Commun. Math. Phys. 151, 67–87 (1993)

Communications in
Mathematical
Physics

Regularity of Weak Solutions to a Two-Dimensional Modified Dirac–Klein–Gordon System of Equations

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Received March 6, 1992

Abstract. We show that solutions to the modified Dirac–Klein–Gordon system in standard notation

$$\begin{cases}(-i\gamma^{\mu}\partial_{\mu}+M)\psi=0\\ (-\Box+m^2)\varphi=g(t)\psi^{\dagger}\gamma^0\psi\end{cases}$$

in two space dimensions with complex-valued initial data $\psi(0,x)\in L^2(\mathbb{R}^2;\mathbb{C}^4)$, real valued $\varphi(0,x)\in W^{1,2}(\mathbb{R}^2)$ and $\varphi_t(0,x)\in L^2(\mathbb{R}^2)$ have regularity

$$\psi^{\dagger}\gamma^0\psi\equiv|\psi_1|^2+|\psi_2|^2-|\psi_3|^2-|\psi_4|^2\in\mathscr{H}^1_{\mathrm{loc}}(\mathbb{R}^3),$$
$$\varphi\in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^1_+;L^2(\mathbb{R}^2)).$$

Here $\mathscr{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ denotes the (local) Hardy space, and $g(t)$ is assumed to be in $C^1(\mathbb{R})$ and $g(0)=0$. Consequently nonlinear terms $\varphi\psi$ which appear in the classical coupled Dirac–Klein–Gordon system (with the modification $g=g(t)\in C^1$ and $g(0)=0$) can then be defined in $L^{\infty}_{\mathrm{loc}}(\mathbb{R}^1_+;L^1(\mathbb{R}^2))$. We hope these results will be useful in establishing the existence of weak solutions to the classical coupled Dirac–Klein–Gordon system in the framework of compensated compactness.

Table of Contents

* Research at MSRI and IAS supported in part by NSF Grant DMS-8505550 and DMS-9100383

1. Introduction

We are interested in establishing the global existence of weak solutions to the Cauchy problem of arbitrary initial data for the (classical) coupled Dirac–Klein–Gordon system of equations

$$\begin{cases} (-i\gamma^\mu \partial_\mu + M)\psi = g\varphi\psi \quad (M, g > 0)\,, \\ (-\Box + m^2)\varphi = g\bar{\psi}\psi \quad \left(\Box = \triangle - \dfrac{\partial^2}{\partial t^2}, m > 0\right). \end{cases} \tag{1}$$

Here φ is a real scalar, ψ belongs to a complex four-dimensional space in three space dimensions. i is such that $i^2 = -1$. $\{\gamma^\mu\}_{\mu=0}^3$ are Dirac matrices. $\bar{\psi} = \psi^\dagger \gamma^0$, where $\psi^\dagger$ denotes the complex-conjugate transpose of ψ and γ^0 is a diagonal 4×4 matrix with diagonal entries $1, 1, -1, -1$. Later, we will be more specific on the notations which are consistent with our references.

The Cauchy problem for system (1) is well-posed in short time for arbitrary initial data and globally well posed for small smooth initial data (see [1, 15, 19]). In [7], Chadam and Glassey found a special set of global solutions in three space dimensions. Bachelot [2–4] recently proved, among other things, global existence of solutions for initial data being perturbations of the special solutions found in [7]. One of the main difficulties in the global existence theory with arbitrary initial data in two or three space dimensions is that the energy estimate for the system is not positive definite. In one space dimension, Chadam [6] established the global existence of a classical solution with arbitrary initial data by a boot-strapping method in Sobolev spaces. A similar boot-strapping method does not seem to work in two or three space dimensions. In two space dimensions, however, the method is borderline (i.e., involves critical Sobolev exponents). By employing the Hardy space $\mathscr{H}^1$, we hope to make the boot-strapping method work within the framework of compensated compactness [22] where the energy estimate is not necessary. We shall address some successful applications of $\mathscr{H}^1$ and BMO of bounded mean oscillation on harmonic maps at the end of this introduction.

There is a conservation of charge

$$\int |\psi|^2 \, dx = \text{const. in time} \tag{2}$$

for the DKG system (see [13], for example). Inverting the coupled KG equation with zero initial datum, we find

$$\sup_{0 \leqq t \leqq T} \int |\varphi|^p dx \leqq C_{T,p} \tag{3}$$

for all $1 \leqq p \leqq 2$ *in two space dimensions, and for* $p = 1$ in three space dimensions. The nonlinear term $\varphi\psi$ would be defined in $L^\infty((0, T), L^1(\mathbb{R}^n))$, $n = 2$ or 3 if φ were in $L^\infty((0, T); L^2(\mathbb{R}^n))$. We prove that φ is indeed in $L^\infty((0, T), L^2(\mathbb{R}^2))$ if we let g depend on t smoothly and $g(0) = 0$ provided that $\bar{\psi}\psi$ is in $\mathscr{H}^1_{\mathrm{loc}}(\mathbb{R}^3_+)$ in two space dimension (Theorem 2). By applying the quantified compensated compactness of Müller [17] and Coifman, Lions, Meyer and Semmes [8], we show that $\bar{\psi}\psi$ is indeed in $\mathscr{H}^1_{\mathrm{loc}}(\mathbb{R}^3_+)$ if ψ satisfies the homogeneous Dirac equations (Theorem 1).

We have not been able to show that $\bar{\psi}\psi$ is in $\mathscr{H}^1_{\mathrm{loc}}(\mathbb{R}^3_+)$ if ψ satisfies the coupled Dirac equations. Instead, what we have done is the regularity of weak solutions to the following modified DKG system:

$$\begin{cases} (-i\gamma^{\mu}\partial_{\mu} + M)\psi = 0 \\ (-\Box + m^2)\varphi = g(t)\bar{\psi}\psi \\ \psi|_{t=0} = \psi_0(x),\ \varphi|_{t=0} = \varphi_0(x),\ \varphi_t|_{t=0} = \varphi_1(x)\,, \end{cases} \tag{4}$$

where $\psi_0(x) \in L^2(\mathbb{R}^2)$, φ_0 and φ_1 are assumed smooth, and $g(t)$ is as stated above. In view of the full system (1), the assumption $\psi_0(x) \in L^2(\mathbb{R}^2)$ is natural and appropriate compared to better assumptions such as $\psi_0 \in W^{1,2}(\mathbb{R}^2)$, since the energy estimate involving the derivatives of ψ are not positive definite, as mentioned earlier. We believe that the assumptions $\psi_0(x) \in L^2(\mathbb{R}^2)$, φ_0 and φ_1 are smooth should be sufficient in establishing the global existence of a weak solution for system (1) by the compensated compactness method. The regularity of $\bar{\psi}\psi \in \mathscr{H}^1_{\text{loc}}(\mathbb{R}^3_+)$ and $\varphi \in L^{\infty}((0, T); L^2(\mathbb{R}^2))$ that we obtain here for (4) should be useful in obtaining a priori estimates for the full system (1) by a boot-strapping method.

It is worth mentioning that our approach here does indeed yield global existence of weak solutions (Theorem 3) for the full system (1) in one space dimension with weaker assumptions on the initial data than Chadam [6].

Finally we point out that $\mathscr{H}^1$ and BMO have been successfully used in some borderline problems of harmonic maps. F. Hélein [12] showed that any weakly harmonic mapping from a two-dimensional surface into a sphere is smooth. Soon afterwards, L.C. Evans [9] generalized Hélein's result to higher dimensions, asserting in effect that a stationary harmonic mapping from an open subset of $\mathbb{R}^n$ ($n \geqq 3$) into a sphere is smooth, except possibly for a closed singular set of $(n-2)$-dimensional Hausdorff measure zero. The key ingredient of their proofs is that the right-hand side of the equations

$$-\Delta u = |\nabla u|^2 u$$

belongs to the Hardy space $\mathscr{H}^1(\mathbb{R}^n)$ when u is constrained to a sphere $|u| = 1$. This fact and Wente's earlier work [24] on inverting the Poisson equation

$$-\Delta u = f \in \mathscr{H}^1(\mathbb{R}^2)$$

imply immediately that u is continuous, and therefore smooth in the case of the harmonic map. For $n \geqq 3$, Evans noted additionally that certain monotonicity inequalities provide bounds for u in BMO and Fefferman's [10] duality theorem $(\mathscr{H}^1)^* = \text{BMO}$ was then used.

2. Preliminaries

In this secrtion we recall the definitions of various spaces and the relevant basic facts which will make our subsequent presentation clearer.

