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Generalized Chiral Potts Models and Minimal Cyclic Representations of $U_q(\hat{\mathfrak{gl}}(n, \mathbf{C}))$

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Abstract. We present for odd N a construction of the N^{n-1}-state generalization of the chiral Potts model proposed recently by Bazhanov et al. The Yang–Baxter equation is proved.

1. Introduction

The discovery of the chiral Potts model [1–4] opened a new phase in the theory of Yang–Baxter equations (YBE). It gave the first example of an R matrix (=solution to YBE) whose spectral parameters live on an algebraic variety other than $\mathbf{P}^1$ or an elliptic curve. Through the latest developments [5–8] it has become apparent that quantum groups at roots of 1 should lead to this type of R matrices. Because of the technical complexity, this program has been worked out so far only in a few simple examples. Besides the chiral Potts model, which is related to $U_q(\hat{\mathfrak{sl}}(2, \mathbf{C}))$, these are the cases corresponding to $U_q(\hat{\mathfrak{sl}}(3, \mathbf{C}))$ ([7] for $q^3 = 1$, [9] for $q^4 = 1$) or $U_q(A_2^{(2)})$ [8]. In a recent paper [10] Bazhanov et al. proposed a generalization of the chiral Potts model related to N^{n-1} dimensional irreducible representations of $U_q(\hat{\mathfrak{sl}}(n, \mathbf{C}))$ at $q^N = 1$. The aim of this paper is to give a proof to their conjecture.

Let us formulate the problem more precisely. Throughout the paper we fix a primitive N^{th} root of unity q, with N an odd integer $\geqq 3$. We shall deal with a Hopf algebra $\tilde{U}_q$ (essentially the quantum double of a "Borel" subalgebra of $U_q(\hat{\mathfrak{gl}}(n, \mathbf{C}))$) [8]. As an algebra $\tilde{U}_q$ is a trivial extension of $U_q(\hat{\mathfrak{gl}}(n, \mathbf{C}))$ by central elements, with the comultiplication being twisted by them. In this paper

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we shall focus on a family of N^{n-1}-dimensional irreducible representations $(W^{(0)}, \pi_\xi)$ of $\tilde{U}_q$ parametrized by $\xi \in (\mathbf{C}^\times)^{3n-1}$. Set

$$\pi_{\xi\bar{\xi}} = (\pi_\xi \otimes \pi_{\bar{\xi}}) \circ \Delta,$$

where Δ denotes the comultiplication. Consider now an intertwiner between the two tensor representations $\pi_{\xi\bar{\xi}}$ and $\pi_{\bar{\xi}\xi}$, namely a linear isomorphism $R(\xi, \bar{\xi}): W^{(0)} \otimes W^{(0)} \xrightarrow{\sim} W^{(0)} \otimes W^{(0)}$ such that

$$R(\xi, \bar{\xi})\pi_{\xi\bar{\xi}}(g) = \pi_{\bar{\xi}\xi}(g)R(\xi, \bar{\xi}) \quad (g \in \tilde{U}_q).$$

It is a common feature of q being a root of 1 [5] that, for the existence of $R(\xi, \bar{\xi})$ the parameters $\xi, \bar{\xi}$ are forced to lie on a common algebraic variety S. As it turns out, in our case S is a finite cover of $\mathscr{C}_\gamma \times \mathscr{C}_\gamma$, where $\mathscr{C}_\gamma$ denotes the algebraic curve,

$$\mathscr{C}_\gamma = \{r = (u, v) \in \mathbf{C}^{2n} \mid u_i^N + \lambda_i = v_j^N + \mu_j \; 0 \leqq i, j < n\},$$

parametrized by $\gamma = (\lambda_i, \mu_i)_{0 \leqq i < n}$. Following the general scheme [8] it can be shown that if $R(\xi, \bar{\xi})$ exists, then it is unique up to a scalar multiple, and that it satisfies YBE. For $n=2$ this construction reproduces the chiral Potts model [5, 8].

Let now $\xi, \bar{\xi} \in S$ and let $(r, r'), (\bar{r}, \bar{r}')$ be the corresponding points on $\mathscr{C}_\gamma \times \mathscr{C}_\gamma$. In the present case the intertwiner is given as a product of four matrices

$$R(\xi, \bar{\xi}) = S_{\bar{r}r'}^{-1} T_{r'\bar{r}'} \bar{T}_{r\bar{r}} S_{r\bar{r}'}, \tag{1.1}$$

and each factor can be described explicitly. However the matrix elements of $R(\xi, \bar{\xi})$ itself are rather cumbersome (if we use the standard comultiplication of $\tilde{U}_q$, see below). At this stage we received a paper by Bazhanov et al. [10] in which they proposed a simple factorized form of the matrix elements. In our notations they read as follows:

$$R(\xi, \bar{\xi})_{lm, jk} = \frac{\rho_{r'\bar{r}'}(j, l)\rho_{\bar{r}r'}(l, m)\rho_{r\bar{r}}(m, k)}{\rho_{r\bar{r}'}(j, k)}, \tag{1.2}$$

where $j, k, l, m \in \mathbf{Z}^n \bmod \mathbf{Z}(1, \dots, 1)$, and the coefficients are given by

$$\rho_{r\bar{r}}(k, l) = q^{P(k,l)}\sigma_{r\bar{r}}(k - l), \quad P(k, l) = \sum_i (k_i l_{i+1} - k_{i+1} l_i),$$

$$\frac{\sigma_{r\bar{r}}(m + v_i)}{\sigma_{r\bar{r}}(m)} = \frac{\delta_{i-1}(q^{m_{i-1i}} u_{i-1}\tilde{v}_{i-1} - q^{-m_{i-1i}}\tilde{u}_{i-1}v_{i-1})}{\delta_i(q^{m_{ii+1}+1}u_i\tilde{v}_i - q^{-m_{ii+1}-1}\tilde{u}_i v_i)},$$

$$v_i = (0, \dots, \overset{i}{1}, \dots, 0), \quad m_{ij} = m_i - m_j, \quad \delta_i^N = \frac{1}{\lambda_i - \mu_i}.$$

Guided by this formula we then noticed that a modification of the comultiplication leads directly to the R matrix (1.1) which differs from the old one by similarity and has factorized matrix elements (1.2) in a natural basis of $W^{(0)}$.

