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String Theory and the Donaldson Polynomial

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Abstract. It is shown that the scattering of spacetime axions with fivebrane solitons of heterotic string theory at zero momentum is proportional to the Donaldson polynomial.

1. Introduction

A p-brane (i.e. an extended object with a $p+1$-dimensional worldvolume) naturally acts as a source of a $p+2$ form field strength F via the relation

$$\nabla^M F_{MN_1 \cdots N_{p+1}} = q \Delta_{N_1 \cdots N_{p+1}} , \tag{1}$$

where Δ is the p-brane volume-form times a transverse δ-function on the p-brane. In d dimensions they can therefore carry a charge

$$q = \int_{\Sigma^{d-p-2}} *F , \tag{2}$$

where the integral is over a $d-p-2$ dimensional hypersurface at spatial infinity. The dual charge

$$g = \int_{\Sigma^{p+2}} F \tag{3}$$

can be carried by a $d-p-4$ brane. A straightforward generalization [1] of Dirac's original argument implies that quantum mechanically the charges must obey a quantization condition of the form

$$qg = n , \tag{4}$$

just as for the special case of electric and magnetic charges in $d=4$. In particular, strings in ten dimensions are dual, in the Dirac sense, to fivebranes. Thus fivebranes are the magnetic monopoles of string theory.

In [2, 3] it was shown that heterotic string theory admits exact fivebrane soliton solutions. The core of the fivebrane consists of an ordinary Yang–Mills instanton. Thus heterotic strings are dual, in the Dirac sense, to Yang–Mills instantons.

This simple connection between Yang–Mills instantons and heterotic string theory raises many possibilities. On the one hand, heterotic string theory might be used as a tool to study the rich mathematical structure of Yang–Mills instantons, or to suggest interesting generalizations. On the other hand, the mathematical results of Donaldson [4] and others on the construction of new smooth invariants for four-manifolds may have direct implications for non-perturbative semi-classical heterotic string theory.

In this paper this connection is elucidated as follows. We consider N parallel fivebranes on the manifold $M^6 \times X$, where M^6 is six-dimensional Minkowski space and X is the four-manifold transverse to the fivebranes. (The consistency of string theory places restrictions on the choice of X, as discussed in the next section.) The quantum ground states of this system are found to be cohomology classes on the N-instanton moduli space $\mathscr{M}_N(X)$. Transitions among these ground states may be induced by scattering with a zero-momentum axion. Such axions are characterized by a harmonic two-form or an element of $H^2(X)$, and the S-matrix then defines a map $H^2(X) \to H^2(\mathscr{M}_N)$. This map turns out to be precisely the Donaldson map. The fact that the scattering is a map between cohomology classes is ultimately a consequence of zero-momentum spacetime supersymmetry. Multiple axion scattering is given by the intersection numbers on $\mathscr{M}_N$ of these elements of $H^2(\mathscr{M}_N)$, which is the Donaldson polynomial.

This representation of the Donaldson map as a string S-matrix element leads to an apparently new geometrical interpretation of Donaldson theory. For any Kähler manifold X there is a Kähler geometry on $H^2(X, C) \times \mathscr{M}_N(X)$ with Kähler potential defined by

$$\mathscr{K} = \frac{1}{2}\int_X E \wedge J - \ln \int_X J \wedge J\,, \tag{5}$$

where J is the Kähler form on X and E is a solution of

$$\operatorname{tr} F \wedge F = i\partial\bar{\partial}E\,. \tag{6}$$

E is an analog of the Chern–Simons form for Kähler geometry. The Donaldson map is then given by a mixed component of the Christoffel connection, computed as the third derivative of $\mathscr{K}$.

It is noteworthy that the final expressions we derive for the Donaldson map and polynomial are similar to those given by Witten [5]. Indeed, the embedding of four-dimensional Yang–Mills instantons into ten-dimensional string theory given by fivebrane solitons seems to produce a structure of zero-momentum fields and symmetries similar, if not identical, to that of Witten's topological Yang–Mills theory. Though we have not done so, it is possible that the complete structure of topological Yang–Mills theory can be derived from zero-momentum string theory in the soliton sector. This is perhaps in contrast to the usual notion [5] that topological field theory is relevant to a short-distance phase of string theory.

We work only to leading order in α' in this paper. An interesting question, which we do not address, is whether or not higher-order or non-perturbative corrections provide a natural deformation of the Donaldson polynomial analogous to the deformation of the cohomology ring provided by string theory.