2.1. Localization of $\mathscr{H}^1(\mathbb{R}^n)$: $\mathscr{H}^1_{\text{loc}}$ and h^1. Let h be in $C_c^{\infty}(\mathbb{R}^n)$, with support in the unit ball and $\int h = 1$. For any $f \in L^1(\mathbb{R}^n)$, we set

$$f^*(x) = \sup_{\infty > r > 0} \left| \frac{1}{r^n} \int f(y) h\left(\frac{x-y}{r}\right) dy \right| .$$

The Hardy space $\mathscr{H}^1(\mathbb{R}^n)$ is defined to be

$$\mathscr{H}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) | f^* \in L^1(\mathbb{R}^n)\}$$

with norm

$$\|f\|_{\mathscr{H}^1(\mathbb{R}^n)} = \|f^*\|_{L^1(\mathbb{R}^n)} .$$

Observe $f \in \mathscr{H}^1(\mathbb{R}^n)$ implies $\int_{\mathbb{R}^n} f\,dx = 0$. This makes it nontrivial to localize $\mathscr{H}^1(\mathbb{R}^n)$ onto a subset of $\mathbb{R}^n$, as compared to the localization of $L^1(\mathbb{R}^n)$. We shall for our purposes define $\mathscr{H}^1_{\text{loc}}$ as

$$\mathscr{H}^1_{\text{loc}}(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) \,\middle|\, \sup_{1>r>0} \left| \frac{1}{r^n} \int f(y) h\left(\frac{x-y}{r}\right) dy \right| \in L^1_{\text{loc}}(\mathbb{R}^n) \right\}.$$

Notice in $\mathscr{H}^1_{\text{loc}}$ the "sup" is taken over $1 > r > 0$ only. A more refined version was introduced by Goldberg [14]. It is the space $h^1(\mathbb{R}^n)$:

$$h^1(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) \,\middle|\, \sup_{1>r>0} \left| \frac{1}{r^n} \int f(y)\, h\left(\frac{x-y}{r}\right) dy \right| \in L^1(\mathbb{R}^n) \right\}.$$

One of the many properties that $h^1(\mathbb{R}^n)$ has is the following duality theorem

Proposition 1. $(h^1)^* = \{b \in \mathrm{BMO} \,|\, \varphi * b \in L^\infty(\mathbb{R}^n)\}$ *for some* $\varphi \in S$ *(the Schwartz class of rapidly decreasing functions) such that*

$$\|f - \varphi * f\|_{\mathscr{H}^1} \leqq c\,\|f\|_{h^1}, \quad \forall f \in h^1. \tag{5}$$

A sufficient condition for (5) is given by

Proposition 2. *If* $\varphi \in S$, $\int \varphi = 1$, *then* (5) *holds.*

Proposition 1 is a direct quotation from Goldberg [14]. Proposition 2 is from Lemma 4 of the same paper restricted to the case $p = 1$, and one uses the fact that $|\alpha| \leqq N = 0$ from the proof of the lemma.

In the next proposition we present a local version of a result of R.R. Coifman, P.L. Lions, Y. Meyer and S. Semmes [8] on $\mathscr{H}^1(\mathbb{R}^n)$ spaces. For completeness, we also present a proof which follows from one of the ideas introduced in [8].

Proposition 3. *Assume* $b \in L^2(\mathbb{R}^n, \mathbb{R}^n)$, $\operatorname{div} b \in L^2(\mathbb{R}^n)$ *and* $B \in W^{1,2}(\mathbb{R}^n)$. *Then* $b \cdot \nabla B \in h^1(\mathbb{R}^n)$ *and*

$$\|b \cdot \nabla B\|_{h^1} \leqq c(\|b\|^2_{L^2} + \|\operatorname{div} b\|^2_{L^2} + \|B\|^2_{W^{1,2}}), \tag{6}$$

where c is a constant independent of b or B.

Proof. We follow the idea of [8] (see also Evans [9]). Fix the h we mentioned at the beginning of this section. We look at

$$\left| \frac{1}{r^n} \int_{\mathbb{R}^n} (b \cdot \nabla B)(y) h\left(\frac{x-y}{r}\right) dy \right| = \left| \frac{1}{r^n} \int_{B(x,r)} [b \cdot \nabla(B - (B)_{x,r})]\, h\left(\frac{x-y}{r}\right) dy \right|,$$

where $B(x,r)$ denotes a ball centered at x with radius r and $(B)_{x,r}$ denotes the average of B on the ball $B(x,r)$. Upon using integration by parts, we find

$$\begin{aligned}
&\left|\frac{1}{r^n}\int_{\mathbb{R}^n}(b\cdot \nabla B)(y)h\left(\frac{x-y}{r}\right)dy\right| \\
&= \left|\frac{1}{r^n}\int_{B(x,r)}\left[-(\operatorname{div} b)(B-(B)_{x,r})h\left(\frac{x-y}{r}\right)\right.\right. \\
&\qquad \left.\left. + \frac{1}{r}(B-(B)_{x,r})b\cdot(\nabla h)\left(\frac{x-y}{r}\right)\right]dy\right| \\
&\leqq \frac{c}{r^{n+1}}\int_{B(x,r)}(r|\operatorname{div} b|+|b|)|B-(B)_{x,r}|\,dy\cdot
\end{aligned}$$

So

$$\begin{aligned}
&\sup_{1>r>0}\left|\frac{1}{r^n}\int_{\mathbb{R}^n}(b\cdot \nabla B)(y)h\left(\frac{x-y}{r}\right)dy\right| \\
&\leqq c\sup_{\infty>r>0}\left\{\frac{1}{r^{n+1}}\int_{B(x,r)}(|\operatorname{div} b|+|b|)|B-(B)_{x,r}|\,dy\right\}.
\end{aligned}$$

Choose any $2<p<2^*=\dfrac{2n}{n-2}\leqq\infty$ and let $1<q\equiv\dfrac{p}{p-1}<2$. Then

$$\begin{aligned}
&\frac{1}{r^{n+1}}\int_{B(x,r)}(|\operatorname{div} b|+|b|)|B-(B)_{x,r}|\,dy \\
&\leqq \frac{1}{r^{n+1}}\left(\int_{B(x,r)}|B-(B)_{x,r}|^p dx\right)^{1/p}\left(\int_{B(x,r)}(|\operatorname{div} b|+|b|)^q dx\right)^{1/q} \\
&\leqq \frac{C}{r^{1+n/p}}\left(\int_{B(x,r)}|B-(B)_{x,r}|^p dx\right)^{1/p}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}(|\operatorname{div} b|+|b|)^q dx\right)^{1/q} \\
&\leqq C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|DB|^s dx\right)^{1/s}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}(|\operatorname{div} b|+|b|)^q dx\right)^{1/q},
\end{aligned}$$

where $p=s^*$, that is, $s=\dfrac{pn}{p+n}<2$. Consequently

$$\begin{aligned}
&\sup_{\infty>r>0}\left\{\frac{1}{r^{n+1}}\int_{B(x,r)}(|\operatorname{div} b|+|b|)|B-(B)_{x,r}|\,dy\right\} \\
&\qquad\leqq C(M(|DB|^s))^{1/s}(M((|\operatorname{div} b|+|b|)^q))^{1/q} \\
&\qquad\leqq C[(M(|DB|^s))^{2/s}+(M((|\operatorname{div} b|+|b|)^q))^{2/q}],
\end{aligned}$$

$M(\cdot)$ denoting the Hardy–Littlewood maximal function. Now $|DB|^s\in L^{2/s}$, $2/s>1$. Thus

$$\|M(|DB|^s)\|_{L^{2/s}}\leqq C\|\,|DB|^s\|_{L^{2/s}},$$

and so

$$\int_{\mathbb{R}^n} (M(|DB|^s))^{2/s}\,dx \leqq C \int_{\mathbb{R}^n} |DB|^2\,dx \cdot$$

Similarly,

$$\int_{\mathbb{R}^n} (M((|\operatorname{div} b| + |b|)^q))^{2/q}\,dx \leqq C \int_{\mathbb{R}^n} (|\operatorname{div} b| + |b|)^2\,dx\,.$$

Consequently we deduce

$$\sup_{1>r>0} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} (b \cdot \nabla B)(y) h\left(\frac{x-y}{r}\right) dy \right| \in L^1$$

and

$$\left\| \sup_{1>r>0} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} (b \cdot \nabla B)(y) h\left(\frac{x-y}{r}\right) dy \right| \right\|_{L^1} \leqq c(\|b\|_{L^2}^2 + \|\operatorname{div} b\|_{L^2}^2 + \|B\|_{W^{1,2}}^2)\,.$$

The proof of Proposition 3 is completed.