The text is organized as follows. In Sect. 2 we describe the minimal cyclic representation, thereby fixing the notations. We introduce the "spectral variety" S arising from necessary conditions for the existence of the intertwiner. In Sect. 3 we solve the intertwining relation for the $R(\xi, \bar{\xi})$. In Appendix A we prove the indecomposability of tensor product representations and that the intertwiner satisfies YBE. Appendix B is devoted to some details of the proof given in Sect. 3.

2. Spectral Varieties for Minimal Cyclic Representations

2.1. $U_q(\widehat{\mathfrak{gl}}(n,\mathbf{C}))$. Let $\mathfrak{h}_{\mathbf{Z}}$ be a free $\mathbf{Z}$ module of rank n spanned by ε_i $(0 \leqq i < n)$. We introduce $\mathfrak{h} = \mathfrak{h}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C}$ and a symmetric bilinear form (,) on $\mathfrak{h}$ such that $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. We extend the definition of ε_i in such a way that $\varepsilon_{i+n} = \varepsilon_i$. We set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.

The quantized universal enveloping algebra $U_q(\widehat{\mathfrak{gl}}(n,\mathbf{C}))$ is a $\mathbf{C}$-algebra generated by the symbols e_i, f_i $(0 \leqq i < n)$ and q^h $(h \in \mathfrak{h}_{\mathbf{Z}})$ with the following relations:

$$q^{h+h'} = q^h q^{h'}, \quad q^0 = 1,$$
$$q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i, \quad [e_i, f_j] = \delta_{ij}\{q^{\alpha_i}\},$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{ij}-k)} = 0, \quad \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)} = 0 \quad i \neq j.$$

Here $a_{ii} = 2$, and $a_{01} = a_{10} = -2$ for $n = 2, a_{ij} = -1$ $(i \equiv j \pm 1 \bmod n)$ or $=0$ (otherwise, for $n > 2$. We also use the notations

$$\{a\} = \frac{a - a^{-1}}{q - q^{-1}}, \quad [k] = \{q^k\}, \quad [k]! = [k] \cdots [1], \quad a^{(k)} = \frac{a^k}{[k]!}.$$

In the following the indices related to simple roots, e.g., i for e_i, should be understood as modulo n.

We add n central elements z_i $(0 \leqq i < n)$ and their inverses z_i^{-1} to $U_q(\widehat{\mathfrak{gl}}(n,\mathbf{C}))$ and denote the extended algebra simply by $\tilde{U}_q$. We use the comultiplication Δ of the form

$$\begin{aligned} \Delta(e_i) &= e_i \otimes q^{-\varepsilon_i} + z_i q^{\varepsilon_i} \otimes e_i, \\ \Delta(f_i) &= f_i \otimes q^{\varepsilon_{i+1}} + z_i^{-1} q^{-\varepsilon_{i+1}} \otimes f_i, \\ \Delta(q^h) &= q^h \otimes q^h, \quad \Delta(z_i) = z_i \otimes z_i. \end{aligned} \tag{2.1}$$

Remark. This differs from the standard comultiplication $\tilde{\Delta}$ for which we have

$$\begin{aligned} \tilde{\Delta}(e_i) &= e_i \otimes 1 + z_i q^{\alpha_i} \otimes e_i, \\ \tilde{\Delta}(f_i) &= f_i \otimes q^{-\alpha_i} + z_i^{-1} \otimes f_i. \end{aligned}$$

Denote by σ the automorphism of $\tilde{U}_q$ such that $\sigma(e_i) = e_{i+1}, \sigma(f_i) = f_{i+1}$, $\sigma(q^{\varepsilon_i}) = q^{\varepsilon_{i+1}}$ and $\sigma(z_i) = z_{i+1}$. We define the root vectors e_{ij} and f_{ij} inductively by

$$\begin{aligned} e_{ii+1} = e_i, \quad f_{i+1i} &= f_i && (0 \leqq i \leqq n-1), \\ e_{ij} &= e_{ik} e_{kj} - q e_{kj} e_{ik} && (0 \leqq i < k < j \leqq n-1), \\ f_{ij} &= f_{ik} f_{kj} - q^{-1} f_{kj} f_{ik} && (0 \leqq j < k < i \leqq n-1), \\ \sigma(e_{ij}) = e_{i+1j+1}, &\quad \sigma(f_{ij}) = f_{i+1j+1}. \end{aligned}$$

Then we have

$$\begin{aligned} \Delta(e_{ij}) = {} & e_{ij} \otimes q^{-\varepsilon_i - \cdots - \varepsilon_{j-1}} \\ & + (1-q^2) \sum_{i<k<j} z_i \cdots z_{k-1} q^{\varepsilon_i + \cdots + \varepsilon_{k-1}} e_{kj} \otimes e_{ik} q^{-\varepsilon_k - \cdots - \varepsilon_{j-1}} \\ & + z_i \cdots z_{j-1} q^{\varepsilon_i + \cdots + \varepsilon_{j-1}} \otimes e_{ij}, \quad (0 \leqq i < j \leqq n-1), \end{aligned}$$

$$\Delta(f_{ij}) = f_{ij} \otimes q^{\varepsilon_i + \cdots + \varepsilon_{j+1}} + (1-q^{-2}) \sum_{i>k>j} (z_{i-1} \cdots z_k)^{-1} q^{-\varepsilon_i - \cdots - \varepsilon_{k+1}} f_{kj} \otimes f_{ik} q^{\varepsilon_k + \cdots + \varepsilon_{j+1}} + (z_{i-1} \cdots z_j)^{-1} q^{-\varepsilon_i - \cdots - \varepsilon_{j+1}} \otimes f_{ij}, \quad (0 \leqq j < i \leqq n-1).$$

