Commun. Math. Phys. 143, 577–589 (1992)

Communications in Mathematical Physics

The Realm of the Vacuum

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Received April 30, 1991

Dedicated to H. J. Borchers and D. Kastler,
who both celebrated their 65th birthday this year

Abstract. The spacelike asymptotic structure of physical states in local quantum theory is analysed. It is shown that this structure can be described in terms of a vacuum state if the theory satisfies a condition of timelike asymptotic abelianess. Theories which violate this condition can have an involved asymptotic vacuum structure as is illustrated by a simple example.

1. Introduction

The analysis of the spacelike asymptotic structure of physical states in the Haag-Kastler framework of local quantum theory [1] is a longstanding problem. Especially the question of whether these states may be interpreted as excitations of some vacuum state is of great interest. The significance of this problem is based on the fact that relevant information on the type of superselection rules, the range of forces, and the possible statistics of particles is encoded in this structure.

In their pioneering investigation of this problem Borchers, Haag, and Schroer [2] argued that any physical state Φ of finite total energy ought to look like a vacuum state Ω at large spacelike distances. More precisely: if $A(a)=U(a)AU(a)^{-1}$, where A is any local observable and $U(a)$ is the unitary operator inducing the space-time translation a, one should expect that

$$\lim_{a}(\Phi, A(a)\Phi)=(\Omega, A\Omega) \tag{1}$$

if a tends to spacelike infinity [2].

After the discovery of soliton states and of topological charges it became clear, however, that relation (1) does not follow from the basic principles of local quantum theory, at least not in low space-time dimensions. There the spacelike asymptotic structure of physical states can be quite involved and the limit in (1) need not exist. It is apparent that such states can not be associated with the realm of some particular vacuum state. In view of these examples one must regard the status of relation (1) in physical space-time as unsettled. With the exception of massive theories with a particle interpretation, where relation (1) has been established in [3], all attempts to clarify the possible asymptotic vacuum structure of physical states within a general setting have failed so far, cf. the remarks in [4, p. 101].

As a contribution towards the solution of this problem we exhibit in the present article an interesting connection between the nature of timelike correlations in a theory and the spacelike asymptotic structure of states. We will show in Sect. 2 that relation (1) holds in more than two space-time dimensions whenever there exists a sufficiently rich set $\mathfrak{D}$ of local observables A which comply with the following condition.

Condition of timelike asymptotic abelianess: For any given $A \in \mathfrak{D}$ there exists some positive number $1 \leq r < s$, s being the dimension of space, such that[1]

$$\sup_{x_0} \int d^s x \, \| [A^*, A(x_0, \mathbf{x})] \Phi \|^r < \infty \tag{2}$$

for all vectors Φ in the underlying Hilbert space.

Because of the spacelike commutativity of local operators, relation (2) holds if the commutator function $\| [A^*, A(x_0, \mathbf{x})] \Phi \|$ decreases at asymptotic times (uniformly in $\mathbf{x}$) like $|x_0|^{-1-\varepsilon}$ for some $\varepsilon > 0$. Hence our condition is linked to the notion of L^1-asymptotic abelianess in time, cf. for example [5]. But it is weaker in two respects: first we do not anticipate a $|x_0|^{-1-\varepsilon}$ decay of the commutator function for all $\mathbf{x}$, this bound may be violated for some set of points $\mathbf{x}$ of sufficiently small measure. And secondly our condition restricts the decay of timelike commutators only in the strong operator topology and not in the uniform topology. Hence it is sensitive to properties of the underlying states of interest.

In theories where the timelike asymptotic behaviour of the commutator function is dominated by contributions due to the exchange of neutral particles between the localization regions of A and $A(x_0, \mathbf{x})$ a rough estimate shows that one has to put $r \geqq 2$ if condition (2) is to be satisfied (cf. the discussion in Sect. 3). Hence our condition seems reasonable in physical space-time, but is of limited use in lower dimensions.

The condition $r < s$, however, is close to being optimal if one wants to derive relation (1). We will exhibit in Sect. 3 a theory in $s = 2$ dimensions, where condition (2) is satisfied for any $r > 2$, but where the limit in (1) does not exist for a single spacelike direction. A brief discussion of our results is deferred to Sect. 4.

We conclude this introduction with a list of assumptions and some notation: we proceed from a Hilbert space $\mathscr{H}$ of physical states and a unital $*$-algebra $\mathfrak{A} \subset \mathscr{B}(\mathscr{H})$ of local observables whose weak closure has trivial center. (The latter assumption means that we restrict our attention to a subset of states belonging to a fixed superselection sector.) On $\mathscr{H}$ there is a continuous unitary representation $a \to U(a)$ of the space-time translations $\mathbb{R}^{s+1}$ whose spectrum lies in the light cone $V_+ = \{ p \in \mathbb{R}^{s+1} : p_0 \geqq |\mathbf{p}| \}$ and which complies with the condition of locality. This

[1] The space and time part of the translation x with respect to a given Lorentz frame will be denoted by $\mathbf{x}$ and x_0, respectively

means that for each $A \in \mathfrak{A}$ there is some finite distance d such that

$$[A^*, A(a)] = 0 \quad \text{for} \quad |\mathbf{a}| \geqq |a_0| + d, \tag{3}$$

where we adhere to the notation $A(a) = U(a)AU(a)^{-1}$. Finally we assume, as already mentioned, that there is some norm dense and $*$-invariant set $\mathfrak{D} \subset \mathfrak{A}$ of operators A which satisfy the condition (2) of timelike asymptotic abelianess.