This paper is organized as follows. In Sect. II we establish our notation and review some properties of instanton moduli space. The collective coordinate expansion leading to the low-energy effective action is derived in some detail, and is then used to characterize the N-fivebrane ground states. In Sect. III.A the collective coordinate expansion is continued to reveal the Donaldson map as a subleading term in the low-energy effective action. Section III.B gives an alternate derivation of the Donaldson map using supersymmetry and Kähler geometry, and derives (5). In Sect. III.C we discuss the representation of the Donaldson map as a period of the second Chern class which may be relevant in the present context. In Sect. III.D the string scattering amplitude which measures the Donaldson polynomial is described. We conclude with discussion in Sect. IV.

II. Instanton Moduli Space and the Collective Coordinate Expansion

The derivation of the Donaldson map and polynomial from ten-dimensional string theory is straightforward though somewhat involved. We begin with the action describing the low-energy limit of heterotic string theory:

$$\begin{aligned} S_{10} = \frac{1}{\alpha'^4}\int d^{10}x\sqrt{-g}\,e^{-2\phi}\Big(&R + 4\nabla_M\phi\nabla^M\phi - \frac{1}{3}H_{MNP}H^{MNP} - \alpha'\operatorname{tr}F_{MN}F^{MN} \\ &-\bar\psi_M\Gamma^{MNP}\nabla_N\psi_P + 2\bar\lambda\Gamma^{MN}\nabla_M\psi_N + \bar\lambda\Gamma^M\nabla_M\lambda - 2\alpha'\operatorname{tr}\bar\chi\Gamma^M D_M\chi \\ &-\alpha'\operatorname{tr}\bar\chi\Gamma^M\Gamma^{NP}F_{NP}\left(\psi_M + \frac{1}{6}\Gamma_M\lambda\right) + 2\bar\psi_N\Gamma^M\Gamma^N\lambda\nabla_M\phi - 2\bar\psi_M\Gamma^M\psi^N\nabla_N\phi \\ &+\frac{1}{12}H_{MNP}(2\alpha'\operatorname{tr}\bar\chi\Gamma^{MNP}\chi - \bar\lambda\Gamma^{MNP}\lambda + \bar\psi^S\Gamma_{[S}\Gamma^{MNP}\Gamma_{T]}\psi^T \\ &+ 2\bar\psi_S\Gamma^{SMNP}\lambda) + \cdots\Big)\,, \end{aligned} \tag{7}$$

where "$+\cdots$" indicates four-fermi as well as higher-order α' corrections, $H = dB + \alpha'\omega_{3L} - \alpha'\omega_{3Y}$, ω_{3L} and ω_{3Y} are the Lorentz and Yang–Mills Chern Simons three-forms respectively, and "tr" is $\frac{1}{30}$ the trace in the adjoint representation of $E_8\times E_8$ or $SO(32)$. A supersymmetric solution of the equations of motion following from (7) is one for which there exists at least one Majorana–Weyl spinor η obeying

$$\begin{aligned} \delta\psi_M &= \nabla_M\eta - \frac{1}{4}H_{MNP}\Gamma^{NP}\eta = 0\,, \\ \delta\lambda &= \frac{1}{6}H_{MNP}\Gamma^{MNP}\eta - \nabla_M\phi\Gamma^M\eta = 0\,, \\ \delta\chi &= -\frac{1}{4}F_{MN}\Gamma^{MN}\eta = 0\,. \end{aligned} \tag{8}$$

The general solution of this form on $X\times M^6$, where X is a Kähler manifold with $c_1 \geqq 0$ and M^6 is flat six dimensional Minkowski space, was found in [6]. The gauge field may be any self-dual connection on X:

$$F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}\,, \tag{9}$$

where μ, ν are indices tangent to X. Let $\hat{g}$ be a Ricci flat Kähler metric on X. Then the dilaton is the solution of

$$\hat{\Box}\, e^{2\phi} = \alpha'(\mathrm{tr}\,\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} - \mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}) \tag{10}$$

and the metric and axion are

$$g_{\mu\nu} = e^{2\phi}\hat{g}_{\mu\nu}, \quad H_{\mu\nu\lambda} = -\,\varepsilon_{\mu\nu\lambda}{}^{\rho}\nabla_{\rho}\phi, \quad g_{ab} = \eta_{ab}\,, \tag{11}$$

where a, b are tangent to M^6. Special cases of this general solution are discussed in [2, 7, 3]. In [8] it was argued for $X = R^4$ that such solutions are in one to one correspondence with exact solutions of heterotic string theory. For $c_2(F) = N$, this may be viewed as a configuration of N fivebranes transverse to X. (It may also be viewed as a "compactification" to six dimensions, though X need not be compact.) Since (given the metric on X) there is one unique solution for every self-dual Yang–Mills connection (9), the space of static N fivebrane solutions is identical to the moduli space $\mathcal{M}_N$ of N-instanton configurations on X.