2.2. The space $\mathrm{BMO}(\mathbb{R}^n)$. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We set

$$\|f\|_* = \sup\left\{ \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - (f)_{x,r}|\,dy \mid x \in \mathbb{R}^n, r > 0 \right\},$$

where $(f)_{x,r}$ denotes the average of f over the ball $B(x,r)$. Then

$$\mathrm{BMO}(\mathbb{R}^n) = \{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \mid \|f\|_* < \infty \}$$

with seminorm $\|f\|_*$.

As an example, we mention that $\log \frac{1}{|x|} \in \mathrm{BMO}(\mathbb{R}^n)$ (see Stein [20]). In Sect. 4 we will need

Proposition 4. *The function*

$$B(y,\tau) = \begin{cases} \log \frac{\tau}{|y|}, & |y| \leqq \tau, 0 \leqq \tau \leqq \infty, y \in \mathbb{R}^2 \\ 0 & \text{otherwise}, \end{cases}$$

is in $\mathrm{BMO}(\mathbb{R}^3)$; i.e., $\|B\|_* < \infty$.

Proposition 5. *Let* $\beta(\tau) \in C^2(\mathbb{R}^1)$ *be a bounded function so that*

$$\beta(\tau) = \begin{cases} 1, & \tau \leqq 0 \\ \textit{smooth and decreasing}, & 0 \leqq \tau \leqq 1 \\ 0, & \tau > 1 \cdot \end{cases}$$

Then

$$\|B(y,\tau)\beta(\tau - t_0)\|_* \leqq c(t_0),$$

where B *is given in Proposition* 4 *and* $c(t_0)$ *is finite for any finite* $t_0 > 2$.

The proofs of Propositions 4 and 5 are computational and are given in the Appendix.

3. An Estimate on Dirac Equations

In the standard notation of [5], the coupled Dirac system is

$$\left\{\begin{aligned}
\frac{\partial\psi_1}{\partial t} &= -\frac{\partial\psi_4}{\partial x_1} + i\frac{\partial\psi_4}{\partial x_2} - \frac{\partial\psi_3}{\partial x_3} - i(M-g\varphi)\psi_1\\
\frac{\partial\psi_2}{\partial t} &= -\frac{\partial\psi_3}{\partial x_1} - i\frac{\partial\psi_3}{\partial x_2} + \frac{\partial\psi_4}{\partial x_3} - i(M-g\varphi)\psi_2\\
\frac{\partial\psi_3}{\partial t} &= -\frac{\partial\psi_2}{\partial x_1} + i\frac{\partial\psi_2}{\partial x_2} - \frac{\partial\psi_1}{\partial x_3} + i(M-g\varphi)\psi_3\\
\frac{\partial\psi_4}{\partial t} &= -\frac{\partial\psi_1}{\partial x_1} - i\frac{\partial\psi_1}{\partial x_2} + \frac{\partial\psi_2}{\partial x_3} + i(M-g\varphi)\varphi_4 .
\end{aligned}\right. \tag{7}$$

For the two space dimensional Dirac system, i.e., when ψ_1, ψ_2, ψ_3, and ψ_4 do not depend on x_3, the four equations decouple into two similar subsystems, one of which is

$$\left\{\begin{aligned}
\frac{\partial\psi_2}{\partial t} &= -\frac{\partial\psi_3}{\partial x_1} - i\frac{\partial\psi_3}{\partial x_2} - i(M-g\varphi)\psi_2\\
\frac{\partial\psi_3}{\partial t} &= -\frac{\partial\psi_2}{\partial x_1} + i\frac{\partial\psi_2}{\partial x_2} + i(M-g\varphi)\psi_3 .
\end{aligned}\right. \tag{8}$$

Let $\psi_2 = u_2 + i\,v_2$ and $\psi_3 = u_3 + i\,v_3$, we find the subsystem in real variables to be

$$\left\{\begin{aligned}
\frac{\partial u_2}{\partial x_0} + \frac{\partial u_3}{\partial x_1} - \frac{\partial v_3}{\partial x_2} &= (M-g\varphi)v_2\\
\frac{\partial v_2}{\partial x_0} + \frac{\partial v_3}{\partial x_1} + \frac{\partial u_3}{\partial x_2} &= -(M-g\varphi)u_2\\
\frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} + \frac{\partial v_2}{\partial x_2} &= -(M-g\varphi)v_3\\
\frac{\partial v_3}{\partial x_0} + \frac{\partial v_2}{\partial x_1} - \frac{\partial u_2}{\partial x_2} &= (M-g\varphi)u_3 ,
\end{aligned}\right. \tag{9}$$

where we let $t = x_0$ for later simplifications, and we will let $x = (x_0, x_1, x_2)$ in what follows in this section.

The conservation of charge in those real variables is

$$\int_{\mathbb{R}^2} (u_2^2 + v_2^2 + u_3^2 + v_3^2)\,dx_1\,dx_2 = \text{const.} \quad \text{in time.} \tag{10}$$

For simplicity, we shall assume $\psi_1 = \psi_4 = 0$. Therefore the quantity

$$\bar{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$$

will reduce to

$$\bar{\psi}\psi = |\psi_2|^2 - |\psi_3|^2 = u_2^2 + v_2^2 - u_3^2 - v_3^2 \,. \tag{11}$$

We shall prove in this section the following:

Theorem 1. *Assume $g = 0$ in (9) and the (initial) total charge is finite*

$$\int_{\mathbb{R}^2} (u_2^2 + v_2^2 + u_3^2 + v_3^2)\, dx_1\, dx_2 \,|_{t=0} < \infty \,. \tag{12}$$

Then $u_2^2 + v_2^2 - u_3^2 - v_3^2 \in \mathscr{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ and for any $\beta \in C_c^\infty(\mathbb{R}^3)$,

$$\| \beta(u_2^2 + v_2^2 - u_3^2 - v_3^2) \|_{h^1(\mathbb{R}^3)} \leqq C_\beta \| u_2^2 + v_2^2 + u_3^2 + v_3^2 \|_{L^1(\mathbb{R}^2 \times \{t=0\})} \,, \tag{13}$$

where C_β is a constant depending on β.

Remark. The system (9) is linear with constant coefficients when $g = 0$. Weak solutions exist for all time $t \in \mathbb{R}$ provided that the initial total charge is finite. It can be seen that the total charge remains finite for weak solutions for all time $t \in \mathbb{R}$.

We prove Theorem 1 by first establishing a lemma. In this lemma we will use the Fourier transform, which we take in the form

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-2\pi i x \cdot \xi}\, dx$$

for all $f(x) \in L^1(\mathbb{R}^3)$. And the related inverse Fourier transform of a function $g(\xi) \in L^1(\mathbb{R}^3)$ that we shall use is

$$g^\vee(x) = \int_{\mathbb{R}^3} g(\xi) e^{2\pi i \xi \cdot x}\, d\xi \,.$$

The properties that we shall use are listed below. For a convenient reference we refer the reader to Stein and Weiss [21].

(1) $(\hat{f})^\wedge = f$ a.e. if $f(x) \in L^2(\mathbb{R}^3)$.

(2) $\bar{\hat{f}}(\xi) = f^\vee(\xi)$ if $f(x)$ is real valued. Over-head bar denotes complex conjugate from now on.

(3) $\int \hat{f} g = \int f \hat{g}$ for $f, g \in L^2$ (Plancherel Identity).

(4) $\left(\dfrac{\partial f}{\partial x_j}\right)^\wedge = 2\pi i \xi_j \hat{f}$.