2. Spacelike Vacuum Structure and Timelike Correlations

We show in this section that the limit $\lim_{\mathbf{a}} (\Phi, A(\mathbf{a})\Phi)$ exists if $|\mathbf{a}|$ tends to infinity, and we also establish its interpretation in terms of a vacuum state.

To begin we proceed to a more convenient formulation of our condition (2) of timelike asymptotic abelianess. According to that condition there exists for any given $A \in \mathfrak{D}$ some positive number $r < s$ such that the family of maps $(M_{x_0})_{x_0 \in \mathbb{R}} : \mathscr{H} \to \mathbb{R}_+$ given by

$$M_{x_0}(\Phi) = \int d^s x \| [A^*, A(x_0, \mathbf{x})]\Phi \|^r \tag{4}$$

is pointwise bounded. Each map M_{x_0} is continuous since the integration in (4) extends only over a bounded region due to locality. Since $\mathscr{H}$ is a complete metric space we conclude that there is some non-empty open subset of $\mathscr{H}$ on which the family M_{x_0} is uniformly bounded [6, Theorem 46.7]. It then follows from the specific form of M_{x_0} that this family is uniformly bounded on the unit ball $\mathscr{H}_1$ of $\mathscr{H}$. Hence condition (2) is equivalent to the statement that for each $A \in \mathfrak{D}$ there is some r, $1 \leqq r < s$ such that

$$\tau_r(A) = \sup \{ (\int d^s x \| [A^*, A(x_0, \mathbf{x})]\Phi \|^r)^{1/2r} : x_0 \in \mathbb{R},\ \Phi \in \mathscr{H}_1 \} < \infty . \tag{5}$$

In the first part of our analysis we will actually rely on a somewhat weaker condition, viz.

$$\sigma_r(A) = \sup \{ (\int d^s x |(\Phi, [A^*, A(x_0, \mathbf{x})]\Phi)|^r)^{1/2r} : x_0 \in \mathbb{R},\ \Phi \in \mathscr{H}(E)_1 \} < \infty . \tag{6}$$

Here $\mathscr{H}(E) \subset \mathscr{H}$ denotes the spectral subspace of the generator of the time translations corresponding to the spectrum in the interval $[0, E]$. (Since E is kept fixed in the following we do not indicate it as an extra label of σ_r.) We will see that the bound (6) already implies that the function $\mathbf{a} \to (\Phi, A(\mathbf{a})\Phi)$ converges if $|\mathbf{a}|$ tends to infinity. Yet in the proof that the limit can be interpreted as expectation value of A in some vacuum state we have to rely on the stronger condition (5).

Our arguments are similar to the reasoning in [7]. The crucial additional step in the present analysis is the demonstration that the bounds on commutators of smoothed-out local operators given in [7] can be improved if the condition of timelike asymptotic abelianess is satisfied.

Let $A \in \mathfrak{D}$ be any local operator and let $f \in \mathscr{S}(\mathbb{R})$, $g \in \mathscr{S}(\mathbb{R}^s)$ be test functions. We consider the operators

$$B = \int dx f(x_0) g(\mathbf{x}) A(x), \tag{7}$$

where the integral is defined in the weak-$*$-topology of $\mathscr{B}(\mathscr{H})$. If $\Phi \in \mathscr{H}(E)_1$ we get

$$\begin{aligned} |(\Phi, [B^*, B]\Phi)| &\leqq \int dx \int dy |\overline{f(x_0)} f(y_0)| \tfrac{1}{2} (|g(\mathbf{x})|^2 + |g(\mathbf{y})|^2) |(\Phi, [A^*(x), A(y)]\Phi)| \\ &\leqq \|g\|_2^2 \int dx_0 \int dy_0 |\overline{f(x_0)} f(y_0)| \sup_{\mathbf{x}} \int d^s y |(U(x)^{-1}\Phi, [A^*, A(y_0 - x_0, \mathbf{y})] U(x)^{-1}\Phi)| \\ &\leqq \|g\|_2^2 \sigma_r(A)^2 \int dx_0 \int dy_0 |\overline{f(x_0)} f(y_0)| (\Omega_s [d + |x_0 - y_0|]^s)^{1 - 1/r} . \end{aligned} \tag{8}$$

Here Ω_s is the volume of the unit ball in $\mathbb{R}^s$ and d is the distance appearing in the locality condition (3). In the final estimate we made use of locality and Hölder's inequality as well as of condition (6) and the fact that $\mathscr{H}(E)$ is invariant under space-time translations.

Next we replace the operator A in (7) by $AA(a)$, where $a \in \mathbb{R}^{s+1}$ is arbitrary. The resulting integral will be denoted by B_a. Proceeding as in (8) and making use of the estimate

$$\begin{aligned}
&|(U(x)^{-1}\Phi, [(AA(a))^*, (AA(a))(y_0 - x_0, \mathbf{y})]U(x)^{-1}\Phi)| \\
&\leqq \|A\|^2 \|\Phi\| \left\{ \|[A^*, A(y_0 - x_0 + a_0, \mathbf{y} + \mathbf{a})]U(x)^{-1}\Phi\| \right. \\
&\quad + \left\| [A, A^*(y_0 - x_0, \mathbf{y})] \frac{1}{\|A\|} A(a) U(x)^{-1}\Phi \right\| \\
&\quad + \left\| [A^*, A(y_0 - x_0, \mathbf{y})] \frac{1}{\|A\|} A^*(-a) U(x+a)^{-1}\Phi \right\| \\
&\quad \left. + \|[A, A^*(y_0 - x_0 - a_0, \mathbf{y} - \mathbf{a})]U(x+a)^{-1}\Phi\| \right\},
\end{aligned} \tag{9}$$

we arrive at the bound

$$\begin{aligned}
|(\Phi, [B_a^*, B_a]\Phi)| \leqq 2\|g\|_2^2 \|A\|^2 (\tau_r(A)^2 + \tau_r(A^*)^2) \int dx_0 \int dy_0 |\overline{f(x_0)} f(y_0)| \\
\times (\Omega_s [d + |a_0| + |x_0 - y_0|]^s)^{1 - 1/r}
\end{aligned} \tag{10}$$

which holds for arbitrary $\Phi \in \mathscr{H}_1$. Note that in the derivation of this explicit bound with regard to a we had to rely on the stronger condition (5).