For $c_1(X) \geqq 0$ and $c_2(R) \geqq c_2(F)$, the metrics g of (11) are geodesically complete, but may be non-compact. If $c_1(X) > 0$, there are geodesically complete but non-compact Ricci-flat metrics with bounded curvature [9]. This means we have effectively removed the divisor D of c_1 and are really studying string theory on the non-compact manifold X minus D. There appears to be no special difficulty in defining string propagation on such non-compact geometries (though they would not be suitable for Kaluza–Klein compactification). Singularities at isolated points may also arise in solving the dilaton equation (10). If $c_2(R) > c_2(F)$, the singularities are of the type studied in [3]: the metric g is geodesically complete, but non-compact. This again produces no difficulties.

On the other hand if $c_1 < 0$ or $c_2(R) < c_2(F)$, the metric in (11) may have real curvature singularities, which could potentially render string theory ill-defined. More work must be done before our methods can be used to directly study these cases, but the validity of our final formulae for all Kähler X suggests that it may be possible to do so. Possible approaches would be to consider the more general case of time-dependent metrics, or to consider the (eight-dimensional) cotangent bundle of X which has $c_1 = 0$.

The solutions of (9), (10) have bosonic zero modes tangent to $\mathcal{M}_N$. To leading order in α' these zero modes involve only the gauge field and will be denoted $\delta_i A_\mu(x)$, where $i = 1, \ldots, m$, $m \equiv \dim(\mathcal{M}_N)$ and x is a coordinate on X. The zero modes obey the linearized self-duality equation

$$D_{[\mu}\delta_i A_{\nu]} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}D_\rho \delta_i A_\sigma\,. \tag{12}$$

For gauge groups larger than $SU(2)$ or for metrics on X which are not "generic" the gauge connection will in general be reducible (there exist non-trivial solutions of $D\phi = 0$). This leads to orbifold singularities in $\mathcal{M}_N$. In what follows we will restrict ourselves to irreducible connections and ignore such subtleties.

If Z^i is a coordinate on $\mathcal{M}_N$, and $A^0_\mu(x, Z)$ a family of self-dual connections, the zero modes are given by

$$\delta_i A_\mu = \partial_i A^0_\mu - D_\mu \varepsilon_i \,, \tag{13}$$

where $\varepsilon_i(x, Z)$ are arbitrary gauge parameters and $\partial_i = \partial/\partial Z^i$. It is convenient to fix ε_i by requiring

$$D^\mu \delta_i A_\mu = 0 \tag{14}$$

so that the zero modes are orthogonal to fluctuations of the gauge field obtained by gauge transformations. The gauge parameter ε_i then defines a natural gauge connection on $\mathcal{M}_N$ with covariant derivative

$$s_i = \partial_i + [\varepsilon_i, \;\;] \tag{15}$$

which has the property

$$[s_i, D_\mu] = \delta_i A_\mu \,. \tag{16}$$

The Jacobi identity for s_i, D_μ and D_ν implies

$$s_i F_{\mu\nu} = 2D_{[\mu}\delta_i A_{\nu]} \,. \tag{17}$$

The Jacobi identity for s_i, s_j and D_μ implies

$$D_\mu \phi_{ij} = -2s_{[i}\delta_{j]} A_\mu \,, \tag{18}$$

where

$$\phi_{ij} = [s_i, s_j] \tag{19}$$

is the curvature associated to s_i. These relations will be useful shortly.