(5) $\left(\dfrac{P_k(x)}{|x|^{k+n-\alpha}}\right)^\wedge = \gamma_{k,\alpha} \dfrac{P_k(x)}{|x|^{k+\alpha}}$ with $\gamma_{k,\alpha} = (-i)^k \pi^{n/2-\alpha} \dfrac{\Gamma\left(\dfrac{k+\alpha}{2}\right)}{\Gamma\left(\dfrac{k+n-\alpha}{2}\right)}$ for any homogeneous harmonic polynomial $P_k(x)$ of degree $k \geqq 1$ and $0 < \alpha < n$ (see Stein [20], p. 73. The difference between the $\gamma_{k,\alpha}$ here and that of Stein

[20] results from different forms of Fourier transform used). In particular, we shall use in $\mathbb{R}^3$,

$$\left(\frac{\xi_0}{|\xi|^2}\right)^\wedge = \gamma_{1,2}\frac{x_0}{|x|^3} = \frac{i}{2}\frac{\partial}{\partial x_0}\left(\frac{1}{|x|}\right). \tag{14}$$

Lemma 1. *We assume (u_2, v_2, u_3, v_3) satisfies the assumptions of Theorem 1. For any $\beta \in C_c^\infty(\mathbb{R}^3)$ and $q \in C_c^\infty(\mathbb{R}^3)$, $q = 1$ on the support of β, we have*

$$-4\pi\beta(u_2^2 + v_2^2 - u_3^2 - v_3^2) = g_1 \cdot \nabla F_1 + g_2 \cdot \nabla F_2 + g_0 , \tag{15}$$

where

$$g_1 = (u_2, u_3, -v_3)\beta ,$$

$$F_1 = (u_2 q) * \frac{\partial}{\partial x_0}\left(\frac{1}{|x|}\right) - (u_3 q) * \frac{\partial}{\partial x_1}\left(\frac{1}{|x|}\right) + (v_3 q) * \frac{\partial}{\partial x_2}\left(\frac{1}{|x|}\right),$$

$$g_2 = (v_2, v_3, u_3)\beta ,$$

$$F_2 = (v_2 q) * \frac{\partial}{\partial x_0}\left(\frac{1}{|x|}\right) - (v_3 q) * \frac{\partial}{\partial x_1}\left(\frac{1}{|x|}\right) - (u_3 q) * \frac{\partial}{\partial x_2}\left(\frac{1}{|x|}\right),$$

$$g_0 = (u_3, u_2, v_2)\beta \cdot (-\partial_{x_0}, \partial_{x_1}, \partial_{x_2})\left\{\frac{1}{|x|} * [-M v_3 q + (u_3, u_2, v_2) \cdot \nabla q]\right\}$$

$$+ (v_3, v_2, u_2)\beta \cdot (\partial_{x_0}, -\partial_{x_1}, \partial_{x_2})\left\{\frac{1}{|x|} * [M u_3 q + (v_3, v_2, -u_2) \cdot \nabla q]\right\}.$$

Proof. For $\xi = (\xi_0, \xi_1, \xi_2) \neq 0$, we define

$$O(\xi) = \frac{1}{|\xi|}\begin{pmatrix} \xi_0 & 0 & \xi_1 & -\xi_2 \\ 0 & \xi_0 & \xi_2 & \xi_1 \\ \xi_1 & \xi_2 & -\xi_0 & 0 \\ -\xi_2 & \xi_1 & 0 & -\xi_0 \end{pmatrix} \tag{16}$$

which is a symmetric orthogonal matrix. For any $\eta(x) \in C_c^\infty(\mathbb{R}^3)$, we let

$$\begin{pmatrix} \Lambda_1 & \Sigma_1 \\ \Lambda_2 & \Sigma_2 \\ \Lambda_3 & \Sigma_3 \\ \Lambda_4 & \Sigma_4 \end{pmatrix} = O(\xi) \cdot \begin{pmatrix} \widehat{u_2\eta}(\xi) & \widehat{u_2 q}(\xi) \\ \widehat{v_2\eta}(\xi) & \widehat{v_2 q}(\xi) \\ \widehat{u_3\eta}(\xi) & -\widehat{u_3 q}(\xi) \\ \widehat{v_3\eta}(\xi) & -\widehat{v_3 q}(\xi) \end{pmatrix}. \tag{17}$$

It follows that

$$\widehat{u_2\eta} \cdot \overline{\widehat{u_2 q}} + \widehat{v_2\eta} \cdot \overline{\widehat{v_2 q}} - \widehat{u_3\eta} \cdot \overline{\widehat{u_3 q}} - \widehat{v_3\eta} \cdot \overline{\widehat{v_3 q}} = \Lambda_1\overline{\Sigma_1} + \Lambda_2\overline{\Sigma_2} + \Lambda_3\overline{\Sigma_3} + \Lambda_4\overline{\Sigma_4} .$$

By Plancherel identity, we obtain

$$\int_{\mathbb{R}^3} (u_2^2 + v_2^2 - u_3^2 - v_3^2)\eta q \, dx = \int_{\mathbb{R}^3} (\Lambda_1\overline{\Sigma_1} + \Lambda_2\overline{\Sigma_2} + \Lambda_3\overline{\Sigma_3} + \Lambda_4\overline{\Sigma_4}) d\xi . \tag{18}$$

Multiplying the first equation in (9) by η and recalling $g = 0$, we find

$$\frac{\partial}{\partial x_0}(u_2\eta) + \frac{\partial}{\partial x_1}(u_3\eta) - \frac{\partial}{\partial x_2}(v_3\eta) = Mv_2\eta + (u_2, u_3, -v_3)\cdot \nabla\eta\,. \tag{19}$$

Applying the Fourier transform, we obtain

$$2\pi i(\xi_0\widehat{u_2\eta} + \xi_1\widehat{u_3\eta} - \xi_2\widehat{v_3\eta}) = M\widehat{v_2\eta} + ((u_2, u_3, -v_3)\cdot\nabla\eta)^\wedge\,.$$

Hence, the first term on the right-hand side of (18) is

$$\begin{aligned}
\int_{\mathbb{R}^3} \Lambda_1\overline{\Sigma_1}\,d\xi &= \int_{\mathbb{R}^3} \frac{1}{|\xi|}(\xi_0\widehat{u_2\eta} + \xi_1\widehat{u_3\eta} - \xi_2\widehat{v_3\eta})\cdot\frac{1}{|\xi|}(\xi_0\overline{\widehat{u_2q}} - \xi_1\overline{\widehat{u_3q}} + \xi_2\overline{\widehat{v_3q}})\,d\xi \\
&= \frac{1}{2\pi i}\int_{\mathbb{R}^3} (M\widehat{v_2\eta} + ((u_2, u_3, -v_3)\cdot\nabla\eta)^\wedge) \\
&\quad\cdot\left(\frac{\xi_0}{|\xi|^2}\overline{\widehat{u_2q}} - \frac{\xi_1}{|\xi|^2}\overline{\widehat{u_3q}} + \frac{\xi_2}{|\xi|^2}\overline{\widehat{v_2q}}\right)d\xi \\
&= \frac{1}{2\pi i}\int_{\mathbb{R}^3} (Mv_2\eta + ((u_2, u_3, -v_3)\cdot\nabla\eta)^\wedge) \\
&\quad\cdot\left(\frac{\xi_0}{|\xi|^2}(u_2q)^\vee - \frac{\xi_1}{|\xi|^2}(u_3q)^\vee + \frac{\xi_2}{|\xi|^2}(v_3q)^\vee\right)dx \\
&= \frac{1}{2\pi i}\int_{\mathbb{R}^3} (Mv_2\eta + (u_2, u_3, -v_3)\cdot\nabla\eta) \\
&\quad\cdot\left(\widehat{\frac{\xi_0}{|\xi|^2}} * (qu_2) - \widehat{\frac{\xi_1}{|\xi|^2}} * (qu_3) + \widehat{\frac{\xi_2}{|\xi|^2}} * (qv_3)\right)dx \\
&= \frac{1}{4\pi}\int_{\mathbb{R}^3} (Mv_2\eta + (u_2, u_3, -v_3)\cdot\nabla\eta) \\
&\quad\cdot\left(\frac{\partial}{\partial x_0}\left(\frac{1}{|x|}\right) * (qu_2) - \frac{\partial}{\partial x_1}\left(\frac{1}{|x|}\right) * (qu_3) + \frac{\partial}{\partial x_2}\left(\frac{1}{|x|}\right) * (qv_3)\right)dx \\
&= \frac{-1}{4\pi}\int \eta(u_2, u_3, -v_3)\cdot\nabla\left(\frac{\partial}{\partial x_0}\left(\frac{1}{|x|}\right) * (qu_2)\right. \\
&\quad\left. - \frac{\partial}{\partial x_1}\left(\frac{1}{|x|}\right) * (qu_3) + \frac{\partial}{\partial x_2}\left(\frac{1}{|x|}\right) * (qv_3)\right)dx\,,
\end{aligned}$$

where we used (19) in the last equality.

