

Generalized Classical BRST Cohomology and Reduction of Poisson Manifolds

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Abstract. In this paper, we formulate a generalization of the classical BRST construction which applies to the case of the reduction of a Poisson manifold by a submanifold. In the case of symplectic reduction, our procedure generalizes the usual classical BRST construction which only applies to symplectic reduction of a symplectic manifold by a coisotropic submanifold, i.e. the case of reducible “first class” constraints. In particular, our procedure yields a method to deal with “second-class” constraints. We construct the BRST complex and compute its cohomology. BRST cohomology vanishes for negative dimension and is isomorphic as a Poisson algebra to the algebra of smooth functions on the reduced Poisson manifold in zero dimension. We then show that in the general case of reduction of Poisson manifolds, BRST cohomology cannot be identified with the cohomology of vertical differential forms.

1. Introduction

Classical BRST cohomology has a long history in the physics literature, e.g. [1]. Although its origins are in the context of quantum field theory, it is now known that classical BRST cohomology is a cohomology theory that contains all of the information of the symplectic reduction of a symplectic manifold by a closed and embedded coisotropic submanifold [2, 3]. In the language of Dirac [4], this corresponds to symplectic reduction arising from (possibly reducible) “first class constraints.” The classical BRST complex is constructed using only purely algebraic properties of the Poisson algebra of smooth functions on the original (unreduced) symplectic manifold and some of its ideals. Furthermore, since the classical BRST complex is a Poisson superalgebra and the differential a Poisson derivation, classical BRST cohomology inherits the structure of a Poisson

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super-algebra. Classical BRST cohomology is isomorphic (as Poisson algebras) in zero dimension to the algebra of smooth functions on the symplectic reduction [2]. When the (symplectic) normal bundle of the coisotropic submanifold is a trivial bundle, classical BRST cohomology in nonnegative dimension is isomorphic to the cohomology of vertical differential forms with respect to the null foliation [5]. The results in [2] and [3] show that this is the case even if the normal bundle is not a trivial bundle.

In this paper, we transcribe the procedure of the reduction of a Poisson manifold (M, P) by a closed and embedded submanifold into the language of Poisson algebras. Inspired by this example, we give an algebraic definition of the reduction of a Poisson algebra by an ideal. In the case of a Poisson manifold, our algebraic definition gives rise to a notion to the reduction of $C^\infty(M)$ by any ideal, whether this ideal arises as the ideal of functions which vanish on a closed and embedded submanifold or not. In particular, we show that for certain special ideals (“coisotropic ideals”), we can generalize classical BRST cohomology. Our construction has some important applications. Consider the case of the reduction of a symplectic manifold by an arbitrary closed and embedded submanifold. In the language of Dirac, the ideal of functions which vanish on this submanifold I is generated by a collection of first class constraints and second class constraints. Our method tells us how to construct the classical BRST complex in this general case. The idea is to replace the original collection of constraints by a new set of “first class constraints.” Although this new set of constraints is guaranteed to exist, our method does not explicitly construct them, in general. However, in the case where the submanifold of the symplectic manifold is itself a symplectic manifold (i.e. I is generated by only second class constraints), there is a method to explicitly construct the new set of constraints which would, in applications to field theory, preserve any Lorentz covariance type properties of the original collection of constraints. An interesting consequence of our generalization is that classical BRST cohomology is not generally isomorphic to the cohomology of vertical differential forms although it is isomorphic as Poisson algebras to the algebra of smooth functions on the reduced Poisson manifold in zero dimension.

This paper is organized as follows. In Sect. 2, we transcribe the procedure of reduction of a Poisson manifold by a submanifold into purely Poisson algebraic terms. In Sect. 3, we review the Koszul–Tate resolution. In Sect. 4, we show that in the special case of symplectic reduction by a symplectic submanifold, our procedure results in infinitely reducible constraints. In Sect. 5, we construct the space of BRST cochains and show that it forms a Poisson superalgebra. In Sect. 6, we construct the BRST charge inductively. In Sect. 7 we compute the cohomology explicitly. In Sect. 8, we explain why BRST cohomology is not vertical cohomology. Finally, Sect. 9 contains some concluding remarks as well as some possible avenues for future research.

2. Reduction of Poisson Manifolds

In this section, we review the reduction of a Poisson manifold by a submanifold. It is a procedure which becomes, in the case where the Poisson manifold is a symplectic manifold, symplectic reduction by a submanifold. This reduction is done by transcription of this geometric procedure into the language of Poisson algebras (see

[5] and [6]). The algebraic formulation generalizes the geometric one since it can be applied to cases where the reduced Poisson manifold is not smooth.

Let (M, P) be an m -dimensional Poisson manifold, i.e. M is a smooth m -dimensional manifold and P is a bivector in $\Lambda^2(TM)$ such that the Schouten bracket of P with itself vanishes. Given a Poisson manifold, the *Poisson bracket* of two smooth functions on M , f and g , is given by

$$[f, g] = P(df, dg) . \tag{2.1}$$

Since P is in $\Lambda^2(TM)$, the Poisson bracket is antisymmetric. Furthermore, since the exterior derivative acts like a derivation on $C^\infty(M)$, we have

$$[f, gh] = [f, g]h + g[f, h] \tag{2.2}$$

for all f, g, h in $C^\infty(M)$. Finally, the fact that the Schouten bracket of P with itself vanishes is equivalent to the Jacobi identity, i.e.

$$[f, [g, h]] = [[f, g], h] + [g, [f, h]] \tag{2.3}$$

for all f, g, h in $C^\infty(M)$. In other words, $C^\infty(M)$ forms a *Poisson algebra*, i.e. $C^\infty(M)$ is an associative and commutative algebra with unit with respect to pointwise multiplication, a Lie algebra with respect to the Poisson bracket, and the two operations intertwine via Eq. (2.2). The Poisson algebra $C^\infty(M)$ completely characterizes the Poisson manifold (M, P) . Furthermore, all Poisson structures on $C^\infty(M)$ arise from endowing M with a suitable Poisson structure.

Given a Poisson manifold (M, P) , there is a map $P_\# : T^*M \rightarrow TM$ given by

$$P_\# \alpha = \iota(\alpha)P \tag{2.4}$$

for all α in T_m^*M and points m in M where $\iota(\alpha)$ is the interior product. $P_\#$ allows us to define the *hamiltonian vector field associated to a function f in $C^\infty(M)$* by $X_f = P_\#(df)$. In terms of Poisson algebras, this definition is equivalent to $X_f = [f, \cdot]$ since Eq. (2.2) insures that $[f, \cdot]$ is a derivation with respect to pointwise multiplication in $C^\infty(M)$ and, hence, a vector field on M . Furthermore, (2.3) implies that

$$X_{[f, g]} = [X_f, X_g] \tag{2.5}$$

for all f, g in $C^\infty(M)$, where the bracket on the right-hand side is the Lie bracket. In other words, the map $f \mapsto X_f$ is a Lie algebra homomorphism from $C^\infty(M) \rightarrow \Gamma(TM)$.

Consider the closed and embedded submanifold $i: M_0 \hookrightarrow (M, P)$ which has codimension k . The submanifold M_0 is completely characterized by its associated algebra of smooth functions $C^\infty(M_0)$. Let us denote the ideal of functions in $C^\infty(M)$ which vanish on M_0 by I . Since M_0 is a closed and embedded submanifold of M , we have the isomorphism of (associative) algebras

$$\frac{C^\infty(M)}{I} \xrightarrow{\cong} C^\infty(M_0) \tag{2.6}$$

given by $[[f]] \mapsto f|_{M_0}$. This map is certainly well-defined. It is injective since the elements in $C^\infty(M)/I$ which give rise to the zero map on M_0 are those which vanish on M_0 and it is surjective since any smooth function on M_0 arises from the restriction of some smooth function on M . We can do a similar construction with vector fields. Any vector field on M_0 arises as the restriction of some vector field on

M . However, the restriction of a vector field on M to M_0 is not, in general, a vector field on M_0 since it need not be tangent to M_0 . The space of all vector fields on M which restricts to a vector field on M_0 is given by

$$\mathcal{N}(I) = \{X \in \Gamma(TM) \mid X(i) \in I \ \forall i \in I\} . \tag{2.7}$$

Of course, two vector fields in $\mathcal{N}(I)$ may restrict to the same vector field on M_0 . This happens only if their difference vanishes on M_0 . Since vector fields in $\mathcal{N}(I)$ which vanish on M_0 are precisely the elements in $I\mathcal{N}(I)$, we have the isomorphism of Lie algebras and of $C^\infty(M)/I \cong C^\infty(M_0)$ -modules

$$\frac{\mathcal{N}(I)}{I\mathcal{N}(I)} \xrightarrow{\cong} \Gamma(TM_0) \tag{2.8}$$

via the map $[[X]] \mapsto X|_{M_0}$. Equation (2.8) has an algebraic interpretation, as well. Vector fields on M and M_0 can be identified with the derivations on $C^\infty(M)$ and $C^\infty(M_0)$, respectively. The isomorphism $C^\infty(M_0) \xrightarrow{\cong} C^\infty(M)/I$ tells us that derivations on $C^\infty(M_0)$ should be induced from derivations on $C^\infty(M)$. The Lie subalgebra $\mathcal{N}(I)$ consists of precisely those derivations in $C^\infty(M)$ which respect the ideal I and, hence, induce derivations on $C^\infty(M)/I$. The ideal $I\mathcal{N}(I)$ consists of those derivations in $\mathcal{N}(I)$ which induce the zero map on $C^\infty(M)/I$.

The hamiltonian vector fields on M are just the inner derivations of $C^\infty(M)$. It is natural to ask when a hamiltonian vector field restricts to a vector field on M_0 . Suppose that X_f restricts to a vector field on M_0 then X_f must belong to $\mathcal{N}(I)$. This means that $X_f(i) = [f, i]$ must belong to I for all i in I . In other words, functions whose hamiltonian vector fields when restricted to M_0 are vector fields on M_0 are those functions in the *normalizer of I* denoted by

$$N(I) = \{f \in C^\infty(M) \mid [f, i] \in I \ \forall i \in I\} . \tag{2.9}$$

Notice that $N(I)$ forms a Poisson subalgebra of $C^\infty(M)$. It will play an important role in what follows.

Under certain conditions, M_0 has an associated involutive distribution such that the space of leaves of its associated foliation inherits the structure of a Poisson manifold. Let us describe this situation in more detail.

Denote the pullback of TM and T^*M to M_0 via the inclusion map by $i^{-1}TM$ and $i^{-1}T^*M$, respectively. The Poisson structure P on M can be pulled back via the inclusion map to an element in $\mathcal{A}^2(i^{-1}T^*M)$ which we shall also denote by P to avoid notational clutter. It allows us to define a rank k subbundle of $i^{-1}T^*M$ whose fibers consist of 1-forms which vanish when evaluated on vectors tangent to M_0 called the *annihilator bundle* (or the *conormal bundle*) of M_0 . It is denoted by $\text{Ann}(TM_0) \rightarrow M_0$ and its fibers are given by

$$\text{Ann}_m(TM_0) = \{\alpha \in i^{-1}T_m^*M \mid \alpha(v) = 0 \ \forall v \in T_mM_0\} \tag{2.10}$$

for all points m in M_0 . At every point m in M_0 , let us define

$$T_mM_0^\perp = P_\# \text{Ann}_m(TM_0) = \{P_\# \alpha \mid \alpha \in \text{Ann}_m(TM_0)\} . \tag{2.11}$$

Let us assume that TM_0^\perp has constant rank so that TM_0^\perp forms a subbundle of $i^{-1}TM$. The *null distribution* of M_0 is given by $m \mapsto V_m$ for all m in M_0 , where

$$V_m = T_mM_0 \cap T_mM_0^\perp . \tag{2.12}$$

Let us assume that V has constant rank so that V forms a subbundle of TM_0 . We will show that the null distribution is an involutive distribution over M_0 .

Let us begin by recalling several facts. First of all, given any i in I , $di|_{M_0}$ belongs to $\Gamma(\text{Ann}(TM_0))$ since for all $[[v]] \in \mathcal{N}(I)/I\mathcal{N}(I)$, $di(v) = v(i)$ which belongs to I by the definition of $\mathcal{N}(I)$ and, therefore, vanishes when restricted to M_0 . Furthermore, the exterior derivative of an element in I^2 always vanishes when restricted to M_0 . Therefore, we have the well-defined map

$$I/I^2 \xrightarrow{\cong} \Gamma(\text{Ann}(TM_0)) \tag{2.13}$$

given by $[[i]] \mapsto di|_{M_0}$. This map is readily seen to be an isomorphism by showing the isomorphism locally and then globalizing it by using partitions of unity. Thus, all sections of TM_0^\perp are restrictions to M_0 of hamiltonian vector fields of elements in I . Since the sections of the null distribution are the sections of $TM_0 \cap TM_0^\perp$ and $N(I)$ consists of all functions on M whose hamiltonian vector fields when restricted to M_0 are vector fields on M_0 , all sections of $TM_0 \cap TM_0^\perp$ are restrictions to M_0 of hamiltonian vector fields of elements in

$$I' = N(I) \cap I . \tag{2.14}$$

Notice that I' is naturally a Poisson subalgebra of $N(I)$ and, therefore,

$$[X_{i_1}, X_{i_2}] = X_{[i_1, i_2]} \tag{2.15}$$

for all i_1, i_2 in I' . This proves that V is an involutive distribution on M_0 .