With this information, which replaces the cruder bounds in Lemma 2.3 of [7], one can proceed exactly as in the proof of Proposition 2.4 of [7]. The corresponding result is the following one.

Lemma 2.1. *Let $A(g) = \int d^s x g(\mathbf{x}) A(\mathbf{x})$, where $A \in \mathfrak{D}$ is any local operator and $g \in \mathscr{S}(\mathbb{R}^s)$ any test function whose Fourier-transform $\tilde{g}$ vanishes in some neighborhood of the origin. Furthermore, let $P(E)$ be the orthogonal projection onto $\mathscr{H}(E)$. Then there holds for small $\varepsilon > 0$,*

$$\|P(E)A(g)P(E)\| \leqq C_\varepsilon \left(\int d^s p \left(\frac{E}{|\mathbf{p}|} \right)^{1+\varepsilon} (|\mathbf{p}|^{s/r - s} + d^{s - s/r}) |\tilde{g}(\mathbf{p})|^2 \right)^{1/2} \sigma_r(A),$$

where C_ε depends only on ε, and d is the distance appearing in the locality condition (3). Similarly,

$$\begin{aligned}
&\|P(E)(AA(a))(g)P(E)\| \\
&\leqq C_\varepsilon \left(\int d^s p \left(\frac{E}{|\mathbf{p}|} \right)^{1+\varepsilon} (|\mathbf{p}|^{s/r - s} + [|a_0| + d]^{s - s/r}) |\tilde{g}(\mathbf{p})|^2 \right)^{1/2} \|A\| (\tau_r(A) + \tau_r(A^*)),
\end{aligned}$$

where $a \in \mathbb{R}^{s+1}$ is arbitrary.

Let us consider now the function $\mathbf{x} \to m(\mathbf{x}) = (\Phi, A(\mathbf{x})\Phi)$, where $A \in \mathfrak{D}$ and $\Phi \in \mathscr{H}(E)_1$ are kept fixed for a moment. We are interested in the regularity properties of the Fourier transform $\tilde{m}$ of m,

$$\tilde{m}(\mathbf{p}) = (2\pi)^{-s/2} \int d^s x \, e^{i\mathbf{p}\mathbf{x}} (\Phi, A(\mathbf{x})\Phi), \tag{11}$$

which is defined in the sense of distributions. Note that $\tilde{m}$ has support in the ball $|\mathbf{p}| \leqq 2E$ because of the spectrum condition. For any operator $A \in \mathfrak{D}$ there exists by assumption some number $r < s$ such that $\sigma_r(A) < \infty$. Thus, by the first part of the preceding lemma, the restriction of $\tilde{m}$ to the region $|\mathbf{p}| \geqq k > 0$ can be represented by some square integrable function (cf. Theorem 2.5 in [7]), and

$$\lim_{k \searrow 0} \int_{|\mathbf{p}| \geqq k} d^s p \ |\mathbf{p}|^{s+1+\varepsilon - s/r} |\tilde{m}(\mathbf{p})|^2 < \infty \tag{12}$$

for any $\varepsilon > 0$. Choosing $0 < \varepsilon < (s/r - 1)$ and taking into account that $\tilde{m}$ has compact support we conclude that also

$$\lim_{k \searrow 0} \int_{|\mathbf{p}| \geqq k} d^s p \ |\tilde{m}(\mathbf{p})| < \infty . \tag{13}$$

Hence $\tilde{m}$ can be split into a Lebesgue integrable part, which we denote by $\tilde{l}$, and a distribution $\tilde{d} = \tilde{m} - \tilde{l}$ which is localized at $\mathbf{p} = 0$. The Fourier transform of any Lebesgue integrable function is continuous and converges to 0 at infinity. Hence $\sup_{\mathbf{x}} |l(\mathbf{x})| < \infty$, and since also $\sup_{\mathbf{x}} |m(\mathbf{x})| \leqq \|A\| \ \|\Phi\|^2 < \infty$ we find that $d(\mathbf{x})$, which can only be a polynominal because of the support properties of $\tilde{d}$, is constant.

Thus we have shown that $\tilde{m}$ is a complex measure which, apart from an atomic part at $\mathbf{p} = 0$, is Lebesgue absolutely continuous. Hence there exists for any $A \in \mathfrak{D}$ (and consequently also for any $A \in \mathfrak{A}$ since $\mathfrak{D}$ is by assumption norm dense in $\mathfrak{A}$) the limit

$$\lim_{|\mathbf{a}| \to \infty} (\Phi, A(\mathbf{a})\Phi) = \omega_0(A), \tag{14}$$

where $\omega_0(A)$ is some state on $\mathfrak{A}$. Since the weak closure of $\mathfrak{A} \subset \mathscr{B}(\mathscr{H})$ is assumed to have trivial center it then follows from locality (cf. [2]) that

$$w - \lim_{|\mathbf{a}| \to \infty} A(\mathbf{a}) = \omega_0(A) \cdot \mathbf{1}. \tag{15}$$