2.2 Invariants. In this paper we consider the case $q = e^{2\pi i/N}$ with $N \geqq 3$ odd. In this case $\tilde{U}_q$ has a large center [11]. We define the following central elements:

$$\begin{aligned}
\alpha_{ij} &= ((1-q^2) e_{ij} q^{\varepsilon_i + \cdots + \varepsilon_{j-1}})^N & &(0 \leqq i < j \leqq n-1),\\
\phi_{ij} &= (z_i \cdots z_{j-1} q^{2(\varepsilon_i + \cdots + \varepsilon_{j-1})})^N & &(0 \leqq i < j \leqq n-1),\\
\beta_{ij} &= ((1-q^{-2}) f_{ij} q^{-\varepsilon_i - \cdots - \varepsilon_{j+1}})^N & &(0 \leqq j < i \leqq n-1),\\
\psi_{ij} &= (z_{i-1} \cdots z_j q^{2(\varepsilon_i + \cdots + \varepsilon_{j+1})})^{-N} & &(0 \leqq j < i \leqq n-1),\\
\alpha_{i+1\, j+1} &= \sigma(\alpha_{ij}), \quad \phi_{i+1\, j+1} = \sigma(\phi_{ij}),\\
\beta_{i+1\, j+1} &= \sigma(\beta_{ij}), \quad \psi_{i+1\, j+1} = \sigma(\psi_{ij}).
\end{aligned}$$

Then we have

$$\Delta(\alpha_{ij}) = \alpha_{ij} \otimes 1 + \sum_{i<k<j} \phi_{ik} \alpha_{kj} \otimes \alpha_{ik} + \phi_{ij} \otimes \alpha_{ij} \quad (0 \leqq i < j \leqq n-1),$$

$$\Delta(\beta_{ij}) = \beta_{ij} \otimes 1 + \sum_{i>k>j} \psi_{ik} \beta_{kj} \otimes \beta_{ik} + \psi_{ij} \otimes \beta_{ij} \quad (0 \leqq j < i \leqq n-1).$$

Consider representations π and π'. Suppose that any central element is represented by a scalar in these representations. We use the comultiplication (2.1) to form the tensor products. In general, two representations $(\pi \otimes \pi') \circ \Delta$ and $(\pi' \otimes \pi) \circ \Delta$ are not equivalent. The reason is as follows. Take an element of the center of $\tilde{U}_q$, say α_{ii+1}. Then we have

$$\Delta(\alpha_{ii+1}) = \alpha_{ii+1} \otimes 1 + \phi_{ii+1} \otimes \alpha_{ii+1}.$$

Therefore, if two representations $(\pi \otimes \pi') \circ \Delta$ and $(\pi' \otimes \pi) \circ \Delta$ are equivalent, the following identity follows.

$$\frac{\pi(\alpha_{ii+1})}{\pi(1-\phi_{ii+1})} = \frac{\pi'(\alpha_{ii+1})}{\pi'(1-\phi_{ii+1})}.$$

Introduce the following element in the field of quotients of the center of $\tilde{U}_q$,

$$\Gamma_{ii+1} = \frac{\alpha_{ii+1}}{1-\phi_{ii+1}}.$$

Then Γ_{ii+1} is *invariant* in the sense that its images under π and π' coincide. In general, if we define Γ_{ij} and $\bar{\Gamma}_{ij}$ inductively by

$$\alpha_{ij} = \sum_{\substack{s \geqq 0 \\ i < k_s < \cdots < k_1 < k_0 = j}} \Gamma_{ik_s} \cdots \Gamma_{k_1 j} (1-\phi_{ik_s}) \quad (0 \leqq i < j \leqq n-1),$$

$$\beta_{ij} = \sum_{\substack{s \geqq 0 \\ i > k_s > \cdots > k_1 > k_0 = j}} \bar{\Gamma}_{ik_s} \cdots \bar{\Gamma}_{k_1 j} (1-\psi_{ik_s}) \quad (0 \leqq j < i \leqq n-1),$$

$$\Gamma_{i+1\, j+1} = \sigma(\Gamma_{ij}), \quad \bar{\Gamma}_{i+1\, j+1} = \sigma(\bar{\Gamma}_{ij}), \tag{2.2}$$

they are invariant: $\pi(\Gamma_{ij}) = \pi'(\Gamma_{ij})$, $\pi(\bar{\Gamma}_{ij}) = \pi'(\bar{\Gamma}_{ij})$.

Remark. From

$$\operatorname{tr}(\pi\otimes\pi')\circ\Delta(e_0\cdots e_{n-1})=\operatorname{tr}(\pi'\otimes\pi)\circ\Delta(e_0\cdots e_{n-1})$$

we have another necessary condition for the equivalence of two representations $(\pi\otimes\pi')\circ\Delta$ and $(\pi'\otimes\pi)\circ\Delta$;

$$\begin{aligned}&(\pi'(z_0\cdots z_{n-1}q^{\varepsilon_0+\cdots+\varepsilon_{n-1}})-\pi'(q^{-\varepsilon_0-\cdots-\varepsilon_{n-1}}))\operatorname{tr}\pi(e_0\cdots e_{n-1})\\&=(\pi(z_0\cdots z_{n-1}q^{\varepsilon_0+\cdots+\varepsilon_{n-1}})-\pi(q^{-\varepsilon_0-\cdots-\varepsilon_{n-1}}))\operatorname{tr}\pi'(e_0\cdots e_{n-1}).\end{aligned}$$

This condition is satisfied if the central element $z_0\cdots z_{n-1}q^{2(\varepsilon_0+\cdots+\varepsilon_{n-1})}$ is represented by 1 in both representations π and π'.