By the Frobenius theorem, associated to the involutive distribution V there exists a foliation of M_0 by maximal connected submanifolds (called *leaves*) such that the tangent space to each leaf is the restriction of V . Let us denote the space of leaves of the foliation by \tilde{M} . Let us assume, furthermore, that conditions are such that the projection map $\pi: M_0 \rightarrow \tilde{M}$ which takes each point on M_0 and projects it into the leaf containing it is a smooth map. In this case, we will show that \tilde{M} has an induced Poisson structure \tilde{P} . We will construct \tilde{P} by inducing a Poisson algebra structure on $C^\infty(\tilde{M})$ from the Poisson algebra structure on $C^\infty(M)$.

The functions in $C^\infty(M)$ which induce smooth functions on \tilde{M} are those which, when restricted to M_0 are constant along each leaf of the foliation. Since each leaf is connected and has tangent vector fields which are restrictions of hamiltonian vector fields of elements in I' , a function f in $C^\infty(M)$ induces a function in $C^\infty(\tilde{M})$ if and only if $\mathcal{L}_{X_i} f|_{M_0} = 0$ for all i in I' . Equivalently, f induces a smooth function on \tilde{M} if and only if f belongs to $N(I', I)$, where

$$N(I', I) = \{f \in C^\infty(M) | [f, i'] \in I, \forall i' \in I'\} . \tag{2.16}$$

Furthermore, all functions in $C^\infty(\tilde{M})$ are induced from functions in $N(I', I)$ since any smooth function \tilde{f} on \tilde{M} , can be extended to a smooth function $\pi^*\tilde{f}$ on M_0 which projects to it. Since two functions in $N(I', I)$ restrict to the same function on M_0 and, therefore, induce the same function on \tilde{M} if and only if they differ by an element of I , we have the isomorphism of associative algebras

$$C^\infty(\tilde{M}) \xrightarrow{\cong} \frac{N(I', I)}{I} . \tag{2.17}$$

This is not obviously an isomorphism of Poisson algebras since $N(I', I)/I$ does not appear to naturally inherit the structure of a Poisson algebra from $C^\infty(M)$ as it

stands. In order to obtain a Poisson algebra for the right-hand side of this equation, we need to delve deeper into the algebraic structure of $N(I', I)$.

Suppose that f is an element of $N(I', I)$, then for all i' in I , $df(X_{i'}) = [i', f]$ belongs to I and, therefore, vanishes when restricted to M_0 . In other words, we have

$$df|_{M_0} \in \Gamma(\text{Ann}(TM_0 \cap TM_0^\perp)) \tag{2.18}$$

since all sections of the null distribution are given by restrictions to M_0 of hamiltonian vector fields of elements in I' . However, it is a general fact from linear algebra that if V is a (finite dimensional) vector space and W_1 and W_2 are subspaces then

$$\text{Ann}(W_1 \cap W_2) = \text{Ann}(W_1) + \text{Ann}(W_2) . \tag{2.19}$$

We can see this as follows. Consider α in $\text{Ann}(W_1)$ and β in $\text{Ann}(W_2)$ then $\alpha + \beta$ certainly lies in $\text{Ann}(W_1 \cap W_2)$ so that $\text{Ann}(W_1) + \text{Ann}(W_2) \subseteq \text{Ann}(W_1 \cap W_2)$. The equality follows from some linear algebra and by counting dimensions. Therefore, we obtain the result

$$df|_{M_0} \in \Gamma(\text{Ann}(TM_0)) + \Gamma(\text{Ann}(TM_0^\perp)) . \tag{2.20}$$

Since $\Gamma(\text{Ann}(TM_0))$ consists of the restriction to M_0 of the exterior derivative of elements in I , there exists an i in I such that

$$(df - di)|_{M_0} \in \Gamma(\text{Ann}(TM_0^\perp)) . \tag{2.21}$$

Let us use another fact. Consider α in $\text{Ann}_m(TM_0^\perp) = \text{Ann}_m(P_\#(\text{Ann}(TM_0)))$, where m is a point in M_0 then $\alpha(P_\# \beta) = 0$ for all β in $\text{Ann}_m(TM_0)$. However, $0 = \alpha(P_\# \beta) = P(\beta, \alpha) = -P(\alpha, \beta) = -\beta(P_\# \alpha)$ for all β in $\text{Ann}_m(TM_0)$ means that $P_\# \alpha$ must belong to $T_m M_0$. In other words, $P_\# \text{Ann}_m(TM_0^\perp) \subseteq T_m M_0$ for all m in M_0 . Applying $P_\#$ to both sides of Eq. (2.21) yields

$$X_{(f-i)}|_{M_0} \in \Gamma(TM_0) . \tag{2.22}$$

This tells us that $f - i$ belongs to $N(I)$. In other words, f belongs to $N(I) + I$. Combining this result with (2.17), we get

$$C^\infty(\tilde{M}) \xrightarrow{\cong} \frac{N(I) + I}{I} . \tag{2.23}$$

However, the right-hand side is still not obviously a Poisson algebra. This can be remedied by using a basic fact from linear algebra. Given two subspaces W_1 and W_2 of a vector space V , there is the canonical isomorphism

$$\frac{W_1 + W_2}{W_2} \xrightarrow{\cong} \frac{W_1}{W_1 \cap W_2} \tag{2.24}$$

defined by $[[w_1 + w_2]] \mapsto [[w_1]]$ for all w_1 in W_1 and w_2 in W_2 . Using this fact on the right-hand side of Eq. (2.23), we obtain the isomorphism $C^\infty(\tilde{M}) \xrightarrow{\cong} N(I)/I'$. The right-hand side is naturally a Poisson algebra since $N(I)$ is a Poisson subalgebra of $C^\infty(M)$ and I' is a *Poisson ideal* of $N(I)$, i.e. I' is an ideal with respect to both the Lie bracket and multiplication operations in $N(I)$. We use this isomorphism to endow $C^\infty(\tilde{M})$ with the structure of a Poisson algebra thereby completing the process of inducing a Poisson structure \tilde{P} on \tilde{M} from the Poisson manifold (M, P) . Therefore,

this isomorphism induces a Poisson structure \tilde{P} on \tilde{M} . We have just shown the following result.

Theorem 2.25. *Let $i: M_0 \hookrightarrow (M, P)$ be a closed and embedded submanifold of a Poisson manifold. Assume that TM_0^\perp and the null distribution have constant rank. Furthermore, let us assume that the canonical projection map of a point in M_0 into the leaf that contains it, $\pi: M_0 \rightarrow \tilde{M}$, is a smooth map. In this case, we have the isomorphism*

$$C^\infty(\tilde{M}) \xrightarrow{\cong} \frac{N(I)}{I'} \tag{2.26}$$

where I is the ideal of functions vanishing on M_0 , $N(I)$ is the normalizer of I in $C^\infty(M)$, and $I' = N(I) \cap I$. Since the right-hand side is naturally a Poisson algebra, the isomorphism defines a Poisson structure \tilde{P} on \tilde{M} . The Poisson manifold (\tilde{M}, \tilde{P}) is said to be the reduction of the Poisson manifold (M, P) by M_0 .

This theorem shows that the process of reduction of a Poisson manifold by a submanifold is essentially an algebraic one. This leads to the following purely algebraic definition.

Definition 2.27. *Let \mathcal{P} be a Poisson algebra, J be an ideal of \mathcal{P} , $N(J)$ be the normalizer of J in \mathcal{P} , and $J' = N(J) \cap J$. We say that the Poisson algebra $N(J)/J'$ is the reduction of the Poisson algebra \mathcal{P} by the ideal J .*

The reduction of the Poisson algebra of functions on a Poisson manifold by the ideal of functions which vanish on a submanifold is well-defined even if the projection map $\pi: M_0 \rightarrow \tilde{M}$ is not smooth. Therefore, this algebraic definition of reduction generalizes the geometric one. Also, notice that I' given in the above definition is generally nonzero since

$$I^2 \subseteq I' \tag{2.28}$$

Consider the special case where the Poisson manifold (M, P) is, in fact, a symplectic manifold. This occurs when P is a nondegenerate bivector. Its inverse, ω , is a 2-form on M which is closed because the Schouten bracket of P with itself vanishes thereby making (M, ω) into a symplectic manifold. Let us call the Poisson algebra $C^\infty(M)$ a *symplectic algebra* if (M, P) is a symplectic manifold. There is, in fact, an algebraic characterization of this fact, i.e. $C^\infty(M)$ is a symplectic algebra if and only if the kernel of the map from $C^\infty(M) \rightarrow \Gamma(TM)$ given by $f \mapsto [f, \cdot]$ consists of the locally constant functions.

The symplectic reduction of (M, ω) by the submanifold M_0 is precisely the procedure of the reduction of the Poisson manifold (M, P) by M_0 . Notice that here TM_0^\perp is the usual symplectic normal bundle to TM_0 and the null distribution on M_0 arises as the null space of the pullback via the inclusion map of ω to M_0 . There are two extremes to symplectic reduction. The first is when M_0 is a coisotropic submanifold of (M, ω) , i.e. $T_m M_0^\perp \subseteq T_m M_0$ for all m in M_0 . In this case, the null distribution is just TM_0^\perp . Let I be the ideal of functions of $C^\infty(M)$ which vanish on M_0 . If M_0 is coisotropic, sections of TM_0^\perp belong to the space of sections of TM_0 . However, all sections of TM_0^\perp are restrictions of hamiltonian vector fields of elements in I to M_0 . Since $N(I)$ consists of all functions on M whose hamiltonian vector fields when restricted to M_0 are sections of TM_0 , we have $I \subseteq N(I)$. This

is equivalent to the statement that the ideal I forms a Poisson subalgebra of $C^\infty(M)$, i.e.

$$[I, I] \subseteq I. \tag{2.29}$$

It is clear that M_0 is a coisotropic submanifold if and only if Eq. (2.29) holds. In this case, the symplectic reduction of (M, P) by M_0 , is algebraically given by

$$C^\infty(\tilde{M}) \xrightarrow{\cong} \frac{N(I)}{I}. \tag{2.30}$$

The geometric procedure inspires the following algebraic definition.

Definition 2.31. *Let \mathcal{P} be a Poisson algebra, J an ideal, and $N(J)$ the normalizer of J in \mathcal{P} . The ideal J is said to be a coisotropic ideal of \mathcal{P} if and only if J is a Poisson subalgebra of \mathcal{P} .*

Clearly, if J is an ideal of $C^\infty(M)$ then J' is a coisotropic ideal of $C^\infty(M)$.

The other extreme occurs when M_0 is a symplectic submanifold of (M, ω) . In this case, since $i^*\omega$ is already nondegenerate, the null distribution vanishes, i.e. $T_m M_0 \cap T_m M_0^\perp = 0$. However, $\Gamma(TM_0 \cap TM_0^\perp)$ are hamiltonian vector fields of elements in I' restricted to M_0 , therefore, for all i in I' we have $X_i|_{M_0} = 0$. However, $X_i = P_\# di$ and $P_\#$ is an isomorphism since (M, P) is a symplectic manifold. Therefore, we have $di|_{M_0} = 0$ for all i in I' . However, the elements in I which satisfy $di|_{M_0} = 0$ are just the elements in I^2 , therefore, $I' \subseteq I^2$. Combining this with Eq. (2.28), we conclude that M_0 is a symplectic submanifold of the symplectic manifold (M, P) if and only if

$$I' = I^2. \tag{2.32}$$

Therefore, if $(\tilde{M}, \tilde{\Omega})$ is the symplectic reduction of (M, ω) by the symplectic submanifold M_0 then we conclude that

$$C^\infty(\tilde{M}) \xrightarrow{\cong} \frac{N(I)}{I^2}. \tag{2.33}$$

Classical BRST cohomology is a cohomology theory which performs the reduction of the Poisson algebra $C^\infty(M)$ of smooth functions on a Poisson manifold (M, P) by the ideal I of functions which vanish on a submanifold in the case where I is a coisotropic ideal of $C^\infty(M)$. However, one might expect to be able to perform the classical BRST construction for the reduction of the Poisson algebra $C^\infty(M)$ by a coisotropic ideal I whether it is the ideal of functions vanishing on some submanifold or not. We will show that the reduction of a Poisson manifold by an arbitrary submanifold can always be thought of as the reduction of Poisson algebras by a suitable coisotropic ideal.

Following [7], let us restrict ourselves to certain interesting ideals.