It is an immediate consequence of the latter relation that ω_0 is invariant under space-time translations, i.e.

$$\omega_0(A(a)) = \omega_0(A) \quad \text{for} \quad A \in \mathfrak{A}, \ a \in \mathbb{R}^{s+1}. \tag{16}$$

If the functions $a \to A(a)$ were norm continuous, as is the case for C^* dynamical systems, one could show without further input that ω_0 is a vacuum state in some positive energy representation of $\mathfrak{A}$. However, in the present more general setting we have to rely on the second part of the preceding lemma, which is based on condition (5). With this input we can show that the functions $a \to \omega_0(AA(a))$ are continuous if $A \in \mathfrak{D}$. To this end we consider the function

$$\mathbf{x} \to l_a(\mathbf{x}) = (\Phi, (AA(a))(\mathbf{x})\Phi) - \omega_0(AA(a)), \tag{17}$$

where $\Phi \in \mathscr{H}(E)_1$ and $E > 0$ is arbitrary. It follows from the second part of the preceding lemma and relation (15) that the Fourier transform $\tilde{l}_a$ of l_a is Lebesgue integrable and satisfies

$$\int d^s p \ |\mathbf{p}|^{s+1+\varepsilon - s/r} \ |\tilde{l}_a(\mathbf{p})|^2 \leqq C_\varepsilon (1 + |a_0|^{s - s/r}) \tag{18}$$

for some $r < s$. Here $\varepsilon > 0$ and $a \in \mathbb{R}^{s+1}$ are arbitrary, and the constant C_ε does not depend on a. On the other hand there holds the straightforward estimate for the radial mean of $l_a(\mathbf{x})$

$$\left(\frac{1}{R}\int_0^R d\varrho\, |l_a(\varrho e)|\right)^2 \leqq \frac{1}{R}\int_0^R d\varrho\, |l_a(\varrho e)|^2$$
$$\leqq \left(\int_{|\mathbf{p}|\leqq 2E} d^s p \int_{|\mathbf{q}|\leqq 2E} d^s q \frac{1}{|\mathbf{p}|^{s-\delta}|\mathbf{q}|^{s-\delta}}\left|\frac{e^{iR\mathbf{e}(\mathbf{p}-\mathbf{q})}-1}{R\mathbf{e}(\mathbf{p}-\mathbf{q})}\right|^2\right)^{1/2} \int d^s p\, |\mathbf{p}|^{s-\delta}\, |\tilde{l}_a(\mathbf{p})|^2, \quad (19)$$

where $\mathbf{e}\in\mathbb{R}^s$ is a fixed unit vector, $\delta>0$, and we made use of the fact that $\tilde{l}_a$ has support in the ball $|\mathbf{p}|\leqq 2E$. Combining this explicit bound in R with the estimate (18) we see that, for some $r<s$,

$$\frac{1}{R}\int_0^R d\varrho\, |l_a(\varrho e)| \leqq o(R)(1+|a_0|^{s-s/r})^{1/2}, \quad (20)$$

where the function $o(R)$ does not depend on a and tends to zero if R approaches infinity.[2] Recalling the definition of l_a we conclude that the sequence of continuous functions

$$a \to \frac{1}{R}\int_0^R d\varrho\, (\Phi, (AA(a))(\varrho e)\Phi) \quad (21)$$

converges to $a\to\omega_0(AA(a))$ uniformly on compact sets of $\mathbb{R}^{s+1}$ if R tends to infinity. Hence the latter function is continuous for $A\in\mathfrak{D}$. Since $\mathfrak{D}$ is norm dense in $\mathfrak{A}$ the same holds true for any $A\in\mathfrak{A}$, and in particular for all hermitian elements. Taking into account that ω_0 is a translational invariant state it follows that $a\to\omega_0(AB(a))$ is continuous for all $A, B\in\mathfrak{A}$.

From the bound (20) it is also apparent that the integration of (21) with any test function $f(a)\in\mathscr{S}(\mathbb{R}^{s+1})$ can be interchanged with the limit $R\to\infty$. Making use of the freedom to choose the energy-momentum support of Φ in relation (21) as well as of relation (15) it then follows from standard arguments (cf. Sect. 5 in [8]) that the Fourier transform of $a\to\omega_0(AA(a))$ has support in the light cone V_+ if $A\in\mathfrak{D}$. This result can be extended to the function $a\to\omega_0(AB(a))$ for arbitrary $A, B\in\mathfrak{A}$ if one notices that $a\to\omega_0(AA(a))$ is, for hermitian A, the Fourier transform of some positive measure as a consequence of the properties of ω_0 established so far.

We finally show that ω_0 has the clustering property, and hence is a pure state on $\mathfrak{A}$. To this end we restrict the sequence of functions (21) to the spacelike plane $a_0=0$. It follows from the bound (20) that this sequence converges to $\mathbf{a}\to\omega_0(AA(\mathbf{a}))$ in the limit $R\to\infty$, uniformly for $\mathbf{a}\in\mathbb{R}^s$. Hence in (21) the limits $|\mathbf{a}|\to\infty$ and $R\to\infty$ can be interchanged, and making use of relation (15) and the dominated convergence theorem we see that $\lim_{|\mathbf{a}|\to\infty}\omega_0(AA(\mathbf{a}))=\omega_0(A)^2$ for $A\in\mathfrak{D}$. This shows that ω_0 has the clustering property for hermitian operators $A\in\mathfrak{A}$, and again this suffices to conclude that $\lim_{|\mathbf{a}|\to\infty}\omega_0(AB(\mathbf{a}))=\omega_0(A)\omega_0(B)$ for arbitrary $A, B\in\mathfrak{A}$.