Similarly

$$\begin{aligned}
\int_{\mathbb{R}^3} \Lambda_2\overline{\Sigma_2}\,d\xi &= \frac{1}{4\pi}\int_{\mathbb{R}^3} (-Mu_2\eta + (v_2, v_3, u_3)\cdot\nabla\eta) \\
&\quad\cdot\left(\frac{\partial}{\partial x_0}\left(\frac{1}{|x|}\right) * (qv_2) - \frac{\partial}{\partial x_1}\left(\frac{1}{|x|}\right) * (qv_3) - \frac{\partial}{\partial x_2}\left(\frac{1}{|x|}\right) * (qu_3)\right)dx
\end{aligned}$$

$$= \frac{-1}{4\pi} \int_{\mathbb{R}^3} \eta(v_2, v_3, u_3) \cdot \nabla\left(\frac{\partial}{\partial x_0}\left(\frac{1}{|x|}\right) * (qv_2) - \frac{\partial}{\partial x_1}\left(\frac{1}{|x|}\right) * (qv_3) - \frac{\partial}{\partial x_2}\left(\frac{1}{|x|}\right) * (qu_3) \right) dx\,.$$

For the third term on the right-hand side of (18), we multiply the third equation in (9) (with $g = 0$) with q and use the Fourier transform to find

$$2\pi i(\xi_0 \widehat{qu_3} + \xi_1 \widehat{qu_2} + \xi_2 \widehat{qv_2}) = -M\widehat{qv_3} + [(u_3, u_2, v_2) \cdot \nabla q]^{\wedge}\,.$$

If we set

$$\Pi_3 = -Mqv_3 + (u_3, u_2, v_2) \cdot \nabla q$$

temporarily, we then have

$$\overline{\Sigma_3} = \frac{1}{|\xi|}(\xi_0 \overline{\widehat{qu_3}} + \xi_1 \overline{\widehat{qu_2}} + \xi_2 \overline{\widehat{qv_2}}) = \frac{i}{2\pi|\xi|} \overline{\widehat{\Pi_3}} = \frac{i}{2\pi|\xi|} \Pi_3{}^{\vee}\,.$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}^3} \Lambda_3 \overline{\Sigma_3}\, d\xi &= \int \frac{1}{|\xi|}(-\xi_0 \widehat{u_3\eta} + \xi_1 \widehat{u_2\eta} + \xi_2 \widehat{v_2\eta}) \frac{i}{2\pi|\xi|} \Pi_3{}^{\vee}\, d\xi \\
&= \frac{i}{2\pi} \int \left(-\frac{\xi_0}{|\xi|^2} \widehat{u_3\eta} + \frac{\xi_1}{|\xi|^2} \widehat{u_2\eta} + \frac{\xi_2}{|\xi|^2} \widehat{v_2\eta} \right) \Pi_3{}^{\vee}\, d\xi \\
&= \frac{i}{2\pi} \int \left[\left(\frac{-\xi_0}{|\xi|^2} \Pi_3{}^{\vee} \right) \widehat{u_3\eta} + \left(\frac{\xi_1}{|\xi|^2} \Pi_3{}^{\vee} \right) \widehat{u_2\eta} + \left(\frac{\xi_2}{|\xi|^2} \Pi_3{}^{\vee} \right) \widehat{v_2\eta} \right] d\xi \\
&= \frac{i}{2\pi} \int \left(\left(\frac{-\xi_0}{|\xi|^2} \right)^{\wedge} * \Pi_3 \right)(u_3\eta) \\
&\quad + \left(\left(\frac{\xi_1}{|\xi|^2} \right)^{\wedge} * \Pi_3 \right)(u_2\eta) + \left(\left(\frac{\xi_2}{|\xi|^2} \right)^{\wedge} * \Pi_3 \right)(v_2\eta)\, dx \\
&= \frac{-1}{4\pi} \int \left[\left(-\frac{\partial}{\partial x_0}\left(\frac{1}{|x|}\right) * \Pi_3 \right)(u_3\eta) \right. \\
&\quad \left. + \left(\frac{\partial}{\partial x_1}\left(\frac{1}{|x|}\right) * \Pi_3 \right)(u_2\eta) + \left(\frac{\partial}{\partial x_2}\left(\frac{1}{|x|}\right) * \Pi_3 \right)(v_2\eta) \right] dx\,.
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_{\mathbb{R}^3} \Lambda_4 \overline{\Sigma_4}\, d\xi = \frac{-1}{4\pi} \int &\left[\left(\frac{\partial}{\partial x_0}\left(\frac{1}{|x|}\right) * \Pi_4 \right)(v_3\eta) - \left(\frac{\partial}{\partial x_1}\left(\frac{1}{|x|}\right) * \Pi_4 \right)(v_2\eta) \right. \\
&\left. + \left(\frac{\partial}{\partial x_2}\left(\frac{1}{|x|}\right) * \Pi_4 \right)(u_2\eta) \right] dx,
\end{aligned}$$

where

$$\Pi_4 = Mqu_3 + (v_3, v_2, -u_2)\cdot \nabla q \cdot$$

Since $\eta \in C_c^\infty$ is arbitrary, we find

$$\begin{aligned}
&-4\pi q(u_2^2 + v_2^2 - u_3^2 - v_3^2) \\
&= (u_2, u_3, -v_3)\cdot \nabla\left(\frac{\partial}{\partial x_0}\left(\frac{1}{|x|}\right) * (qu_2)\right. \\
&\quad \left. - \frac{\partial}{\partial x_1}\left(\frac{1}{|x|}\right) * (qu_3) + \frac{\partial}{\partial x_2}\left(\frac{1}{|x|}\right) * (qv_3)\right) \\
&\quad + (v_2, v_3, u_3)\cdot \nabla\left(\frac{\partial}{\partial x_0}\left(\frac{1}{|x|}\right) * (qv_2) - \frac{\partial}{\partial x_1}\left(\frac{1}{|x|}\right) * (qv_3) - \frac{\partial}{\partial x_2}\left(\frac{1}{|x|}\right) * (qu_3)\right) \\
&\quad + (u_3, u_2, v_2)\cdot(-\partial_{x_0}, \partial_{x_1}, \partial_{x_2})\left(\left(\frac{1}{|x|}\right) * (-Mqv_3 + (u_3, u_2, v_2)\cdot \nabla q)\right) \\
&\quad + (v_3, v_2, u_2)\cdot(\partial_{x_0}, -\partial_{x_1}, \partial_{x_2})\left(\left(\frac{1}{|x|}\right) * (Mqu_3 + (v_3, v_2, -u_2)\cdot \nabla q)\right).
\end{aligned}$$

Multiplying the last identity by β on both sides, we complete the proof of Lemma 1.

Proof of Theorem 1. From Lemma 1, we have

$$\| (u_2^2 + v_2^2 - u_3^2 - v_3^2)\beta \|_{h^1(\mathbb{R}^3)} \leqq \frac{1}{4\pi}(\| g_1 \cdot \nabla F_1 \|_{h^1} + \| g_2 \cdot \nabla F_2 \|_{h^1} + \| g_0 \|_{h^1}) .$$

We see easily that $g_1, g_2 \in L^2(\mathbb{R}^3)$, and

$$\begin{aligned}
\operatorname{div} g_1 &= M\beta v_2 + (u_2, u_3, -v_3)\cdot \nabla\beta \in L^2(\mathbb{R}^3) , \\
\operatorname{div} g_2 &= -M\beta u_2 + (v_2, v_3, u_3)\cdot \nabla\beta \in L^2(\mathbb{R}^3) .
\end{aligned}$$

By simple elliptic (potential) theory, we find

$$\begin{aligned}
\| F_1 \|_{W^{1,2}(\mathbb{R}^3)} &\leqq C \| (u_2, u_3, v_3) q \|_{L^2(\mathbb{R}^3)}, \\
\| F_2 \|_{W^{1,2}(\mathbb{R}^3)} &\leqq C \| (v_2, v_3, u_3) q \|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