Definition 2.34. *Let J be an ideal in the Poisson algebra \mathcal{P} . J is said to be an associative ideal in \mathcal{P} if and only if*

$$f^2 \in J \Rightarrow f \in J. \tag{2.35}$$

Notice that if I is the ideal of functions which vanish on a submanifold M_0 in a Poisson algebra $C^\infty(M)$ then I is an associative ideal since if $f(p)^2 = 0$ then

$f(p) = 0$ for all points p in M_0 . Associative ideals are interesting because of the following result from [7].

Proposition 2.36. *Let \mathcal{P} be a Poisson algebra and J an associative ideal. Furthermore, let $J' = N(J) \cap J$, where $N(J)$ is the normalizer of J . If $N(J')$ is the normalizer of J' then*

$$N(J) = N(J') . \tag{2.37}$$

Proof. Suppose that f belongs to $N(J')$ for J an associative ideal of $C^\infty(M)$ then for all i in J , we have the equality

$$[[i^2, f], f] = 2[i, f]^2 + 2i[[i, f], f] . \tag{2.38}$$

Since i^2 belongs to $J^2 \subseteq J'$, $[i^2, f]$ belongs to J' which in turn implies that $[[i^2, f], f]$ belongs to $J' \subseteq J$. In other words, the left-hand side of Eq. (2.38) belongs to J . Since in addition $2i[[i, f], f]$ belongs to J , Eq. (2.38) implies that $[i, f]^2$ belongs to J but since J is an associative ideal, this means that $[i, f]$ belongs to J . In other words, f belongs to $N(J)$ thereby proving that $N(J') \subseteq N(J)$.

Conversely, suppose that f belongs to $N(J)$, then $[f, i']$ belongs to J for all i' in J' . We need only show that $[f, i']$ belongs to $N(J')$ for all i' in J' to establish that $N(J) \subseteq N(J')$. Notice that for all i in J and i' in J , we have $[[f, i'], i] = [[f, i], i'] + [f, [i, i']]$ but $[f, i]$ belongs to J and $[i, i']$ belongs to I therefore, we conclude that f belongs to $N(J)$. This proves Eq. (2.37). ■

Therefore, if \mathcal{P} is a Poisson algebra and I is an associative ideal, then the reduction of \mathcal{P} by I is $\frac{N(I')}{I'}$ which is naturally a Poisson algebra since I' is a Poisson ideal of $N(I')$. In other words, we have shown that the reduction of \mathcal{P} by an associative ideal I is the same as the reduction of \mathcal{P} by the ideal coisotropic I' .

In the case where (\tilde{M}, \tilde{P}) is the reduction of the Poisson manifold (M, P) by M_0 and I is the ideal of functions which vanish on M_0 then we have the isomorphism of Poisson algebras

$$C^\infty(\tilde{M}) \xrightarrow{\cong} \frac{N(I')}{I'} . \tag{2.39}$$

The usual classical BRST construction occurs when (M, P) gives rise to a symplectic manifold and I is the ideal of functions which vanish on a closed and embedded coisotropic submanifold of M . In this case, $I' = I$ and I is generated by a collection of so-called “first-class constraints” with respect to which the classical BRST complex is constructed.

The program that we wish to follow is now apparent. We will replace the role of I by I' in doing the classical BRST construction. It differs from the usual classical BRST construction since I' is not generally the ideal of functions which vanish on some submanifold of M . Since I' is a coisotropic ideal in $C^\infty(M)$, we expect many of the usual constructions in classical BRST to generalize. Although satisfactory from a purely homological standpoint since the role of generators for I is just replaced by generators in I' , it may not be satisfactory in certain physical applications. After all, in physical applications, we are usually given constraints which generate I and not I' and, in general, there is no natural way to get from a collection of constraints in I to a collection of constraints in I' . Furthermore, it is often desirable to continue

working with the constraints in I because these constraints might possess some desirable covariance property that one is trying to preserve. However, things are not quite as bad as they might seem. For example, in the case where (M, P) gives rise to a symplectic manifold and M_0 is a symplectic submanifold, then we have the isomorphism

$$C^\infty(\tilde{M}) \xrightarrow{\cong} \frac{N(I^2)}{I^2}. \tag{2.40}$$

In this case, a collection of generators for I , say $\Phi = (\phi_1, \dots, \phi_k)$, naturally gives rise to a collection of (first class) constraints in I^2 namely $\{\phi_i \phi_j | 1 \leq i \leq j \leq k\}$. These generators for I^2 would preserve any ‘‘Lorentz covariance’’ type properties of the original constraints. However, we will see that the constraints which generate I^2 will be infinitely reducible. We will be careful to take this into account.

3. The Koszul–Tate Resolution Revisited

The Koszul–Tate resolution is a complex which has nontrivial homology only in zero dimension where it is isomorphic to $C^\infty(M)/J$, where J is any ideal in $C^\infty(M)$. This complex performs the first step in the reduction of $C^\infty(M)$ by an ideal J – namely going from $C^\infty(M)$ to $C^\infty(M)/J$. The Koszul–Tate resolution is a generalization of the Koszul resolution due to Tate [8] which is performed by adjoining additional variables to the space of Koszul chains. These additional variables will turn out to be the antighost sector of the ‘‘ghosts for ghosts’’ in the BRST complex. Of particular interest is the case where $J = I'$ for some ideal I' of functions which vanish on a closed and embedded submanifold of a Poisson manifold (M, P) . In the next section, we will see that when (M, P) is a symplectic manifold and I is the ideal of functions which vanish on a closed and embedded symplectic submanifold M_0 then $I' = I^2$ is always generated by infinitely reducible constraints. In this section, we follow [2] in the construction of a Koszul–Tate complex but for general ideals J in $C^\infty(M)$ placing special emphasis upon the case of infinitely reducible constraints.

Let us review the construction of the Koszul–Tate complex. Let J be an ideal of $C^\infty(M)$ generated by the elements $\Psi = (\psi_1, \psi_2, \dots, \psi_{m_0})$ (called *constraints*). The usual Koszul complex [9] is constructed by introducing a free $C^\infty(M)$ -module V_1 with a basis given by $\{b_{a_0}^{(0)} | a_0 = 1, \dots, m_0\}$ whose elements are called the *antighosts of level 0*. In other words, V_1 consists of elements of the form $\sum_{a_0=1}^{m_0} f^{a_0} b_{a_0}^{(0)}$, where f^{a_0} belongs to $C^\infty(M)$ for all $a_0 = 1, \dots, m_0$. The free module V_1 is given a subscript 1 to indicate that its elements are assigned a \mathbb{Z} -grading 1 called the *antighost number*. The space of Koszul chains is given by $\mathcal{K}^{(0)} = \mathcal{S}_{C^\infty(M)}(V_1)$, the symmetric superalgebra over V_1 . In other words, $\mathcal{K}^{(0)}$ consists of all polynomials with coefficients in $C^\infty(M)$ over the antighosts where we regard these antighosts as being anticommuting variables. Another way to put it is that $\mathcal{K}^{(0)} = \bigoplus_{b=0}^{m_0} \mathcal{K}_b^{(0)}$ forms a commutative and associative superalgebra with unit graded by antighost number b freely generated by the antighosts. The Koszul differential is defined to be a $C^\infty(M)$ -linear graded derivation $\delta^{(0)}: \mathcal{K}_{b+1}^{(0)} \rightarrow \mathcal{K}_b^{(0)}$ such that

$$\delta^{(0)} b_{a_0}^{(0)} = \psi_{a_0}. \tag{3.1}$$

The homology of this complex, $H(\mathcal{K}^{(0)})$, is $C^\infty(M)/J$ in zero dimension.

We say that the constraints Ψ are *irreducible* if λ^{a_0} belongs to $C^\infty(M)$ and

$$\sum_{a_0=1}^{m_0} \lambda^{a_0} \psi_{a_0} = 0 \Rightarrow \lambda^{a_0} = \sum_{b_0=1}^{m_0} \gamma^{a_0 b_0} \psi_{b_0}, \quad \forall a_0 = 1, \dots, m_0. \tag{3.2}$$

for some $\gamma^{a_0 b_0}$ in $C^\infty(M)$ antisymmetric in its indices. If Ψ are irreducible, then Ψ forms a regular sequence in $C^\infty(M)$ and, therefore, the homology of the Koszul complex vanishes for nonzero antighost number. However, if Ψ are not a set of irreducible constraints, then we say that the constraints Ψ are *reducible*. In this case, there will be nontrivial cycles at nonzero antighost number.

If Ψ are a collection of reducible constraints then there exists a collection of functions on M , $Z^{(1)a_0}_{a_1}$ (for $a_0 = 1, \dots, m_0$ and $a_1 = 1, \dots, m_1$ for some m_1), which do not belong to $J \setminus 0$ such that

$$\sum_{a_0=1}^{m_0} Z^{(1)a_0}_{a_1} \psi_{a_0} = 0 \tag{3.3}$$

for all a_1 , and, for all functions λ^{a_0} , we have

$$\sum_{a_0=1}^{m_0} \lambda^{a_0} \psi_{a_0} = 0 \Rightarrow \lambda^{a_0} = \sum_{a_1=1}^{m_1} Z^{(1)a_0}_{a_1} \rho^{a_1} + \chi^{a_0 b_0} \psi_{b_0} \tag{3.4}$$

for some functions ρ^{a_1} and $\chi^{a_0 b_0}$ antisymmetric in its indices. If $Z^{(1)a_0}_{a_1}$ exist which satisfy Eq. (3.3) then $H_1(\mathcal{K}^{(0)})$ is nonzero since it contains homologically nontrivial cycles of the form

$$\sum_{a_0=1}^{m_0} Z^{(1)a_0}_{a_1} b_{a_0}^{(0)} \tag{3.5}$$

for all $a_1 = 1, \dots, m_1$. Furthermore, Eq. (3.4) means that the space of all nontrivial cycles in $\mathcal{K}_1^{(0)}$ are generated by such elements. We shall now utilize the method of Tate [8] to remove the nontrivial cycles in $\mathcal{K}_1^{(0)}$. Introduce a free $C^\infty(M)$ -module V_2 which has a basis $\{b_{a_1}^{(1)} | a_1 = 1, \dots, m_1\}$. The elements in this basis are called antighosts of level 1 and are assigned antighost number 2. A new space of chains is constructed from the old by adjoining these new antighosts. We define $\mathcal{K}^{(1)} = \mathcal{S}_{C^\infty(M)}(V_1 \oplus V_2)$, where $\mathcal{S}_{C^\infty(M)}(V_1 \oplus V_2)$ is the symmetric superalgebra over the free $C^\infty(M)$ -module $(V_1 \oplus V_2)$, i.e. $\mathcal{K}^{(1)}$ consists of all polynomials with coefficients in $C^\infty(M)$ over the commuting variables $b_{a_1}^{(1)}$ and the anticommuting variables $b_{a_0}^{(0)}$. This makes $\mathcal{K}^{(1)} = \bigoplus_{b=0}^\infty \mathcal{K}_b^{(1)}$ into a commutative and associative superalgebra with unit freely generated by the antighosts graded by antighost number. The new differential, $\delta^{(1)}: \mathcal{K}_{b+1}^{(1)} \rightarrow \mathcal{K}_b^{(1)}$ is a $C^\infty(M)$ -linear graded derivation which extends $\delta^{(0)}$ by mapping $b_{a_0}^{(0)}$ into the nontrivial cycles in Eq. (3.5), i.e.

$$\delta^{(1)} b_{a_1}^{(1)} = \sum_{a_0=1}^{m_0} Z^{(1)a_0}_{a_1} b_{a_0}^{(0)}. \tag{3.6}$$

Equation (3.3) insures that $\delta^{(1)2} = 0$. These new antighosts kill off the nontrivial cycles in $\mathcal{K}_1^{(0)}$ while leaving the homology in zero dimension unchanged, i.e. $H_0(\mathcal{K}^{(1)}) = C^\infty(M)/J$ and $H_1(\mathcal{K}^{(1)}) = 0$. However, there may be nontrivial cycles in $\mathcal{K}_2^{(1)}$ either because they were there to begin within the Koszul complex or because we have introduced them by choosing overcomplete $Z^{(1)a_0}_{a_1}$.

This procedure can be carried out for higher antighost numbers [2]. Suppose that there exists a collection of functions $Z^{(i)}_{a_i}{}^{a_i-1}$ which do not belong to $J \setminus 0$ where $i = 1, \dots, L$ and $a_i = 1, \dots, m_i$ that satisfy Eqs. (3.3) and (3.4) as well as the equations

$$\sum_{a_{j-1}=1}^{m_{j-1}} Z^{(j)}_{a_j}{}^{a_j-1} Z^{(j-1)}_{a_{j-1}}{}^{a_{j-1}-1} = 0 \pmod J \tag{3.7}$$

and

$$\sum_{a_{j-1}=1}^{m_{j-1}} \lambda^{a_{j-1}} Z^{(j-1)}_{a_{j-1}}{}^{a_{j-1}-1} = 0 \pmod J \Rightarrow \lambda^{a_{j-1}} = \sum_{a_j=1}^{m_j} \rho^{a_j} Z^{(j)}_{a_j}{}^{a_j-1} \pmod J \tag{3.8}$$

all $j = 2, \dots, L$ and $a_j = 1, \dots, m_j$. The number L is called the *order of reducibility* of this system. It is defined to be the last i for which $Z^{(i)}_{a_i}{}^{a_i-1}$ is nonzero. It is possible for L to be infinite, but let us assume that L is finite for now.