2.3 Minimal Representations. We call a representation of $\tilde{U}_q$ *cyclic* if e_i^N, f_i^N are represented by non-zero scalars. Recently cyclic representations of the quantized universal enveloping algebras have been investigated by several authors [11–13]. In this paper we consider the following family of N^{n-1} dimensional cyclic representations with the parameters $\xi=((x_i,a_i)_{0\leqq i<n},(c_i/c_{i+1})_{0\leqq i<n-1})\in(\mathbf{C}^\times)^{3n-1}$ [14, 7]. Consider $W=\bigotimes_{i=0}^{n-1}V_i$, where $V_i\cong\mathbf{C}^N$. Let Z_i, X_i be invertible linear operators on W such that

$$Z_i=1\otimes\cdots\otimes\overset{i\text{-th}}{Z}\otimes\cdots\otimes 1,$$

$$X_i=1\otimes\cdots\otimes\overset{i\text{-th}}{X}\otimes\cdots\otimes 1,$$

where $X,Z\in\operatorname{End}(\mathbf{C}^N)$, $ZX=qXZ$, $Z^N=1$, $X^N=1$. Set

$$W^{(0)}=\{w\in W\,|\,Z_0\cdots Z_{n-1}w=w\}.$$

Note that $\dim W^{(0)}=N^{n-1}$. We fix the canonical bases $\{u_i\}\subset\mathbf{C}^N$, $\{w_m\}\subset W^{(0)}$ as follows.

$$Zu_i=u_{i-1},\quad Xu_i=q^iu_i,$$

$$w_m=\sum_{k=0}^{N-1}u_{m_0+k}\otimes\cdots\otimes u_{m_{n-1}+k},\quad m=(m_0,\ldots,m_{n-1}).$$

Consider the following representations on $W^{(0)}$ with the parameter $\xi=((x_i,a_i)_{0\leqq i<n},(c_i/c_{i+1})_{0\leqq i<n-1})$.

$$\pi_\xi(e_i)=x_i\{a_iZ_i\}X_iX_{i+1}^{-1},$$

$$\pi_\xi(f_i)=x_i^{-1}\{a_{i+1}Z_{i+1}\}X_i^{-1}X_{i+1},$$

$$\pi_\xi(q^{\varepsilon_i})=a_iZ_i,\quad \pi_\xi(z_i)=\frac{c_i}{c_{i+1}a_ia_{i+1}}.$$

This representation is irreducible for generic ξ. This choice of $\pi_\xi(z_i)$ satisfies the condition in the Remark at the end of Subsect. 2.2. The expressions of the root vectors e_{ij} and f_{ij} in this representation are given by

$$\pi_\xi(e_{ij})=x_i\cdots x_{j-1}\{a_iZ_i\}(a_{i+1}Z_{i+1}\cdots a_{j-1}Z_{j-1})^{-1}X_iX_j^{-1}\quad(0\leqq i<j\leqq n-1),$$

$$\pi_\xi(f_{ij})=(x_{i-1}\cdots x_j)^{-1}\{a_iZ_i\}a_{i-1}Z_{i-1}\cdots a_{j+1}Z_{j+1}X_iX_j^{-1}\quad(0\leqq j<i\leqq n-1).$$

The weight space

$$W^{(0)}_{m_0,\dots,m_{n-1}}=\{w\in W^{(0)}|Z_i w=q^{m_i}w\ (0\leqq i\leqq n-1)\},$$

where $m_0+\cdots+m_{n-1}\equiv 0 \bmod N$, is one dimensional. For this reason we call this representation the *minimal* cyclic representation.

The quantum R matrix is an invertible linear operator on $W^{(0)}\otimes W^{(0)}$ which intertwines two representations $\pi_{\xi\tilde{\xi}}$ and $\pi_{\tilde{\xi}\xi}$:

$$R(\xi,\tilde{\xi})\pi_{\xi\tilde{\xi}}(g)=\pi_{\tilde{\xi}\xi}(g)R(\xi,\tilde{\xi}),\quad g\in\tilde{U}_q. \tag{2.3}$$

As was discussed previously, for arbitrary ξ and $\tilde{\xi}$ there is no such intertwiner. The invariants $\Gamma_{ij},\bar{\Gamma}_{ij}\ (0\leqq i\neq j\leqq n-1)$ should have the common value for π_ξ and $\pi_{\tilde{\xi}}$. For the minimal cyclic representation we have

$$\begin{aligned}
\pi_\xi(\alpha_{ij})&=(1-a_i^{2N})(x_ix_{i+1}\cdots x_{j-1})^N,\\
\pi_\xi(\phi_{ij})&=\left(\frac{c_ia_i}{c_ja_j}\right)^N,\\
\pi_\xi(\beta_{ij})&=(1-a_i^{-2N})(x_{i-1}x_{i-2}\cdots x_j)^{-N},\\
\pi_\xi(\psi_{ij})&=\left(\frac{c_ia_j}{c_ja_i}\right)^N.
\end{aligned}\tag{2.4}$$

Fix $(\Gamma^0_{ij},\bar{\Gamma}^0_{ij})_{0\leqq i\neq j\leqq n-1}\in(\mathbf{C}^\times)^{2n(n-1)}$. Consider a subvariety (maybe void)

$$\mathscr{S}=\{\xi\in(\mathbf{C}^\times)^{3n-1}|\pi_\xi(\Gamma_{ij})=\Gamma^0_{ij},\quad \pi_\xi(\bar{\Gamma}_{ij})=\bar{\Gamma}^0_{ij}\}.$$

If it is not void, we call it a spectral variety. If an intertwiner (2.3) exists, then ξ and $\tilde{\xi}$ should lie on the same spectral variety.

Set

$$K_i=\pi_\xi(\Gamma_{i\,i+1})\pi_\xi(\bar{\Gamma}_{i+1\,i}),\quad H_i=\frac{\pi_\xi(\Gamma_{i\,i+2})}{\pi_\xi(\Gamma_{i\,i+1})\pi_\xi(\Gamma_{i+1\,i+2})}.$$

These are rational functions of $A_i=a_i^N\ (0\leqq i\leqq n-1)$ and $C_i=(c_i/c_{i+1})^N\ (0\leqq i\leqq n-2)$.

Lemma 2.1. *For generic A_i, C_i, the Jacobian of the map*

$$(A_0,\dots,A_{n-1},C_0,\dots,C_{n-2})\mapsto(K_0,\dots,K_{n-2},H_0,\dots,H_{n-3})$$

has rank $2n-3$.

