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Signature Characters for A_2 and B_2

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Abstract. The signatures of the inner product matrices on a Lie algebra's highest weight representation are encoded in the representation's signature character. We show that the signature characters of a finite-dimensional Lie algebra's highest weight representations obey simple difference equations that have a unique solution once appropriate boundary conditions are imposed. We use these results to derive the signature characters of all A_2 and B_2 highest weight representations. Our results extend, and explain, signature patterns analogous to those observed by Friedan, Qiu and Shenker in the Virasoro algebra's representation theory.

1. Introduction

The theory of non-unitary highest weight Lie algebra representations is still relatively undeveloped. An interesting new approach was suggested by the work of Friedan, Qiu and Shenker (FQS) who, in one of the foundational papers [1] of conformal field theory, used the Virasoro algebra's determinant formula [2, 3] to analyse the unitarity of its highest weight representations. They established the unitarity of a continuum of the representations, and the non-unitarity of all other representations except for an infinite discrete series. Drawing on evidence from statistical physics, as well as computation, they conjectured that the representations in the discrete series are unitary. This was later proven by another method [4].

The relevance of FQS's work to the representation theory of finite-dimensional Lie algebras is not immediately obvious, since the unitary highest weight representations of these algebras can be classified simply by studying the induced representations of the embedded $su(2)$ subalgebras. However, as FQS recognised,

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the emergence of the discrete series suggests very strongly that there are definite patterns in the dependence of the inner product matrix signatures on a representation's location amongst the vanishing curves, and that the unitarity of the discrete series can be understood as a consequence of these patterns. Thus one should look for results that describe and explain these patterns and hence give explicit expressions for all inner product matrix signatures of all highest weight representations.

Here, we carry this through for the Lie algebras A_2 and B_2. (These are the simplest non-trivial cases, since the relevant vector spaces for A_1 are one-dimensional and their inner product matrix signatures are easily calculable.) We obtain explicit (and intriguingly simple) expressions for these algebras' signature characters (generating functions in which the inner product matrix signatures are encoded). Our arguments apply to many other cases of interest, including all finite-dimensional Lie algebras.

2. Notation and Basic Results

We follow the conventions of Kac and Kazhdan [5] with modifications. Let g be a complex semisimple Lie algebra of rank r with symmetrisable Cartan matrix $A=(a_{ij})$, that is, $A=DA_{\mathrm{sym}}$, where $D=\mathrm{diag}(d_1,\dots,d_r)$, $A_{\mathrm{sym}}=(a_{ij}^{\mathrm{sym}})$ is symmetric and the d_i are non-zero. Let h be a Cartan subalgebra of g; let $\{\alpha_1,\dots,\alpha_r\}$ be a basis of simple roots with respect to h; Δ_+ the set of positive roots with respect to this basis; Δ_- the set of negative roots. Define the positive and negative root semilattices to be $\Lambda_\pm=\left\{\pm\sum_{i=1}^{r} n_i\alpha_i : n_i\in\mathbb{Z}_+\right\}$. Let g_α be the root subspace of g corresponding to the root α, and define the subalgebras $n_+=\sum_{\alpha\in\Delta_+} g_\alpha$, $n_-=\sum_{\alpha\in\Delta_-} g_\alpha$, so that $g=n_-\oplus h\oplus n_+$. We choose a set of generators $\{E_{\alpha_i}, E_{-\alpha_i}, H_i : 1\leqq i\leqq r\}$ for g; here $\{H_1,\dots,H_r\}$ is a basis for h, and $E_{\pm\alpha_i}\in g_{\pm\alpha_i}$. The Killing form $(,)$ on g is the unique symmetric invariant non-degenerate bilinear form on g with the property that $(H_i,H_j)=a_{ij}^{\mathrm{sym}}$. The Killing form, restricted to h, induces a form on h^* which we also denote by $(,)$. We have $(\alpha_i,\alpha_j)=a_{ij}^{\mathrm{sym}}$ and $a_{ij}=2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)$. Finally, we define $\rho\in h^*$ by $(\rho,\alpha_i)=\frac{1}{2}(\alpha_i,\alpha_i)$ for α_i a simple root. The generators obey the relations

$$\begin{aligned}
&[H_i,H_j]=0,\\
&[H_i,E_{\pm\alpha_j}]=\pm a_{ij}E_{\pm\alpha_j},\\
&[E_{\alpha_i},E_{-\alpha_j}]=\delta_{ij}\frac{2(E_{\alpha_i},E_{-\alpha_i})}{(\alpha_i,\alpha_i)}H_i.
\end{aligned}\tag{2.1}$$

Denote by $U(a)$ the universal enveloping algebra of the Lie algebra a. For $\lambda\in h^*$, the Verma module representation $V(\lambda)$ of g is the representation of g containing a vector $|\lambda\rangle$ such that

$$U(n_+)|\lambda\rangle=0,\quad H_i|\lambda\rangle=\frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}|\lambda\rangle,\tag{2.2}$$

and such that if $x,y\in U(n_-)$ then $x|\lambda\rangle=y|\lambda\rangle$ only if $x=y$. We have the decomposition $V(\lambda) = \bigoplus_{\mu} V(\lambda)_{\mu}$, where

$$V(\lambda)_{\mu} = \left\{ v \in V(\lambda) : H_i v = \frac{2(\alpha_i, \lambda - \mu)}{(\alpha_i, \alpha_i)} v \right\}. \tag{2.3}$$

We consider below representations $V(\lambda)$ for λ such that (α_i, λ) is real.

Define an adjoint map $\dagger$ on g to be the unique algebra anti-automorphism $\dagger : g \to g$ such that $(H_i)^{\dagger} = H_i, (E_{\alpha_i})^{\dagger} = E_{-\alpha_i}$ and $(E_{-\alpha_i})^{\dagger} = E_{\alpha_i}$, and extend this to an algebra anti-automorphism on $U(g)$, also denoted by $\dagger$. The Shapovalov form on $V(\lambda)$ is the unique bilinear form $\langle , \rangle$ such that $\langle \lambda, \lambda \rangle = 1$ and $\langle x\lambda, y\lambda \rangle = \langle \lambda, (x)^{\dagger} y\lambda \rangle = \langle (y)^{\dagger} x\lambda, \lambda \rangle$ for $x, y \in U(g)$. We have that $\langle V(\lambda)_{\mu}, V(\lambda)_{\nu} \rangle = 0$ if $\mu \neq \nu$. Denote by $M_{\mu}(\lambda)$ the real symmetric matrix defining (in some choice of basis) the inner product restricted to $V(\lambda)_{\mu}$. For $\lambda \in h^*$ we write $\lambda = -\sum_{i=1}^{r} \lambda^i \alpha_i$.

