with respect to the variable τ along the pure imaginary semiaxis $[i,i\infty)$ with the differential 1-form

$$(\tau/i)^N\exp\left[-2\pi i\tau^{-1}\frac{(D-26)}{24}\right]((\theta_1'(0|\tau))^8/2\pi)^{\frac{(2-D)}{24}}(1/i)d\tau \tag{71}$$

provides the N-point amplitude of the open bosonic strings for D-dimensional space [3, Vol. 2, formula (8.1.55)].

5. Higher Genus Riemann Surfaces

A compact Riemann surface M of genus g is homeomorphic to a polygon whose sides are identified according to $A_1B_1A_1^{-1}B_1^{-1}\dots A_gB_gA_g^{-1}B_g^{-1}$ [12, p. 17]. If $z=x+iy$ is a local complex coordinate on M the differential 1-form $f(z)dz+g(z)d\bar{z}$

is said to be holomorphic when $g(z)=0$ and a function $f(z)$ is holomorphic. Due to [12, Propositions 3.2.7, 3.2.8] the vector space of holomorphic 1-forms on a compact Riemann surface M of genus g has the dimension g and there exists the unique basis $\omega_1=\omega_1(z)dz,\ldots,\omega_g=\omega_g(z)dz$ such that

$$\int_{A_j}\omega_k=\delta_{jk}. \tag{72}$$

Furthermore, for this basis the complex $g\times g$ matrix

$$\int_{B_j}\omega_k=\tau_{jk} \tag{73}$$

is symmetric with positive definite imaginary part. On the surface M we introduce the Riemannian metric

$$ds^2=1/g\sum_{k,j=1}^{g}(\operatorname{Im}\tau)^{-1}_{kj}\omega_k(z)\bar{\omega}_j(z)(dx^2+dy^2) \tag{74}$$

and the corresponding canonical 2-form

$$\varrho_M=i/2g\sum_{k,j=1}^{g}(\operatorname{Im}\tau)^{-1}_{kj}\omega_k\wedge\bar{\omega}_j. \tag{75}$$

It follows from the relations (72), (73), and [12, Proposition 3.2.3] that $\operatorname{vol}(M)=1$ [see (35)]. We introduce a g-dimensional Riemann's theta function

$$\theta(z,\tau=\sum_{n\in(\mathbf{Z})^g}\exp[2\pi i[(1/2)(n,\tau n)+(n,z)]], \tag{76}$$

where a column $z\in\mathbf{C}^g$ and a symmetric complex $g\times g$ matrix τ has a positive definite imaginary part. For $\varepsilon,\varepsilon'\in\mathbf{Z}^g$ we define the first order theta function with integer characteristic ε,ε'

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z,\tau)=\exp[2\pi i[(\varepsilon,\tau\varepsilon)/8+(\varepsilon,z)/2+(\varepsilon,\varepsilon')/4]]\theta((1/2)\varepsilon'+(1/2)\tau\varepsilon+z,\tau). \tag{77}$$

In view of [12, Proposition 6.1.5]

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(-z,\tau)=\exp[\pi i(\varepsilon,\varepsilon')]\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z,\tau). \tag{78}$$

The integer characteristic ε,ε' is said to be even (odd) if $(\varepsilon,\varepsilon')\equiv 0\,(\operatorname{mod}2)$ $[(\varepsilon,\varepsilon')\equiv 1\,(\operatorname{mod}2)]$.

Lemma 5.1. *For every point ξ of a compact Riemann surface of genus g there exists an odd integer characteristic $\varepsilon,\varepsilon'\in\mathbf{Z}^g$ with the property that for some point η*

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\left(\int_{\xi}^{\eta}\omega,\tau\right)\neq 0, \tag{79}$$

where a vector $\omega=(\omega_1,\ldots,\omega_g)$ is a basis for the space of holomorphic differential 1-forms on a surface M.

Proof. Let the function (79) be identically zero. Differentiating it at the point $\eta=\xi$ we have

$$\sum_{j=1}^{g}\frac{\partial}{\partial z_j}\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau)\omega_j(\xi)=0. \tag{80}$$

Let a lattice $L \subset \mathbf{Z}^{2g}$ be given. Let us define an orthogonal lattice

$$L^{\perp} = \{(x_1; x_2) \in \mathbf{Q}^g \times \mathbf{Q}^g \mid \exp[2\pi i[(x_1, a_2) - (x_2, a_1)]] = 1 \ \forall (a_1; a_2) \in L\},$$

where $\mathbf{Q}$ is the rational number field. In particular, $(2\mathbf{Z}^{2g})^{\perp} = (1/2)\mathbf{Z}^{2g}$ or

$$2\mathbf{Z}^{2g} = 4(2\mathbf{Z}^{2g})^{\perp}. \tag{81}$$

With a symmetric complex $g \times g$ matrix τ we relate a lattice $L_\tau \subset \mathbf{C}^g$ in the following way: $L_\tau = \{\tau x + y \in \mathbf{C}^g \mid (x; y) \in L\}$. Due to [12, Chap. 6, formula (1.4.6)] the theta function (77) is quasiperiodic with respect to the lattice $(\mathbf{Z}^{2g})_\tau$, namely for any $\mu, \mu' \in \mathbf{Z}^g$,

$$\begin{aligned}\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}&(z + \mu' + \tau\mu, \tau)\\ &= \exp[2\pi i[-(\mu, \tau\mu)/2 - (\mu, z) + (\mu', \varepsilon)/2 - (\mu, \varepsilon')/2]]\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z, \tau).\end{aligned} \tag{82}$$