Suppose that we have constructed the Koszul–Tate resolution up to level i where $i < L$. That is, for all $j = 0, \dots, i$, we have introduced free $C^\infty(M)$ -modules V_{j+1} which are assigned antighost number $j + 1$ and are spanned by the antighosts of level j , $\{b_{a_j}^{(j)} | a_j = 1, \dots, m_j\}$, such that the space of chains is $\mathcal{K}^{(i)} = \mathcal{S}_{C^\infty(M)}(\mathbb{V}^{(i)})$, where $\mathbb{V}^{(i)} = \bigoplus_{j=1}^{i+1} V_j$. Furthermore, we have defined the differential $\delta^{(i)}: \mathcal{K}_{b+1}^{(i)} \rightarrow \mathcal{K}_b^{(i)}$ by Eq. (3.6) and

$$\delta^{(i)} b_{a_j}^{(j)} = \sum_{a_{j-1}=1}^{m_{j-1}} Z^{(j)}_{a_j}{}^{a_j-1} b_{a_{j-1}}^{(j-1)} + M_{(j)a_j} \tag{3.9}$$

for all $j = 1, \dots, i$. The homology of the complex $\delta^{(i)}: \mathcal{K}_{b+1}^{(i)} \rightarrow \mathcal{K}_b^{(i)}$ vanishes for antighost number $b = 1, \dots, i$ since, by construction, we have removed all of the nontrivial cycles up to antighost number i . The nontrivial cycles in $\mathcal{K}_{i+1}^{(i)}$ are generated by

$$\sum_{a_i=1}^{m_i} Z^{(i+1)}_{a_{i+1}}{}^{a_i} b_{a_i}^{(i)} + M_{(i+1)a_{i+1}}, \tag{3.10}$$

where $M_{(i)a_i}$ has antighost number $i + 1$ and contains only antighosts of level less than i . As before, $M_{(i+1)a_{i+1}}$ is arbitrary up to a boundary. We introduce a free $C^\infty(M)$ -module V_{i+1} which has antighost number $i + 2$ and a basis $\{b_1^{(i+1)}, b_2^{(i+1)}, \dots, b_{m_{i+1}}^{(i+1)}\}$. Let us define the level $i + 1$ Koszul–Tate chains by $\mathcal{K}^{(i+1)} = \mathcal{S}_{C^\infty(M)}(\mathbb{V}^{(i+1)})$. The differential on this complex is a $C^\infty(M)$ -linear graded derivation $\delta^{(i+1)}: \mathcal{K}_{b+1}^{(i+1)} \rightarrow \mathcal{K}_b^{(i+1)}$ which extends $\delta^{(i)}$ via

$$\delta^{(i+1)} b_{a_{i+1}}^{(i+1)} = \sum_{a_i=1}^{m_i} Z^{(i+1)}_{a_{i+1}}{}^{a_i} b_{a_i}^{(i)} + M'_{(i+1)a_{i+1}}, \tag{3.11}$$

for some $M'_{(i+1)a_{i+1}}$ consisting of antighosts with level less than i such that $\delta^{(i+1)^2} b_{a_{i+1}}^{(i+1)} = 0$. It is easy to verify that any such differential must have $M'_{(i+1)a_{i+1}} = M_{(i+1)a_{i+1}}$ up to a $\delta^{(i)}$ boundary [2] and, therefore, its homology satisfies

$$H_0(\mathcal{K}^{(i+1)}) = C^\infty(M)/J \quad \text{and} \quad H_b(\mathcal{K}^{(i+1)}) = 0, \quad \forall b = 1, \dots, i + 1. \tag{3.12}$$

In other words, any differential $\delta^{(i+1)}$ which satisfies (3.11) and agrees when acting upon lower antighosts with $\delta^{(i)}$, is a bona fide Koszul–Tate differential, i.e. its associated homology satisfies (3.12). We will use this result later when constructing the BRST charge.

This construction proceeds until i reaches the level L . At this point, the complex $\mathcal{K}^{(L)} = \mathcal{S}_{C^\infty(M)}(\mathbb{V}^{(L)})$ with differential $\delta^{(L)}: \mathcal{K}_{b+1}^{(L)} \rightarrow \mathcal{K}_b^{(L)}$ forms an acyclic resolution of $C^\infty(M)/J$.

In the case where the system is infinitely reducible, this procedure is iterated an infinite number of times introducing an infinite level of antighosts. This gives rise to the space of Koszul–Tate chains $\mathcal{K}^{(\infty)} = \mathcal{S}_{C^\infty(M)}(\mathbb{V}^{(\infty)})$ and the differential $\delta^{(\infty)}: \mathcal{K}_{b+1}^{(\infty)} \rightarrow \mathcal{K}_b^{(\infty)}$ is a graded derivation which acts upon each antighost via Eq. (3.11). The Koszul–Tate complex still forms an acyclic resolution $C^\infty(M)/J$ since all of the nontrivial cycles at any given antighost number have been removed by the same construction.

4. Infinite Reducibility

Let $i: M_0 \hookrightarrow M$ be a closed and embedded submanifold of codimension k where $k \geq 2$ and I be the ideal of functions in $C^\infty(M)$ which vanish on M_0 . In this section, we present a particular collection of elements which generate I^2 (essentially the collection of products of constraints) that are necessarily infinitely reducible. This is of particular interest in the case where M_0 is a symplectic submanifold of M , e.g. it arises as the zero locus of a collection of so-called “second class constraints.” In this case, we recall that $I' = I^2$ is a coisotropic ideal of $C^\infty(M)$ – the ideal with respect to which we would construct the Koszul–Tate complex and, eventually, the BRST complex.

Let us begin with a special case Let $i: M_0 \hookrightarrow M$ be a closed and embedded submanifold of codimension 2 and I be the ideal of functions which vanish on M_0 . Let us assume that I is generated by the irreducible constraints $\Phi = (\phi_1, \phi_2)$, i.e. Φ are irreducible constraints and

$$I = \{ \phi_1 f + \phi_2 g \mid f, g \in C^\infty(M) \}. \tag{4.1}$$

In this case, the ideal I^2 is generated by the elements $\Psi \equiv (\psi_1, \psi_2, \psi_3) \equiv (\phi_1^2, \phi_1 \phi_2, \phi_2^2)$, i.e.

$$I^2 = \left\{ \sum_{a_0=1}^3 \alpha^{a_0} \psi_{a_0} \mid \alpha^{a_0} \in C^\infty(M) \right\}. \tag{4.2}$$

We will show that Ψ are necessarily infinitely reducible.

Suppose that

$$\lambda^1 \phi_1^2 + \lambda^2 \phi_1 \phi_2 + \lambda^3 \phi_2^2 = 0 \tag{4.3}$$

for some functions λ^{a_0} then $\lambda^1 \phi_1^2$ belongs to the ideal generated by ϕ_2 . We would like to conclude that $\lambda^1 = \rho^1 \phi_2$ for some function ρ^1 . As usual, this is done first locally and then extended globally.

Definition 4.4. Let $i: M_0 \rightarrow M$ be a closed and embedded codimension k submanifold of an n -dimension manifold M such that M_0 is the zero locus of a map $\Phi = (\phi_1, \dots, \phi_k): M \rightarrow \mathbb{R}^k$. A regular coordinate chart $x: U \rightarrow \mathbb{R}^n$ of M_0 in M with respect to Φ is a coordinate chart $x: U \rightarrow \mathbb{R}^n$ such that $U \cap M_0 \neq \emptyset$, $x = (\phi_1, \dots, \phi_k, y_1, \dots, y_{n-k})$, and $y = (y_1, \dots, y_{n-k}): M_0 \cap U \rightarrow \mathbb{R}^{n-k}$ is a coordinate chart for $U \cap M_0$.

Let $\mathcal{U} = \{U_\alpha, M \setminus M_0\}$ be a cover of M , where $\{x_\alpha: U_\alpha \rightarrow \mathbb{R}^n\}$ is a collection of regular coordinate charts of M_0 in M with respect to Φ and let $\{\sigma_\alpha, \sigma'\}$ be a partition of unity subordinate to this cover. In the regular neighborhood U_α , the fact that $\lambda^1 \phi_1^2$ belongs to the ideal generated by ϕ_2 implies that $\lambda^1 = h_\alpha \phi_2$ for some h_α in $C^\infty(U_\alpha)$. This is the case since the ϕ_i are 2 of the coordinates of U_α . We can extend h_α to a function on M by using partitions of unity, i.e. $\rho^1 \equiv \sum_\alpha h_\alpha \sigma_\alpha + h' \sigma'$, where h' is any function on $M \setminus M_0$ so that $\lambda^1 = \rho^1 \phi_2$ globally. Actually, this result is nothing more than the fact that Φ forms a regular sequence in $C^\infty(M)$.

Similarly, (4.3) tells us that $\lambda^3 \phi_2^2$ belongs to the ideal generated by ϕ_1 and, therefore, $\lambda^2 = \rho^2 \phi_1$ for some function ρ^2 . Plugging this into (4.3), we obtain $\phi_1 \phi_2 (\rho^2 \phi_1 + \lambda^2 + \rho^1 \phi_2) = 0$. Working in a regular cover and globalizing the result, we conclude that $\lambda^2 = -\rho^2 \phi_1 - \rho^1 \phi_2$ and we define

$$\begin{aligned} Z^{(1)1}_1 &= \phi_2, & Z^{(1)2}_1 &= -\phi_1, & Z^{(1)3}_1 &= 0, \\ Z^{(1)1}_2 &= 0, & Z^{(1)2}_2 &= -\phi_2, & Z^{(1)3}_2 &= \phi_1, \end{aligned} \tag{4.5}$$

which satisfies $\lambda^{a_0} = \sum_{a_1=1}^3 Z^{(1)a_0}_{a_1} \rho^{a_1}$ for all $a_0 = 1, 2, 3$. Since $Z^{(1)a_0}_{a_1}$ do not belong to $I^2 \setminus 0$ and they satisfy (3.3) and (3.4), this concludes the analysis of reducibility at level one.

What of the second level? Suppose that $\sum_{a_1=1}^3 \lambda^{a_1} Z^{(1)a_0}_{a_1}$ belongs to $I^2 \setminus 0$ for all a_0 for some functions λ^{a_1} . Plugging in $a_0 = 1$, we see that $\lambda^1 \phi_2$ belongs to I^2 and, therefore, λ^1 must belong to I . Similarly, plugging in $a_0 = 3$ implies that λ^2 belongs to I . Finally, if λ^1 and λ^2 both belong to I then the equation which results from setting $a_0 = 2$ is automatically satisfied. In other words, there exists a collection of functions ρ^{a_2} where $a_2 = 1, \dots, 4$ such that

$$\lambda^1 = \rho^1 \phi_1 + \rho^2 \phi_2 \quad \text{and} \quad \lambda^2 = \rho^3 \phi_1 + \rho^4 \phi_2. \tag{4.6}$$

Let us now define a collection of functions $Z^{(2)a_1}_{a_2}$, where $a_2 = 1, \dots, 4$ which do not belong to $I^2 \setminus 0$ (but belong to I) via

$$Z^{(2)1}_1 = Z^{(2)2}_3 = \phi_1, \quad Z^{(2)1}_2 = Z^{(2)2}_4 = \phi_2, \tag{4.7}$$

where all others vanish. Equation (4.6) is then $\lambda^{a_1} = Z^{(2)a_1}_{a_2} \rho^{a_2}$ and this completes the analysis of reducibility at level two.

A similar computation for level three gives us the $Z^{(3)a_2}_{a_3}$, where $a_3 = 1, \dots, 8$ are given by

$$\begin{aligned} Z^{(3)1}_1 &= Z^{(3)2}_3 = Z^{(3)3}_5 = Z^{(3)4}_7 = \phi_1, \\ Z^{(3)1}_2 &= Z^{(3)2}_4 = Z^{(3)3}_6 = Z^{(3)4}_8 = \phi_2, \end{aligned} \tag{4.8}$$

and all other $Z^{(3)a_2}_{a_3}$ vanish. This pattern continues for an infinite number of levels. The $Z^{(i)a_{i-1}}_{a_i}$ (for $i \geq 2$) will be a collections of functions where $a_i = 1, \dots, 2^i$ and $a_{i-1} = 1, \dots, 2^{i-1}$ which belong to I but not to $I^2 \setminus 0$ given by

$$\begin{aligned} Z^{(i)1}_1 &= Z^{(i)2}_3 = \dots = Z^{(i)2^i-1}_{2^i-1} = \phi_1, \\ Z^{(i)1}_2 &= Z^{(i)2}_4 = \dots = Z^{(i)2^i-1}_{2^i-1} = \phi_2, \end{aligned} \tag{4.9}$$

and all other $Z^{(i)a_{i-1}}_{a_i}$ vanish. This concludes our proof of the infinite reducibility of Ψ .