Now the character of $V(\lambda)$ can be defined as

$$\chi(V(\lambda)) = (x_1)^{\lambda^1} \cdots (x_r)^{\lambda^r} \sum_{\mu \in \Lambda_-} \dim(V(\lambda)_{\mu})(x_1)^{\mu^1} \cdots (x_r)^{\mu^r}. \tag{2.4}$$

In physicists' notation, defining the fundamental co-weights $h_i \in h$ so that $\alpha_j(h_i) = \delta_{ij}$, letting $x_i = \exp(-i\theta_i)$, $\theta = (\theta_1, \ldots, \theta_r)$ and $H = (h_1, \ldots, h_r)$, and writing $\theta \cdot H = \sum_{i=1}^{r} \theta_i h_i$, we have

$$\chi(V(\lambda)) = \mathrm{Tr}_{V(\lambda)}(e^{i\theta \cdot H}). \tag{2.5}$$

By analogy, the *signature character* of $V(\lambda)$ is defined as

$$\chi^{\mathrm{sig}}(V(\lambda)) = (x_1)^{\lambda^1} \cdots (x_r)^{\lambda^r} \sum_{\mu \in \Lambda_-} \mathrm{sig}(M_{\mu}(\lambda))(x_1)^{\mu^1} \cdots (x_r)^{\mu^r}, \tag{2.6}$$

where $\mathrm{sig}(M)$ denotes the signature of the matrix M; that is, if $M = SDS^T$, where S is non-singular and $D = \mathrm{diag}(+1, \ldots, +1, 0, \ldots, 0, -1, \ldots, -1)$, then $\mathrm{sig}(M) = \mathrm{tr}(D)$. This can be rewritten by defining a linear operator P on $V(\lambda)$ with the property that if $v \in V(\lambda)_{\mu}$ is an eigenvector of $(M_{\mu}(\lambda))$ with eigenvalue a_v then

$$Pv = \mathrm{sgn}(a_v)v, \tag{2.7}$$

where the sign function is given by

$$\mathrm{sgn}(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t = 0, \\ -1 & \text{for} \quad t < 0. \end{cases} \tag{2.8}$$

Then

$$\chi^{\mathrm{sig}}(V(\lambda)) = \mathrm{Tr}_{V(\lambda)}(Pe^{i\theta \cdot H}). \tag{2.9}$$

We shall work with normalised signature characters

$$\sigma(\lambda) \equiv e^{-i\theta \cdot H(\lambda)} \chi^{\mathrm{sig}}(V(\lambda)). \tag{2.10}$$

The fundamental result we shall need is Shapovalov's determinant formula [6, 5],

$$\det(M_{\mu}(\lambda)) = C \prod_{\alpha \in \Delta_+} \prod_{n > 0} \left((\alpha, \lambda + \rho) - n \frac{(\alpha, \alpha)}{2} \right)^{P(\mu + n\alpha)}, \tag{2.11}$$

where if the weight decomposition of $U(n_-)$ is given by $\bigoplus_\mu U(n_-)_\mu$ then $P(\mu) = \dim(U(n_-)_\mu)$, and C is a non-zero basis-dependent constant. We call the planes $P(n,\alpha) \equiv \{\lambda \in h^* : (\alpha, \lambda + \rho) = n\}$ the *Shapovalov vanishing planes* of g.

3. General Properties of Signature Characters for Finite-Dimensional Lie Algebras

For the remainder of the paper we take g to be a finite-dimensional Lie algebra. Since the matrices $M_\mu(\lambda)$ are real and symmetric, we have that

$$\mathrm{sig}(M_\mu(\lambda)) = \sum_{j=1}^{P(\mu)} \mathrm{sgn}(t_j), \tag{3.1}$$

where $\{t_1, \ldots, t_{P(\mu)}\}$ are the eigenvalues of $M_\mu(\lambda)$. It follows that $\mathrm{sig}(M_\mu(\lambda))$ is constant on any connected region of h^* in which $\det(M_\mu(\lambda)) \neq 0$, and in particular that $\sigma(V(\lambda))$ is constant on connected regions R in which $\det(M_\mu(\lambda)) \neq 0$ for any $\mu \in \Lambda_-$ and any $\lambda \in R$. Our first result is the observation that the change of the signature characters between two such neighbouring regions is given by a simple difference equation.

We consider a point $\lambda_0 \in h^*$ lying on precisely one of the vanishing planes $P(n_0, \alpha_0)$. Choose a basis $\{\kappa_i : 1 \leqq i \leqq r\}$ for h^* such that $(\kappa_i, \alpha_0) = 0$ for $1 \leqq i \leqq (r-1)$ and $\kappa_r = \alpha_0$, and define coordinates $(t_1, \ldots, t_{r-1}, \varepsilon) \equiv (\underline{t}, \varepsilon)$ for $\lambda \in h^*$ by the equation $\lambda = \lambda_0 + \sum_{i=1}^{r-1} t_i \kappa_i + \varepsilon \kappa_r$. We write $V(\underline{t}, \varepsilon)$ for $V(\lambda)$ and $|\underline{t}, \varepsilon\rangle$ for $|\lambda\rangle$, and we adopt the convention that any vector written in the form $w(\underline{t}, \varepsilon)$ belongs to $V(\underline{t}, \varepsilon)$. Define the neighbourhood $N(\delta)$ of λ_0 as the coordinate region with $|\underline{t}| < \delta$ and $|\varepsilon| < \delta$, and choose δ_0 sufficiently small such that $N(\delta_0)$ intersects no vanishing plane other than $P(n_0, \alpha_0)$. Shapovalov's formula implies that, for $|\underline{t}| < \delta_0$, the Verma module $V(\underline{t}, 0)$ contains a unique descendent highest weight vector $v(\underline{t}, 0) \in V(\underline{t}, 0)_{n_0\alpha_0}$. We can express $v(\underline{t}, 0)$ as $a(\underline{t})|\underline{t}, 0\rangle$, where $a(\underline{t}) \in U(n_-)$ is defined for $|\underline{t}| < \delta_0$. Define the vectors $v(\underline{t}, \varepsilon) \in V(\underline{t}, \varepsilon)_{n_0\alpha_0}$ by $v(\underline{t}, \varepsilon) = a(\underline{t})|\underline{t}, \varepsilon\rangle$.

If $\mu + n_0\alpha_0 \notin \Lambda_-$ then the inner product matrix $M_\mu(\lambda)$ is non-singular in $N(\delta_0)$ and its signature is constant throughout the neighbourhood. Now we consider $M_\mu(\lambda)$ for μ such that $\mu + n_0\alpha_0 \in \Lambda_-$. Reducing δ_0 if necessary, we choose $a_i \in U(n_-)_{\mu + n_0\alpha_0}$ for $1 \leqq i \leqq P(\mu + n_0\alpha_0)$ and $b_i \in (U(n_-))_\mu$ for $P(\mu + n_0\alpha_0) + 1 \leqq i \leqq P(\mu)$ such that, setting

$$v_i(\underline{t}, \varepsilon) = \begin{cases} a_i v(\underline{t}, \varepsilon) & \text{for} \quad 1 \leqq i \leqq P(\mu + n_0\alpha_0), \\ b_i|\underline{t}, \varepsilon\rangle & \text{for} \quad P(\mu + n_0\alpha_0) + 1 \leqq i \leqq P(\mu), \end{cases} \tag{3.2}$$

the set $\{v_i(\underline{t}, \varepsilon) : 1 \leqq i \leqq r\}$ forms a basis of $V(\underline{t}, \varepsilon)_\mu$ throughout $N(\delta_0)$.

Now

$$\langle v_i(\underline{t}, \varepsilon), v_j(\underline{t}, \varepsilon)\rangle = O(\varepsilon) \tag{3.3}$$

for $1 \leqq i \leqq P(\mu + n_0\alpha_0)$ or $1 \leqq j \leqq P(\mu + n_0\alpha_0)$. Thus in this basis the inner product matrix M_μ has the form

$$\begin{pmatrix} M' & X \\ X^T & M'' \end{pmatrix}, \tag{3.4}$$

where

$$
\begin{aligned}
(M')_{ij} &= M_{ij} && \text{for} \quad 1 \leqq i, j \leqq P(\mu + n_0\alpha_0), \\
(M'')_{ij} &= M_{i+P(\mu+n_0\alpha_0), j+P(\mu+n_0\alpha_0)} && \text{for} \quad 1 \leqq i, j \leqq P(\mu) - P(\mu + n_0\alpha_0),
\end{aligned}
\tag{3.5}
$$

and $M'(\underline{t}, \varepsilon)$ and $X(\underline{t}, \varepsilon)$ are $O(\varepsilon)$ in the neighbourhood $N(\delta_0)$.