Proof. In the neighborhood of $C_i=0\ (0\leqq i\leqq n-2)$ we have

$$K_i=C_i(A_i-A_i^{-1})(A_{i+1}-A_{i+1}^{-1})+O(C^2),\quad H_i=\frac{A_{i+1}^2}{1-A_{i+1}^2}+O(C).$$

At $C_i=0$ the Jacobian matrix

$$J=\frac{\partial(K_0,\dots,K_{n-2},H_0,\dots,H_{n-3})}{\partial(C_0,\dots,C_{n-2},A_1,\dots,A_{n-2})}$$

is upper triangular with nonzero diagonal. This shows rank $J=2n-3$. □

Define the projections

$$p_1:(\mathbf{C}^\times)^{3n-1}\to(\mathbf{C}^\times)^{2n-1},\quad p_2:(\mathbf{C}^\times)^{2n-1}\to(\mathbf{C}^\times)^{2n-1},$$
$$p_1(\xi)=((x_i,a_i)_{0\leqq i<n},(c_i/c_{i+1})_{0\leqq i<n-1}),$$
$$p_2(((a_i)_{0\leqq i<n},(c_i/c_{i+1})_{0\leqq i<n-1}))=((A_i)_{0\leqq i<n},(C_i)_{0\leqq i<n-1}).$$

Then $p_1|_{\mathscr{S}}, p_2$ are finite maps, and $p_2\circ p_1(\mathscr{S})$ is contained in the variety $\{K_i=\text{const.}, H_i=\text{const.}\}$. Lemma 2.1 shows that the latter (more precisely every irreducible component of it passing through a point near $C_i=0$) has dimension $\leqq(2n-1)-(2n-3)=2$. In fact there is a two dimensional component of $p_1(\mathscr{S})$ given by an explicit parametrization. Fix $\tilde{\gamma}=(\kappa_i,\lambda_i,\mu_i)_{0\leqq i<n}\in(\mathbf{C}^\times)^n\times\mathbf{C}^{2n}$. Define a two dimensional subvariety $S_{\tilde{\gamma}}$ in $(\mathbf{C}^\times)^{3n}$ with coordinates $(x_i,a_i,c_i)_{0\leqq i<n}$ by the following substitutions:

$$\left(\frac{a_i}{c_i}\right)^N=\frac{s-\lambda_i}{s'-\lambda_i},\quad (a_ic_i)^N=\frac{s'-\mu_{i-1}}{s-\mu_{i-1}},\quad x_i^N=\kappa_i\frac{s'-\lambda_i}{s'-\mu_i}.$$

Then the invariants are constant on $S_{\tilde{\gamma}}$:

$$\pi_\xi(\Gamma_{ij})=\left(\prod_{i\leqq l\leqq j-1}\kappa_l\frac{\lambda_l-\mu_{j-1}}{\mu_{l-1}-\mu_{j-1}}\right)\frac{\mu_{i-1}-\lambda_i}{\lambda_i-\mu_{j-1}},$$
$$\pi_\xi(\bar{\Gamma}_{ij})=\left(\prod_{j\leqq l\leqq i-1}\kappa_l^{-1}\frac{\mu_l-\lambda_j}{\lambda_{l+1}-\lambda_j}\right)\frac{\lambda_i-\mu_{i-1}}{\mu_{i-1}-\lambda_j}.$$

We introduce new parameters u_i,v_i,u_i',v_i' $(0\leqq i<n)$ in such a way that

$$u_i^N=s-\lambda_i,\quad v_i^N=s-\mu_i,$$
$$u_i'^N=s'-\lambda_i,\quad v_i'^N=s'-\mu_i,$$
$$\frac{a_i}{c_i}=\frac{u_i}{u_i'},\qquad a_ic_i=\frac{v_{i-1}'}{v_{i-1}},\quad x_i=\kappa_i^{1/N}\frac{u_i'}{v_i'}.$$

Note that $r=(u_i,v_i)_{0\leqq i<n}, r'=(u_i',v_i')_{0\leqq i<n}$ lie on the curve

$$\mathscr{C}_\gamma=\{(u_i,v_i)_{0\leqq i<n}\in(\mathbf{C}^\times)^{2n}|u_i^N+\lambda_i=v_j^N+\mu_j\,(0\leqq i,j<n)\},$$

where $\gamma=(\lambda_i,\mu_i)_{0\leqq i<n}$. Thus $S_{\tilde{\gamma}}$ is a finite covering of the product of curves $\mathscr{C}_\gamma\times\mathscr{C}_\gamma$.

The κ_i,λ_i,μ_i $(0\leqq i<n)$ are the parameters of moduli and r,r' are the spectral parameters. If we fix the moduli parameters, the R matrix (if it ever exists) depends on two sets of spectral parameters: $R=R(r,r',\tilde{r},\tilde{r}')$. In Appendix A we show that for a generic choice of $\tilde{\gamma}$, (r,r') and $(\tilde{r},\tilde{r}')$ the R is unique up to a scalar multiple.

There is some redundancy in the moduli parameters. The change of κ_i makes no change in R (see 3.1). Furthermore, the simultaneous projective transformation of $s,s',\tilde{s},\tilde{s}'$ and λ_i,μ_i $(0\leqq i<n)$ also preserves R. Therefore the number of essential moduli parameters is $2n-3$.

3. Intertwiner for Minimal Cyclic Representations

In this section we shall give an explicit solution to the intertwining relation (2.3).

3.1. The Case $g=e_i$. First we solve (2.3) with $g=e_i$ for $i=0,\dots,n-1$. In terms

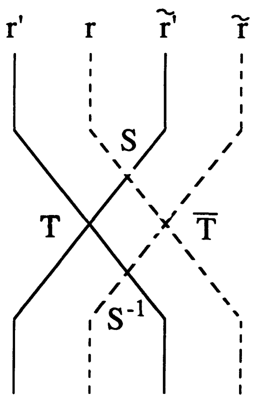

Fig. 1. R matrix factorized into four operators

of u_i, v_i and $\kappa_i, \pi_{\xi\tilde{\xi}}(e_i)$ is given by

$$\begin{aligned}\eta_i \pi_{\xi\tilde{\xi}}(e_i) &= v'_{i-1}\tilde{v}_{i-1}u_i\tilde{v}'_i Z_i X_i X_{i+1}^{-1} \otimes Z_i^{-1} - v_{i-1}\tilde{v}_{i-1}u'_i\tilde{v}'_i Z_i^{-1} X_i X_{i+1}^{-1} \otimes Z_i^{-1} \\ &\quad + v'_{i-1}\tilde{v}'_{i-1}v_i\tilde{u}_i Z_i \otimes Z_i X_i X_{i+1}^{-1} - v'_{i-1}\tilde{v}_{i-1}v_i\tilde{u}'_i Z_i \otimes Z_i^{-1} X_i X_{i+1}^{-1},\end{aligned}$$