From Proposition 3, we find

$$\begin{aligned}
\| g_1 \cdot \nabla F_1 \|_{h^1} + \| g_2 \cdot \nabla F_2 \|_{h^1} &\leqq C(\| g_1 \|_{L^2}^2 + \| \operatorname{div} g_1 \|_{L^2}^2 + \| F_1 \|_{W^{1,2}}^2 \\
&\quad + \| g_2 \|_{L^2}^2 + \| \operatorname{div} g_2 \|_{L^2}^2 + \| F_2 \|_{W^{1,2}}^2) \\
&\leqq C(\| u_2^2 + v_2^2 + u_3^2 + v_3^2 \|_{L^1(\mathbb{R}^2)}) .
\end{aligned}$$

where C depends on β and M. For g_0, it can be seen that

$$\begin{aligned}
\| g_0 \|_{L^{3/2}} &\leqq \| (u_3, u_2, v_2)\beta \|_{L^2} \left\| \nabla\left(\frac{1}{|x|} * [-Mqv_3 + (u_3, u_2, v_2)\cdot \nabla q]\right) \right\|_{L^6} \\
&\quad + \| (v_3, v_2, u_2)\beta \|_{L^2} \left\| \nabla\left(\frac{1}{|x|} * [Mqu_3 + (v_3, v_2, -u_2)\cdot \nabla q]\right) \right\|_{L^6}
\end{aligned}$$

$$\leqq C\|(u_2, v_2, u_3, v_3)\|_{L^2(\mathbb{R}^2)}(\| -Mqv_3 + (u_3, u_2, v_2)\cdot \nabla q\|_{L^2(\mathbb{R}^3)}$$
$$+ \|Mqu_3 + (v_3, v_2, -u_2)\cdot \nabla q\|_{L^2(\mathbb{R}^3)}) \leqq C\|(u_2, v_2, u_3, v_3)\|^2_{L^2(\mathbb{R}^2)}\,.$$

Let $R > 0$ be so large that the support of β is contained in the ball $B(0, R-1)$. And let $M(g)(x)$ denote the Hardy–Littlewood maximal function of g. Thus we have

$$\sup_{1>r>0}\left|\frac{1}{r^3}\int g_0(x-y)h\left(\frac{y}{r}\right)dy\right| \leqq \begin{cases} M(|g_0|)(x), & x \in B(0,R)\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

So,

$$\begin{aligned}\|g_0\|_{h^1(\mathbb{R}^3)} &= \left\|\sup_{1>r>0}\left|\frac{1}{r^3}\int g_0(x-y)h\left(\frac{y}{r}\right)dy\right|\right\|_{L^1(\mathbb{R}^3)} \\ &\leqq \|M(|g_0|)(x)\|_{L^1(B(0,R))} \\ &\leqq C\|M(|g_0|)(x)\|_{L^{3/2}(B(0,R))} \\ &\leqq C\|g_0\|_{L^{3/2}(\mathbb{R}^3)}.\end{aligned}$$

Therefore

$$\|(u_2^2 + v_2^2 - u_3^2 - v_3^2)\beta\|_{h^1(\mathbb{R}^3)} \leqq C_\beta\|u_2^2 + v_2^2 + u_3^2 + v_3^2\|_{L^1(\mathbb{R}^2)}\,.$$

The proof of Theorem 1 is completed.

4. An Estimate on the Klein–Gordon Equation

Consider

$$\begin{cases} \varphi_{tt} - \varphi_{x_1x_1} - \varphi_{x_2x_2} + m^2\varphi = f(t, x_1, x_2) \\ \varphi(0, x) = \varphi_t(0, x) = 0\,. \end{cases} \tag{20}$$

Suppose $f \in h^1(\mathbb{R}^3)$ and $f = 0$, $t \leqq 0$. We show next

Theorem 2. *The solution φ of* (20) *satisfies the estimate*

$$\|\varphi(t,\cdot)\|^2_{L^2(\mathbb{R}^2)} \leqq C_T\|f\|_{L^1(\mathbb{R}^2\times[0,t])}\|f\|_{h^1(\mathbb{R}^3)} \tag{21}$$

for all $t \in [0, T]$, $T > 0$.

Proof. Without loss of generality, we assume $m = 0$. Introduce $u(t, x_1, x_2)$ such that

$$\begin{cases} u_{tt} - u_{x_1x_1} - u_{x_2x_2} = F(t, x_1, x_2) \equiv \int_0^t f(s, x_1, x_2)\,ds \\ u(0, x) = u_t(0, x) = 0\,. \end{cases} \tag{22}$$

We observe that

$$\varphi = u_t\,.$$

The idea is to obtain the L^2-estimate on u_t which will follow as an energy estimate of the new Eq. (22) for u. To do so, we need an L^∞-estimate on u.

Step 1. L^∞-estimate on u. We have formula

$$u(t,x) = \int_0^t \iint_{|y| \leqq \tau} \frac{F(t-\tau, x-y)}{\sqrt{\tau^2 - |y|^2}}\, dy\, d\tau$$

$$= \iint_{|y| \leqq t} \left(\int_{|y|}^t \frac{F(t-\tau, x-y)}{\sqrt{\tau^2 - |y|^2}}\, d\tau \right) dy$$

$$= \iint_{|y| \leqq t} \left[\int_{|y|}^t F(t-\tau, x-y)\, d\left(\log \frac{\tau + \sqrt{\tau^2 - |y|^2}}{|y|} \right) \right] dy$$

$$= \iint_{|y| \leqq t} \left[\left(F(t-\tau, x-y) \log \frac{\tau + \sqrt{\tau^2 - |y|^2}}{|y|} \right) \Bigg|_{|y|}^t \right.$$

$$\left. + \int_{|y|}^t \log \frac{\tau + \sqrt{\tau^2 - |y|^2}}{|y|} f(t-\tau, x-y)\, d\tau \right] dy$$

$$= \int_0^t \iint_{|y| \leqq \tau} f(t-\tau, x-y) \log \frac{\tau + \sqrt{\tau^2 - |y|^2}}{|y|}\, dy\, d\tau$$

$$= \int_{\mathbb{R}^3} f(t-\tau, x-y)\, W(y,\tau)\, \beta(\tau - t)\, dy\, d\tau\,,$$

where we used that $f(t,x) = 0$ for $t < 0$, $\beta(\tau - t)$ is as in Proposition 5, and $W(y,\tau)$ is

$$W(y,\tau) = \begin{cases} \log \dfrac{\tau + \sqrt{\tau^2 - |y|^2}}{|y|}, & |y| \leqq \tau, 0 \leqq \tau < \infty \\ 0, & \text{otherwise}\,. \end{cases}$$

Now W can be split into two parts: $W = B + Z$, where B is as in Proposition 4 and

$$Z(y,\tau) = \begin{cases} \log\left(1 + \sqrt{1 - \dfrac{|y|^2}{\tau^2}} \right), & |y| \leqq \tau, 0 \leqq \tau < \infty \\ 0, & \text{otherwise}\,, \end{cases}$$

so that $\|Z\|_{L^\infty} \leqq \log 2$. Hence $W\beta(\tau - t)$ is in $\mathrm{BMO}(\mathbb{R}^3)$ and $L^1(\mathbb{R}^3)$ for any $t > 0$, and therefore, $\varphi * (W\beta) \in L^\infty$ for $\varphi \in S$, $\int \varphi = 1$. By Goldberg's local version

[14] of Fefferman's duality theorem [10], we find

$$\| u(t,\cdot) \|_{L^\infty(\mathbb{R}^2)} \leqq C(t) \| f \|_{h^1(\mathbb{R}^3)} .$$

Step 2. Multiplying (18) by u_t and using integration by parts, we find

$$\frac{d}{dt} \int_{\mathbb{R}^2} \left(\frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 \right) dx = \frac{d}{dt} \int_{\mathbb{R}^2} F(t,x) u(t,x) \, dx - \int_{\mathbb{R}^2} f(t,x) u \, dx .$$

Integrating in the t-direction and using the initial condition, we obtain

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^2} (u_t^2 + |\nabla u|^2) \, dx &= \int_{\mathbb{R}^2} F(t,x) u \, dx - \int_0^t \int_{\mathbb{R}^2} f u \, dx \, ds \\ &\leqq \| u(t,\cdot) \|_{L^\infty(\mathbb{R}^2)} \| f \|_{L^1(\mathbb{R}^2 \times (0,t))} \\ &\quad + \sup_{0 \leqq s \leqq t} \| u(s,\cdot) \|_{L^\infty(\mathbb{R}^2)} \cdot \| f \|_{L^1(\mathbb{R}^2 \times (0,t))} \\ &\leqq C_t \| f \|_{h^1(\mathbb{R}^3)} \cdot \| f \|_{L^1(\mathbb{R}^2 \times (0,t))} . \end{aligned}$$

Thus

$$\| \varphi(t,\cdot) \|_{L^2}^2 \leqq C_t \| f \|_{h^1(\mathbb{R}^3)} \| f \|_{L^1(\mathbb{R}^2 \times (0,t))}$$

for all $t \geqq 0$. As a consequence, we find

$$\| \varphi(t,\cdot) \|_{L^2} \leqq C_t \| f \|_{h^1(\mathbb{R}^3)} .$$

The proof is completed.