If $\mu, \mu' \in 2\mathbf{Z}^g$ the multiplier in the right-hand side of Eq. (82) does not depend on the characteristic $\varepsilon, \varepsilon' \in \mathbf{Z}^g$. Therefore, the relation

$$\phi_{2(\mathbf{Z})^{2g}}(z) = \left(\ldots, \theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z, \tau), \ldots\right), \tag{83}$$

where $\varepsilon, \varepsilon'$ run all 4^g pairs of the vectors whose components are equal to 0, 1, defines a mapping $\phi_{2(\mathbf{Z})^{2g}} : \mathbf{C}^g/(2\mathbf{Z}^{2g})_\tau \to \mathbf{CP}^{4^g - 1}$. By the Lefschetz theorem [14, Chap. 2, Theorem 1.3] the relation (81) implies that the mapping (83) is a holomorphic imbedding. Hence the vectors $\left\{\frac{\partial}{\partial z_j}\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0, \tau)\right\}$ generate the vector space $\mathbf{C}^g$. In view of Eq. (78) these vectors are non-zero only for the odd integer characteristics. For every point ξ of a compact Riemann surface M of genus g the vector $(\omega_1(\xi), \ldots, \omega_g(\xi))$ is not zero [12, p. 81]. There exists therefore an odd integer characteristic $\varepsilon, \varepsilon'$ such that the function (79) is not identically zero.

We denote by ξ a point of a compact Riemann surface M and a corresponding local complex coordinate on M.

Proposition 5.2. *Let a point ξ of a compact Riemann surface M of genus g and an odd integer characteristic $\varepsilon, \varepsilon' \in \mathbf{Z}^g$ satisfy the condition* (79). *Then the function on $M \times M$*

$$\begin{aligned}f(\xi_1, \xi_2) = &\left|\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\left(\int_{\xi}^{\xi_2}\omega - \int_{\xi}^{\xi_1}\omega, \tau\right)\right|\\ &\times \exp\left[-\pi\sum_{i,j=1}^{g}\mathrm{Im}\left(\int_{\xi}^{\xi_2}\omega_i - \int_{\xi}^{\xi_1}\omega_i\right)(\mathrm{Im}\,\tau)_{ij}^{-1}\,\mathrm{Im}\left(\int_{\xi}^{\xi_2}\omega_j - \int_{\xi}^{\xi_1}\omega_j\right)\right]\end{aligned} \tag{84}$$

is not identically zero and it does not depend on a choice of the paths of integration in (84). *The function*

$$\begin{aligned}G(\xi_1, \xi_2) = &-1/2\pi \log f(\xi_1, \xi_2) + 1/2\pi \int_M (\log f(\xi_1, \eta) + \log f(\eta, \xi_2))\varrho_M(\eta)\\ &-1/2\pi \int_{M^{\times 2}} (\log f(\eta_1, \eta_2))\varrho_M(\eta_1) \wedge \varrho_M(\eta_2)\end{aligned} \tag{85}$$

satisfies Eq. (34) *for a compact Riemann surface M of genus g endowed with the Riemannian metric* (74) *and it does not depend on a choice of a point $\xi \in M$ and an odd* *integer characteristic* $\varepsilon, \varepsilon' \in \mathbf{Z}^g$ *satisfying the condition* (79). *The function* $G(\xi_1, \xi_2)$ *is continuous everywhere on* $M \times M$ *except for the diagonal points where it has the singularity* $-1/2\pi \log|\xi_2 - \xi_1|$.

Proof. Let a closed contour homological zero be added to a path of integration from a point ξ to a point ξ_1. Then the integral $\int_{\xi}^{\xi_1} \omega$ is not changed since the holomorphic differential 1-forms ω_i, $i=1, \ldots, g$, are closed [12, Proposition 1.3.8, Theorem 1.3.11]. Let a closed contour $\sum_k (\mu'_k A_k + \mu_k B_k)$, where $\mu, \mu' \in \mathbf{Z}^g$, be added to a path of integration from a point ξ to a point ξ_1. Then in view of the relations (72), (73) the vector $\mu' + \tau\mu$ is added to the vector $\int_{\xi}^{\xi_1} \omega$. It follows from Eq. (82) that this addition does not change the function (84). Since the closed loops $A_1, \ldots, A_g$, $B_1, \ldots, B_g$ are the generators of the fundamental group of the Riemann surface M [12, p. 18] the function (84) is independent of a choice the paths of integration in (84).