For $(\underline{t}, \varepsilon) \in N(\delta_0)$, Eq. (2.11) implies that

$$\det(M(\underline{t}, \varepsilon)) = A(\underline{t})\varepsilon^{P(\mu+n_0\alpha_0)} + O(\varepsilon^{P(\mu+n_0\alpha_0)+1}) \tag{3.6}$$

for some non-zero function $A(\underline{t})$. As $M''(\underline{t}, \varepsilon) = M''(\underline{t}, 0) + O(\varepsilon)$ and $\det(M(\underline{t}, \varepsilon))$ is non-zero for $(\underline{t}, \varepsilon) \in N(\delta_0)$ if $\varepsilon \neq 0$, $M''(\underline{t}, 0)$ must be non-singular. Hence, again reducing δ_0 if necessary, we can assume $M''(\underline{t}, \varepsilon)$ is non-singular throughout $N(\delta_0)$.

Thus there is a new basis in which M_μ has the form

$$\begin{pmatrix} M' - X(M'')^{-1}X^T & 0 \\ 0 & M'' \end{pmatrix}. \tag{3.7}$$

Likewise $M'''(\underline{t}) \equiv \lim\limits_{\varepsilon \to 0} \frac{1}{\varepsilon} M'(\underline{t}, \varepsilon)$ must be non-singular. Hence, for sufficiently small ε,

$$
\begin{aligned}
\operatorname{sig}(M(\underline{t}, \varepsilon)) &= \operatorname{sig}(M'(\underline{t}, \varepsilon)) + \operatorname{sig}(M''(\underline{t}, \varepsilon)) \\
&= \operatorname{sgn}(\varepsilon) \operatorname{sig}(M'''(\underline{t})) + \operatorname{sig}(M''(\underline{t}, 0)).
\end{aligned}
\tag{3.8}
$$

Now, the Poincaré–Birkhoff–Witt theorem implies that $(a_i)^\dagger a_j$ can be expressed as a sum of products $\sum\limits_k r_k s_k t_k$ with $r_k \in U(n_-)$, $s_k \in U(h)$ and $t_k \in U(n_+)$. Given such an expression, we define $H_{ij} = \sum\limits_{\{k : r_k = t_k = 1\}} s_k$. The definition is in fact independent of the expression chosen, since $U(n_\pm)$ and $U(h)$ are mutually disjoint subalgebras. Now if $a \in n_+$ then $av(\underline{t}, 0) = 0$ and so $av(\underline{t}, \varepsilon) = O(\varepsilon)$. Hence

$$\langle v_i(\underline{t}, \varepsilon), v_j(\underline{t}, \varepsilon) \rangle = \langle v(\underline{t}, \varepsilon), H_{ij} v(\underline{t}, \varepsilon) \rangle + O(\varepsilon^2). \tag{3.9}$$

Thus

$$\varepsilon M'''(\underline{t}) = \langle v(\underline{t}, \varepsilon), v(\underline{t}, \varepsilon) \rangle M_{\mu+n_0\alpha_0}(\lambda_0 - n_0\alpha_0) + O(\varepsilon^2), \tag{3.10}$$

where the latter inner product matrix is taken in the basis

$$\{a_i | \lambda_0 - n_0\alpha_0 \rangle : 1 \leqq i \leqq P(\mu + n_0\alpha_0)\}.$$

This establishes the following result:

Theorem 1. *Let g, h, λ_0, be as above. Then, in the notation previously defined*

$$\lim_{\varepsilon \to 0^+} \sigma(\underline{t}, \varepsilon) - \lim_{\varepsilon \to 0^-} \sigma(\underline{t}, \varepsilon) = 2 \operatorname{sgn}(\langle v(\underline{t}, \varepsilon), v(\underline{t}, \varepsilon) \rangle |_{\varepsilon = 0^+}) e^{-in_0\theta \cdot H(\alpha_0)} \sigma(\lambda - n_0\alpha_0). \tag{3.11}$$

This, the main result of the section, we refer to as the *signature character difference equation.* We also have, from Eq. (3.8):

Lemma 2. *Let g, h, λ_0, be as above. Then, in the notation previously defined,*

$$\sigma(\lambda_0) = \frac{1}{2}\left(\lim_{\varepsilon \to 0^+} \sigma(\underline{t}, \varepsilon) + \lim_{\varepsilon \to 0^-} \sigma(\underline{t}, \varepsilon) \right). \tag{3.12}$$

The next result requires some preliminary notation. Suppose that a formal power series valued function $f: R \to \mathbb{C}[[e^{-i\theta_1}, \ldots, e^{-i\theta_r}]]$ is defined on a subset R of h^* that includes almost all the points $t\rho$ and $-t\rho$ for real t. Define functions f_α on R by $f(\lambda) = \sum_{\alpha \in \Lambda_-} f_\alpha(\lambda) e^{i\theta \cdot H(\alpha)}$ for $\lambda \in R$. Then we define $\lim_{\lambda \to \infty} f(\lambda)$ to be

$$\sum_{\alpha \in \Lambda_-} \lim_{t \to \infty} f_\alpha(t\rho) e^{i\theta \cdot H(\alpha)}, \tag{3.13}$$

whenever these latter limits exist. We define $\lim_{\lambda \to -\infty} f(\lambda)$ similarly. A partial ordering is defined on Λ_- by setting $\alpha < \beta$ if $\beta - \alpha \in \Lambda_+$.

Theorem 3. *Let g be a finite-dimensional Lie algebra of rank r; let h be a Cartan subalgebra of g; let R be the subset of h^* comprising all points λ for which no Shapovalov determinant* $\det(M_\mu(\lambda))$ *vanishes. Let $f: R \to \mathbb{C}[[e^{-i\theta_1}, \ldots, e^{-i\theta_r}]]$ be a power series valued function on R with the properties that*

(i) $\lim_{\lambda \to -\infty} f(\lambda) = \lim_{\lambda \to -\infty} \sigma(\lambda)$;

(ii) $\lim_{\lambda \to \infty} f(\lambda) = \lim_{\lambda \to \infty} \sigma(\lambda)$;

(iii) *if λ_0 is a point on precisely one of the Shapovalov vanishing planes $P(n_0, \alpha_0)$, and $(\underline{t}, \varepsilon)$ are the coordinates defined above in a neighbourhood $N(\delta)$ of λ which intersects precisely one vanishing plane, then f obeys the equation*

$$\lim_{\varepsilon \to 0^+} f(\underline{t}, \varepsilon) - \lim_{\varepsilon \to 0^-} f(\underline{t}, \varepsilon) = \pm 2 e^{-in_0\theta \cdot H(\alpha_0)} f(\lambda_0 - n_0\alpha_0); \tag{3.14}$$

(iv) *Each function f_α is constant on the union of any connected components of R that are not separated by any vanishing plane $P(n, \beta)$ with $n\beta < \alpha$.*

Then $f(\lambda) = \sigma(\lambda)$ for all $\lambda \in R$.