With this information on the functional ω_0 it is clear now that the GNS-representation of $\mathfrak{A}$ induced by ω_0 has all properties required from a vacuum representation. We collect these results in the following theorem.

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[2] It is noteworthy that $o(R)$ does not depend on the specific choice of $\Phi\in\mathscr{H}(E)_1$ either. We comment on this observation in Sect. 4

Theorem 2.2. *Let $\mathfrak{A}$ be an algebra of local observables on some Hilbert space $\mathscr{H}$ which satisfies the assumptions stated in the Introduction. Then there exists for any $A \in \mathfrak{A}$ the limit*

$$w-\lim_{|\mathbf{a}|\to\infty} A(\mathbf{a}) = \omega_0(A)\cdot \mathbf{1},$$

where ω_0 is some pure state on $\mathfrak{A}$. In the (irreducible) GNS*-representation $(\pi_0, \mathscr{H}_0)$ induced by ω_0 there exists a continuous unitary representation $a \to U_0(a)$ of the space-time translations with spectrum in the light cone V_+, and*

$$U_0(a)\pi_0(A)U_0(a)^{-1} = \pi_0(A(a)) \quad \textit{for} \quad A \in \mathfrak{A}.$$

The GNS*-vector $\Omega \in \mathscr{H}_0$ corresponding to ω_0 is invariant under the action of the operators $U_0(a)$ and hence represents a (within this representation unique) vacuum state.*

We conclude this section with a remark pertaining to relativistic covariance. Up to this point we have considered a fixed Lorentz frame. Yet it seems natural to assume that the condition (2) of timelike asymptotic abelianess holds in any frame. In that case one finds that the limit (15) exists in all frames and that the resulting vacuum states ω_0 coincide. This is so since in more than two space-time dimensions the intersection of any two spacelike planes having a point in common contains some spacelike ray. The asymptotic vacua in different Lorentz frames can then be identified by using relation (15) along that ray.

We note that the vacuum state ω_0 may not be Lorentz invariant, however. Noninvariant vacuum (ground) states appear in free quantum electrodynamics, for example. They are obtained by adding to the Fock-vacuum a homogeneous and stationary electromagnetic background field. If, however, the original representation of $\mathfrak{A}$ is Lorentz invariant, then the resulting vacuum state is Lorentz invariant too, as can be seen from relation (15).

3. Some Instructive Examples

In this section we will provide, on one hand, some evidence to the effect that our condition of timelike asymptotic abelianess is a reasonable assumption in quantum field theories based on physical space-time. On the other hand we will exhibit a theory in three space-time dimensions which only barely violates our condition, but which admits states with an involved asymptotic vacuum structure.

Both questions will be discussed in the theory of a free, scalar and neutral quantum field in $s+1$ space-time dimensions. We begin by recalling the formulation of this theory in terms of Weyl-operators. The basic building block is the Hilbert space of single particle (momentum space) wave functions f, g. These functions are equipped with the scalar product

$$\langle f, g\rangle = \int \frac{d^s p}{2p_0}\overline{f(\mathbf{p})}g(\mathbf{p}), \tag{22}$$

where $p_0 = (\mathbf{p}^2 + m^2)^{1/2}$ and $m \geqq 0$ is the mass of the underlying particle.[3] On this space there acts a continuous unitary representation of the space-time translations

[3] We assume that $s > 1$ if $m = 0$

given by

$$(V(a)f)(\mathbf{p}) = e^{ia \cdot p} f(\mathbf{p}), \tag{23}$$

where $a \cdot p = a_0 p_0 - \mathbf{ap}$. Of particular interest is the real subspace $\mathscr{L}$ of "locally generated wave functions"

$$\mathscr{L} = \tilde{\mathscr{D}}(\mathbb{R}^s) + ip_0 \tilde{\mathscr{D}}(\mathbb{R}^s). \tag{24}$$

Here $\tilde{\mathscr{D}}(\mathbb{R}^s)$ is the space of Fourier transforms of all real test functions which have compact support in configuration space $\mathbb{R}^s$. The space $\mathscr{L}$ is invariant under the action of the translations V.

The Weyl-algebra $\mathfrak{W}$ over $\mathscr{L}$ is the (abstract) algebra generated by unitary operators $W(f)$, $f \in \mathscr{L}$ which satisfy the Weyl-relations

$$W(f)W(g) = e^{-i\operatorname{Im}\langle f, g\rangle} W(f+g). \tag{25}$$

As is well known (cf. [5, Chap. 5.2]) one can equip $\mathfrak{W}$ with a C^*-norm, and we will subsequently make use of this fact. The action of the space-time translations on $\mathfrak{W}$ is given by the automorphisms

$$W(f)(a) := W(V(a)f), \quad a \in \mathbb{R}^{s+1}. \tag{26}$$

From the definition of $\mathscr{L}$ and the Weyl-relations it follows that the elements of $\mathfrak{W}$ are local in the sense of relation (3).