A natural metric $\mathcal{G}$ on $\mathcal{M}_N$ is induced from the metric g on X:

$$\mathcal{G}_{ij} = \int_X d^4x \sqrt{g}\, e^{-2\phi} g^{\mu\nu} \operatorname{tr}(\delta_i A_\mu \delta_j A_\nu) \,. \tag{20}$$

In addition there is a complex structure $\mathcal{T}$ on $\mathcal{M}_N$ induced from the complex structure J on X:

$$\mathcal{T}_i{}^j = \int_X d^4x \sqrt{g}\, e^{-2\phi} J_\mu{}^\nu \operatorname{tr}(\delta_i A_\lambda \delta_k A_\nu) g^{\mu\lambda} \mathcal{G}^{kj} \,. \tag{21}$$

It is easily seen that the zero modes are related by

$$\mathcal{T}_i{}^j \delta_j A_\mu = -J_\mu{}^\nu \delta_i A_\nu \,. \tag{22}$$

In addition to bosonic zero modes, there are fermionic zero modes of the superpartner χ of A_M. These zero modes are paired with the bosonic zero modes by the unbroken supersymmetry [10] and are given by

$$\chi_i = \delta_i A_\mu \Gamma^\mu \varepsilon' \,, \tag{23}$$

where ε' is the four-dimensional chiral spinor obeying

$$J_{\mu\nu} = -i\varepsilon'^\dagger \Gamma_{\mu\nu}\varepsilon', \quad J^\mu{}_\nu \Gamma^\nu \varepsilon' = i\Gamma^\mu \varepsilon', \quad \varepsilon^\dagger \varepsilon = 1 \,. \tag{24}$$

It is easy to check, using (14) and (12), that $\Gamma^\mu D_\mu \chi_i = 0$.

Equation (23) would appear to give m zero modes, where m is the dimension of $\mathcal{M}_N$, but we know from the index theorem that these are not linearly independent.

Using (24) and (22) one finds

$$\mathscr{T}_i{}^j \chi_j = i\chi_i \,. \tag{25}$$

This gives $m/2$ independent zero modes, as implied by the index theorem.

The low-energy dynamics of N fivebranes in $X \times M^6$ is best described by an effective action S_{eff}. This action can be derived by a (super) collective coordinate expansion which begins

$$\begin{aligned} A_\mu(x, \sigma) &= A^0_\mu(x, Z(\sigma)) + \cdots \\ \chi(x, \sigma) &= \lambda^i(\sigma)\chi_i(x, Z(\sigma)) + \cdots \,, \end{aligned} \tag{26}$$

where (x, σ) is a coordinate on $X \times M^6$ and the bosonic (fermionic) collective coordinates Z^i (λ^i) are dynamical fields on the soliton worldbrane. $\lambda^i = \lambda^i_+ + \lambda^i_-$ is a doublet of six-dimensional Weyl fermions obeying $\lambda^j = -i\mathscr{T}_k{}^j \lambda^k$. Under $SO(5, 1)$ worldbrane Lorentz transformations, the λ^i's transform into one another. It is possible to assemble the λ^i's into $m/2$ six-dimensional symplectic Majorana–Weyl spinors transforming covariantly under $SO(5, 1)$. However $SO(5, 1)$ covariance is not necessary for our purposes, and the supersymmetric $SO(5, 1)$ covariant formulation introduces a number of extraneous complications which obscure the connection with Donaldson theory. Our expressions will have manifest invariance under two-dimensional super-Poincaré transformations which we take to act in the σ^0, σ^1 plane. The subscripts on $\lambda^i_\pm$ denote the corresponding two-dimensional chirality. In accord with this and as a further simplification, we henceforth restrict λ^i and Z^i to depend only on σ^0 and σ^1.

The effective action S_{eff} can be expanded in powers of inverse length. Taking Z^i to be dimensionless and λ^i to have dimensions of $(\text{length})^{-1/2}$ this is an expansion in the parameter $n = n_\partial + n_f/2$ with n_∂ the number of σ derivatives and n_f the number of fermion fields. The expansion (26) solves the spacetime equations of motion to order $n = 0$, while the leading terms in S_{eff} are $n = 2$. To have a consistent action we must still solve the spacetime equations to order $n = 1$. This requires that the component of the gauge field tangent to the worldbrane acquires the term

$$A_a = \nabla_a Z^i \varepsilon_i - \frac{1}{2}\phi_{ij}\bar{\lambda}^i \Gamma_a \lambda^j \tag{27}$$

with ϕ_{ij} given by (19).