Due to $\mathrm{vol}(M)=1$ the function $-1/2\pi \log f(\xi_1, \xi_2)$ in (85) may be replaced by the function

$$f_1(\xi_1, \xi_2) = -1/2\pi \log \left| \theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \left(\int_{\xi}^{\xi_2} \omega - \int_{\xi}^{\xi_1} \omega, \tau \right) \right| - \sum_{i,j=1}^{g} \mathrm{Im} \left(\int_{\xi}^{\xi_1} \omega_i \right) (\mathrm{Im}\,\tau)^{-1}_{ij} \, \mathrm{Im} \left(\int_{\xi}^{\xi_2} \omega_j \right).$$

The functions $\mathrm{Im} \int_{\xi}^{\xi_1} \omega_i, \ldots, i=1, \ldots, g$, are locally harmonic. By the condition (79) a locally holomorphic function $F(\xi_1, \xi_2) = \theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \left(\int_{\xi}^{\xi_2} \omega - \int_{\xi}^{\xi_1} \omega, \tau \right)$ is not identically zero. Then [14, Chap. 2, Lemma 3.4] implies that there exist $2g-2$ points $\eta_1, \ldots, \eta_{g-1}, \zeta_1, \ldots, \zeta_{g-1} \in M$ such that the zeros of the function $F(\xi_1, \xi_2)$ counting multiplicities are equal to the sum of the diagonal points $\{(\eta; \eta) \mid \eta \in M\}$ and the points $\{\eta_i\} \times M$, $M \times \zeta_i$, $i=1, \ldots, g-1$. In view of Eq. (78) it is possible to consider $\zeta_i = \eta_i$, $i=1, \ldots, g-1$. Hence the function $f_1(\xi_1, \xi_2)$ is locally harmonic except for the singularities $-1/2\pi \log|\xi_2 - \xi_1|$; $-1/2\pi \log|\xi_1 - \eta_i|$, $-1/2\pi \log|\xi_2 - \eta_i|$, $i=1, \ldots, g-1$. Due to $\mathrm{vol}(M)=1$ the last two sets of the singularities are canceled out of (85) and the function $G(\xi_1, \xi_2)$ has the singularity $-1/2\pi \log|\xi_2 - \xi_1|$. It follows from Eq. (42) that the function $G(\xi_1, \xi_2)$ satisfies the first and the second equations (34) for a compact Riemann surface endowed with the Riemannian metric (74). The last two equations (34) are verified immediately. If we choose another point $\tilde{\xi} \in M$ and another odd characteristic $\tilde{\varepsilon}, \tilde{\varepsilon}' \in \mathbf{Z}^g$ satisfying the condition (79) it is possible to define another Green's function $\tilde{G}(\xi_1, \xi_2)$ by means of Eqs. (84), (85). In view of the first equation (34) the function $G(\xi_1, \xi_2) - G(\xi_1, \xi_2)$ is harmonic with respect to the variable ξ_1 and, therefore, it is constant [6, Sect. 31, Corollaire 4]. Now the third equation (34) implies that this function equals zero. Thus the Green's function $G(\xi_1, \xi_2)$ is independent of a choice of a point $\xi \in M$ and of an odd characteristic $\varepsilon, \varepsilon' \in \mathbf{Z}^g$ satisfying the condition (79).

By the definitions (84), (85) the Green's function $G(\xi_1, \xi_2)$ depends on a basis $\omega_1, \ldots, \omega_g$ for the space of holomorphic differential 1-forms on the compact Riemann surface M of genus g. In order to find a symmetry relation similar to the relation (54) we introduce due to [16] the Teichmüller space and the space of moduli of Riemann surfaces. Let M be a smooth compact connected orientable surface whose homology group $H_1(M,\mathbf{Z})$ is isomorphic to $\mathbf{Z}^{2g}$. Two orientation preserving homeomorphisms of Riemann surfaces of genus g onto M: $f_1: M_1 \to M$ and $f_1: M_1 \to M$ are called equivalent if there is a commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f_2} & M_{\lambda} \\ \phi\downarrow & & \downarrow\psi \\ M_2 & \xrightarrow{f_2} & M, \end{array} \tag{86}$$

where ϕ is a conformal mapping onto M_2 and ψ is a homeomorphism homotopic to the identity. The equivalence classes $[f_1: M_1 \to M]$ are called the points of the Teichmüller space T_g. For the space of moduli M_g a homeomorphism ψ is not obliged to be homotopic to the identity. As an example of a space of moduli we consider the tori $M = \mathbf{T}(1,i)$, $M_1 = \mathbf{T}(1,\tau')$, $M_2 = \mathbf{T}(1,\tau)$, where $\tau \in H_1$, $\tau' = M(g)(\tau)$ and g is the matrix (53). Choose in the diagram (86) the mappings $f_1(z) = \alpha_{\tau'}^{-1}(z)$, $f_2(z) = \alpha_\tau^{-1}(z)$, where the homeomorphism α_τ^{-1} is given by the relation (49). The mapping $\phi(z) = (c\tau + d)z$ defines the conformal equivalence of the tori $\mathbf{T}(1,\tau')$ and $\mathbf{T}(1,\tau)$ since the system of the equivalence relations $z \sim z + a\tau + d$, $z \sim z + c\tau + d$ is equivalent to the system of the equivalence relations $z \sim z + \tau$, $z \sim z + 1$. The mapping $\psi(x+iy) = by + dx + i(ay + cx)$ [see (54)] is a homeomorphism of the torus $\mathbf{T}(1,i)$ onto itself. It is easy to verify that $\psi \circ f_1 = f_2 \circ \phi$ or the diagram (86) is commutative.