where $\eta_i = (q-q^{-1})a_i\tilde{a}_i v_{i-1}\tilde{v}_{i-1}v'_i\tilde{v}'_i/(\kappa_i)^{1/N}$. Therefore $R(\xi,\tilde{\xi})$ can be chosen independently of κ_i. Set

$$\begin{aligned}&\delta_i^N = 1/(\lambda_i - \mu_i), \quad \Omega_i = (X_i X_{i+1}^{-1} \otimes X_i^{-1} X_{i+1})^{(1-N)/2}, \\ &C_i = (Z_i^2 \otimes 1)(\Omega_i \Omega_{i-1})^{-1} = (\Omega_i \Omega_{i-1})^{-1}(Z_i^2 \otimes 1), \\ &\bar{C}_i = (1 \otimes Z_i^{-2})\Omega_i\Omega_{i-1} = \Omega_i\Omega_{i-1}(1 \otimes Z_i^{-2}).\end{aligned}$$

Proposition 3.1. *Suppose S, T and $\bar{T}$ satisfy the following equations for all i:*

$$\begin{aligned}&S_{r\tilde{r}}(\Omega)C_i\delta_i(u_i\tilde{v}_i\Omega_i - \tilde{u}_i v_i \Omega_i^{-1}) \\ &\quad = \delta_{i-1}(u_{i-1}\tilde{v}_{i-1}\Omega_{i-1} - \tilde{u}_{i-1}v_{i-1}\Omega_{i-1}^{-1})C_i S_{r\tilde{r}}(\Omega),\end{aligned} \tag{3.1a}$$

$$\begin{aligned}&T_{r\tilde{r}}(C)(q^{-1}\delta_{i-1}\tilde{u}_{i-1}v_{i-1}C_i + \delta_i u_i \tilde{v}_i)\Omega_i^2 \\ &\quad = (q^{-1}\delta_{i-1}u_{i-1}\tilde{v}_{i-1}C_i + \delta_i\tilde{u}_i v_i)\Omega_i^2 T_{r\tilde{r}}(C),\end{aligned} \tag{3.1b}$$

$$\begin{aligned}&\bar{T}_{r\tilde{r}}(\bar{C})(q\delta_{i-1}u_{i-1}\tilde{v}_{i-1}\bar{C}_i + \delta_i\tilde{u}_i v_i)\Omega_i^{-2} \\ &\quad = (q\delta_{i-1}\tilde{u}_{i-1}v_{i-1}\bar{C}_i + \delta_i u_i\tilde{v}_i)\Omega_i^{-2}\bar{T}_{r\tilde{r}}(\bar{C}).\end{aligned} \tag{3.1c}$$

Then

$$R(\xi,\tilde{\xi}) = S_{\tilde{r}r'}(\Omega)^{-1}\bar{T}_{r\tilde{r}}(\bar{C})T_{r'\tilde{r}'}(C)S_{r\tilde{r}'}(\Omega)$$

satisfies (2.3) with $g = e_i$ for all i.

The proof is left to Appendix B.

The solutions to (3.1) are given as follows. First note that Ω_i, C_i act on the base elements $w_m \in W^{(0)}$ as

$$\begin{aligned}&\Omega_i w_{2k} \otimes w_{2l} = q^{(k-l)_{i\,i+1}} w_{2k} \otimes w_{2l}, \\ &C_i w_{2k} \otimes w_{2l} = \underset{i}{q}^{-(k-l)_{i-1\,i+1}} w_{2(k-\nu_i)} \otimes w_{2l}, \\ &\nu_i = (0,\ldots,1,\ldots,0), \quad k_{ij} = k_i - k_j.\end{aligned}$$

Set

$$S_{r\tilde{r}}(\Omega)w_{2k}\otimes w_{2l} = \sigma_{r\tilde{r}}(k-l)^{-1}w_{2k}\otimes w_{2l}.$$

Then (3.1a) is reduced to the recurrence relations

$$\frac{\sigma_{r\tilde{r}}(m+v_i)}{\sigma_{r\tilde{r}}(m)} = \frac{\delta_{i-1}(q^{m_{i-1i}}u_{i-1}\tilde{v}_{i-1} - q^{-m_{i-1i}}\tilde{u}_{i-1}v_{i-1})}{\delta_i(q^{m_{ii+1}+1}u_i\tilde{v}_i - q^{-m_{ii+1}-1}\tilde{u}_i v_i)}, \tag{3.2}$$

which determine the $\sigma_{r\tilde{r}}(m)$ uniquely up to an overall scalar multiple. Next let

$$T_{r\tilde{r}}(C) = \sum_m \sigma_{r\tilde{r}}(m)\prod_{i=0}^{n-1}(Z_i^{2m_i}\otimes 1)\prod_{i=0}^{n-1}(\Omega_i\Omega_{i-1})^{-m_i},$$

$$\bar{T}_{r\tilde{r}}(\bar{C}) = \sum_m \sigma_{r\tilde{r}}(m)\prod_{i=0}^{n-1}(1\otimes Z_i^{-2m_i})\prod_{i=0}^{n-1}(\Omega_i\Omega_{i-1})^{m_i}.$$

Substitute the above expression for T into (3.1b) and equate the coefficients of $\prod_{j=0}^{n-1}(Z_j^{2m_j}\otimes 1)\prod_{j=0}^{n-1}(\Omega_j\Omega_{j-1})^{-m_j}\times\Omega_i^2$; do likewise for $\bar{T}$ and (3.1c). Then we find that (3.1b, c) are reduced to the same relation (3.2).