The leading order worldbrane action may now be derived by substitution of the expansion (26), (27) of A and χ into the ten dimensional action (7) and integration over X, the transverse space. Using

$$F_{a\mu} = \nabla_a Z^i \delta_i A_\mu - s_{[i}\delta_{j]} A_\mu \bar{\lambda}^i \Gamma_a \lambda^j \,, \tag{28}$$

one has the bosonic term

$$\begin{aligned} S^b_{\text{eff}} &= -\frac{2}{\alpha'^3}\int d^4x \sqrt{g}\, e^{-2\phi} \int d^6\sigma \operatorname{tr}(\delta_i A_\mu \delta_j A_\nu) g^{\mu\nu} \nabla_a Z^i \nabla^a Z^j \\ &= -\frac{2}{\alpha'^3}\int d^6\sigma \mathscr{G}_{ij} \nabla^a Z^i \nabla_a Z^j \,. \end{aligned} \tag{29}$$

Including the fermionic terms gives the $d = 6$ supersymmetric sigma model with target space $\mathcal{M}_N$:

$$S_{\text{eff}} = -\frac{2}{\alpha'^3}\int d^6\sigma \mathcal{G}_{ij}(\nabla^a Z^i \nabla_a Z^j + 2\bar{\lambda}^i \Gamma^a(\nabla_a \lambda^j + \nabla_a Z^k \Gamma^j_{kl}\lambda^l)) + (\text{fermi})^4 \, . \tag{30}$$

Because we have maintained only an $SO(1, 1)$ subgroup of $SO(5, 1)$ as a manifest symmetry of (30), only two of the eight supersymmetries are manifest.

For values of Z^i corresponding to N widely separated instantons, (30) is approximated by N separate terms describing the dynamics of each of the N fivebranes. The full action (30) includes additional fivebrane interaction terms.

Classically, there is one static ground state for each point $Z^i \in \mathcal{M}_N$. However quantum mechanical groundstates involve a superposition over Z^i eigenstates. As explained by Witten [11] the supercharges of the supersymmetric sigma model (30) act at zero momentum as the exterior derivative on the target space $\mathcal{M}_N$, and the general supersymmetric ground state can be written in the form

$$|\mathcal{O}^s\rangle = \mathcal{O}^s|0\rangle, \quad \mathcal{O}^s = \mathcal{O}^s_{i_1\cdots i_p}(Z)\psi^{i_1}\cdots\psi^{i_p} \tag{31}$$

where $\mathcal{O}^s_{i_1\cdots i_p}$ is a harmonic form on $\mathcal{M}_N$, $\psi^i = \mathrm{Re}\lambda^i_+ + i\mathrm{Re}\lambda^i_-$ and the state $|0\rangle$ is chosen so that

$$(\psi^i)^*|0\rangle = 0 \, . \tag{32}$$

In summary, the low-energy dynamics of N-fivebranes is described by a supersymmetric sigma-model with target space $\mathcal{M}_N$ and the ground states of this system are cohomology classes on $\mathcal{M}_N$.

III. The Donaldson Map and Polynomial

In addition to the leading term (30) in S_{eff}, there are a number of terms representing interactions between spacetime fields which are not localized on the fivebrane and the localized worldbrane fields appearing in S_{eff}. This corresponds to the fact that the state of the fivebrane can be perturbed by scattering with spacetime fields. For the special limiting case of zero-momentum spacetime fields, energy conservation implies that scattering can only induce transitions among the groundstates. Zero momentum spacetime axions are characterized by harmonic forms on X, so axion scattering is a map involving $H(X)$ and $H(\mathcal{M}_N(X))$. This strongly suggests that the scattering should be given by the Donaldson map. In the following two subsections we demonstrate that this is indeed the case by two separate methods. Subsection (A) contains a straightforward continuation of the collective coordinate expansion. In subsection (B), it is observed that the Donaldson map has a geometric interpretation as a certain connection coefficient derivable from a Kähler potential. Its form is then deduced in a few lines using supersymmetry.

A. Derivation by Collective Coordinate Expansion. Consider the interaction of a low-momentum spacetime axion with the worldbrane fermions. Other spacetime fields can be treated in an analogous fashion. Spacetime axions are described by the potential

$$B_{\mu\nu} = Y(\sigma)T_{\mu\nu} \, , \tag{33}$$

where T is a harmonic two form on X, and Y depends only on σ^0, σ^1. The ten-dimensional coupling

$$\mathscr{L}'_{\text{int}} = \frac{1}{2\alpha'^3} e^{-2\phi} \partial_M B_{NP} \operatorname{tr} \bar{\chi} \Gamma^{MNP} \chi \tag{34}$$

appearing in (7) descends to a coupling in S_{eff} between one spacetime axion and two worldbrane fermions. Substituting (33) and (26) into (34) and integrating out the zero mode wave function one finds