Remarks.
(1) The result is sharp, as can be seen by taking

$$f = \delta(x,t) .$$

f is not in $h^1(\mathbb{R}^3)$, but only slightly so. And a solution of (20) with $m = 0$ is

$$\varphi = \frac{1}{\sqrt{(t^2 - |x|^2)_+}} ,$$

which is not in $L^\infty_{\mathrm{loc}}((0, \infty), L^2(\mathbb{R}^2))$.
(2) Similar method works to prove $\varphi \in L^\infty(0, T; L^2(\mathbb{R}^2))$ for

$$\begin{cases} \Box \varphi = 0 \\ \varphi|_{t=0} = 0 \\ \varphi_t|_{t=0} = f(x) \in h^1(\mathbb{R}^2) , \end{cases}$$

and

$$\| \varphi(t,\cdot) \|_{L^2} \leqq C_t \| f \|_{h^1(\mathbb{R}^2)}, \quad \forall t \in \mathbb{R} .$$

(3) For related $L^p - L^q$ estimates on KG equations, we refer the reader to Peral [18] and Marshall etc. [16].

5. Some Remarks

(i) Existence of weak solutions in 1-D. In one space dimension, the Dirac–Klein–Gordon system takes the form:

$$\begin{cases} \dfrac{\partial u_2}{\partial t} + \dfrac{\partial u_3}{\partial x_1} = (M - g\varphi)v_2 \\ \dfrac{\partial v_2}{\partial t} + \dfrac{\partial v_3}{\partial x_1} = -(M - g\varphi)u_2 \\ \dfrac{\partial u_3}{\partial t} + \dfrac{\partial u_2}{\partial x_1} = -(M - g\varphi)v_3 \\ \dfrac{\partial v_3}{\partial t} + \dfrac{\partial v_2}{\partial x_1} = (M - g\varphi)u_3 \\ \varphi_{tt} - \varphi_{x_1 x_1} + m^2\varphi = g(u_2^2 + v_2^2 - u_3^2 - v_3^2)\,. \end{cases}$$

As mentioned before, Chadam [6] proved that there exists a global unique solution to its Cauchy problem with initial data $(u_2, v_2, u_3, v_3)|_{t=0} \in H^1(\mathbb{R}^1)$ and $\varphi(0, x) \in H^1$, $\varphi_t(0, x) \in L^2$. We can now prove

Theorem 3. *There exists a global weak solution to the Cauchy problem of the* 1-D DKG *system with* $(u_2, v_2, u_3, v_3)|_{t=0} \in L^2(\mathbb{R}^1)$, $\varphi(0, x) \in H^1$ *and* $\varphi_t(0, x) \in L^2$.

Sketch of proof. We mollify the initial data to find a sequence of exact classical solutions $\{\varphi^k, u_2^k, v_2^k, u_3^k, v_3^k\}_{k=1}^{\infty}$ from Chadam [6] with the estimates

$$\sup_{0 \leqq t \leqq T} \|(u_2^k, v_2^k, u_3^k, v_3^k)\|_{L^2(\mathbb{R}^2)} \leqq C$$

and

$$\sup_{0 \leqq t \leqq T} \|\varphi^k\|_{L^\infty(\mathbb{R}^2)} \leqq C_T.$$

Therefore

$$\|\varphi^k(v_2^k, u_2^k, v_3^k, u_3^k)\|_{L^2_{\mathrm{loc}}(\mathbb{R}^3)} \leqq C\,.$$

By Tartar's [22] compensated compactness, $(u_2^k)^2 + (v_2^k)^2 - (u_3^k)^2 - (v_3^k)^2$ is weakly continuous. We therefore have no difficulty to pass the limit through this term. The other nonlinear terms $\varphi^k u_2^k$ etc. are also weakly continuous since $\{\varphi^k\}$ is compact in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ (see e.g. Peral [18]). The sketch of proof is completed.

(ii) 2-D classical coupled DKG system.

To investigate the difficulty of establishing the existence of weak solutions of nonlinear equations, it is a common technique (see some papers of DiPerna, Lions and Majda) to see how the nonlinear terms of the equations behave with respect to weakly convergent sequences of exact solutions in a suitable space naturally related to the equation. The question of how to produce approximate solutions with the estimates that are satisfied by the exact solutions can be handled in a much easier way in most cases than that of how to deal with the passage of the limit through the nonlinear terms. Here for our system (1), we shall similarly assume that we have a sequence of exact solutions $\{\varphi^k, \psi^k\}_{k=1}^{\infty}$ which satisfies the natural estimate

$$\sup_{0 \leqq t \leqq T} \int_{\mathbb{R}^2} |\psi^k|^2 \, dx \leqq C_T .$$

For $\{\varphi^k\}_{k=1}^{\infty}$, we depend on the KG equation to give its estimate. For simplicity, we shall assume $\varphi^k = 0$ and $\varphi_t^k = 0$ at $t = 0$. We obtain

$$\sup_{0 \leqq t \leqq T} \int_{\mathbb{R}^2} |\varphi^k|^p \, dx \leqq C_{pT}, \quad \forall p < 2 .$$

Furthermore, we assume

$$\psi^k \rightharpoonup \psi \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^3) \text{ weakly,}$$

$$\varphi^k \to \varphi \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^3), \forall p < 2 .$$

The strong convergence of $\{\varphi^k\}$ follows from the $W^{\alpha, p}_{\text{loc}}(\mathbb{R}^3)$ estimate for some $0 < \alpha < 1$ (see Peral [18], for example).

In order to define $\{\varphi, \psi\}$ to be a weak solution, we need $\varphi\psi$ to be defined in $L^1_{\text{loc}}(\mathbb{R}^3)$. For this purpose, it is sufficient to have estimate $\int_0^T \|\varphi^k\|_{L^2(\mathbb{R}^2)} dt \leqq C_T$. Further, we need $\{\varphi^k \psi^k\}_{k=1}^{\infty}$ to be compact in $W^{-1,2}_{\text{loc}}(\mathbb{R}^3)$ in order for $\{\overline{\psi^k} \psi^k\}$ to be weakly continuous by the standard compensated compactness of Tartar [22]. In terms of the L^p-estimate, we need $\{\varphi^k\}_{k=1}^{\infty}$ to be in $L^p((0, T), L^{\frac{2p}{2-p}}(\mathbb{R}^2))$ for some $p > \frac{6}{5}$. Unfortunately, artificial examples (with $f = \bar{\psi}\psi \in \mathscr{H}^1_{\text{loc}}(\mathbb{R}^3) \cap L^\infty((0, T); L^1(\mathbb{R}^2))$ in problem (20)) show that $\{\varphi^k\}_{k=1}^{\infty}$ need not lie in that space. However, since we know $\{\overline{\psi^k} \psi^k\}_{k=1}^{\infty}$ is actually in $L^\infty((0, T); L^1(\mathbb{R}^2))$, we hope that a modified version of Tartar's compensated compactness will require only that $\{\varphi^k \psi^k\}_{k=1}^{\infty}$ be compact in $L^\infty((0, T); W^{-1,2}_{\text{loc}}(\mathbb{R}^2))$.

If this is the case, we need only

$$\varphi \in L^1((0, T); L^p(\mathbb{R}^2))$$

for some $p > 2$. Along this line we observe that the estimate

$$\sup_{0 \leqq t \leqq T} \|\varphi\|_{L^{3,\infty}(\mathbb{R}^2)} \leqq C_T \|f\|_{L^\infty((0, T); L^1(\mathbb{R}^2))}$$

for problem (20) is probably true and sharp. $L^{3,\infty}$ denotes the weak L^3 space.

In conclusion, to establish existence of weak solutions to system (1) we may need to prove an estimate of the form

$$\sup_{0 \leqq t \leqq T} \|\varphi\|_{L^p(\mathbb{R}^2)} \leqq C_T \sup_{0 \leqq t \leqq T} \|f\|_{L^1(\mathbb{R}^2)}$$

for some $p \in [2, 3)$ for problem (16), and establish a modified version of compensated compactness of Tartar so that only $\{\varphi\psi\} \in$ compact set of $L^\infty((0, T), W^{-1,2}_{\text{loc}}(\mathbb{R}^2))$ is required for $\{\bar{\psi}\psi\}$ to be weakly continuous.

Appendix

In this appendix we shall prove Propositions 4 and 5.