A point of the Teichmüller space is determined by a space of holomorphic differential 1-forms on a surface M and by a basis of the homology group $H_1(M,\mathbf{Z})$ satisfying the conditions (36). In fact, there exists the unique basis $\omega = (\omega_1, \ldots, \omega_g)$ of the space of holomorphic differential 1-forms on M satisfying the condition (72) [12, Proposition 3.2.8]. The matrix τ is defined by means of the relations (73). We call $J(M) = \mathbf{C}^g/(\mathbf{Z}^{2g})_\tau$ the Jacobian variety of the surface M. Taking a point $P_0 \in M$ we define a mapping $\phi: M \to J(M)$, $\phi(P) = \int_{P_0}^{P} \omega \bmod (\mathbf{Z}^{2g})_\tau$. Due to [12, Proposition 3.6.1] the image of this mapping is a compact Riemann surface of genus g with a basis $(\omega_1, \ldots, \omega_g)$ for the space of holomorphic differential 1-forms and with a basis $(A_1, \ldots, A_g, B_1, \ldots, B_g)$ for the hmology group $H_1(M,\mathbf{Z})$ which satisfy the conditions (36) and (72). Thus the point of the Teichmüller space is determined by the pair of bases $\{(A_1, \ldots, A_g, B_1, \ldots, B_g), (\omega_1, \ldots, \omega_g)\}$ satisfying the conditions (36) and (72). In view of [12, Corollary 3.2.1] for a homology basis $(A_1, \ldots, A_g, B_1, \ldots, B_g)$ there is the unique dual basis $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ for the space of real harmonic 1-forms on the surface M. By the Hodge theorem [10, Chap. 6, Theorem 3.4] on a compact Riemann surface there is the unique harmonic 1-form with given periods. Hence the relations (72) and (73) imply that

$$\omega_i = \alpha_i + \sum_{j=1}^{g} \tau_{ij}\beta_j, \qquad i = 1, \ldots, g. \tag{87}$$

Therefore, the pair of bases $\{(A,B),\omega\}$ may be replaced by the triple $\{(A,B),(\alpha,\beta),\tau\}$, where (α,β) is the dual basis of real harmonic 1-forms on M for a basis (A,B) satisfying the condition (36) and τ is a symmetric $g \times g$ complex matrix with positive definite imaginary part. The space of such matrices is called the

Siegel upper half space H_g of genus g. Choose another basis for $H_1(M,\mathbf{Z})$,

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \tag{88}$$

where a,b,c,d are $g\times g$ matrices. In order (A',B') to be a basis for $H_1(M,\mathbf{Z})$ it is necessary

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2g\cdot\mathbf{Z}), \qquad \det G = \pm 1. \tag{89}$$

A basis (A',B') ought to satisfy the conditions (36). Hence $G\in Sp(2g,\mathbf{Z})$ where the group $Sp(2g,\mathbf{Z})$ consists of the invertible matrices (89) satisfying the relation

$$G^t J_0 G = J_0, \qquad J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{90}$$

By using the equality (90) we show that for a basis (A',B') the dual basis (α',β') of the real harmonic 1-forms on the surface M has the following form:

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} -c & d \\ a & -b \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{91}$$

A basis $\omega'=(\omega'_1,\dots,\omega'_g)$ of holomorphic 1-forms on M which satisfy the conditions (72) for a basis (A',B') is given by $\omega'=((c\tau+d)^{-1})^t\omega$. Now by the definition (73) we have

$$\tau' = (a\tau+b)(c\tau+d)^{-1}. \tag{92}$$

Therefore, the Teichmüller space T_g is a set of the triples $\{(A,B),(\alpha,\beta),\tau\}$ with the equivalence relations (88), (91), (92) where a matrix (89) belongs to the group $Sp(2g,\mathbf{Z})$.