We will consider various representations of $\mathfrak{W}$ in which the space-time translations are unitarily implemented and satisfy the relativistic spectrum condition. The familiar Fock-representation, for example, is induced by the vacuum state ω_0 fixed by

$$\omega_0(W(f)) = e^{-\langle f, f\rangle/2}, \quad f \in \mathscr{L}. \tag{27}$$

In order to check whether this model complies with our condition of timelike asymptotic abelianess we have to estimate the norm of the commutator of Weyl-operators. Making use of the Weyl-relations we get

$$\|[W(f), W(g)]\| = |e^{2i\operatorname{Im}\langle f, g\rangle} - 1| \leqq 2|\operatorname{Im}\langle f, g\rangle|. \tag{28}$$

On the other hand we have

$$\begin{aligned} \int d^s x |\operatorname{Im}\langle f, V(x_0, \mathbf{x})g\rangle|^2 &\leqq \int d^s x\, |\langle f, V(x_0, \mathbf{x})g\rangle|^2 \\ &= (2\pi)^s \int \frac{d^s p}{(2p_0)^2} |f(\mathbf{p})|^2 |g(\mathbf{p})|^2. \end{aligned} \tag{29}$$

Thus, combining (28) and (29), we obtain the bound

$$\int d^s x \|[W(f), W(g)(x_0, \mathbf{x})]\|^2 \leqq 4(2\pi)^s \int \frac{d^s p}{(2p_0)^2} |f(\mathbf{p})|^2 |g(\mathbf{p})|^2. \tag{30}$$

The right-hand side is finite for any $f, g \in \mathscr{L}$ and any mass $m \geqq 0$ if $s \geqq 3$. Since the elements of $\mathfrak{W}$ are finite sums of Weyl-operators we arrive at

Lemma 3.1. *Let $s \geqq 3$. Then there holds for any $W \in \mathfrak{W}$,*

$$\sup_{x_0} \int d^s x \|[W^*, W(x_0, \mathbf{x})]\|^2 < \infty.$$

This result shows that in free field theories in physical space-time one may put $r = 2$ in condition (2), i.e. there holds a strengthened form of our condition of timelike asymptotic abelianess. One may therefore expect that this condition is also reasonable in the presence of interaction. We recall in this context that condition (2) is weaker than the requirement of L^1-asymptotic abelianess in time. To illustrate this fact, consider the theory of a massless particle in $s=3$ dimensions. There the commutators $\|[W^*, W(x_0, \mathbf{x})]\|$ decrease like $|x_0|^{-1}$ in lightlike directions $|\mathbf{x}|=|x_0|$, hence the integral over x_0 diverges logarithmically. Yet since the set of points $\mathbf{x}$, where one has such a slow decay of the commutator, is of measure zero condition (2) still holds in this case.

Condition (2) does not hold, however, if $s<3$. We consider in the remainder of this section the theory of a massless particle in $s=2$ dimensions. There one finds that for $f, g \in \mathscr{L}$ and large $|x_0|$

$$\int d^2x \, |\mathrm{Im}\langle f, V(x_0, \mathbf{x})g\rangle|^2 \leqq c \ln|x_0| \tag{31}$$

and

$$\sup_{\mathbf{x}} |\langle f, V(x_0, \mathbf{x})g\rangle| \leqq c|x_0|^{-1/2}. \tag{32}$$

Thus it follows from relation (28) that for any $W \in \mathfrak{W}$ and any $r>2$,

$$\sup_{x_0} \int d^2x \| W^*, W(x_0, \mathbf{x})]\|^r < \infty . \tag{33}$$

Hence condition (2) is only mildly violated in this case. Nevertheless this fact allows for a complicated asymptotic vacuum structure of the underlying states, as we shall demonstrate now.

We consider special coherent states φ on the Weyl-algebra $\mathfrak{W}$ of the form

$$\varphi(W(f)) = e^{il(f) - \langle f, f\rangle/2}, \quad f \in \mathscr{L}. \tag{34}$$

Here $l(\cdot)$ is a real linear functional on $\mathscr{L}$ which we choose according to

$$l(f) = \mathrm{Im} \int_{|\mathbf{p}| \leqq 1/2} \frac{d^2p}{2|\mathbf{p}|} \frac{i}{|\mathbf{p}|(-\ln|\mathbf{p}|)^{\kappa}} (f(\mathbf{0}) - f(\mathbf{p})), \tag{35}$$

where $1/2 < \kappa < 1$ is kept fixed in the following. This expression is well defined since $|f(0) - f(\mathbf{p})| \leqq c|\mathbf{p}|$.

With the methods expounded in [2] (cf. also [9]) one can show that φ induces an irreducible representation of $\mathfrak{W}$ in which the translations (26) are unitarily implemented and satisfy the spectrum condition. (For the latter result it is crucial that $\kappa > 1/2$, cf. Proposition 3 in [9].) We call such states *elementary states* for short. Note that φ is not a vector state in the Fock-representation.

Let us turn now to the analysis of the behaviour of the state φ under large translations. To this end we introduce the (continuous) function

$$\lambda(a) = \int_{|\mathbf{p}| \leqq 1/2} d^2p \frac{1 - \cos a \cdot p}{2 \cdot |\mathbf{p}|^2(-\ln|\mathbf{p}|)^{\kappa}}, \quad a \in \mathbb{R}^{2+1}, \tag{36}$$

This function diverges if $|a| \to \infty$ since $\kappa < 1$. On the other hand we have

$$\lim_{|a| \to \infty} (l(V(a)f) - \lambda(a) f(\mathbf{0})) = 0 \tag{37}$$

by the Riemann-Lebesgue lemma. Taking into account that the scalar product $\langle f, f\rangle$ is invariant under the action of the translations $V(a)$ it follows that the expectation value $\varphi(W(f)(a))$ does not converge in the limit $|a| \to \infty$ if $f(\mathbf{0}) \neq 0$. In fact, since $f(\mathbf{0})$ can be any real number, there does not exist any subsequence of translations a_n tending to infinity such that this expectation value converges for all $f \in \mathscr{L}$. Introducing the notation $(a)\varphi$, $a \in \mathbb{R}^{2+1}$ for the translated states φ, i.e. $(a)\varphi(W) := \varphi(W(-a))$ for $W \in \mathfrak{W}$, we arrive at

Fact 1. *There exist elementary states φ such that the family $(a)\varphi$, $a \in \mathbb{R}^{2+1}$ does not contain any weak-*-convergent subsequence if $|a|$ tends to infinity.*

Because of the weak-*-compactness of the unit ball of the dual space of $\mathfrak{W}$ the family of states $(a)\varphi$, $a \in \mathbb{R}^{2+1}$ has weak limit points, however.[4] It is instructive to explore the properties of these limit points.