Proof. For suppose $f \neq \sigma$. Let α be such that $f_\alpha(\lambda) \neq \sigma_\alpha(\lambda)$ for some $\lambda \in R$ and such that if $f_\beta(\lambda') \neq \sigma_\beta(\lambda')$ for some $\lambda' \in N$ then $\beta \geqq \alpha$. We have the component form of Eq. (3.14):

$$\lim_{\varepsilon \to 0^+} f_\alpha(\underline{t}, \varepsilon) - \lim_{\varepsilon \to 0^-} f_\alpha(\underline{t}, \varepsilon) = \pm 2 f_{\alpha + n_0\alpha_0}(\lambda_0 - n_0\alpha_0). \tag{3.15}$$

Now, since $\lim_{\lambda \to \infty} f_\alpha(\lambda) = \lim_{\lambda \to \infty} \sigma_\alpha(\lambda)$, there must be some point λ_0 that lies on precisely one Shapovalov vanishing plane $P(n_0, \alpha_0)$ and such that in the neighbourhood of λ_0 the functions f_α and σ_α satisfy Eq. (3.15) with opposite signs on the right-hand side. Since the same sign holds for all components of f crossing the plane $P(n_0, \alpha_0)$ at λ_0, and since properties (i) and (iii) imply that $f_0(\lambda) = 1$ for all λ, we have that

$$\lim_{\varepsilon \to 0^+} f_{n_0\alpha_0}(\underline{t}, \varepsilon) - \lim_{\varepsilon \to 0^-} f_{n_0\alpha_0}(\underline{t}, \varepsilon) = \pm 2, \tag{3.16}$$

where the sign on the right-hand side is the opposite to that in the analogous equation satisfied by $\sigma_{n_0\alpha_0}$. So since $f_\beta = \sigma_\beta$ for all $\beta < \alpha$, we must have that $n_0\alpha_0 = \alpha$, and that f satisfies Eq. (3.11) across all sections of all vanishing planes $P(n, \beta)$ such that $n\beta < \alpha$. Now $f_\alpha(\lambda) = \sigma_\alpha(\lambda)$ asymptotically as $\lambda \to \pm\infty$. Moreover, by definition, f_α and σ_α are constant on any set of regions in R that are not separated by a plane

$P(n,\beta)$ with $n\beta<\alpha$. But a well known property of the root systems of finite-dimensional Lie algebras is that if α and β are positive roots and m and n are positive integers such that $m\alpha=n\beta$ then $m=n$ and $\alpha=\beta$. Hence $P(n_0,\alpha_0)$ is the only vanishing plane $P(n,\beta)$ such that $n\beta=n_0\alpha_0$. But now we have established that $f_\alpha=\sigma_\alpha$ for all points in R above $P(n_0,\alpha_0)$, and likewise for all points in R below $P(n_0,\alpha_0)$. Thus $f_\alpha=\sigma_\alpha$ throughout R, which is a contradiction.

We now obtain explicitly the asymptotic signature characters for all semi-simple finite dimensional Lie algebras g.

Lemma 4. *Let $\sigma(\lambda)$ be the signature character of the highest weight representation $V(\lambda)$ of the Lie algebra g. Then, with the above definitions,*

$$\lim_{\lambda\to\infty}\sigma(\lambda)=\prod_{\alpha\in\Delta_+}(1-e^{-i\theta\cdot H(\alpha)})^{-1},$$

$$\lim_{\lambda\to-\infty}\sigma(\lambda)=\prod_{\alpha\in\Delta_+}(1+e^{-i\theta\cdot H(\alpha)})^{-1}. \tag{3.17}$$

Proof. By the Poincaré–Birkhoff–Witt theorem, we may take a basis of $U(n_-)_\mu$ to be lexicographically ordered monomials in the lowering operators of g. Each monomial corresponds to some ordered partition $\{\beta\}=\{\beta_1,\dots,\beta_k\}$ of μ in terms of negative roots. We denote the set of such partitions by $\Pi(\mu)$. Take generators $E_\alpha\in g_\alpha$. We consider the inner product matrix $M_\mu(\lambda)$ in the basis $\{v_i(\lambda):1\leqq i\leqq P(\mu)\}$, where the vector v_i corresponds to the i^{th} partition $\{\beta_1,\dots,\beta_k\}$ of μ, and

$$v_i=E_{\beta_1}\cdots E_{\beta_k}|\lambda\rangle. \tag{3.18}$$

As $\lambda\to\pm\infty, M_\mu(\lambda)$ is asymptotically diagonalised, in the sense that

$$\lim_{\lambda\to\pm\infty}\operatorname{sig}(M_\mu(\lambda))=\sum_{i=1}^{P(\mu)}\lim_{\lambda\to\pm\infty}\operatorname{sgn}(M_\mu(\lambda)_{ii}). \tag{3.19}$$

Now

$$\lim_{t\to\infty}\operatorname{sgn}(\langle v_{\{\beta\}}(\pm t\rho),v_{\{\beta\}}(\pm t\rho)\rangle)=(\pm 1)^{l(\{\beta\})}, \tag{3.20}$$

where the length $l(\{\beta\})$ of a partition is the number of roots which appear in the partition. Hence

$$\lim_{\lambda\to\pm\infty}\operatorname{sig}(M_\mu(\lambda))=\sum_{\{\beta\}\in\Pi(\mu)}(\pm 1)^{l(\{\beta\})}. \tag{3.21}$$

Thus the signature characters $\sigma(\lambda)$ obey

$$\lim_{\lambda\to\pm\infty}\sigma(\lambda)=\sum_{\mu\in\Lambda_-}\sum_{\{\beta\}\in\Pi(\mu)}(\pm 1)^{l(\{\beta\})}e^{i\theta\cdot H(\mu)}$$

$$=\prod_{\alpha\in\Delta_+}(1\mp e^{-i\theta\cdot H(\alpha)})^{-1}. \tag{3.22}$$

This completes the proof.

4. Signature Characters of A_2

We now use the results of the last section to obtain signature characters for the highest weight representations of A_2. In the notation defined above, the A_2 Cartan matrix is $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and the positive root set $\Delta_+ = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}$. The A_2 partition function is

$$P(\mu) = \begin{cases} \min(m+1, n+1) & \text{if} \quad \mu = -m\alpha_1 - n\alpha_2 \quad \text{for integers } m \text{ and } n \geqq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

From the Shapovalov formula (2.11), it follows that R splits into a union of connected subregions:

$$R = \bigcup_{(a_1, a_2, a_3) \in I} R(a_1, a_2, a_3), \tag{4.2}$$

where

$$I = \{(a_1, a_2, a_3) : a_i \text{ non-negative integers with } a_1 + a_2 = a_3 \text{ or } a_1 + a_2 + 1 = a_3 \text{ or } a_1 = 0 \text{ and } a_2 > a_3 \text{ or } a_2 = 0 \text{ and } a_1 > a_3\} \tag{4.3}$$

and

$$R(a_1, a_2, a_3) = \{\lambda \in h^* : a_i = \max(0, \lfloor (\alpha_i, \lambda + \rho) \rfloor)\}. \tag{4.4}$$

We set $e^{-i\theta_j} = x_j$ for $j = 1, 2$. We shall define a formal power series valued function $f : R \to \mathbb{C}[[x_1, x_2]]$ that is constant on the connected components $R(a_1, a_2, a_3)$ of R and is asymptotically equal to the signature character $\sigma(\lambda)$ of an A_2 highest weight representation $V(\lambda)$ as $\lambda \to \pm\infty$. We then show that f satisfies Eq. (3.14). It will then follow, from Theorem 3, that $f(\lambda)$ is precisely the signature character $\sigma(\lambda)$ for $\lambda \in R$.

We define f on R in terms of a function $\tilde{f}$ defined on I, so that

$$f(\lambda) = \frac{\tilde{f}(a_1, a_2, a_3)}{(1 - x_1^2)(1 - x_2^2)(1 - (x_1 x_2)^2)} \quad \text{if} \quad \lambda \in R(a_1, a_2, a_3). \tag{4.5}$$

We take

$$\begin{aligned} \tilde{f}(a_1, a_2, a_3) = {} & (1 + x_1)(1 + x_2)(1 + x_1 x_2) - 2(x_1)^{a_1 + 1}(1 + x_2)(1 + x_1 x_2) \\ & - 2(x_2)^{a_2 + 1}(1 + x_1)(1 + x_1 x_2) - 2(x_1 x_2)^{a_3 + 1}(1 + x_1)(1 + x_2) \\ & + 4(x_1)^{a_1 + 1}(x_1 x_2)^{\min(a_2, a_3) + 1}(1 + x_2) \\ & + 4(x_2)^{a_2 + 1}(x_1 x_2)^{\min(a_1, a_3) + 1}(1 + x_1) \\ & + 4(x_1)^{\max(a_1, a_3) + 1}(x_2)^{\max(a_2, a_3) + 1}(1 + x_1 x_2) \\ & - 8(x_1)^{\min(a_2, a_3) + a_1 + 2}(x_2)^{\min(a_1, a_3) + a_2 + 2}. \end{aligned} \tag{4.6}$$

The following two results are easily verified from Eqs. (4.5–4.6).