3.2. Remaining Cases. The above $R(\xi,\tilde{\xi})$ clearly satisfies (2.3) with $g=q^{\varepsilon_i}, z_i$ $(i=0,\dots,n-1)$. Finally we consider (2.3) with $g=f_i$ $(i=0,\dots,n-1)$. Let

$$R'_j = R(\xi,\tilde{\xi})^{-1}(R(\xi,\tilde{\xi})\pi_{\xi\tilde{\xi}}(f_j) - \pi_{\tilde{\xi}\xi}(f_j)R(\xi,\tilde{\xi})).$$

We can easily show that this R'_j satisfies the following relations:

$$[\pi_{\xi\tilde{\xi}}(e_i), R'_j] = 0,$$
$$\pi_{\xi\tilde{\xi}}(q^{\varepsilon_i})R'_j = q^{\delta_{ij+1}-\delta_{ij}}R'_j\pi_{\xi\tilde{\xi}}(q^{\varepsilon_i}).$$

Then from Proposition A.2 it follows that R'_j vanishes.

Therefore the obtained $R(\xi,\tilde{\xi})$ is the intertwiner of the two representations $\pi_{\xi\tilde{\xi}}$ and $\pi_{\tilde{\xi}\xi}$. We shall show in Appendix A the following

Theorem 3.2. *The intertwiner R satisfies the Yang–Baxter equation,*

$$(R(\eta,\zeta)\otimes 1)(1\otimes R(\xi,\zeta))(R(\xi,\eta)\otimes 1) = (1\otimes R(\xi,\eta))(R(\xi,\zeta)\otimes 1)(1\otimes R(\eta,\zeta)). \tag{3.3}$$

In the base $\{w_m\}$ this R matrix has factorized matrix elements:

$$R(\xi,\tilde{\xi})w_{2j}\otimes w_{2k} = \sum_{l,m}\frac{\rho_{r'\tilde{r}'}(j,l)\rho_{\tilde{r}r'}(l,m)\rho_{r\tilde{r}}(m,k)}{\rho_{r\tilde{r}'}(j,k)}w_{2l}\otimes w_{2m},$$

where

$$\rho_{r\tilde{r}}(k,l) = q^{P(k,l)}\sigma_{r\tilde{r}}(k-l),\quad P(k,l) = \sum_{i=0}^{n-1}(k_i l_{i+1} - k_{i+1}l_i)$$

3.3. Symmetries. In this section we shall give certain symmetries which simplify some of the computations in the previous sections.

Define $\tilde{U}_q(\hat{\mathfrak{gl}}(n,\mathbf{Q}))$ to be an associative algebra over $\mathbf{Q}(q)$ $(q=e^{2\pi i/N})$ with the generators $e_i, f_i, q^{\pm\varepsilon_i}, z_i^{\pm 1}$ $(0\leq i<n)$ and the defining relations given in 2.1. Let θ be a $\mathbf{Q}$-linear involutive automorphism of $\tilde{U}_q(\hat{\mathfrak{gl}}(n,\mathbf{Q}))$ such that

$$\theta(e_i) = f_{n-i},\quad \theta(q^{\varepsilon_i}) = q^{-\varepsilon_{n-i+1}},\quad \theta(z_i) = z_{n-i}^{-1},\quad \theta(q) = q^{-1}.$$

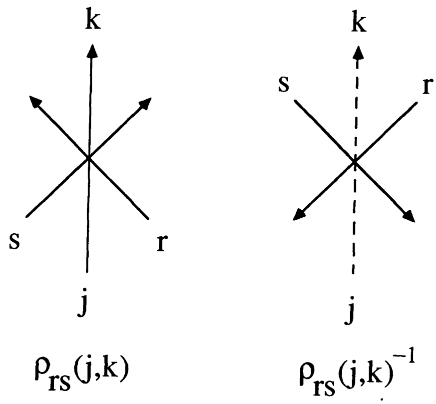

Fig. 2. Boltzmann weights of the generalized chiral Potts model

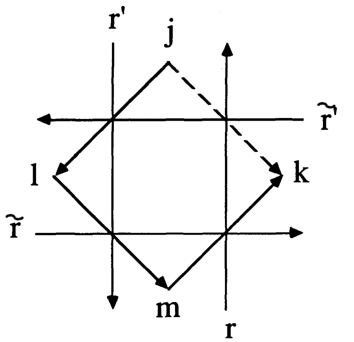

Fig. 3. Matrix element of the R matrix

Then we have

$$(\theta \otimes \theta) \circ \Delta = \Delta \circ \theta.$$

Recall the definition of $W^{(0)}$ in 2.3. Let us denote by $W^{(0)\prime}$ the $\mathbf{Q}(q)$ vector space defined similarly with $\mathbf{C}$ replaced by $\mathbf{Q}(q)$.

Denote by A the rational function field over $\mathbf{Q}(q)$ in the variables

$$(\lambda_i, \mu_i, \kappa_i^{1/N}, \delta_i, x_i, \tilde{x}_i, a_i, \tilde{a}_i, c_i, \tilde{c}_i, u_i, \tilde{u}_i, v_i, \tilde{v}_i, u_i', \tilde{u}_i', v_i', \tilde{v}_i')_{0 \leqq i < n}.$$

Let J be the ideal of A generated by the following relations.

$$\begin{aligned}
&\delta_i^N(\lambda_i - \mu_i) = 1, \\
&u_i^N + \lambda_i = v_j^N + \mu_j, \quad u_i'^N + \lambda_i = v_j'^N + \mu_j, \\
&\tilde{u}_i^N + \lambda_i = \tilde{v}_j^N + \mu_j, \quad \tilde{u}_i'^N + \lambda_i = \tilde{v}_j'^N + \mu_j, \\
&\quad a_i u_i' = c_i u_i, \quad a_i c_i v_{i-1} = v_{i-1}', \quad x_i v_i' = \kappa_i^{1/N} u_i', \\
&\quad \tilde{a}_i \tilde{u}_i' = \tilde{c}_i \tilde{u}_i, \quad \tilde{a}_i \tilde{c}_i \tilde{v}_{i-1} = \tilde{v}_{i-1}', \quad \tilde{x}_i \tilde{v}_i' = \kappa_i^{1/N} \tilde{u}_i'.
\end{aligned}$$