Let us now consider the more general case where $i: M_0 \hookrightarrow M$ is a closed and embedded submanifold with codimension k , where $k > 2$ but where I is still

generated by a collection of irreducible constraints $\Phi = (\phi_1, \dots, \phi_k)$. In this case, I^2 will be generated by the collection of elements $\Psi = \{\phi_i \phi_j | i \geq j = 1, \dots, k\}$. Since Ψ contains $\{\phi_1^2, \phi_1 \phi_2, \phi_2^2\}$ as a subset and we have shown that subset is infinitely reducible, Ψ is itself infinitely reducible. After all, the introduction of the additional generators does not remove the reducibility of the original set of generators.

Let us now relax the assumption that I is generated by irreducible constraints. If I is generated by the reducible constraints $\Phi = (\phi_1, \phi_2, \dots, \phi_l)$ for some $l > k$, then I^2 will still be generated by the elements $\Psi = \{\phi_i \phi_j | i \leq j = 1, \dots, l\}$. The fact that Φ are reducible will only mean that there are more relations between the various elements in $\phi_i \phi_j$, not less. Therefore, Ψ will still be an infinitely reducible set of constraints. Another way to see this is that about every point in M_0 , there exists an open neighborhood in M containing it, U , such that a subset of k elements in Φ are regular constraints in $C^\infty(U)$. These regular constraints are locally infinitely reducible following the argument given above. Suppose it were true that these constraints are globally finitely reducible then this would imply that the constraints would locally be finitely reducible which would be a contradiction.

We have just shown the following theorem:

Theorem 4.10. *Let $i: M_0 \hookrightarrow M$ be a closed and embedded submanifold of codimension $k \geq 2$. If I is the ideal of functions which vanish on M_0 generated by elements $(\phi_1, \phi_2, \dots, \phi_k)$ then the collection of elements which generate I^2 given by $\{\phi_i \phi_j | i \leq j = 1, \dots, k\}$ is necessarily infinitely reducible.*

5. The BRST Complex

In this section, we construct the BRST complex extending the Koszul–Tate complex through the introduction of ghosts. We show that the space of BRST cochains forms a Poisson superalgebra which is graded by an integer called the ghost number. The BRST differential is exhibited as an inner derivation by an element with ghost number 1 called the BRST charge. Therefore, the associated cohomology inherits the structure of a Poisson superalgebra graded by ghost number. If J is a coisotropic ideal of $C^\infty(M)$ then we will see that BRST cohomology is isomorphic as Poisson algebras in zero dimension to $N(J)/J$.

Let J be generated by elements ψ_{a_0} , where $a_0 = 1, \dots, m_0$. Furthermore, let us assume that this system has order of reducibility L . Suppose that the Koszul–Tate complex has been constructed up to level $i < L$, i.e. we have introduced the free $C^\infty(M)$ -module graded by antighost number $\mathbb{V}^{(i)} = \bigoplus_{j=1}^{i+1} V_j$, where V_{j+1} is spanned by $\{b_{a_j}^{(j)} | a_j = 1, \dots, m_j\}$, the antighosts at level j . Let the dual free $C^\infty(M)$ -module of $\mathbb{V}^{(i)}$ be denoted by $\mathbb{V}^{(i)*} = \bigoplus_{j=1}^{i+1} V_j^*$, where each V_{j+1}^* has a basis dual to the antighosts at level j called the *ghosts* at level j which will be denoted by $\{c_{a_j}^{(j)} | a_j = 1, \dots, m_j\}$. In other words, if $\langle \cdot, \cdot \rangle: \mathbb{V}^{(i)*} \otimes \mathbb{V}^{(i)} \rightarrow C^\infty(M)$ is the dual pairing then

$$\langle c_{a_j}^{(j)}, b_{a_i}^{(i)} \rangle = \delta_j^i \delta_{a_i}^{a_j} \tag{5.1}$$

for all $j, l = 0, \dots, i, a_j = 1, \dots, m_j$, and $d_i = 1, \dots, m_i$. There ghosts have their own \mathbb{Z} -grading called the c -number. The c -number of $c_{a_j}^{(j)}$ is defined to be $j + 1$ and the c -number of $b_{a_j}^{(j)}$ is defined to be 0. This is similar to the antighost number, which we will now call the b -number, since the b -number of $b_{a_j}^{(j)}$ is $j + 1$ and we will

define the b -number of $c_{(j)}^{a_j}$ to be zero. Finally, the *ghost number* is a \mathbb{Z} -grading defined to be the difference of the c -number and the b -number. We will often denote the ghost number of an object u by $|u|$.

The grading by ghost number makes $\mathbb{V}^{(i)} \oplus \mathbb{V}^{(i)*}$ into a \mathbb{Z} -graded free $C^\infty(M)$ -module. Consider $\mathcal{S}_{C^\infty(M)}(\mathbb{V}^{(i)} \oplus \mathbb{V}^{(i)*})$, the associative and commutative superalgebra (graded by ghost number) freely generated by the ghosts and antighosts over $C^\infty(M)$. It can be endowed with the structure of a Poisson superalgebra where the Poisson bracket satisfies $[a, b] = -(-)^{|a||b|} [b, a]$ for all a and b in $\mathcal{S}(\mathbb{V}^{(i)} \oplus \mathbb{V}^{(i)*})$ with definite ghost number. This Poisson bracket extends the Poisson bracket on $C^\infty(M)$ via the dual pairing i.e.

$$[c_{(i)}^{a_i}, c_{(j)}^{d_j}] = [b_{a_i}^{(i)}, b_{d_j}^{(j)}] = [c_{(i)}^{a_i}, f] = [b_{a_i}^{(i)}, f] = 0, \tag{5.2}$$

$$[c_{(i)}^{a_i}, b_{d_j}^{(j)}] = \delta_j^l \delta_{d_j}^{a_i} \tag{5.3}$$

for all $j, l = 0, \dots, i, a_j = 1, \dots, m_j, d_l = 1, \dots, m_l$, and f in $C^\infty(M)$ as well as the intertwining relation

$$[a, bc] = [a, b]c + (-1)^{|a||b|} b[a, c] \tag{5.4}$$

for all a, b, c in $\mathcal{S}_{C^\infty(M)}(\mathbb{V}^{(i)} \oplus \mathbb{V}^{(i)*})$ with definite ghost numbers. This bracket preserves the ghost number grading but does not preserve the (c, b) -number bigrading. Therefore, $\mathcal{S}_{C^\infty(M)}(\mathbb{V}^{(i)} \oplus \mathbb{V}^{(i)*})$ is a Poisson superalgebra graded by ghost number.

Let us think of $\mathcal{S}_{C^\infty(M)}(\mathbb{V}^{(i)} \oplus \mathbb{V}^{(i)*})$ as the space of polynomials with coefficients in $C^\infty(M)$ in the \mathbb{Z}_2 -graded variables $\{b_{a_j}^{(j)}, c_{(j)}^{a_j} | j = 1, \dots, i; a_j = 1, \dots, m_j\}$. The space of BRST cochains of reducibility level i with ghost number $g, K_{(i)}^g$, consists of the space of all *infinite formal sums* with coefficients in $C^\infty(M)$ over the \mathbb{Z}_2 -graded variables $\{b_{a_j}^{(j)}, c_{(j)}^{a_j} | j = 1, \dots, i; a_j = 1, \dots, m_j\}$ which have ghost number g . The BRST cochains at various ghost numbers for a certain reducibility level i assemble into $K_{(i)} = \bigoplus_{g \in \mathbb{Z}} K_{(i)}^g$, i.e. finite sums of elements at different ghost numbers. $K_{(i)}$ is endowed with the structure of a Poisson superalgebra which naturally extends the one defined above for $\mathcal{S}_{C^\infty(M)}(\mathbb{V}^{(i)} \oplus \mathbb{V}^{(i)*})$. $K_{(i)}$ forms a Poisson superalgebra graded by ghost number but the Poisson bracket does not preserve the (c, b) -bigrading. For a system of reducibility level L , the total space of BRST cochains is given by the Poisson superalgebra $K_{(L)}$ graded by ghost number.

The reason that we allow certain infinite formal sums in the space of BRST cochains is so that the BRST differential can be given as an inner derivation by an element Q with ghost number 1. In the case of finite reducibility, Q must be the sum of an infinite number of monomials. Furthermore, imposing that $K_{(i)}$ consists of only finite sums of elements with different ghost number insures that in the infinitely reducible case, $K_{(\infty)}$ has a well-defined Poisson bracket. For example, consider the Poisson bracket of infinite sums of elements at different ghost number, e.g.

$$\sum_{j=0}^{\infty} \sum_{a_j=1}^{m_j} [f^{(j, a_j)} b_{a_j}^{(j)}, g_{(l, d_l)} c_{(l)}^{d_l}] = \sum_{j=0}^{\infty} \sum_{a_j=1}^{m_j} (f^{(j, a_j)} g_{(j, a_j)} + [f^{(j, a_j)}, g_{(l, d_l)}] b_{a_j}^{(j)} c_{(l)}^{d_l}). \tag{5.5}$$

The first term on the right-hand side need not converge. Therefore, $K_{(\infty)}$ would not form a Poisson algebra if all infinite formal sums of elements at different ghost

numbers were allowed. This problem does not arise if we define $K_{(\infty)}$ to consist of only finite sums of elements at different ghost numbers. Of course, it may be possible that $K_{(\infty)}$ could be allowed to contain some subset of infinite sums of elements at different ghost numbers. After all, the first term on the right-hand side of (5.5) will converge for certain choices of functions $f^{(j, a_j)}$ and $g_{(l, a_l)}$, e.g. it may be possible to redefine the space of BRST cochains to be the completion of $\mathcal{L}_{C^\infty(M)}(\mathbb{V}^{(i)} \oplus \mathbb{V}^{(i)*})$ with respect to some norm. This resulting space, if it can be shown to respect the constructions of this paper, would then be a possible space of BRST cochains. However, for our purposes, it is sufficient to consider the case where the space of BRST cochains is a finite sum of elements at different ghost numbers.

Let $K_{(i)}^{c,b}$ be the set of all elements in $K_{(i)}^{c-b}$ which have the correct (c, b) -number bigrading. In fact, we can think of elements in $K_{(i)}^g$ as infinite formal sums of elements in $K_{(i)}^{c,b}$, where $c - b = g$. As stated before, $K_{(i)}^g$ forms a Poisson algebra graded by ghost number but the (c, b) -number bigrading of $K_{(i)}$ is not respected by the Poisson bracket. Nonetheless, this bigrading does provide an additional structure. We can define a filtration of $K_{(i)}$ by $F^p K_{(i)} = \bigoplus_{c \geq p, b \geq 0} K_{(i)}^{c,b}$ so that

$$K_{(i)} = F^0 K_{(i)} \supseteq F^1 K_{(i)} \supseteq F^2 K_{(i)} \supseteq F^3 K_{(i)} \supseteq \dots \tag{5.6}$$

Since our Poisson bracket satisfies $[F^p K_{(i)}, F^q K_{(i)}] \subseteq F^{p+q} K_{(i)}$ and similarly for the associative multiplication, $K_{(i)}$ is a *filtered Poisson superalgebra*. Let us denote the space of elements in $F^p K_{(i)}$ with ghost number g by $F^p K_{(i)}^g$.

Suppose that there exists a sequence of maps

$$\dots \xrightarrow{D^{(0)}} K_{(i)}^{g-1} \xrightarrow{D^{(0)}} K_{(i)}^g \xrightarrow{D^{(0)}} K_{(i)}^{g+1} \xrightarrow{D^{(0)}} \dots \tag{5.7}$$

These maps naturally break up under the filtration degree into $D^{(0)} = \delta_0^{(0)} + \delta_1^{(0)} + \delta_2^{(0)} + \dots$, where $\delta_j^{(0)}: K_{(i)}^{c,b} \rightarrow K_{(i)}^{c+j, b+j-1}$ for all $j \geq 0$. We will find this decomposition to be useful in the next section.

It remains to introduce the BRST differential $D: K_{(L)}^g \rightarrow K_{(L)}^{g+1}$ which is a certain Poisson derivation, i.e. D should satisfy

$$D[u, v] = [Du, v] + (-1)^{|u|}[u, Dv], \tag{5.8}$$

where u and v are elements in $K_{(L)}$ with definite ghost number. The reason that this property is desirable is that such a differential insures that the BRST cohomology H_D forms a Poisson superalgebra graded by ghost number. It can be shown [5] that any Poisson derivation on $K_{(L)}$ which increases ghost number by 1 is an inner Poisson derivation, i.e. there exists an element $Q \in K_{(L)}^1$ such that $D = [Q, \cdot]$. The fact that $D^2 = 0$ is equivalent to asserting that $[Q, Q]$ lies in the center of $K_{(L)}$. The BRST charge is a particular element Q in $K_{(L)}^1$ which satisfies $[Q, Q] = 0$ such that its associated cohomology in zero dimension is isomorphic as Poisson superalgebras to $N(J)/J$. We will construct Q in the next section.