To do this we have to determine the limit points of the family of functions $\eta_a \in \mathbb{T}^{\mathbb{R}}$, $|a| \to \infty$, given by

$$\eta_a(u) = e^{i\lambda(a)u}, \qquad u \in \mathbb{R}. \tag{38}$$

The topology which is of relevance here is the weak (Tychonoff) topology on the Cartesian product $\mathbb{T}^{\mathbb{R}}$. All limit points of this family satisfy the functional equation

$$\eta(u)\eta(u') = \eta(u+u'). \tag{39}$$

Its continuous solutions of modulus one are $\eta(u) = e^{i\lambda u}$, $u \in \mathbb{R}$ for fixed $\lambda \in \mathbb{R}$. That these functions are indeed limit points of the family η_a can be seen as follows: given $u_1, \ldots, u_n \in \mathbb{R}$ and $\varepsilon > 0$ there exists some (arbitrarily large) number $q \in \mathbb{N}$ and integers $p_1, \ldots, p_n \in \mathbb{Z}$ such that $|qu_k - p_k| < \varepsilon$ for $k = 1, \ldots, n$ [10, Theorem 201]. Moreover, since $\lambda(a)$ runs continuously through all positive numbers if $|a|$ tends to infinity there exists some $a' \in \mathbb{R}^{2+1}$ such that $\lambda(a') = 2\pi q + \lambda$. In fact one can find such a vector a' on any given ray in $\mathbb{R}^{2+1}$. Thus

$$|\eta_{a'}(u_k) - e^{i\lambda u_k}| = |e^{2\pi i(qu_k - p_k)} - 1| \leqq 2\pi\varepsilon \tag{40}$$

for $k = 1, \ldots, n$, proving that $e^{i\lambda u}$, $u \in \mathbb{R}$ is a weak limit point of the family of functions η_a.

The functions η_a, however, also have limit points which are nowhere continuous. To verify this assertion we pick a sequence of vectors $a_n \in \mathbb{R}^{2+1}$ such that $\lambda(a_n) = 2\pi 2^n$. Let η be a weak limit point of the corresponding sequence η_{a_n} (such limit points exist according to Tychonoff's theorem). Since for any $p, q \in \mathbb{Z}$ and sufficiently large n there holds $\eta_{a_n}(p/2^q) = 1$ it follows that $\eta(p/2^q) = 1$. On the other hand the sequence $\eta_{a_n}(1/3)$ alternates between $e^{2\pi i/3}$ and $e^{4\pi i/3}$, and consequently $\eta(1/3) \neq 1$. This shows that the function η is discontinuous at $u = 1/3$ and therefore (by the functional equation) discontinuous everywhere.

With this information and the help of relations (34) and (37) one can show that the family of states $(a)\varphi$, where $a \in \mathbb{R}^{2+1}$ approaches infinity along any given ray, has on one hand as limit points all states ω_λ, $\lambda \in \mathbb{R}$ given by

$$\omega_\lambda(W(f)) = e^{i\lambda f(\mathbf{0}) - \langle f, f \rangle/2}. \tag{41}$$

These are the familiar vacuum states in massless free field theory whose existence may be traced to the presence of the gauge transformations $W(f) \to e^{i\lambda f(\mathbf{0})} W(f)$. On the other hand there appear also limit states of the form

$$\omega(W(f)) = \eta(f(\mathbf{0})) e^{-\langle f, f \rangle/2}, \tag{42}$$

[4] This fact was misinterpreted in [2, Note added in proof] and [8, Theorem 1]

where η is a discontinuous solution of Eq. (39). These states may still be regarded as vacuum states since they are invariant under space-time translations and are ground states in their respective GNS-representations. But for elements $f \in \mathscr{L}$ with $f(\mathbf{0}) \neq 0$ the functions $u \in \mathbb{R} \to \omega(W(uf))$ are discontinuous, hence the states ω are not regular. It follows that the states ω and φ are not locally normal with respect to each other. More precisely, the restrictions of φ and ω to any local algebra $\mathfrak{W}(\mathbf{O})$, $\mathbf{O} \subset \mathbb{R}^s$ (which is generated by Weyl-operators $W(f)$ with $f \in \tilde{\mathscr{D}}(\mathbf{O}) + ip_0 \tilde{\mathscr{D}}(\mathbf{O})$) induce disjoint representations of $\mathfrak{W}(\mathbf{O})$. We thus place on record

Fact 2. There exist elementary states φ such that the sequence $(a)\varphi$, where a approaches infinity along any given ray, has weak-*-limit points which are locally singular relative to φ.

The preceding results show that it is impossible to describe the asymptotic properties of φ in terms of limit vacuum states.[5] In view of this fact the example of the state φ might appear artificial. Yet we will see that φ can be approximated by states of uniformly bounded energy in the Fock-representation. It is therefore conceivable that states such as φ can appear as asymptotic configurations in certain interacting theories.