To study a space of moduli it is necessary to investigate a group $\operatorname{Aut}M$ of conformal automorphisms of a compact Riemann surface M of genus g. By the Schwartz theorem [12, Corollary 5.1.2.2] for a compact Riemann surface M of genus $g>1$ a group $\operatorname{Aut}M$ is finite. Let an automorphism $T\in\operatorname{Aut}M$ and $(A_1,\dots,A_g,B_1,\dots,B_g)$ be a basis for the homology group $H_1(M,\mathbf{Z})$ satisfying the conditions (36). Then $(TA_1,\dots,TA_g,TB_1,\dots,TB_g)$ is also a basis for the homology group $H_1(M,\mathbf{Z})$ satisfying the conditions (36). Hence two bases are related to each other by the equality (88) where the matrix $G(T)\in Sp(2g,\mathbf{Z})$. By [12, Theorem 5.3.1] this representation $G:\operatorname{Aut}M\to Sp(2g,\mathbf{Z})$ is faithful for a genus $g>1$. The action of an automorphism $T\in\operatorname{Aut}M$ on a differential 1-form $\alpha=\alpha_z dz+\alpha_{\bar z}d\bar z$ is given by

$$T\alpha = \alpha_z(T^{-1}(z))\left(\frac{\partial}{\partial z}T^{-1}(z)\right)dz + \alpha_{\bar z}\left(\frac{\partial}{\partial \bar z}\bar T^{-1}(z)\right)d\bar z. \tag{93}$$

If α is a real harmonic 1-form then $T\alpha$ is also a real harmonic 1-form. In view of [12, Theorem 5.3.2] if (A,B) is a basis for the homology group $H_1(M,\mathbf{Z})$ and (α,β) is the dual basis for the space of real harmonic 1-forms on M then $(T\alpha, T\beta)$ is related to a basis (α,β) by means of Eq. (91) where the matrix $G(T)$ is defined above. The substitution of the relation (87) into the right-hand side of the equality (75) yields

$$\varrho_M = 1/g\sum_{j=1}^{g}\alpha_j\wedge\beta_j. \tag{94}$$

Since a matrix $G(T) \in Sp(2g, \mathbf{Z})$ the equality (91) implies that the 2-form (94) is invariant under the automorphisms from the group $\mathrm{Aut}\, M$.

The Green's function $G_0(\tau, (\alpha, \beta); \xi_1, \xi_2)$ is defined by using the substitution of the relations (87), (94) into definitions (84), (85). Now the relations (93), (91) and the invariance of the 2-form (94) under the automorphisms from the group $\mathrm{Aut}\, M$ provides the following

Proposition 5.3. *Let the matrix*

$$G(T) = \begin{pmatrix} a(T) & b(T) \\ c(T) & d(T) \end{pmatrix} \tag{95}$$

define the above faithful representation $G: \mathrm{Aut}\, M \to Sp(2g, \mathbf{Z})$. *Then for any automorphism* $T \in \mathrm{Aut}\, M$

$$G_0(\tau, (\alpha, \beta); T^{-1}\xi_1, T^{-1}\xi_2) = G_0(\tau, (a(T)\alpha - b(T)\beta, -c(T)\alpha + d(T)\beta); \xi_1, \xi_2). \tag{96}$$

Now we establish the following

Proposition 5.4. *For any matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2g \cdot \mathbf{Z}) \tag{97}$$

the Green's function satisfies the relation

$$G_0((a\tau + b)(c\tau + d)^{-1}, (a\alpha - b\beta, -c\alpha + d\beta); \xi_1, \xi_2) = G_0(\tau, (\alpha, \beta); \xi_1, \xi_2). \tag{98}$$

Proof. It follows from the definitions (77), (84), (85) and the relation (87) that the function $G_0(\tau, (\alpha, \beta); \xi_1, \xi_2)$ may be rewritten in the form (85) where the function $f(\xi_1, \xi_2)$ is replaced by the function

$$\theta^\alpha \begin{bmatrix} \int_\xi^{\xi_2} \beta - \int_\xi^{\xi_1} \beta + \varepsilon/2 \\ \int_\xi^{\xi_2} \alpha - \int_\xi^{\xi_1} \alpha + \varepsilon'/2 \end{bmatrix} (\tau) \ . \tag{99}$$

The modified theta function of the variables $x_1, x_2 \in \mathbf{R}^g$ is given by

$$\theta^\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (\tau) = \exp[i\pi(x_1, \tau x_1) + i\pi(x_1, x_2)] \theta(\tau x_1 + x_2, \tau). \tag{100}$$

In view of [14, Chap. 2, formula (5.3′)] the function (99) is invariant under the shifts $\varepsilon \to \varepsilon + 2\nu$, $\varepsilon' \to \varepsilon' + 2\nu'$ for $\nu, \nu' \in \mathbf{Z}^g$. We consider therefore $\varepsilon_1, \varepsilon_2 \in (\mathbf{Z}/2\mathbf{Z})^g$. On the space of vectors $(\varepsilon_1; \varepsilon_2) \in (\mathbf{Z}/2\mathbf{Z})^g \times (\mathbf{Z}/2\mathbf{Z})^g$ we introduce a quadratic form $Q((\varepsilon_1; \varepsilon_2)) = (\varepsilon_1, \varepsilon_2) \pmod 2$. Let us define due to J.-I. Igusa the subgroup $\Gamma_{1,2} = \{G \in Sp(2g, \mathbf{Z}) \mid Q(Gx) = Q(x)\}$. It consists of such elements of the group $Sp(2g, \mathbf{Z})$ which preserve the parity of the integer characteristic of the first order theta function (77). By [14, Chap. 2, Proposition 5.5] for any matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{1,2}, \tag{101}$$

$$\left| \theta^\alpha \begin{bmatrix} -cx_2 + dx_1 \\ ax_2 - bx_1 \end{bmatrix} ((a\tau + b)(c\tau + d)^{-1}) \right| = |\det(c\tau + d)|^{1/2} \left| \theta^\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (\tau) \right|. \tag{102}$$