$$\begin{aligned}\mathscr{L}'_{\text{int}} &= \frac{1}{2\alpha'^3} \int d^4x \sqrt{g} e^{-2\phi} T_{\mu\nu} \operatorname{tr}(\bar{\lambda}^i \chi_i^\dagger \Gamma^a \Gamma^{\mu\nu} \lambda^j \chi_j) \nabla_a Y \\ &= \frac{2}{\alpha'^3} \mathcal{O}'_{ij} \bar{\lambda}^i \Gamma^a \lambda^j \nabla_a Y ,\end{aligned} \tag{35}$$

where

$$\mathcal{O}'_{ij}(T) \equiv \frac{1}{4} \int d^4x \sqrt{g} e^{-2\phi} \chi_i^\dagger \Gamma^{\mu\nu} T_{\mu\nu} \chi_j . \tag{36}$$

The lambda bilinear appearing in (35) can be seen, using (32), to be equivalent to $\psi^i \psi^j$ when acting on a vacuum state. Substituting the formula (23) for χ_i one has, after some algebra,

$$\mathcal{O}'_{ij}(T) = \int_X \operatorname{tr}(\delta_i A \wedge \delta_j A) \wedge T_+ , \tag{37}$$

where T_+ is the self-dual part of T. It is easily checked that $\mathcal{O}'_{ij}$ is not closed and so does not represent a cohomology class on $\mathscr{M}_N$.

This is remedied by the observation that (34) is not the only term in S_{10} which gives rise to the coupling of a spacetime axion to worldbrane fermions. Because of the bilinear term in the expansion (27) for A_a, such couplings also arise from the ten-dimensional term

$$\frac{2}{\alpha'^3} e^{-2\phi} \partial_M B_{NP} \omega_{3Y}^{MNP} . \tag{38}$$

From the expansion for A_a, the relevant term in ω_{3Y} is

$$\omega_{a\mu\nu}^{3Y} = -\frac{1}{2} \operatorname{tr}(\phi_{ij} \bar{\lambda}^i \Gamma_a \lambda^j F_{\mu\nu}) . \tag{39}$$

This formula may then be used to reduce (38) to a coupling in S_{eff}. The result may be added to (35) to give the total coupling of a single spacetime axion of the form (33) to two worldbrane fermions:

$$\mathscr{L}_{\text{int}} = \frac{2}{\alpha'^3} \mathcal{O}_{ij} \bar{\lambda}^i \Gamma^a \lambda^j \nabla_a Y , \tag{40}$$

where

$$\mathcal{O}_{ij}(T) = \int_X \operatorname{tr}(\delta_i A \wedge \delta_j A - \phi_{ij} F) \wedge T_+ . \tag{41}$$

(41) has several important properties. The first is that $\mathcal{O}$ is closed on $\mathcal{M}_N$:

$$\partial_{[i}\mathcal{O}_{jk]} = -\int_X \mathrm{tr}(D\phi_{[ij} \wedge \delta_{k]}A + \phi_{[ij}s_{k]}F) \wedge T_+ = 0\,, \tag{42}$$

upon integration by parts on X. Secondly, if T_+ is trivial in $H^2(X)$ so that $T_+ = dU$ one has

$$\begin{aligned}\mathcal{O}_{ij}(dU) &= \int_X \mathrm{tr}(D\delta_{[i}A \wedge \delta_{j]}A + s_{[i}\delta_{j]}A \wedge F) \wedge U \\ &= -2\partial_{[i}\int_X \mathrm{tr}(\delta_{j]}A \wedge F) \wedge U\,,\end{aligned} \tag{43}$$

i.e. the image of an exact form on X is an exact form on $\mathcal{M}_N$. Thus (41) gives a map from the cohomology of X into the cohomology of $\mathcal{M}_N$. Using Poincaré duality (41) may be written:

$$\mathcal{O}_{ij}(\Sigma) = \int_\Sigma \mathrm{tr}(\delta_i A \wedge \delta_j A - \phi_{ij}F)\,, \tag{44}$$

where Σ is the surface Poincaré dual to T_+. This is a standard expression [4] for the Donaldson map from $H_2(X) \to H^2(\mathcal{M}_N(X))$ in terms of differential forms, and is identical to that derived in the context of topological quantum field theory by Witten [5].