6. Construction of the BRST Charge

In this section, we construct the BRST charge using a refinement of the methods in [3] and [10]. It is a BRST cochain Q with ghost number 1 which satisfies

$$[Q, Q] = 0. \tag{6.1}$$

The BRST differential $D^{(L)}: K_{(L)}^g \rightarrow K_{(L)}^{g+1}$ is given by the inner derivation by this element

$$D^{(L)} = [Q, \cdot] . \tag{6.2}$$

Notice that $D^{(L)^2} = 0$ because of Eq. (6.1) and the Jacobi identity. The BRST differential $D^{(L)}$ will often be denoted by D for short.

The main result of this section can be summarized by the following.

Theorem 6.3. *Let J be a coisotropic ideal of $C^\infty(M)$, $\Psi = (\psi_1, \dots, \psi_{m_0})$ be elements which generate J , and $K_{(L)}$ be the space of BRST cochains with respect to these constraints. There exists an element Q in $K_{(L)}^1$ satisfying Eq. (6.1) such that*

$$Q = \sum_{a_0=1}^{m_0} c_{(0)}^{a_0} \psi_{a_0} + \sum_{i=1}^L \sum_{a_i=1}^{m_i} \sum_{a_{i-1}=1}^{m_{i-1}} c_{(i)}^{a_i} Z^{(i)}_{a_i-1} b_{a_{i-1}}^{(i-1)} + \text{etc.} , \tag{6.4}$$

where etc. consists of terms with at least two ghosts and one antighost or terms with at least two antighosts and one ghost. Furthermore, we can replace $Q \mapsto Q + \delta_0^{(L)} \lambda$ for any λ in $K_{(L)}^2$ and still satisfy Eqs. (6.1) and (6.4), where $\delta_0^{(L)}$ is the Koszul–Tate differential.

Let us begin by observing that the filtration of $K_{(i)}$ defined in Eq. (5.6) is unbounded, in general. Any element x in $K_{(i)}^g$ can be written as the sum $x = \sum_{j=0}^\infty x_j$, where x_j has c -number j . In particular, we can decompose the BRST charge, if it exists, into the (possibly infinite) sum

$$Q = Q_0 + Q_1 + Q_2 + \dots . \tag{6.5}$$

We will construct Q inductively by constructing the Q_{i+1} from Q_0, \dots, Q_i . Let us begin with the definition.

$$Q_0 = \sum_{a_0=1}^{m_0} c_{(0)}^{a_0} \psi_{a_0} , \tag{6.6}$$

then we see that Q_0^2 belongs to $F^2 K_{(0)}^2$ since

$$Q_0^2 = \sum_{j_0, k_0, l_0=1}^{m_0} f_{j_0 k_0}^{l_0} c_{(0)}^{j_0} c_{(0)}^{k_0} \psi_{l_0} , \tag{6.7}$$

where the structure functions $f_{j_0 k_0}^{l_0}$ are defined by

$$[\psi_{j_0}, \psi_{k_0}] = \sum_{l_0=1}^{m_0} f_{j_0 k_0}^{l_0} \psi_{l_0} \tag{6.8}$$

for all $j_0, k_0 = 1, \dots, m_0$.

The right-hand side of this equation looks like the image under the Koszul differential of the Q_1 -term in the BRST charge for irreducible constraints. This observation forms the basis for the inductive construction of Q that follows.

Suppose that Q_0 is defined as above and there exists $Q_j \in K_{(j)}^{j+1, j}$ for all $j = 1, \dots, \mu_i$, where $r_i = \min(i, L)$ such that

$$Q_j = \sum_{a_j=1}^{m_j} c_{(j)}^{a_j} \left(\sum_{a_{j-1}=1}^{m_{j-1}} Z^{(j)}_{a_j-1} b_{a_{j-1}}^{(j-1)} + M_{(j) a_j} \right) + N_j \tag{6.9}$$

such that $N_j \in K_{(j-1)}$ contains at least two c 's and the $M_{(j)a_j}$ are the same ones which appear in Eq. (3.10) and which contain at least two b 's. Furthermore, let us suppose that for all $j = 0, \dots, i$, we have

$$[R_j, R_j] \in F^{j+2} K_{(j)}^2, \tag{6.10}$$

where

$$R_j = \sum_{l=0}^j Q_l. \tag{6.11}$$

Define $D^{(i)}: K_{(\mu_i)}^g \rightarrow K_{(\mu_i)}^{g+1}$ by $D^{(i)} = [R_i, \cdot]$ and decompose it by c -number, i.e.

$$D^{(i)} = \sum_{j=0}^{\infty} \delta_j^{(i)}, \tag{6.12}$$

where $\delta_j^{(i)}: K_{(\mu_i)}^{c,b} \rightarrow K_{(\mu_i)}^{c+j, b+j-1}$. In particular, we have

$$\begin{aligned} \delta_0^{(i)} = & \sum_{a_0=1}^{m_0} [c_{(0)}^{a_0}, \cdot] \psi_{a_0} + \sum_{j=1}^{\mu_i} \sum_{a_j=1}^{m_j} [c_{(j)}^{a_j}, \cdot] \left(\sum_{a_{j-1}=1}^{m_{j-1}} Z^{(j)}_{a_j}{}^{a_{j-1}} b_{a_{j-1}}^{(j-1)} \right. \\ & \left. + M_{(j)a_j} \right) (-1)^{(j+1)|\cdot|. \end{aligned} \tag{6.13}$$

This implies that

$$\delta_0^{(i)} c_{(l)}^{a_l} = 0 \quad \text{and} \quad \delta_0^{(i)} b_{a_0}^{(0)} = \psi_{a_0} \tag{6.14}$$

for all $a_l = 1, \dots, m_l; l = 0, \dots, \mu_i$; and

$$\delta_0^{(i)} b_{a_j}^{(j)} = \sum_{a_{j-1}=1}^{m_{j-1}} Z^{(j)}_{a_j}{}^{a_{j-1}} b_{a_{j-1}}^{(j-1)} + M_{(j)a_j}, \tag{6.15}$$

for all $a_j = 1, \dots, m_j$ and $j = 1, \dots, \mu_i$. It is just the level μ_i Koszul–Tate differential acting upon $K_{(r)}$ extended to act trivially upon the ghosts. From the construction of the Koszul complex, we have a differential complex for each c -number and for all values of $j = 0, \dots, L$ given by

$$\dots \xrightarrow{\delta_0^{(j)}} K_{(j)}^{c, b+1} \xrightarrow{\delta_0^{(j)}} K_{(j)}^{c, b} \xrightarrow{\delta_0^{(j)}} K_{(j)}^{c, b-1} \xrightarrow{\delta_0^{(j)}} \dots \tag{6.16}$$

whose homology $H(K_{(j)}^{c, \cdot}; \delta_0^{(j)})$ satisfies

$$H_b(K_{(j)}^{c, \cdot}, \delta_0^{(j)}) = \begin{cases} 0 & \text{for } b = 1, \dots, j \\ K_{(j)}^{c, 0} & \text{for } b = 0. \end{cases} \tag{6.17}$$

If $j = L$ then we have the acyclicity condition

$$H(K_{(L)}^{c, \cdot}, \delta_0^{(L)}) = \begin{cases} 0 & \text{for } b > 0 \\ K_{(L)}^{c, 0} & \text{for } b = 0. \end{cases} \tag{6.18}$$

In order to avoid annoying factors of two which will otherwise arise, define for elements O in $K_{(L)}$ with odd ghost number

$$O^2 = \frac{1}{2}[O, O], \tag{6.19}$$

then $D^{(i)}R_i^2 = [R_i, R_i^2] = 0$. However, Eq. (6.12) implies that

$$\delta_0^{(i)}R_i^2 = -\delta_1^{(i)}R_i^2 - \delta_2^{(i)}R_i^2 - \delta_2^{(i)}R_i^2 - \cdots. \tag{6.20}$$

Since R_i^2 is in $F^{i+2}K_{(i)}^2$ the filtration degrees of the $\delta_j^{(i)}$ which appear on the right-hand side of Eq. (6.20), imply that

$$\delta_0^{(i)}R_i^2 \in F^{i+3}K_{(i)}^3. \tag{6.21}$$

Let $X_i \in K_{(i)}^{i+2,i}$ be the piece of R_i^2 which belongs to $K_{(i)}^{i+2,i}$ then the previous equation implies that

$$\delta_0^{(i)}X_i = 0. \tag{6.22}$$

Since $H_i(K_{(r)}^{i+2,\cdot}, \delta_0^{(i)}) = 0$, there exists an element Y_i in $K_{(i)}^{i+2,i+1}$ such that

$$X_i = -\delta_0^{(i)}Y_i. \tag{6.23}$$

Of course, we notice that the Y_i which we have chosen is hardly unique. In fact, Eq. (6.23) is invariant under the shift $Y_i \mapsto Y_i + U_i$, where U_i is a $\delta_0^{(i)}$ closed cycle in $K_{(\mu_i)}^{i+2,i+1}$.

There are two cases to consider here, The first case occurs when $\mu_i = L$. In this case, since U_i is a $\delta_0^{(i)}$ closed cycle in $K_{(L)}^{i+2,i+1}$ and $\delta_0^{(i)}$ is just the level L Koszul–Tate differential, Eq. (6.18) implies that U_i must be a $\delta_0^{(i)}$ boundary. Define

$$Q_{i+1} = Y_i \tag{6.24}$$

keeping in mind that it is arbitrary up to a $\delta_0^{(i)}$ boundary U_i . Since Y_i belongs to $K_{(L)}^{i+2,i+1}$ but only contains c 's (ghosts) of level less than or equal to L , it must contain at least two ghosts and an antighost. Similarly, the fact that the b -number is $i + 1 > L$ insures that each monomial in U_i contains at least two antighosts and one ghost.

The other case occurs when $\mu_i = i$. In this case, we cannot define $Q_{i+1} = Y_i$ as in the previous case if we are to satisfy Eq. (6.4) for $j = i + 1$. However, the inclusion $K_{(i)} \hookrightarrow K_{(i+1)}$ and Eq. (6.23) implies that

$$X_i = -\delta_0^{(i+1)}Y_i. \tag{6.25}$$

Although Y_i does not satisfy Eq. (6.4), the previous equation is invariant under the shift $Y_i \mapsto Y_i + U'_i$, where U'_i is any $\delta_0^{(i+1)}$ cocycle in $K_{(i+1)}^{i+2,i+1}$ so the question arises as to whether U'_i can be chosen so that $Y_i + U'_i$ satisfies the boundary conditions. Since the general form for $\delta_0^{(i+1)}$ cocycles in $K_{(i+1)}^{i+2,i+1}$ is given by Eq. (3.10) and we must satisfy the boundary conditions in Eq. (6.9), we conclude that

$$Q_{i+1} = Y_i + \sum_{a_{i+1}=1}^{m_{i+1}} c_{(i+1)}^{a_{i+1}} \left(\sum_{a_i=1}^{m_i} Z^{(i+1)a_i}_{a_{i+1}} b_{a_i}^{(i)} + M_{(i+1)a_{i+1}} \right). \tag{6.26}$$

As in the previous case, the Q_{i+1} is arbitrary up to $\delta_0^{(i)}$ boundaries and each monomial in Y_i contains at least two ghosts and one antighost. Also, each monomial in $M_{(i+1)a_{i+1}}$ contains at least two antighosts since $M_{(i+1)a_{i+1}}$ belongs to $K^{0,i+1}$ and only contains antighosts $b_a^{(j)}$, where $j \leq i - 1$. Therefore, Q_{i+1} is of the form

$$Q_{i+1} = \sum_{a_{i+1}=1}^{m_{i+1}} c_{(i+1)}^{a_{i+1}} \left(\sum_{a_i=1}^{m_i} Z^{(i+1)a_i}_{a_{i+1}} b_{a_i}^{(i)} + M_{(i+1)a_{i+1}} \right) + \text{etc.}, \tag{6.27}$$

where etc. consists of terms with at least two antighosts and one ghost or at least two ghosts and one antighost.