In view of [14, Chap. 2, Proposition A.4] the group $\Gamma_{1,2}$ is generated by the following matrices

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}, \quad \begin{pmatrix} I & b \\ 0 & I \end{pmatrix} \tag{103}$$

for all $a \in GL(g, \mathbf{Z})$, $\det a = \pm 1$ and for all symmetric $b \in GL(g, \mathbf{Z})$ whose diagonal matrix elements are even. By the Proposition 5.2 the function $G_0(\tau, (\alpha, \beta); \xi_1, \xi_2)$ is independent of a choice of an odd integer characteristic $\varepsilon, \varepsilon'$. Thus the relation (102) implies that the relation (98) holds for any matrix (103) from the group $Sp(2g, \mathbf{Z})$. By using the definition (76) we obtain for any symmetric $b \in GL(g, \mathbf{Z})$ the following relation $\theta(z, \tau + b) = \theta(z + (b)/2, \tau)$ where the components $(b)_i$ of the vector (b) coincide with the diagonal matrix elements b_{ii}. Now the definition (100) implies that for any symmetric matrix $b \in GL(2, \mathbf{Z})$,

$$\left| \theta^\alpha \begin{bmatrix} x_1 + \varepsilon/2 \\ x_2 - bx_1 + \varepsilon'/2 \end{bmatrix} (\tau + b) \right| = \left| \theta^\alpha \begin{bmatrix} x_1 + \varepsilon/2 \\ x_2 + (1/2)(\varepsilon' + b\varepsilon + (b)) \end{bmatrix} (\tau) \right| . \tag{104}$$

Since $m^2 \equiv m \pmod 2$ for every integer m it is easy to show that for any symmetric matrix $b \in GL(2, \mathbf{Z})$ the quadratic form $Q(\varepsilon; \varepsilon' + b\varepsilon + (b)) = Q(\varepsilon; \varepsilon')$. Hence the relation (104) and the independence of the Green's function (85) of a choice of an odd integer characteristic imply the relation (98) for the third matrix (103) where b is an arbitrary symmetric matrix from the group $GL(g, \mathbf{Z})$. Due to [14, Proposition A.5] namely these matrices (103) generate the group $Sp(2g, \mathbf{Z})$.

The Propositions 5.3 and 5.4 imply the following

Corollary 5.5. *Let for a compact Riemann surface M of genus $g > 1$ the matrix* (95) *define the faithful representation $G : \mathrm{Aut}\, M \to Sp(2g, \mathbf{Z})$. Then for any automorphism $T \in \mathrm{Aut}\, M$,*

$$G_0(\tau, (\alpha, \beta); T\xi_1, T\xi_2) = G_0((a(T)\tau + b(T))(c(T)\tau + d(T))^{-1}, (\alpha, \beta); \xi_1, \xi_2) . \tag{105}$$

It follows from the Proposition 5.2 that all assumptions of the Proposition 2.2 are fulfilled. Then under the conditions (26), (30) the amplitude (31) for a compact Riemann surface M of genus $g > 1$ has a form

$$\begin{aligned} &\int_{M^{\times N}} \left(\prod_{j=1}^{N} \varrho_M(z_j) \right) A_N(k|z, \tau) f(z) \\ &= \int_{M^{\times N}} \left(\prod_{j=1}^{N} \varrho_M(z_j) \right) \left[\prod_{1 \leq i < j \leq N} (\exp[-4\pi G_0(\tau, (\alpha, \beta); z_i, z_j)])^{\frac{\alpha^2(k_i, k_j)}{4\pi}} \right] f(z) . \end{aligned} \tag{106}$$

The relation (105) implies that for any automorphism $T \in \mathrm{Aut}\, M$,

$$A_N(k|Tz, \tau) = A_N(k|z, (a(T)\tau + b(T))(c(T)\tau + d(T))^{-1}) , \tag{107}$$

where a matrix G [see (95)] defines the faithful representation $G : \mathrm{Aut}\, M \to Sp(2g, \mathbf{Z})$.