Lemma 5. *With the above definitions and conventions, let λ_0 be a point lying on precisely one A_2 vanishing line $P(n_0, \alpha_0)$. Then*

$$\lim_{\varepsilon \to 0^+} f(\underline{t}, \varepsilon) - \lim_{\varepsilon \to 0^-} f(\underline{t}, \varepsilon) = e(\lambda_0) 2 x_1^{-n_0 \alpha_0^1} x_2^{-n_0 \alpha_0^2} f(\lambda_0 - n_0 \alpha_0); \tag{4.7}$$

where the sign function $e(\lambda_0)$ *is given by*

$$e(\lambda_0)=\begin{cases}+1 & \text{if} \quad \alpha_0=\alpha_1 \text{ or } \alpha_2, \\ +1 & \text{if} \quad \alpha_0=\alpha_3 \quad \text{and} \quad \lambda_0+\rho\notin C, \\ -1 & \text{if} \quad \alpha_0=\alpha_3 \quad \text{and} \quad \lambda+\rho\in C,\end{cases} \tag{4.8}$$

and where C *is the fundamental Weyl chamber of* A_2, *that is,*

$$C=\{\lambda\in h^*:(\alpha_1,h)\geqq 0 \text{ and } (\alpha_2,h)\geqq 0\}.$$

Lemma 6. *With the above definitions,*

$$\lim_{\lambda\to\infty} f(\lambda)=(1-x_1)^{-1}(1-x_2)^{-1}(1-x_1x_2)^{-1},$$

$$\lim_{\lambda\to-\infty} f(\lambda)=(1+x_1)^{-1}(1+x_2)^{-1}(1+x_1x_2)^{-1}. \tag{4.9}$$

But now we have derived the signature characters for almost all A_2 highest weight representations.

Theorem 7. *In the above notation, the normalised* A_2 *signature characters for* $V(\lambda)$ *are given by*

$$\sigma(\lambda)=f(\lambda). \tag{4.10}$$

Proof. This follows from Theorem 3 and Lemmas 4, 5 and 6.

Thus we have signature characters for $V(\lambda)$ for all $\lambda\in R$. We proceed to deduce the signature characters for the remaining A_2 highest weight representations.

Theorem 8. *If* λ *lies on a vanishing line section bounding (only) the two regions* $R(a_1^{(1)},a_2^{(1)},a_3^{(1)})$ *and* $R(a_1^{(2)},a_2^{(2)},a_3^{(2)})$, *the normalised* A_2 *signature character for* $V(\lambda)$ *is*

$$\sigma(\lambda)=\frac{\sum_{i=1}^{2}\tilde{f}(a_1^{(i)},a_2^{(i)},a_3^{(i)})}{2(1-x_1^2)(1-x_2^2)(1-(x_1x_2)^2)}. \tag{4.11}$$

Proof. This follows from Lemma 2 and Theorem 7.

Now we calculate $\sigma(\lambda)$ for λ lying on the intersection of two or more vanishing lines. Any such λ clearly lies either on one of the $P(n,\alpha_1)$ or on one of the $P(n,\alpha_2)$ (possibly both).

Theorem 9. *If* λ *lies on the intersection of two or more vanishing lines, one of which is* $P(n_i,\alpha_i)$ *for* $i=1$ *or* 2 *and some* n_i, *let the regions that have* λ *as a corner point and a section of* $P(n_i,\alpha_i)$ *as an edge be* $R(a_1^{(j)},a_2^{(j)},a_3^{(j)})$ *for* $j=1,2,3,4$. *Then the normalised* A_2 *signature character for* $V(\lambda)$ *is*

$$\sigma(\lambda)=\frac{\sum_{j=1}^{4}\tilde{f}(a_1^{(j)},a_2^{(j)},a_3^{(j)})}{4(1-x_1^2)(1-x_2^2)(1-(x_1x_2)^2)}. \tag{4.12}$$

Proof. First let us suppose that λ_0 lies on the intersection of two or more vanishing lines, one of which is $P(n_1,\alpha_1)$ for some n_1. If λ_0 lies on the intersection of precisely two vanishing lines, denote the second vanishing line by $P(n_0,\alpha_0)$; if λ_0 lies at a triple intersection, denote by $P(n_0,\alpha_0)$ the line of the form $P(n_2,\alpha_2)$ on which λ_0 lies.

The following facts are known about the representation $V(\lambda_0)$. Besides the vector $|\lambda_0\rangle$, $V(\lambda_0)$ contains highest weight vectors $v_0 \in V(\lambda_0)_{n_0\alpha_0}$ and $v_1 \in V(\lambda_0)_{n_1\alpha_1}$. Other than v_0, $V(\lambda_0)$ contains no highest weight vectors not lying in $U(n_-)v_1$. $U(n_-)v_0$ forms a representation of g isomorphic to $V(\lambda_0 - n_0\alpha_0)$. We set

$$\tilde{P}(\mu;\lambda_0) = \dim\left(\frac{V(\lambda_0)_\mu \cap (U(n_-)v_0)}{V(\lambda_0)_\mu \cap (U(n_-)v_0) \cap (U(n_-)v_1)}\right). \tag{4.13}$$

Define the fundamental weights Λ_1 and Λ_2 by $(\Lambda_i, \alpha_j) = \delta_{ij}$; define coordinates (t, ε) for $\lambda \in h^*$ by $\lambda = \lambda_0 + t\Lambda_1 + \varepsilon\Lambda_2$. We write $V(t,\varepsilon)$ for $V(\lambda)$ and $|t,\varepsilon\rangle$ for $|\lambda\rangle$, and we adopt the convention that any vector written in the form $w(t,\varepsilon)$ belongs to $V(t,\varepsilon)$. Define the neighbourhood $N(\delta)$ of λ_0 as the coordinate region with $|t| < \delta$ and $|\varepsilon| < \delta$, and choose δ_0 such that for non-zero $\varepsilon < \delta_0$ the representation $V(0,\varepsilon)$ contains a unique descendent highest weight vector $v_1(0,\varepsilon) \in V(0,\varepsilon)_{n_1\alpha_1}$; set $v_1(0,0) = v_1$. We can express $v_0(0,0)$ as $a|0,0\rangle$, where $a \in U(n_-)$. Define the vectors $v_0(0,\varepsilon) \in V(0,\varepsilon)_{n_0\alpha_0}$ by $v_0(0,\varepsilon) = a|0,\varepsilon\rangle$.

If $\mu + n_1\alpha_1 \notin \Lambda_-$ and $\mu + n_0\alpha_0 \notin \Lambda_-$ then the inner product matrix $M_\mu(\lambda)$ is non-singular in $N(\delta_0)$ and its signature is constant throughout the neighbourhood. Now we consider $M_\mu(\lambda)$ for μ such that $\mu + n_1\alpha_1 \in \Lambda_-$ or $\mu + n_0\alpha_0 \in \Lambda_-$. Reducing δ_0 if necessary, we choose

$$\begin{aligned} &a_i \in (U(n_-))_{\mu+n_1\alpha_1} && \text{for} \quad 1 \leqq i \leqq P(\mu + n_1\alpha_1), \\ &b_i \in (U(n_-))_{\mu+n_0\alpha_0} && \text{for} \quad P(\mu + n_1\alpha_1) + 1 \leqq i \leqq \tilde{P}(\mu;\lambda_0) + P(\mu + n_1\alpha_1), \\ &c_i \in (U(n_-))_\mu && \text{for} \quad \tilde{P}(\mu + \lambda_0) + P(\mu + n_1\alpha_1) + 1 \leqq i \leqq P(\mu), \end{aligned} \tag{4.14}$$

such that, setting

$$w_i(0,\varepsilon) = \begin{cases} a_i v_1(0,\varepsilon) & \text{for} \quad 1 \leqq i \leqq P(\mu + n_1\alpha_1), \\ b_i v_0(0,\varepsilon) & \text{for} \quad P(\mu + n_1\alpha_1) + 1 \leqq i \leqq \tilde{P}(\mu;\lambda_0) + P(\mu + n_1\alpha_1), \\ c_i|0,\varepsilon\rangle & \text{for} \quad \tilde{P}(\mu + \lambda_0) + P(\mu + n_1\alpha_1) + 1 \leqq i \leqq P(\mu), \end{cases} \tag{4.15}$$

the set $\{w_i(0,\varepsilon) : 1 \leqq i \leqq P(\mu)\}$ forms a basis of $V(0,\varepsilon)_\mu$ for $\varepsilon \leqq \delta_0$.