This takes care of the induction for Eq. (6.9) but we still have to perform the induction on (6.10). That is, we need to show that R_{i+1}^2 belongs to $F^{i+3} K_{(i+1)}$. Let us begin by noting that the definition of X_i and Q_{i+1} yields

$$R_i^2 + \delta_0^{(i+1)} Q_{i+1} \in F^{i+3} K_{(i)}^2 \subseteq F^{i+3} K_{(i+1)}^2. \tag{6.28}$$

Since

$$\begin{aligned} R_{i+1}^2 &= Q_{i+1}^2 + R_i^2 + [R_i, Q_{i+1}] \\ &= -Q_{i+1}^2 + R_i^2 + D^{(i+1)} Q_{i+1} \\ &= -Q_{i+1}^2 + R_i^2 + \delta_0^{(i+1)} Q_{i+1} + \delta_1^{(i+1)} Q_{i+1} + \delta_2^{(i+1)} Q_{i+1} + \dots, \end{aligned} \tag{6.29}$$

and Eq. (6.28) holds, we need only show that

$$-Q_{i+1}^2 + \delta_1^{(i+1)} Q_{i+1} + \delta_2^{(i+1)} Q_{i+1} + \dots \in F^{i+3} K_{(i+1)}. \tag{6.30}$$

First of all, it is easy to see that $\delta_1^{(i+1)} Q_{i+1} + \delta_2^{(i+1)} Q_{i+1} + \dots$ belongs to $F^{i+3} K_{(i+1)}$ using the fact that Q_{i+1} belongs to $F^{i+2} K_{(i+1)}$ and the filtration degrees of $\delta_j^{(i+1)}$ for all $j > 0$. Furthermore, Q_{i+1}^2 belongs to $F^{i+3} K_{(i+1)}$ since the monomial in Q_{i+1}^2 with the lowest c -number arises by taking a Poisson bracket of a level i ghost with a level i antighost in computing Q_{i+1}^2 resulting in a term in Q_{i+1}^2 with c -number $2(i+2) - (i+1) = i+3$. (The reason that the commutator of a level $i+1$ ghost with a level $i+1$ antighost does not appear in computing Q_{i+1}^2 is that there are no level $i+1$ antighosts in Q_{i+1} .) Therefore, we conclude that Q_{i+1}^2 belongs to $F^{i+3} K_{(i+1)}$.

This concludes the construction of the BRST charge.

7. Classical BRST Cohomology

In this section we will show that classical BRST cohomology vanishes for negative ghost number and is isomorphic to the E_2 term in the spectral sequence associated to the filtration by c -number by constructing the explicit isomorphism extending the one given in [10]. Furthermore, at zero ghost number, we will show that classical BRST cohomology is isomorphic as Poisson algebras to $N(J)/J$, where J is the coisotropic ideal with respect to which the BRST complex was constructed. We will discuss BRST cohomology at positive ghost number in the next section.

Let us begin by stating the basic result of this section.

Theorem 7.1. *Let $C^\infty(M)$ be a Poisson algebra, J be an associative ideal generated by elements $\Psi = (\psi_1, \dots, \psi_{m_0})$, and K be the space of BRST cochains constructed relative to the constraints Ψ and $D: K^g \rightarrow K^{g+1}$ be the BRST differential given by the inner derivation $D = [Q, \cdot]$, where Q is the BRST charge. Let D be decomposed by c -number*

$$D = \sum_{i=0}^{\infty} \delta_i, \tag{7.2}$$

where $\delta_i: K^{c,b} \rightarrow K^{c+i, b+i-1}$. If $H_D(K)$ is BRST cohomology then there is an isomorphism of associative algebras

$$H_D^g(K) \cong \begin{cases} 0 & \text{for } g < 0 \\ H_{\delta_1}^g(H_{\delta_0}^0(K)) & \text{for } g \geq 0 \end{cases} \tag{7.3}$$

If $g \geq 0$ and $\llbracket x \rrbracket$ is an element of H_D^g then the isomorphism $\chi: H_D^g(K) \xrightarrow{\cong} H_{\delta_1}^g(H_{\delta_0}^0(K))$ is given by

$$\llbracket x \rrbracket \mapsto \llbracket \langle x_0 \rangle \rrbracket, \tag{7.4}$$

where x_0 is the component of x in $K^{g,0}$, $\langle x_0 \rangle$ is an element in H_{δ_0} , and $\llbracket \langle x_0 \rangle \rrbracket$ is an element in $H_{\delta_1}^g(H_{\delta_0}^0(K))$. In particular, at zero ghost number

$$H_D^0(K) \cong \frac{N(J)}{J}. \tag{7.5}$$

The isomorphism in Eq. (7.3) can be used to define a Poisson superalgebra structure on $H_{\delta_1}^g(H_{\delta_0}^0(K))$ which agrees with the Poisson algebra structure of $N(J)/J$ at zero ghost number.

Before starting with the proof, note that $D^2 = 0$ is equivalent to the string of equations

$$\sum_{i=0}^p \delta_i \delta_{p-i} = 0, \tag{7.6}$$

for each $p \geq 0$. In particular, plugging in $p = 0$ and 1, we obtain

$$\delta_0^2 = 0 \tag{7.7}$$

and

$$\delta_0 \delta_1 + \delta_1 \delta_0 = 0. \tag{7.8}$$

We know that δ_0 is just the Koszul–Tate differential from the construction of the BRST charge in the previous section of Eq. (7.7) is not too surprising.

Equation (7.3) arises from the fact that the BRST complex is a complex filtered by c -number (see (5.6)). This filtration has an associated spectral sequence whose E_0 term is just the Koszul–Tate complex so that the E_1 term is just $H_{\delta_0}(K)$. However, the E_2 term in the spectral sequence is the cohomology of the complex $H_{\delta_0}(K)$ with the differential induced by δ_1 which we shall also denote by δ_1 . Therefore, $E_2^{c,b}$ is just $H_{\delta_1}^c(H_{\delta_0}^b(K))$. The spectral sequence degenerates at this point because of the acyclicity of the Koszul–Tate complex. Notice that if $b > 0$ then $E_2^{c,b}$ vanishes because of the acyclicity of the Koszul–Tate complex. If the constraints Ψ are irreducible, then the filtration is bounded since the BRST complex is finite dimensional. In this case, we know that the E_2 term is isomorphic to H_D and the fact that $E_2^{c,b} = 0$ for $b > 0$ implies that $H_D^g = 0$ for all $g < 0$. However, in the case where Ψ are reducible constraints, this filtration is no longer bounded and, therefore, it is not immediately clear if the E_2 term is isomorphic to H_D . We will first show that H_D vanishes for negative ghost number directly and then show that the map (7.4) between H_D and the E_2 term is, in fact, an isomorphism.

Let us now assume that x is a BRST cocycle in K^g , where $g < 0$. We can decompose x by c -number to get $x = \sum_{i=0}^{\infty} x_i$, where x_i belongs to $K^{i, i-g}$. Decomposing the equation $Dx = 0$ by c -number yields

$$\sum_{j=0}^i \delta_j x_{i-j} = 0 \tag{7.9}$$

for all $i \geq 0$. Plugging in $i = 0$ and 1, for example, yields

$$\delta_0 x_0 = 0, \tag{7.10}$$

and

$$\delta_1 x_0 + \delta_0 x_1 = 0. \tag{7.11}$$

We will show that there exist a y in K^{1-g} such that $x = Dy$. We decompose y by c -number to get $y = \sum_{i=0}^{\infty} y_i$, where y_i belongs to $K^{i, 1-g+i}$ for all $i \geq 0$. Decomposing the equation $x = Dy$ by c -number tells us that such a y exists if and only if there exist y_i in $K^{i, 1-g+i}$ which satisfy

$$x_p = \sum_{j=0}^p \delta_j y_{p-j} \tag{7.12}$$

for all $p \geq 0$.

The existence of such y_i is a consequence of the acyclicity of the Koszul–Tate complex. For example, Eq. (7.10) tells us that x_0 is a δ_0 closed cycle and, therefore, δ_0 exact from the acyclicity of the Koszul–Tate complex since x_0 has antighost number of at least one. In other words, there exists y_0 in $K^{0, 1-g}$ such that $x_0 = \delta_0 y_0$ which is just Eq. (7.12) where $p = 0$. Similarly, Eqs. (7.11) and (7.8) implies that

$$0 = \delta_0 x_1 + \delta_1 x_0 = \delta_0 x_1 + \delta_1 \delta_0 y_0 = \delta_0(x_1 - \delta_1 y_0). \tag{7.13}$$

The acyclicity of the Koszul–Tate complex tell us that there exists y_1 in $K^{1, 2-g}$ such that $x_1 - \delta_1 y_0 = \delta_0 y_1$ which is just Eq. (7.12) with $p = 1$.

The construction of the higher terms in y proceeds by induction. Suppose that there exists y_0, y_1, \dots, y_i satisfying Eq. (7.12) for all $p = 0, \dots, i$ then

$$\begin{aligned} \delta_0 \left(x_{i+1} - \sum_{j=1}^{i+1} \delta_j y_{i+1-j} \right) &= \delta_0 x_{i+1} + \sum_{j=1}^{i+1} \sum_{l=1}^l \delta_l \delta_{j-l} y_{i+1-j} \quad \text{using} \tag{7.6} \\ &= \delta_0 x_{i+1} + \sum_{l=1}^{i+1} \sum_{j=l}^{i+1} \delta_l \delta_{j-l} y_{i+1-j} \\ &= \delta_0 x_{i+1} + \sum_{l=1}^{i+1} \delta_l \sum_{s=0}^{i+1-l} \delta_s y_{i+1-s-l} \\ &= \delta_0 x_{i+1} + \sum_{l=1}^{i+1} \delta_l x_{i+1-l} \quad \text{from induction hypothesis} \\ &= \sum_{l=0}^{i+1} \delta_l x_{i+1-l} = 0, \tag{7.14} \end{aligned}$$

where we have used that fact that x is a D cocycle in the last step. Since $x_{i+1} - \sum_{j=1}^{i+1} y_{i+1-j}$ is a δ_0 closed cycle with antighost number $i + 1 - g > 0$, the

acyclicity of the Koszul–Tate complex tells us that there exists a y_{i+1} in $K^{i+1, 2-g+i}$ such that

$$x_{i+1} = \sum_{j=1}^{i+1} \delta_j y_{i+1-j} + \delta_0 y_{i+1} = \sum_{j=0}^{i+1} \delta_j y_{i+1-j}. \tag{7.15}$$

This completes the induction. Therefore, BRST cohomology vanishes for negative ghost number.

Let us now assume that $g \geq 0$. We will show that the map (7.4) is an isomorphism of associative algebras. We first check that the map is well defined. Consider any $y \in K^g$ for $g \geq 0$ then we have the decomposition $y = y_0 + y_1 + \dots$, where $y_i \in K^{g+i, i}$ for all $i \geq 0$. In this case, Dy belongs to K^{g+1} and the component of Dy in $K^{g+1, 0}$ is given by $\delta_1 y_0 + \delta_0 y_1$. We see that $\chi(\llbracket Dy \rrbracket) = \langle \llbracket \delta_1 y_0 + \delta_0 y_1 \rrbracket \rangle = \langle \llbracket \delta_1 y_0 \rrbracket \rangle = 0$ and, therefore, the map is well-defined.

Is χ injective? Consider $x \in K^g$ which decomposes into $x = x_0 + x_1 + x_2 + \dots$, where $x_i \in K^{g+i, i}$ such that $\chi(\llbracket x \rrbracket) = \langle \llbracket x_0 \rrbracket \rangle = 0$. We need to show that there exists $y \in K^{g-1}$ such that $x = Dy$. Decomposition of the previous equation is equivalent the existence of $y_i \in K^{g-1+i, i}$ such that for all $p \geq 0$,

$$x_p = \sum_{i=0}^{p+1} \delta_i y_{p+1-i}. \tag{7.16}$$

The fact that $\langle \llbracket x_0 \rrbracket \rangle = 0$ implies that $\llbracket x_0 \rrbracket = \delta_1 \llbracket y_0 \rrbracket$ with some $y_0 \in K^{g-1, 0}$ or $\llbracket x_0 - \delta_1 y_0 \rrbracket = 0$. Therefore, $x_0 - \delta_1 y_0 - \delta_0 y_1 = 0$ for some $y_1 \in K^{g, 1}$. In other words,

$$x_0 = \delta_0 y_1 + \delta_1 y_0 \tag{7.17}$$

for some y_0 and y_1 . This is just (7.16) for $p = 0$. We now proceed to show (7.16) by induction. Suppose that there exist $y_i \in K^{g-1+i, i}$ for all $i = 0, \dots, r$ which satisfy (7.16) for all $p = 0, \dots, r - 1$, then a similar argument to Eq. (7.14) yields

$$\delta_0 \left(x_r - \sum_{j=1}^{r+1} \delta_j y_{r+1-j} \right) = 0. \tag{7.18}$$

Therefore, there exists an element $y_{r+1} \in K^{g+r, r+1}$ which satisfies

$$x_r = \sum_{j=0}^{r+1} \delta_j y_{r+1-j}. \tag{7.19}$$

This completes our induction. The proof of the surjectivity of χ proceeds similarly.