To compare the amplitude (106) with the amplitude obtained in [4] we consider the function (79) of the variables $\xi, \eta \in M$. By Lemma 5.1 it is not identically zero. It follows from [14, Chap. 2, Lemma 3.4] and the relation (78) that there are $g - 1$ points $\eta_1, \ldots, \eta_{g-1} \in M$ such that the zeros of the function (79) counting multiplicities are equal to the sum of the diagonal points $\{(\eta, \eta) \mid \eta \in M\}$ and the points $\{\eta_i\} \times M, M \times \{\eta_i\}, i = 1, \ldots, g - 1$. Setting $\xi = \eta_i$ and differentiating the function (79) at the point $\eta=\eta_i$ we obtain (80) at the point $\xi=\eta_i$. In view of the relation (78) for an odd integer characteristic $\varepsilon,\varepsilon'\in \mathbf{Z}^g$ we have $\frac{\partial^2}{\partial z_i \partial z_j}\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau)=0$, $i,j=1,\dots,g$. Thus differentiating the function (79) twice and setting $\xi=\eta=\eta_i$ we show that the derivative of the function (80) equals zero at the point $\xi=\eta_i$. Hence the function (80) has zero of order two at every point $\xi=\eta_i$, $i,j=1,\dots,g-1$. Due to [12, Corollary 3.4.9.2] the coefficients of the holomorphic differential 1-form on a compact Riemann surface of genus g has exactly $2g-2$ zeros counting multiplicities. Let us denote the coefficient (80) by $\zeta(\xi)$. We have established that $\zeta(\xi)$ has $g-1$ zeros of order two. Therefore, there exists the square root $(\zeta(\xi))^{1/2}$ which is locally holomorphic function on the surface M as the coefficient $\zeta(\xi)$. Let $|\varrho_M(z)| = ds^2(dx^2+dy^2)^{-1}$ be the multiplier in the Riemannian metric (74). Since the matrix $\operatorname{Im}\tau$ is positive definite and the vector $(\omega_1(z),\dots,\omega_g(z))$ is not zero at any point of a Riemann surface M [12, p. 81] the multiplier $|\varrho_M(z)|>0$. We substitute the equalities (84), (85) into the right-hand side of the relation (106) and we assume that the coupling constant $\alpha^2=4\pi$, the tachyon masses $(k_j,k_j)=2$, $j=1,\dots,N$ and the distribution function

$$\prod_{j=1}^{N}|\zeta(z_j)|^2|\varrho_M(z_j)|^{-1}\exp\left[-4\int_M \varrho_M(w)\log f(z_j,w)\right]. \tag{108}$$

It is easy to show that this function has no singularities. Then the amplitude (106) has the following form:

$$\begin{aligned}C&\int_{M\times N}\left(\prod_{j=1}^{N}(i/2)dz_j\wedge d\bar z_j\right)\prod_{1\leq i<j\leq N}|E(z_i,z_j)|^{2(k_i,k_j)}\\ &\times\exp\left[-2\pi\sum_{1\leq i<j\leq N}\sum_{m,l=1}^{g}(k_i,k_j)\operatorname{Im}\left(\int_\xi^{z_j}\omega_m-\int_\xi^{z_i}\omega_m\right)(\operatorname{Im}\tau)^{-1}_{ml}\operatorname{Im}\left(\int_\xi^{z_j}\omega_l-\int_\xi^{z_i}\omega_l\right)\right],\end{aligned} \tag{109}$$

$$E(z,w)=\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\left(\int_z^w\omega,\tau\right)(\zeta(z))^{-1/2}(\zeta(w))^{-1/2}. \tag{110}$$

The expression (110) is the coefficient of the Prime form [14, Chap. 3b, Sect. 1]. In our case the path of integration from the point z to the point w passes through the fixed point ξ. The expression (109) is similar to the N-point amplitude [4] as the radius of the compactification torus tends to infinity.

In order to integrate over the Teichmüller space it is necessary to introduce the coordinates. A compact Riemann surface M of genus $g>1$ is topologically a polygon whose $4g$ sides are identified according to $A_1B_1A_1^{-1}B_1^{-1}\dots A_gB_gA_g^{-1}B_g^{-1}$ [12, p. 17]. Due to [12, p. 18] the fundamental group $\pi_1(M)$ is generated by the $2g$ closed loops $A_1,\dots,A_g$, $B_1,\dots,B_g$ subject to the conditions (36) and the single relation

$$\prod_{j=1}^{g}A_jB_jA_j^{-1}B_j^{-1}=1. \tag{111}$$

Thus for $g>1$ the fundamental group $\pi_1(M)$ is not abelian. In this case the holomorphic universal covering space of a compact Riemann surface is the upper half plane H_1 [12, Theorems 4.6.1, 4.6.3, 4.6.4]. The group of the conformal automorphisms of the upper half plane is the group of the linear fractional transformations with the real coefficients. It is isomorphic to the quotient group $SL(2,\mathbf{R})/\pm I$. Consider a set of all faithful representations $\chi:\pi_1(M)\to SL(2,\mathbf{R})/\pm I$ such that the group $\chi(\pi_1(M))$ acts properly discontinuously on the upper half plane: for every point $z\in H_1$ the isotropy subgroup $(\chi(\pi_1(M)))_z$ at z is finite and there exists a neighbourhood U of z which is invariant under all $L\in(\chi(\pi_1(M)))_z$ and $L(U)\cup U=\emptyset$ for all $L\in\chi(\pi_1(M))\backslash\chi(\pi_1(M)))_z$. Two representations χ_1 and χ_2 are called equivalent if there exists a matrix $L\in SL(2,\mathbf{R})$ such that for any loop $C\in\pi_1(M)$ the relation $\chi_2(C)=L\chi_1(C)L^{-1}$ holds. The equivalence classes of such representations are called the points of the Fricke space F_g [17, 18]. It is easy to see that the Fricke space F_g has the real dimension $6g-6$. To introduce the complex structure on a surface M we define a Riemann surface of genus g as the quotient space $H_1/\chi(\pi_1(M))$. The Poincaré metric $ds^2=y^{-2}(dx^2+dy^2)$ on H_1 is invariant under the linear fractional transformations with the real coefficients. It induces therefore the Riemannian metric on $H_1/\chi(\pi_1(M))$. By means of the relations (32), (33) this metric defines the complex structure. Due to [18, Chap. 1, Sect. 5, Theorem] the Fricke space F_g is homeomorphic to the Teichmüller space T_g. The Fricke coordinates are simplest ones on the Teichmüller space. Nevertheless, the holomorphic differential 1-forms $\omega_1,\ldots,\omega_g$ are unknown as the explicit functions of the Fricke coordinates. Therefore, the Green's function (84), (85) and the scattering amplitude (106) are unknown explicitly.