In this basis the inner product matrix $M_\mu(0,\varepsilon)$ has the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon M'(\varepsilon) & \varepsilon A(\varepsilon) \\ 0 & \varepsilon A^T(\varepsilon) & M''(\varepsilon) \end{pmatrix}. \tag{4.16}$$

We have that $M''(\varepsilon) = M''(0) + O(\varepsilon)$ and we know that $M''(0)$ is non-singular, since if it were not $V(0,0)$ would contain a highest weight vector not descended from v_0 or v_1. Hence there is a new basis in which $M_\mu(0,\varepsilon)$ has the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon M'(0) + O(\varepsilon^2) & 0 \\ 0 & 0 & M''(0) + O(\varepsilon) \end{pmatrix}. \tag{4.17}$$

Now a generalisation of the Shapovalov formula obtained by one of us (A.K.)

implies that the submatrix

$$\begin{pmatrix} \varepsilon M'(0) + O(\varepsilon^2) & 0 \\ 0 & M''(0) + O(\varepsilon) \end{pmatrix} \tag{4.18}$$

has determinant of order $\varepsilon^{\tilde{P}(\mu;\lambda_0)}$. (For completeness, a proof of this last result is given in an appendix.) Hence $M'(0)$ must be non-singular, and we have that

$$\sigma(0,0) = \frac{1}{2}\left(\lim_{\varepsilon \to 0^+} \sigma(0,\varepsilon) + \lim_{\varepsilon \to 0^-} \sigma(0,\varepsilon) \right). \tag{4.19}$$

Equation (4.12) follows.

When λ lies on the intersection of two or more vanishing lines, one of which is $P(n_2, \alpha_2)$ for some n_2, the argument is similar. This completes the proof.

The case when λ lies on a triple intersection of vanishing lines is particularly interesting. Here, Theorem 9 gives us two expressions for $\sigma(\lambda)$. Moreover, since these representations are unitary, the (unnormalised) signature character equals the ordinary character, which has a well-known expression given by the Weyl character formula. However, it is easy to see that the three expressions are equal. Explicitly, if $\lambda = (r_1 \Lambda_1 + r_2 \Lambda_2)$, for non-negative integers r_1 and r_2, then Theorem 9 implies that

$$\begin{aligned} \sigma(\lambda) = {} & (1 - x_1)^{-1}(1 - x_2)^{-1}(1 - x_1 x_2)^{-1}(1 - x_1^{r_1+1} - x_2^{r_2+2} + x_1^{r_1+1}(x_1 x_2)^{r_2+1} \\ & + x_2^{r_2+1}(x_1 x_2)^{r_1+1} - (x_1 x_2)^{r_1+r_2+2}). \end{aligned} \tag{4.20}$$

This gives us an alternative proof of the well known result that $V(\lambda)$ is unitary if $\lambda = (r_1 \Lambda_1 + r_2 \Lambda_2)$, for non-negative integers r_1 and r_2.

Corollary 10. *Let $\lambda = (r_1 \Lambda_1 + r_2 \Lambda_2)$ for non-negative integers r_1 and r_2, and let $V'(\lambda)$ be the maximal proper submodule of $V(\lambda)$. Then the bilinear form $\langle\,,\rangle'$ induced on the representation $V(\lambda)/V'(\lambda)$ by $\langle\,,\rangle$ is positive definite. (In other words, the irreducible representation with highest weight λ is unitary.)*

Proof. The quotient module $V'(\lambda)$ is the submodule of $V(\lambda)$ generated by all the states $v \in V(\lambda)$ such that $\langle v, w \rangle = 0$ for all $w \in V(\lambda)$. So the representation $V(\lambda)/V'(\lambda)$ has the same signature character with respect to $\langle\,,\rangle'$ as $V(\lambda)$ does with respect to $\langle\,,\rangle$. In other words,

$$\begin{aligned} \chi^{\mathrm{sig}}(V(\lambda)/V'(\lambda)) &= (1 - x_1)^{-1}(1 - x_2)^{-1}(1 - x_1 x_2)^{-1} \\ &\quad \cdot x_1^{r_1} x_2^{r_2}(1 - x_1^{r_1+1} - x_2^{r_2+2} + x_1^{r_1+1}(x_1 x_2)^{r_2+1} \\ &\quad + x_2^{r_2+1}(x_1 x_2)^{r_1+1} - (x_1 x_2)^{r_1+r_2+2}) \\ &= \chi(V(\lambda)/V'(\lambda)), \end{aligned} \tag{4.21}$$

the last equality following from the Weyl character formula. This completes the proof.

Finally, we note that the signature characters on R can be re-expressed in a suggestively simple way. Denote the Weyl reflection corresponding to $\alpha \in \Delta_+$ by r_α and let $C_0 \equiv C$ be the fundamental Weyl chamber for A_2. Denote the other Weyl chambers for A_2 by $C_1 = r_{\alpha_1} C$, $C_2 = r_{\alpha_2} C$, $C_3 = r_{\alpha_1} r_{\alpha_2} C$, $C_4 = r_{\alpha_2} r_{\alpha_1} C$, $C_5 = r_{\alpha_3} C$.

Then

$$f(\lambda)=\frac{\tilde{f}_i(a_1,a_2,a_3)}{(1-x_1^2)(1-x_2^2)(1-(x_1x_2)^2)} \tag{4.22}$$

for $\lambda\in R(a_1,a_2,a_3)$ such that $\lambda+\rho\in C_i$, where

$$\begin{aligned}
\tilde{f}_0(a_1,a_2,a_3)&=(1+x_1)(1+x_2)(1+x_1x_2)-2x_1^{a_1+1}(1+x_2)(1+x_1x_2)\\
&\quad-2x_2^{a_2+1}(1+x_1)(1+x_1x_2)+4x_1^{a_1+1}(x_1x_2)^{a_2+1}(1+x_2)\\
&\quad+4x_2^{a_2+1}(x_1x_2)^{a_1+1}(1+x_1)-8(x_1x_2)^{a_1+a_2+2}\\
&\quad+2(x_1x_2)^{a_3+1}(1-x_1)(1-x_2),\\
\tilde{f}_1(a_1,a_2,a_3)&=(1-x_1)(1+x_2)(1+x_1x_2)-2x_2^{a_2+1}(1-x_1x_2)(1+x_1)\\
&\quad-2(x_1x_2)^{a_3+1}(1-x_1)(1+x_2)+4x_1^{a_3+1}x_2^{a_2+1}(1-x_1x_2),\\
\tilde{f}_2(a_1,a_2,a_3)&=(1-x_2)(1+x_1)(1+x_1x_2)-2x_1^{a_1+1}(1-x_1x_2)(1+x_2)\\
&\quad-2(x_1x_2)^{a_3+1}(1-x_2)(1+x_1)+4x_2^{a_3+1}x_1^{a_1+1}(1-x_1x_2),\\
\tilde{f}_3(a_1,a_2,a_3)&=(1-x_1)(1-x_1x_2)(1+x_2)-2x_2^{a_2+1}(1-x_1)(1-x_1x_2),\\
\tilde{f}_4(a_1,a_2,a_3)&=(1-x_2)(1-x_1x_2)(1+x_1)-2x_1^{a_1+1}(1-x_2)(1-x_1x_2),\\
\tilde{f}_5(a_1,a_2,a_3)&=(1-x_1)(1-x_2)(1-x_1x_2).
\end{aligned} \tag{4.23}$$