Finally, it is clear that χ is an isomorphism of associative algebras since if $x \in K^g$ and $y \in K^h$ which decompose into sums of $x_i \in K^{g+i, i}$ and $y_i \in K^{h+i, i}$, respectively, then $\chi(\llbracket x \rrbracket \llbracket y \rrbracket) = \chi(\llbracket xy \rrbracket) = \langle \llbracket x_0 y_0 \rrbracket \rangle = \chi(\llbracket x \rrbracket) \chi(\llbracket y \rrbracket)$. We can endow $H_{\delta_1}(H_{\delta_0}(K))$ with the structure of a Poisson superalgebra by defining the Poisson bracket on elements $x, y \in H_{\delta_1}(H_{\delta_0}(K))$ by

$$[x, y] = \chi(\llbracket \chi^{-1}(x), \chi^{-1}(y) \rrbracket). \tag{7.20}$$

8. Vertical Cohomology is not BRST Cohomology

In this section, we show that in the case of the reduction of a Poisson manifold by a submanifold, BRST cohomology is not generally isomorphic to the cohomology

of vertical differential forms with respect to the null foliation. We prove this by looking at a simple counterexample.

Let N be a smooth manifold and $\mathcal{F} \rightarrow N$ be a smooth involutive rank k subbundle of the tangent bundle $TN \rightarrow N$. Let us denote the space of leaves of the associated foliation by \tilde{N} and the canonical projection map, which need not be smooth, by $\pi: N \rightarrow \tilde{N}$. The vectors in \mathcal{F}_n for all points n in N are called the *vertical vectors at n with respect to the foliation* $\pi: N \rightarrow \tilde{N}$, e.g. if π is a smooth map then \mathcal{F}_n are the vectors in T_nN in the kernel of π_* . The space of *vertical differential forms with respect to the foliation* $\pi: N \rightarrow \tilde{N}$, $\Omega_{\mathcal{F}}(N)$, is defined to be sections of the bundle $\Lambda\mathcal{F} \rightarrow N$. There exists a natural map $\Omega^p(N) \rightarrow \Omega_{\mathcal{F}}^p(N)$, denoted by $\gamma \mapsto \bar{\gamma}$, by restricting γ to act upon only vectors in \mathcal{F} . This map is surjective. A consequence is that the exterior derivative $d: \Omega^p(N) \rightarrow \Omega^{p+1}(N)$ induces a derivative operator (called the *vertical derivative*) $d_{\mathcal{F}}: \Omega_{\mathcal{F}}^p(N) \rightarrow \Omega_{\mathcal{F}}^{p+1}(N)$ via $d_{\mathcal{F}}\bar{\gamma} = \overline{d\gamma}$. The vertical derivative $d_{\mathcal{F}}$ is well-defined if \mathcal{F} is involutive [11]. This is the case because the kernel of the above map $\Omega^p(N) \rightarrow \Omega_{\mathcal{F}}^p(N)$ forms a differential ideal in $\Omega(N)$. The cohomology of this complex $H_{\mathcal{F}}(N)$ is called the *cohomology of vertical differential forms with respect to the foliation* $\pi: N \rightarrow \tilde{N}$.

Let (\tilde{M}, \tilde{P}) be the reduction of the Poisson manifold (M, P) by the submanifold M_0 then the space of vertical vectors at m in M_0 with respect to foliation $M_0 \rightarrow \tilde{M}$ is just $V_m = T_mM_0 \cap T_mM_0^\perp$. Clearly, $V \rightarrow M_0$ forms an involutive subbundle of TM_0 and we have $d_V: \Omega_V^p(M_0) \rightarrow \Omega_V^{p+1}(M_0)$, the complex of vertical differential forms with respect to the foliation $\pi: M_0 \rightarrow \tilde{M}$. (Notice that π need not be smooth here.) More explicitly, the vertical derivative is given by

$$(d_V f)(X) = X(f) , \tag{8.1}$$

$$(d_V \alpha)(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) , \tag{8.2}$$

and

$$d_V(\omega \wedge \psi) = (d_V \omega) \wedge \psi + (-1)^{|\omega|} \omega \wedge (d_V \psi) \tag{8.3}$$

for all f in $C^\infty(M_0)$, X, Y in $\Gamma(TM_0^\perp)$, α in $\Omega_V^1(M_0)$, and ω, ψ in $\Omega_V(M_0)$ with definite degree. The cohomology of this complex is denoted by $H_V(M_0)$.

Let us now consider the special case where (M, P) is a symplectic manifold and I is the ideal of functions in $C^\infty(M)$ which vanish on M_0 . Recall that $\Gamma(V) = \Gamma(TM_0^\perp \cap TM_0)$ consists of the restriction of hamiltonian vector fields of elements in I' to M_0 . In other words, there is a surjective homomorphism of lie algebras $I' \rightarrow \Gamma(V)$ given by $i \mapsto X_i|_{M_0}$. Furthermore, this map has a kernel since $0 = X_i|_{M_0} = P_\# di|_{M_0}$ but $P_\#: \text{Ann}(M_0) \rightarrow TM_0^\perp$ is invertible since (M, P) is symplectic. Therefore, $di|_{M_0} = 0$ but $di|_{M_0}$ is a section of the annihilator bundle of M_0 . The isomorphism in Eq. (2.13) implies that i must be an element of I^2 . In other words, we have the isomorphism of Lie algebras

$$\frac{I'}{I^2} \xrightarrow{\cong} \Gamma(V) \tag{8.4}$$

which is also a $C^\infty(M)/I \cong C^\infty(M_0)$ module isomorphism. Therefore, we can make the identification

$$\Omega_V(M_0) \cong \Lambda_{C^\infty(M)/I}(I'/I^2)^* , \tag{8.5}$$

where $\Lambda_{C^\infty(M)/I}^{p_0}(I'/I^2)^*$ consists of all alternating $C^\infty(M)/I$ -linear p -forms from I to $C^\infty(M)/I$. The vertical differential in this setting can be identified with

$$(\delta_V[f])(\langle\phi_1\rangle) = [[f, \phi_1]], \tag{8.6}$$

$$(\delta_V\beta)(\langle\phi_1\rangle, \langle\phi_2\rangle) = [[[\phi_1], \beta(\langle\phi_2\rangle)]] - [[[\phi_2], \beta(\langle\phi_1\rangle)]] - \beta(\langle[\phi_1, \phi_2]\rangle), \tag{8.7}$$

for all β in $(I'/I^2)^*$, $[f]$ in $C^\infty(M)/I$ and $\langle\phi_1\rangle, \langle\phi_2\rangle$ in I'/I^2 and then extending it via derivation to $\Lambda_{C^\infty(M)/I}(I'/I^2)^*$. This algebraic formulation of vertical cohomology can also be extended to the case of the reduction of Poisson manifolds by defining $\mathcal{C}(I) = \{i \in I' \mid [i, g] \in I, \forall g \in C^\infty(M)\}$ and then replacing I^2 by $\mathcal{C}(I)$ in the above since $\mathcal{C}(I)$ contains all of the elements i in I' such that $X_i|_{M_0} = 0$. Such a complex is well-defined for any ideal J in $C^\infty(M)$. Let us call this algebraic complex $\delta_V: \Lambda_{C^\infty(M)/\mathcal{C}(J)}^{p_0}(J'/\mathcal{C}(J))^* \rightarrow \Lambda_{C^\infty(M)/\mathcal{C}(J)}^{p_0+1}(J'/\mathcal{C}(J))^*$ the vertical complex of $C^\infty(M)$ with respect to J . Let us call the cohomology of this complex H_{δ_V} the vertical cohomology of $C^\infty(M)$ with respect to J .

In the very special case where (\tilde{M}, \tilde{P}) is the symplectic reduction of the symplectic manifold (M, P) by the coisotropic submanifold M_0 then we have $I' = I$ and $\mathcal{C}(I) = I^2$. In other words, the cochains in vertical cohomology are given by $\Lambda_{C^\infty(M)/I}(I'/I^2)^*$ and δ_V is still given by the formulas above. This gives rise to yet another algebraic cohomology theory. Let J be a coisotropic ideal in $C^\infty(M)$ then we can define the complex $\bar{\delta}: \Lambda_{C^\infty(M)/J}^{p_0}(J/J^2) \rightarrow \Lambda_{C^\infty(M)/J}^{p_0+1}(J/J^2)$, where the differential $\bar{\delta}$ is defined by Eqs. (8.6) and (8.7) (where δ_V is replaced by $\bar{\delta}$) and then extended as a graded derivation. Let us denote the cohomology of this complex by $H_{\bar{\delta}}$. $H_{\bar{\delta}}$ is an example of Rinehart cohomology [12].

How does all this relate to BRST cohomology, H_D ? We showed in the previous section that H_D is isomorphic to $H_{\delta_1}(H_{\delta_0})$ which is isomorphic to $H_V(M_0)$ in the case of the reduction of a symplectic manifold (M, P) by a closed and embedded coisotropic submanifold M_0 provided that the normal bundle of M_0 is trivial [5, 13, 10]. The results of [2] and [14] shows that this is the case even if the normal bundle is nontrivial. However, it is not true that for the case of reduction of a general Poisson manifold, BRST cohomology can be identified with vertical cohomology. Consider the example where (M, P) is a Poisson manifold with the trivial Poisson structure $P = 0$ and M_0 is a closed and embedded submanifold of M . In this case, $\Gamma(TM_0^\perp) = 0$ by definition so that there are no vertical vectors so that $\Omega_V(M_0) = C^\infty(M_0)$. Let $J = I$ be the ideal of functions of $C^\infty(M)$ which vanish on M_0 . Let us furthermore assume that I is an ideal generated by a collection of irreducible constraints $\Psi = (\psi_1, \dots, \psi_{m_0})$. In this case, the BRST operator is given by

$$Q = \sum_{i=1}^{m_0} c_{(0)}^i \psi_i, \tag{8.8}$$

then the BRST differential is given by

$$D = \sum_{i=1}^{m_0} \psi_i [c_{(0)}^i, \cdot] \tag{8.9}$$

which is precisely the Koszul differential, i.e. $D = \delta_0$ and $\delta_i = 0$ for all $i > 0$. Therefore, BRST cohomology is given by

$$H_D^g \cong \begin{cases} 0 & \text{for } g < 0 \\ \mathcal{S}_{C^\infty(M)/J}^g(V_1)^* & \text{for } g \geq 0 \end{cases}, \tag{8.10}$$

where V_1 is the m_0 -dimensional free $C^\infty(M)/I$ module of level zero antighosts and $\mathcal{S}_{C^\infty(M)/J}^g(V_1)^*$ is the space of ghost number g elements of the symmetric superalgebra over V_1 . Therefore, it is not correct to identify vertical cohomology with BRST cohomology here. However, it would still be correct to make the identification

$$H_D^g \cong H_\delta^g \tag{8.11}$$

if $g \geq 0$ since if J is generated by irreducible elements Ψ , then by Eq. (3.2), J/J^2 is a free m_0 -dimensional module over $C^\infty(M)/J$ so that $\mathcal{A}_{C^\infty(M)/J}(J/J^2)^* \xrightarrow{\cong} \mathcal{S}_{C^\infty(M)/J}^g(V_1)^*$. Of course, this isomorphism is true since \mathcal{S} is the symmetric superalgebra which is, in this case, an exterior (nonsuper) algebra since elements in V_1 have antighost number 1.

The question then arises as to whether this correspondence holds in general. That is, if H_D is the BRST cohomology associated to the reduction of the Poisson algebra $C^\infty(M)$ by a coisotropic ideal J then is it true the isomorphism in Eq. (8.11) holds? This correspondence is certainly true in the case of symplectic reduction of $C^\infty(M)$ by a coisotropic ideal $J = I$ of functions which vanish on a coisotropic submanifold as well as for the simple example given above. It remains to be seen whether it is true in general.

9. Conclusion

In this paper, we have generalized classical BRST cohomology to the more general framework of the reduction of a Poisson algebra $C^\infty(M)$ by a coisotropic ideal. This setting encompasses the reduction of Poisson manifolds. Let us make a few remarks.

First of all, it is not known what classical BRST cohomology computes for positive ghost numbers, in general. It is isomorphic to this cohomology theory of Rinehart?

Secondly, many of the constructions in this paper extend to the case where \mathcal{P} is a Poisson algebra which is a Noetherian ring under associative multiplication, i.e. when all ideals of \mathcal{P} are finitely generated. This is not the case, for example, when \mathcal{P} is the Poisson algebra of smooth functions on an infinite-dimensional poisson manifold. An extension of classical BRST cohomology to this case would be enlightening especially in applications to classical field theory.

Thirdly, BRST cohomology has a version which appears in quantum theory, usually in the context of Lie algebra cohomology [15, 16, 17]. There has been much work relating quantum BRST cohomology to classical BRST using methods of quantization inspired by geometric quantization, e.g. [18, 19, 20, 21, 22, 23]. An extension of the results in this paper to the case of geometric quantization is in progress [24].

Finally, an extension of these techniques to the case of the reduction of a Poisson supermanifold would be useful in certain physical applications, e.g. the covariant quantization of the superstring.

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