B. Derivation from Kähler Geometry. In this subsection we will provide an alternate derivation of (40) which is less direct, but shorter and provides some geometrical insight. For these purposes it is convenient to view the solution (9)–(11) not as N fivebranes on X, but as a "compactification" from ten to six dimensions. The low-energy action then contains, in addition to Z^i, complex massless moduli fields Y^α that parameterize the complexified Kähler cone (a subset of $H^2(X, C)$). The imaginary part of Y^α is the axion associated to the harmonic two form T_α on X. (The α index was suppressed in the previous subsection.) Six-dimensional supersymmetry then implies that the metric appearing in the kinetic term for the moduli fields is Kähler, or equivalently in complex coordinates,

$$J_{I\bar{J}} = i\partial_I\partial_{\bar{J}}\mathcal{K}\,. \tag{45}$$

To give an expression for $\mathcal{K}$, we note that on a Kähler manifold a closed (p, p) form $H_{p,p}$ is locally the curl of a $2p-1$ form:

$$H_{p,p} = dG_{2p-1} = (\partial + \bar{\partial})(G_{p,p-1} + G_{p-1,p})\,. \tag{46}$$

Since the left-hand side of (46) is of type (p, p),

$$\partial G_{p,p-1} = \bar{\partial} G_{p-1,p} = 0\,, \tag{47}$$

so that locally $G_{p,p-1} = \partial F_{p-1,p-1}$. We conclude that locally a closed (p, p) form can always be written in the form

$$H_{p,p} = i\partial\bar{\partial}F_{p-1,p-1}\,. \tag{48}$$

F is real if H is, and is determined up to a closed $(p-1, p-1)$ form.

$\mathcal{K}$ is then given by

$$\mathcal{K} = \frac{1}{2}\int_X E \wedge J - \ln\int_X J \wedge J\,, \tag{49}$$

where J is the Kähler form on X and E is a solution of

$$\mathrm{tr}\, F \wedge F = i\partial\bar{\partial}E \,. \tag{50}$$

E is related to the two-dimensional WZW action and can not be simply expressed as a function of A. A formula for $\mathscr{K}$ as a conformal field theory correlation function is given in [12].

The second variation of $\mathscr{K}$ can be computed by noting that

$$\partial_I \partial_{\bar{J}} \mathrm{tr}\, F \wedge F = 2\bar{\partial}\partial\, \mathrm{tr}(\delta_I A \wedge \delta_{\bar{J}} A - \phi_{I\bar{J}} F) \,. \tag{51}$$

This determines the second variation of E up to a closed two-form on X times a closed two-form on $\mathscr{M}_N$. The ambiguity in the definition of E may thus be fixed so that

$$\begin{aligned} i\partial_I \partial_{\bar{J}} \mathscr{K} &= -\int_X \mathrm{tr}(\delta_I A \wedge \delta_{\bar{J}} A - \phi_{I\bar{J}} F) \wedge J \\ &= \int \sqrt{g}\, J^{\mu\nu} \mathrm{tr}\, \delta_I A_\mu \delta_{\bar{J}} A_\nu \\ &= \mathscr{T}_{I\bar{J}} \,, \end{aligned} \tag{52}$$

where in the last line we have used $J \cdot F = 0$. The coupling of Y to two λ''s is then determined by supersymmetry to be proportional to the mixed Christoffel connection (as in (30) with an index lowered) on $H^2(X, C) \times \mathscr{M}_N$:

$$\mathscr{L}_{\mathrm{int}} = \frac{-2i}{\alpha'^3} V_a Y^\alpha \bar{\lambda}'^{\bar{J}} \Gamma^a \lambda'^I \Gamma_{\bar{J}I\alpha} \,. \tag{53}$$

In Kähler geometry, the Christoffel connection is given by

$$\Gamma_{\bar{J}I\alpha} = \mathscr{K}_{,\bar{J}I\alpha} \,. \tag{54}$$

Differentiating (52) one more time and using $\partial_\alpha J = T_\alpha$ we easily recover the formula (41) of the previous section

$$-i\Gamma_{\bar{J}I\alpha} \doteq \int_X \mathrm{tr}(\delta_I A \wedge \delta_{\bar{J}} A - \phi_{I\bar{J}} F) \wedge T_\alpha = \mathscr{O}_{\bar{J}I\alpha} \tag{55}$$

except for the absence of a projection on to the self-dual part of T_α. This difference can be accounted for if λ' is related to λ of the previous section by the field redefinition

$$\lambda' = e^{-iX_\alpha Y^\alpha} \lambda \,, \tag{56}$$

where $X_\alpha \equiv \int T_\alpha \wedge J / \int J \wedge J$.