5. Signature Characters of B_2

The B_2 Cartan matrix and symmetrised Cartan matrix are given by

$$A=\begin{pmatrix}2&-2\\-1&2\end{pmatrix},\quad A^{\mathrm{sym}}=\begin{pmatrix}1&-1\\-1&2\end{pmatrix}. \tag{5.1}$$

With this choice the simple roots are α_1,α_2 with α_1 the short root. The other positive roots are $\alpha_3=\alpha_1+\alpha_2$ and $\alpha_4=2\alpha_1+\alpha_2$. The Weyl group of B_2 is the dihedral group D_4 and there are thus eight Weyl chambers. We denote these as follows.

$$\begin{aligned}
C_0&=C, & C_1&=r_1C,\\
C_2&=r_1r_2C, & C_3&=r_1r_2r_1C,\\
C_4&=r_2r_1r_2r_1C, & C_5&=r_2r_1r_2C,\\
C_6&=r_2r_1C, & C_7&=r_2C.
\end{aligned} \tag{5.2}$$

The element $\rho\in h^*$ is given by $\rho=\frac{1}{2}(4\alpha_1+3\alpha_2)$. The region R splits into a union of connected subspaces

$$R=\cup R(a_1,a_2,a_3,a_4) \tag{5.3}$$

where

$$R(a_1,a_2,a_3,a_4)=\left\{\lambda\in h^*:a_i=\max\left(0,\left\lfloor\frac{2(\alpha_i,\lambda+\rho)}{(\alpha_i,\alpha_i)}\right\rfloor\right)\right\}. \tag{5.4}$$

If $\lambda \in R(a_1, a_2, a_3, a_4)$ and $\lambda + \rho \in C_i$ then the normalised signature character of $V(\lambda)$ is given by

$$\sigma(\lambda) = \frac{\tilde{f}_i(a_1, a_2, a_3, a_4)}{(1-x_1^2)(1-x_2^2)(1-(x_1x^2)^2)(1-(x_1^2x_2)^2)}, \tag{5.5}$$

where the $\tilde{f}_i$ are defined as follows.

$$\begin{aligned}
\tilde{f}_0(a_1, a_2, a_3, a_4) = {} & (1+x_1)(1+x_2)(1+x_1x_2)(1+x_1^2x_2) \\
& - 2x_1^{a_1+1}(1+x_2)(1+x_1x_2)(1+x_1^2x_2) \\
& - 2x_2^{a_2+1}(1+x_1)(1+x_1x_2)(1+x_1^2x_2) \\
& + 2(x_1x_2)^{a_3+1}(1+x_1)(1-x_2)(1-x_1^2x_2) \\
& + 2(x_1^2x_2)^{a_4+1}(1-x_1)(1+x_2)(1-x_1x_2) \\
& - 4(x_1^2x_2)^{a_4+1}(x_2)^{a_2+1}(1-x_1)(1-x_1x_2) \\
& - 4(x_1x_2)^{a_3+1}x_1^{a_1+1}(1-x_2)(1-x_1^2x_2) \\
& + 4(x_1x_2)^{a_1+1}x_2^{a_2+1}(1+x_1)(1+x_1^2x_2) \\
& + 4x_1^{a_1+1}(x_1^2x_2)^{a_2+1}(1+x_2)(1+x_1x_2) \\
& - 8(x_1x_2)^{2a_2+a_1+3}(1+x_1) - 8(x_1^2x_2)^{a_1+a_2+2}(1+x_2) \\
& + 16(x_1x_2)^{2a_2+2}(x_1^2x_2)^{a_1+1},
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
\tilde{f}_1(a_1, a_2, a_3, a_4) = {} & (1-x_1)(1+x_2)(1+x_1x_2)(1+x_1^2x_2) \\
& - 2x_2^{a_2+1}(1+x_1)(1-x_1x_2)(1+x_1^2x_2) \\
& - 2(x_1^2x_2)^{a_4+1}(1-x_1)(1+x_2)(1+x_1x_2) \\
& + 2(x_1x_2)^{a_3+1}(1-x_1)(1-x_2)(1-x_1^2x_2) \\
& + 4x_2^{a_2+1}(x_1^2x_2)^{a_4+1}(1+x_1)(1-x_1x_2) \\
& + 4(x_1x_2)^{a_2+1}(x_1)^{a_4+1}(1+x_2)(1-x_1^2x_2) \\
& - 8(x_1x_2)^{a_2+a_4+2}(1-x_1^2x_2),
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
\tilde{f}_2(a_1, a_2, a_3, a_4) = {} & (1-x_1)(1+x_2)(1+x_1x_2)(1-x_1^2x_2) \\
& - 2x_2^{a_2+1}(1+x_1)(1-x_1x_2)(1-x_1^2x_2) \\
& - 2(x_1x_2)^{a_3+1}(1-x_1)(1+x_2)(1-x_1^2x_2) \\
& + 4x_1^{a_3+1}x_2^{a_2+1}(1-x_1x_2)(1-x_1^2x_2),
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
\tilde{f}_3(a_1, a_2, a_3, a_4) = {} & (1+x_2)(1-x_1)(1-x_1x_2)(1-x_1^2x_2) \\
& - 2x_2^{a_2+1}(1-x_1)(1-x_1x_2)(1-x_1^2x_2),
\end{aligned} \tag{5.9}$$

$$\tilde{f}_4(a_1, a_2, a_3, a_4) = (1-x_1)(1-x_2)(1-x_1x_2)(1-x_1^2x_2), \tag{5.10}$$

$$\begin{aligned}
\tilde{f}_5(a_1, a_2, a_3, a_4) = {} & (1+x_1)(1-x_2)(1-x_1x_2)(1-x_1^2x_2) \\
& - 2x_1^{a_1+1}(1-x_2)(1-x_1x_2)(1-x_1^2x_2),
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
\tilde{f}_6(a_1, a_2, a_3, a_4) = {} & (1+x_1)(1-x_2)(1-x_1x_2)(1+x_1^2x_2) \\
& - 2x_1^{a_1+1}(1+x_2)(1-x_1x_2)(1-x_1^2x_2) \\
& - 2(x_1^2x_2)^{a_4+1}(1+x_1)(1-x_2)(1-x_1x_2) \\
& + 4x_1^{a_1+1}x_2^{a_4+1}(1-x_1x_2)(1-x_1^2x_2),
\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}\tilde{f}_7(a_1,a_2,a_3,a_4) = &(1+x_1)(1-x_2)(1+x_1x_2)(1+x_1^2x_2)\\ &-2x_1^{a_1+1}(1+x_2)(1+x_1x_2)(1-x_1^2x_2)\\ &+2(x_1^2x_2)^{a_4+1}(1-x_2)(1-x_1)(1-x_1x_2)\\ &-2(x_1x_2)^{a_3+1}(1+x_1)(1-x_2)(1+x_1^2x_2)\\ &+4x_1^{a_1+1}(x_1x_2)^{a_3+1}(1+x_2)(1-x_1^2x_2)\\ &+4x_2^{a_5+1}x_1^{a_1+1}(1+x_1)(1-x_1x_2)(1+x_1x_2^{\varepsilon})\\ &-8x_2^{a_5+1}x_1^{a_1+a_3+2}(1-x_1x_2)(1+x_1x_2^{\varepsilon}),\end{aligned} \tag{5.13}$$

where

$$a_5 = \left\lfloor \frac{a_1+a_3+1}{2} \right\rfloor \tag{5.14}$$

and

$$\varepsilon = 2a_5 - a_1 - a_3 - 1. \tag{5.15}$$

A. Appendix

Here we establish formulae for the determinants of certain submatrices of the inner product matrices of A_2 highest weight representations. We consider the inner product matrix $M_\mu(\lambda)$ on $V(\lambda)_\mu$, where $\mu = -(m_1\alpha_1 + m_2\alpha_2)$ is an A_2 weight, in the basis $\{v_i(\lambda): 1 \leqq i \leqq \min(m_1, m_2) + 1\}$, where

$$v_i(\lambda) = (E_{-\alpha_1})^{m_1 - i + 1}(E_{-\alpha_1-\alpha_2})^{i-1}(E_{-\alpha_2})^{m_2-i+1}|\lambda\rangle. \tag{A.1}$$