For the proof that φ can be approximated in the way indicated we introduce the sequence of regularized functionals

$$l_n(f) = \operatorname{Im} \int \frac{d^2p}{2|\mathbf{p}|} \, \frac{is_n(\mathbf{p})}{|\mathbf{p}|(-\ln|\mathbf{p}|)^\kappa} f(p), \qquad f \in \mathscr{L}. \tag{43}$$

Here s_n is a step function defined by $s_n(\mathbf{p}) = -1$ for $1/n \leqq |\mathbf{p}| \leqq 1/2$, $s_n(\mathbf{p}) = 1$ for $1/v \leqq |\mathbf{p}| \not< 1/n$, and $s_n(\mathbf{p}) = 0$ elsewhere. For given n the number $v > 0$ is adjusted in such a way that the expression (43) vanishes if $f(\mathbf{p})$ is replaced by a constant.

With these preparations we achieve two things. First, it follows from the support properties of s_n that $|l_n(f)|^2 \leqq c_n \langle f, f \rangle$ for certain constants c_n which do not depend on f. Hence the coherent states φ_n on $\mathfrak{W}$ given by

$$\varphi_n(W(f)) = e^{il_n(f) - \langle f, f \rangle / 2}, \qquad f \in \mathscr{L} \tag{44}$$

are vector states in the Fock-representation [2]. Second, the value of (43) does not change if one replaces $f(\mathbf{p})$ by $f(\mathbf{p}) - f(\mathbf{0})$. Applying the dominated convergence theorem one finds that $l_n(f)$ converges to $l(f)$ for all $f \in \mathscr{L}$, and consequently

$$\lim_n \varphi_n(W) = \varphi(W) \quad \text{for} \quad W \in \mathfrak{W}. \tag{45}$$

It remains to control the energy of the approximating states φ_n. As is well known (cp. [2]) the expectation value of the energy in a coherent state is equal to the energy of its mean field. Hence we obtain for the mean energy E_n in state φ_n

$$E_n = \int \frac{d^2p}{2|\mathbf{p}|} |\mathbf{p}| \left| \frac{is_n(\mathbf{p})}{|\mathbf{p}|(-\ln|\mathbf{p}|)^\kappa} \right|^2, \tag{46}$$

and this expression stays bounded in the limit of large n since $\kappa > 1/2$. Thus we have established

[5] Yet it is noteworthy that the vacua $\omega_{\lambda(a)}$, $a \in \mathbb{R}^{2+1}$ approximate the states $(a)\varphi$, $a \in \mathbb{R}^{2+1}$ for large $|a|$ in the weak-*-topology. It is an interesting problem whether such approximating families of vacua exist also in general

Fact 3. There exist elementary states φ with asymptotic properties described above which are the weak-$*$-limit of a sequence of vector states in the Fock-representation with uniformly bounded mean energy.

This result shows that an energetically mild perturbation of a state may completely distort its spacelike asymptotic structure. As will be discussed in the subsequent section, this cannot happen in theories satisfying our condition of timelike asymptotic abelianess.

4. Concluding Remarks

The simple model discussed in the preceding section exhibits certain features which make difficult a general analysis of the spacelike asymptotic structure of physical states. In the presence of massless particles and long range forces this asymptotic structure can be quite involved and the heuristic picture that physical states are excitations of some vacuum state may fail. This phenomenon originates from the fact that a small amount of energy can give rise to large effects (fields etc.).

There are no indications that this interesting theoretical possibility is realized in physics. We have discussed in our paper an assumption which could explain for this fact: the dynamical law is such that correlations between timelike separated observables decay sufficiently rapidly. Under these circumstances it follows that all physical states approach some fixed vacuum state at large spacelike distances.

Let us indicate another interesting consequence of this assumption: all states in the energy-connected component of a given representation of $\mathfrak{A}$[6] have the same asymptotic limit vacuum state ω_0. To prove this one proceeds as in Sect. 2 and considers the radial mean

$$\frac{1}{R}\int_0^R d\varrho\,((\Phi, A(\varrho e)\Phi)-\omega_0(A)), \tag{47}$$

where $A\in\mathfrak{D}$ is kept fixed and $\Phi\in\mathscr{H}(E)_1$ is arbitrary. It follows from Lemma 2.1 that this expression converges to 0, uniformly for $\Phi\in\mathscr{H}(E)_1$, if R tends to infinity (cf. footnote 2). Now let φ be any weak-$*$-limit point of the set of vector states generated by $\Phi\in\mathscr{H}(E)_1$. Since the integral in (47) can be approximated by finite Riemann sums, uniformly for $\Phi\in\mathscr{H}(E)_1$, one can interchange the integration with the weak-$*$-limit. Hence, by the preceding remark, one gets

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R d\varrho\,\varphi(A(\varrho e))=\omega_0(A). \tag{48}$$

By a similar reasoning as in the proof of relation (14) one finds on the other hand that $\varphi(A(\mathbf{a}))$ converges if $|\mathbf{a}|\to\infty$, and combining this information with Eq. (48) one arrives at the desired relation

$$\lim_{|\mathbf{a}|\to\infty}\varphi(A(\mathbf{a}))=\omega_0(A). \tag{49}$$

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[6] The energy-connected component of a representation is the set of states which can be approximated in the weak-$*$-topology by vector states with uniformly bounded energy relative to the given representation, cf. [11, Sect. 4] and [12]. The energy-connected component of a vacuum representation, for example, contains all charged states which can be generated from the vacuum with a finite amount of energy

It is an immediate consequence of this result that each energy connected component contains only one vacuum state. Hence if there exist several vacuum states in a theory complying with our assumptions they are separated by an infinite amount of energy. For a more detailed discussion of this issue cf. [11, Sect. 4] and [12].

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Communicated by H. Araki