C. The Donaldson Map as the Second Chern Class. It is known [4, see also 13, 14] that $\mathscr{O}$ can be written as integrals of $\mathrm{tr}\, \mathscr{F}^2$ for a certain curvature $\mathscr{F}$. The fact that $\mathscr{O}$ couples to axions then strongly suggests that the observations in this paper are connected with the structure of anomalies in string theory. While we do not understand this connection, in the hope that it might be understood later we record here this representation of $\mathscr{O}$. Introduce a connection $\mathscr{D}$ (on the universal bundle over $\mathscr{M}_N$) by

$$\mathscr{D} = dZ^i s_i + dx^\mu D_\mu \,. \tag{57}$$

Then the associated curvature $\mathscr{F} = \mathscr{D}^2$ has components

$$\mathscr{F}_{ij} = \phi_{ij}, \quad \mathscr{F}_{i\mu} = \delta_i A_\mu, \quad \mathscr{F}_{\mu\nu} = F_{\mu\nu} . \tag{58}$$

Now consider the integral $c_2(\Sigma)$ of the second Chern class of $\mathscr{F}$ over a four surface Σ in $\mathscr{M}_N \times X$,

$$c_2(\Sigma) = \frac{1}{8\pi^2} \int_{\Sigma} \operatorname{tr} \mathscr{F} \wedge \mathscr{F} . \tag{59}$$

If Σ is a product of a two surface $\Sigma_{\mathscr{M}}$ in $\mathscr{M}_N$ with a two surface Σ_X in X one finds

$$c_2(\Sigma_{\mathscr{M}} \times \Sigma_X) = -\frac{1}{8\pi^2} \int_{\Sigma_{\mathscr{M}}} \mathcal{O}_{ij}(\Sigma_X) dZ^i \wedge dZ^j , \tag{60}$$

i.e. the Donaldson map $H_2(X) \to H^2(\mathscr{M}_N)$ is a period of the second Chern class. A similar result holds for the maps $H_\alpha(X) \to H^{4-\alpha}(\mathscr{M}_N)$.

D. The Donaldson Polynomial. A physical realization of the Donaldson polynomial may be obtained by considering multiple axion scattering. Let $|m\rangle$ be the ground state corresponding to the top rank form on $\mathscr{M}_N$:

$$|m\rangle = \varepsilon_{i_1 \cdots i_m} \lambda^{i_1} \cdots \lambda^{i_m} |0\rangle . \tag{61}$$

The amplitude for scattering p axions associated to the classes $T_1 \ldots T_p$ off the state $|0\rangle$ and winding up in the state $|m\rangle$ is proportional to

$$A(T_1, \ldots, T_p) = \langle m| \mathcal{O}^{T_1} \cdots \mathcal{O}^{T_p} |0\rangle . \tag{62}$$

It is easily seen that this reduces to

$$A(T_1, \ldots, T_p) = \int_{\mathscr{M}_N} \mathcal{O}(T_1) \wedge \cdots \wedge \mathcal{O}(T_p) \tag{63}$$

which is the Donaldson polynomial.

While our derivation from string theory of (41, 63) was only valid for $c_1(X) \geqq 0$, it is known [4] that (41) and (63) are representations of the Donaldson map and polynomial for any algebraic X. It would be interesting to try to extend our derivation to the more general case.

IV. Conclusion

We have shown that the Donaldson map appears explicitly as a coupling in the low-energy action for heterotic string theory in the soliton sector. This implies the Donaldson polynomial can be measured by scattering massless fields and solitons. This realization leads to concrete formulae for the Donaldson map and polynomial which are equivalent to, and provide a new perspective on, formulae derived by Witten in the framework of topological Yang–Mills theory. It also led to an interpretation of the Donaldson map as a Kähler–Christoffel connection on $H^2(X, C) \times \mathscr{M}_N(X)$.

The fact that this scattering is a map between cohomology classes was insured by zero-momentum worldbrane supersymmetry, which acts like the exterior derivative on $\mathscr{M}_N$. This should be contrasted with topological Yang–Mills theory where the exterior derivative on the instanton moduli space is constructed in terms of a BRST operator.

Our work suggests a number of generalizations and applications. Perhaps this connection can provide new insights into, or stringy interpretations of, the various theorems on the structure of four-manifolds which follow from Donaldson's work. Alternately, the remarkable properties of the Donaldson polynomial may translate into interesting properties of the fivebrane-axion S-matrix, or even have implications for the closely related problem of instanton-induced supersymmetry breaking in string theory.

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