The quadratic holomorphic differentials provide another coordinates on the Teichmüller space. A quadratic holomorphic differential is an invariant under the conformal mappings form $\phi(z)dz^2$, where z is a local complex coordinate on a Riemann surface M and the function $\phi(z)$ is holomorphic. In view of [12, Proposition 3.5.2] the complex dimension of the space of the quadratic holomorphic differentials is $3g-3$. By the norm of the quadratic holomorphic differential $\phi(z)dz^2$ we mean the integral $\int\limits_M |\phi(z)|(i/2)dz\wedge d\bar{z}$. The unique ball with respect to this norm in the space of the quadratic holomorphic differentials is homeomorphic to the Teichmüller space [18, Chap. 1, Sect. 5, Theorem]. By using a basis in the space of the quadratic holomorphic differentials and a basis in the space of the holomorphic differential 1-forms on a compact Riemann surface it is possible to define a canonical form on the space of moduli [19]. This form induces a canonical measure on the space of moduli. In the paper [20] by means of a choice of the special basis for the space of holomorphic differential 1-forms on a compact Riemann surface the canonical form and measure are calculated explicitly. It is possible to integrate formally the amplitude (106) with this canonical measure over the space of moduli.

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References

1. Polyakov, A.M.: Quantum geometry of bosonic strings. Phys. Lett. **103** B, 207–210 (1981)
2. Belavin, A.A., Knizhnik, V.A.: Complex geometry and theory of quantum strings. ZETP **91**, 364–390 (1986)
3. Green, M.B., Schwarz, J.H., Witten, E.: Superstring theory, Vols. 1, 2. Cambridge: Cambridge University Press 1987

4. Dijgraaf, R., Verlinde, E., Verlinde, H.: $C=1$ conformal field theories on Riemann surfaces. Commun. Math. Phys. **115**, 649–690 (1988)
5. Zinoviev, Yu.M.: Duality in the abelian gauge lattice theories. Theor. Math. Phys. **43**, 481–490 (1980)
6. de Rham, G.: Variétés différentiables. Paris: Hermann 1955
7. Zinoviev, Yu.M.: **R**-gauge theories. Theor. Math. Phys. **50**, 135–143 (1982)
8. Gel'fand, I.M., Shilov, G.E.: Generalized functions, Vol. 1. Operations on them. New York: Academic Press 1964
9. Faltings, G.: Calculus on arithmetic surfaces. Ann. Math. **119**, 387–424 (1984)
10. Kra, I.: Automorphic forms and Kleinian groups. New York: Benjamin 1972
11. Wells, R.O.: Differential analysis on complex manifolds. Englewood Cliffs, N.J.: Prentice-Hall 1973
12. Farkas, H.M., Kra, I.: Riemann surfaces. Berlin, Heidelberg, New York: Springer 1980
13. Griffiths, P., Harris, J.: Principles of algebraic geometry. New York: Wiley 1978
14. Mumford, D.: Tata lectures on theta, 1, 2. Boston, Basel, Stuttgart: Birkhäuser 1983, 1984
15. Serre, J.-P.: Cours d'arithmétique. Paris: Presses Univ. de France 1970
16. Bers, L.: Finite-dimensional Teichmüller spaces and generalizations. Bull. Am. Math. Soc. **5**, 131–172 (1981)
17. Keen, L.: On Fricke moduli. In: Advances in the theory of Riemann surfaces (Ann. of Math. Studies, Vol. 66, 205–204). Princeton: Princeton University Press 1971
18. Abikoff, W.: The real analytic theory of Teichmüller space. Lecture Notes in Mathematics, Vol. 820. Berlin, Heidelberg, New York: Springer 1980
19. Mumford, D.: Stability of projective varieties. Enseign. Math. **23**, 39–110 (1977)
20. Beilinson, A.A., Manin, Yu.I.: The Mumford form and the Polyakov measure in string theory. Commun. Math. Phys. **107**, 359–376 (1986)

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