For r_1 and r_2 positive integers with $r_1 \leqq m_1$ and $r_2 \leqq m_2$, define the submatrices $M_\mu^{\alpha_1,r_1}(\lambda)$ and $M_\mu^{\alpha_2,r_2}(\lambda)$ of $M_\mu(\lambda)$ by

$$\begin{aligned}(M_\mu^{\alpha_1,r_1}(\lambda))_{ij} &= \langle v_{m_1-r_1+1+i}, v_{m_1-r_1+1+j}\rangle \quad \text{for} \quad 1 \leqq i, j \leqq \min(m_1,m_2) - m_1 + r_1,\\ (M_\mu^{\alpha_2,r_2}(\lambda))_{ij} &= \langle v_{m_2-r_2+1+i}, v_{m_2-r_2+1+j}\rangle \quad \text{for} \quad 1 \leqq i, j \leqq \min(m_1,m_2) - m_2 + r_2,\end{aligned} \tag{A.2}$$

with the convention that the matrices are null if the ranges of i and j are empty.

The result quoted in the proof of Theorem 9 is a corollary of the following.

Lemma 11. *In the above notation, consider $M_\mu^{\alpha_1,r_1}(r_1\Lambda_1 + r_2\Lambda_2)$, for fixed positive integer r_1, as a matrix of polynomials in r_2. Similarly, consider $M_\mu^{\alpha_2,r_2}(r_1\Lambda_1 + r_2\Lambda_2)$, for fixed positive integer r_2, as a matrix of polynomials in r_1. Then*

$$\det(M_\mu^{\alpha_1,r_1}((r_1-1)\Lambda_1 + r_2\Lambda_2)) = C_1\left(\prod_{p=1}^{m_2}(r_2 - p + 1)^{A(m_1,m_2,r_1,p)}\right) \cdot \left(\prod_{p=1}^{r_1-1}(r_2+p)^{B(m_1,m_2,r_1,p)}\right) \tag{A.3}$$

and

$$\det(M_\mu^{\alpha_2,r_2}(r_1\Lambda_1+(r_2-1)\Lambda_2))=C_2\left(\prod_{p=1}^{m_1}(r_1-p+1)^{A(m_2,m_1,r_2,p)}\right)\cdot\left(\prod_{p=1}^{r_2-1}(r_1+p)^{B(m_2,m_1,r_2,p)}\right) \tag{A.4}$$

where C_1 and C_2 are non-zero constants and

$$\begin{aligned}A(m_1,m_2,r_1,p)&=\max(\min(m_1,m_2-p)+1,0)\\&\quad-\max(\min(m_1-r_1-p,m_2-p)+1,0)\\&\quad-\max(\min(m_1-r_1,m_2-r_1-p)+1,0)\\&\quad+\max(\min(m_1-r_1-p,m_2-r_1-p)+1,0),\\B(m_1,m_2,r_1,p)&=\min(\max(m_1-p+1,0),\max(m_2-p+1,0))\\&\quad-\min(\max(m_1-r_1+1,0),\max(m_2-p+1,0)).\end{aligned} \tag{A.5}$$

Proof. We prove Eq. (A.3); the proof of (A.4) is similar. Fix r_1 and consider λ of the form $(r_1-1)\Lambda_1+r_2\Lambda_2$ (for any r_2). The vector $v(\lambda)=(E_{-\alpha_1})^{r_1}|\lambda\rangle$ is a highest weight vector in $V(\lambda)$; it generates a submodule $U(n_-)v(\lambda)$ of $V(\lambda)$; we write $U_\mu(\lambda)=U(n_-)v(\lambda)\cap V(\lambda)_\mu$. Now the submodule embeddings of the modules $V(\lambda)$ are known to be such that for $r_2=p-1$ and p a positive integer with $1\leqq p\leqq m_2$ the null space of $M_\mu(\lambda)$ has dimension $\dim(U_\mu(\lambda))+A(m_1,m_2,r_1,p)$; that for $r_2=-p$ and p a positive integer with $1\leqq p\leqq(r_1-1)$ the null space of $M_\mu(\lambda)$ has dimension $\dim(U_\mu(\lambda))+B(m_1,m_2,r_1,p)$; and that for all other r_2 the null space of $M_\mu(\lambda)$ has dimension $\dim(U_\mu(\lambda))$. The vectors in the set $\{v_i(\lambda):m_1-r_1+2\leqq i\leqq \min(m_1,m_2)+1\}$ span a subspace of $(V(\lambda))_\mu$ complementary to $U_\mu(\lambda)$, and $M_\mu^{\alpha_1,r_1}(\lambda)$ is the matrix of their inner products. Let p be a positive integer with $1\leqq p\leqq m_2$. Then we have that $M_\mu^{\alpha_1,r_1}((r_1-1)\Lambda_1+r_2\Lambda_2)$ has a null space of dimension $A(m_1,m_2,r_1,p)$ at $r_2=(p-1)$. So $\det(M_\mu^{\alpha_1,r_1}((r_1-1)\Lambda_1+(p-1+\varepsilon)\Lambda_2))$ is at least of order $\varepsilon^{A(m_1,m_2,r_1,p)}$. Thus $\det(M_\mu^{\alpha_1,r_1}((r_1-1)\Lambda_1+r_2\Lambda_2))$ is divisible by $(r_2-p+1)^{A(m_1,m_2,r_1,p)}$. A similar argument shows that it is also divisible by $(r_2+p)^{B(m_1,m_2,r_1,p)}$ if p is a positive integer with $1\leqq p\leqq(r_1-1)$. Hence the polynomial

$$\left(\prod_{p=1}^{m_2}(r_2-p+1)^{A(m_1,m_2,r_1,p)}\right)\left(\prod_{p=1}^{r_1-1}(r_2+p)^{B(m_1,m_2,r_1,p)}\right) \tag{A.6}$$

divides $\det(M_\mu^{\alpha_1,r_1}((r_1-1)\Lambda_1+(p-1)\Lambda_2))$. Now the order of the determinant as a polynomial in r_2 is at most $\max(m_2(\min(m_1,m_2)-m_1+r_1),0)$, since this is the order of the product of the diagonal elements, which dominates all other contributions. But it is easy to verify that

$$\left(\sum_{p=1}^{m_2}A(m_1,m_2,r_1,p)\right)+\left(\sum_{p=1}^{r_1-1}B(m_1,m_2,r_1,p)\right)=\max(m_2(\min(m_1,m_2)-m_1+r_1),0). \tag{A.7}$$

Thus Eq. (A.3) must hold for some constant C_1. Finally, since the matrix $M_\mu^{\alpha_1,r_1}((r_1-1)\Lambda_1+r_2\Lambda_2)$ is non-singular for non-integer r_2, C_1 must be non-zero.

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