Proof of Proposition 4. We prove this proposition by verifying that for any cylinder $Q \subset \mathbb{R}^3$ centered at any point (y_0, τ_0) with middle cross section $B(y_0, \tau_0; R)$ and height $2R$, there exists a number a_Q (not necessarily the mean of B over Q) such that

$$\sup_Q \frac{1}{|Q|} \int_Q |B - a_Q| \, dy \, d\tau \leqq C . \tag{A1}$$

For the equivalence of this condition to the aforementioned definition of BMO, we refer the reader to Torchinsky [23], Stein [20] or Fefferman and Stein [11].

Before we get involved in heavy computation, we notice that $\log \frac{1}{|\tau|} \in BMO(\mathbb{R}^1)$ and $\log \frac{1}{|y|} \in \mathrm{BMO}(\mathbb{R}^2)$ (see Stein [20], for example) and therefore $\log \frac{1}{|y|} \in \mathrm{BMO}(\mathbb{R}^3)$. But none of the functions $\beta_{\mathscr{C}} \log \frac{1}{|y|}$ and $\beta_{\mathscr{C}} \log \frac{1}{|\tau|}$, where $\beta_{\mathscr{C}}$ denotes the characteristic function for the cone

$$\mathscr{C} = \{(y, \tau) \in \mathbb{R}^3 \,|\, |y| \leqq \tau\}$$

is in $\mathrm{BMO}(\mathbb{R}^3)$.

We verify (A1) by considering each of the cases:

Case (i). $\tau_0 \leqq 0$. In this case we take $a_Q = 0$. Notice $B \geqq 0$ and suppose $R > -\tau_0$ (the case $R \leqq -\tau_0$ is trivial). Then the right-hand side of (A1) becomes

$$\begin{aligned}
\frac{1}{|Q|} \int_Q B \, dy \, d\tau &\leqq \frac{1}{|Q|} \int_{\mathscr{C} \cap \{\tau \leqq R + \tau_0\}} B \, dy \, d\tau \\
&\leqq \frac{1}{2\pi R^3} \int_0^{R+\tau_0} \int_r^{R+\tau_0} B \, d\tau \, r dr \, 2\pi \\
&= \frac{1}{R^3} \left\{ \int_0^{R+\tau_0} \left[(R + \tau_0 - r) \log \frac{1}{r} + (R + \tau_0) \log (R + \tau_0) \right.\right. \\
&\qquad \left.\left. - (R + \tau_0) - r \log r + r \right] r \, dr \right\} \\
&= \frac{1}{R^3} \left\{ \int_0^{R+\tau_0} \left\{ (R + \tau_0) r \log \frac{1}{r} + r^2 + r[(R + \tau_0) \log (R + \tau_0) - (R + \tau_0)] \right\} dr \right\} \\
&= \frac{1}{12} \left(1 + \frac{\tau_0}{R} \right)^3 \leqq \frac{1}{12}.
\end{aligned}$$

Case (ii). $\tau_0 > 0$, $|y_0| > \tau_0$, $R \leqq \dfrac{2|y_0| - \tau_0}{3}$. We also take $a_Q = 0$. Then

$$B \leqq \log 2 \quad \text{in } Q.$$

This is because Q does not intersect with the τ axis. More specifically, $\log\frac{\tau}{|y|}$ takes its maximum in Q at the corner $(|y_0|-R, \tau_0+R)$ and the maximum is an increasing function of R in $\left[0, \frac{2|y_0|-\tau_0}{3}\right]$. At $R=\frac{2|y_0|-\tau_0}{3}$, we have

$$\frac{\tau_0+R}{|y_0|-R}=2\,,$$

so

$$\frac{1}{|Q|}\int_Q B\,dy\,d\tau \leqq \log 2\,.$$

Case (iii). $\tau_0>0$, $|y_0|>\tau_0$, $R>\frac{2|y_0|-\tau_0}{3}$. So $R>\frac{\tau_0}{3}$. We take $a_Q=0$ again. Similarly to (i), we find

$$\begin{aligned}\frac{1}{|Q|}\int_Q B\,dy\,d\tau &\leqq \frac{1}{|Q|}\int_0^{\tau_0+R}\int_{|y|\leqq\tau} B\,dy\,d\tau\\ &=\frac{1}{12}\left(1+\frac{\tau_0}{R}\right)^3\leqq\frac{16}{3}.\end{aligned}$$

Case (iv). $\tau_0>0$, $\tau_0/2<|y_0|<\tau_0$, $R<\tau_0/4$. Similar to (ii) we find

$$B=\log\frac{\tau}{|y|}\leqq C \quad \text{in } Q.$$

Case (v). $\tau_0>0$, $\tau_0/2<|y_0|<\tau_0$, $R\geqq\tau_0/4$. Similar to (iii), we have

$$\frac{1}{|Q|}\int_Q B\,dy\,d\tau\leqq\frac{1}{12}\left(1+\frac{\tau_0}{R}\right)^3\leqq\frac{125}{12}.$$

Case (vi). $\tau_0>0$, $|y_0|<\tau_0/2$, $R<\tau_0/4$. In this case we have

$$Q\subset\mathscr{C}.$$

So

$$\sup_Q\frac{1}{|Q|}\int_Q|B-a_Q|\,dy\,d\tau\leqq\left\|\log\frac{1}{|\tau|}\right\|_{\mathrm{BMO}(\mathbb{R}^1)}+\left\|\log\frac{1}{|y|}\right\|_{\mathrm{BMO}(\mathbb{R}^2)}$$

Case (vii). $\tau_0>0$, $|y_0|<\tau_0/2$, $R\geqq\tau_0/4$. It is similar to (i). So the proof of Proposition 4 is completed.

Proof of Proposition 5. We verify that

$$\frac{1}{|Q|}\int_Q|B\beta-(B\beta)_Q|\,dy\,d\tau\leqq C(t_0)$$

for any cylinder Q with cross section $B(y_0,\tau_0;R)$ and height $2R$.

Case (i). $R > \frac{1}{2}$. We have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |B\beta - (B\beta)_Q|\,dy\,d\tau &\leqq \frac{2}{|Q|}\int_Q |B\beta|\,dy\,d\tau \\
&\leqq \frac{1}{\pi R^3}\int_0^{\tau_0+R}\int_{|y|\leqq\tau} \beta(\tau - t_0)\log\frac{\tau}{|y|}\,dy\,d\tau \\
&\leqq \frac{8}{\pi}\int_0^{t_0+1}\int_{|y|\leqq\tau} \log\frac{\tau}{|y|}\,dy\,d\tau \\
&= C(t_0)\,.
\end{aligned}$$

Case (ii). $R < \frac{1}{2}$, $\tau_0 > 1$. So $\tau_0 - R > \frac{1}{2}$. We have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |B\beta - (B\beta)_Q|\,dy\,d\tau \\
&= \frac{1}{|Q|}\int_Q \left|\beta\log\frac{1}{|y|} + \beta\log\tau - \left(\beta\log\frac{1}{|y|}\right)_Q - (\beta\log\tau)_Q\right|dy\,d\tau \\
&\leqq \frac{1}{|Q|}\int_Q \left|\beta\log\frac{1}{|y|} - \left(\beta\log\frac{1}{|y|}\right)_Q\right|dy\,d\tau + 2\log 2 \\
&= \frac{1}{|Q|}\int_Q \left|\beta\log\frac{1}{|y|} - (\beta)_Q\left(\log\frac{1}{|y|}\right)_Q\right|dy\,d\tau + 2\log 2 \\
&\leqq \frac{1}{|Q|}\int_Q \left(\left|\beta\log\frac{1}{|y|} - \beta\left(\log\frac{1}{|y|}\right)_Q\right| + \left|\beta\left(\log\frac{1}{|y|}\right)_Q\right.\right. \\
&\qquad \left.\left. - (\beta)_Q\left(\log\frac{1}{|y|}\right)_Q\right|\right)dy\,d\tau + 2\log 2 \\
&\leqq C + \frac{1}{|Q|}\int_Q |\beta - (\beta)_Q|\,dy\,d\tau\left|\left(\log\frac{1}{|y|}\right)_Q\right| + 2\log 2 \\
&\leqq C + \frac{CR}{|B(y_0, \tau_0; R)|}\int_B \log\frac{1}{|y|}\,dy + 2\log 2 \\
&= C'\,.
\end{aligned}$$

Case (iii). $R < \frac{1}{2}$ and $\tau_0 < 1$. Thus $R + \tau_0 < 2$. This case is then covered by Proposition 4. The proof of Proposition 5 is completed.

Acknowledgement. The author wishes to thank L.C. Evans, P.L. Lions and W. Strauss for helpful conversations.

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Communicated by A. Jaffe