Set $B = A/J$. We denote by E the B subalgebra of $B \otimes_{\mathbf{Q}(q)} \mathrm{End}\, W^{(0)\prime}$ generated by $(Z_i, X_i)_{0 \leqq i < n}$. Define a $\mathbf{Q}$-linear involutive automorphism $*$ of E by

$$\begin{array}{llll} Z_i^* = Z_{n-i+1}^{-1}, & X_i^* = X_{n-i+1}, & q^* = q^{-1}, & \\ \lambda_i^* = \mu_{n-i}, & \kappa_i^{1/N*} = \kappa_{n-i}^{-1/N}, & \delta_i^* = -\delta_{n-i}, & \\ x_i^* = x_{n-i}^{-1}, & a_i^* = a_{n-i+1}^{-1}, & c_i^* = c_{n-i+1}, & \\ \tilde{x}_i^* = \tilde{x}_{n-i}^{-1}, & \tilde{a}_i^* = \tilde{a}_{n-i+1}^{-1}, & \tilde{c}_i^* = \tilde{c}_{n-i+1}, & \\ u_i^* = v_{n-i}, & u_i'^* = v_{n-i}', & \tilde{u}_i^* = \tilde{v}_{n-i}, & \tilde{u}_i'^* = \tilde{v}_{n-i}'. \end{array}$$

Note that π_ξ $(\xi = (x_i, a_i, c_i)_{0 \leqq i < n})$ and $\pi_{\tilde{\xi}}$ $(\tilde{\xi} = (\tilde{x}_i, \tilde{a}_i, \tilde{c}_i)_{0 \leqq i < n})$ are $\mathbf{Q}(q)$-linear homomorphisms

$$\pi_\xi, \pi_{\tilde{\xi}} \colon \tilde{U}_q(\hat{\mathfrak{gl}}(n, \mathbf{Q})) \to E.$$

The following symmetry is valid.

$$(\pi_\xi \circ \theta(g))^* = \pi_\xi(g), \quad (\pi_{\tilde{\xi}} \circ \theta(g))^* = \pi_{\tilde{\xi}}(g), \quad \text{for } g \in \tilde{U}_q(\hat{\mathfrak{gl}}(n, \mathbf{Q})).$$

Suppose that $R \in E \otimes E$ satisfies

$$R(\pi_\xi \otimes \pi_{\tilde{\xi}}) \circ \Delta(g) = (\pi_{\tilde{\xi}} \otimes \pi_\xi) \circ \Delta(g) R,$$

for some $g \in \tilde{U}_q(\hat{\mathfrak{gl}}(n, \mathbf{Q}))$. Then we have

$$\begin{aligned} (R(\pi_\xi \otimes \pi_{\tilde{\xi}}) \circ \Delta(g))^* &= R^*((\pi_\xi \circ \theta \otimes \pi_{\tilde{\xi}} \circ \theta) \circ \Delta \circ \theta(g))^* \\ &= R^*(\pi_\xi \otimes \pi_{\tilde{\xi}}) \circ \Delta \circ \theta(g) \\ &= (\pi_{\tilde{\xi}} \otimes \pi_\xi) \circ \Delta \circ \theta(g) R^*. \end{aligned}$$

It is easy to check that

$$R(\xi, \tilde{\xi})^* = R(\xi, \tilde{\xi}).$$

Therefore, the intertwining equation (2.3) for f_i follows from that for e_i.

Appendix A. Proof of the Yang–Baxter Equation

The goal of this appendix is to prove that the intertwiner of Sect. 3 satisfies YBE (3.3).

A.1. Trigonometric Limit. We begin with the discussion of the minimal cyclic representations in the trigonometric limit. This means the case where the moduli $\kappa_i, \lambda_i, \mu_i$, and hence $a_i = a$, $c_i = c$, $x_i = x$, are all independent of i. In fact c does not enter the representation. Denoting this representation by π_x we have

$$\begin{aligned} \pi_x(e_i) &= x\{aZ_i\} X_i X_{i+1}^{-1}, \\ \pi_x(f_i) &= x^{-1}\{aZ_{i+1}\} X_i^{-1} X_{i+1}, \\ \pi_x(q^{\varepsilon_i}) &= aZ_i, \quad \pi_x(z_i) = a^{-2}. \end{aligned} \tag{A.1}$$

The following Proposition will be of use later.

Proposition A.1. *Let $(W^{(0)}, \pi_x)$ be as above, and let (V', π') be a finite dimensional representation of $\tilde{U}_q$. Consider the linear equation for $F \in \mathrm{End}(W^{(0)} \otimes V')$:*

$$[(\pi_x \otimes \pi')\Delta(e_i), F] = 0 \quad \textit{for all} \quad i,$$

$$(\pi_x \otimes \pi')\Delta(q^{\varepsilon_i})F = q^{m_i}F(\pi_x \otimes \pi')\Delta(q^{\varepsilon_i}) \quad \textit{for all} \quad i. \tag{A.2}$$

Here the m_i *are given integers satisfying* $\sum_i m_i = 0$. *Then, for generic* x, F *has the form* $F = \prod_i Z_i^{k_i} \otimes F'$, $F' \in \mathrm{End}(V')$, *with* $k_i - k_{i+1} + m_i = 0$ *and*

$$[\pi'(e_i), F'] = 0 \quad \textit{for all } i,$$

$$\pi'(q^{\varepsilon_i})F' = q^{m_i}F\pi'(q^{\varepsilon_i}) \quad \textit{for all} \quad i. \tag{A.3}$$

Proof. Clearly F of the form (A.3) satisfies (A.2). Therefore it is sufficient to show that the only solutions are of this form at some special value $x = x_0$. We shall take $x_0 = \infty$.

First consider the case $n > 2$, and define

$$A_\pm = [x^{-1}(\pi_x \otimes \pi')\Delta(e_i), x^{-1}(\pi_x \otimes \pi')\Delta(e_{i+1})]_{q^{\pm 1}},$$
$$B = [x^{-1}(\pi_x \otimes \pi')\Delta(e_{i-1}), A_+ A_-^{-1}]/x^{-1}(1-q^2).$$

Here $[\alpha, \beta]_{q^{\pm 1}} = \alpha\beta - q^{\pm 1}\beta\alpha$. Clearly (A.2) imply the equations

$$[A_\pm, F] = [B, F] = 0. \tag{A.4}$$

Substituting (A.1) one finds after some calculation that

$$A_+ A_-^{-1} = (aZ_{i+1})^{-2} \otimes 1 + O(x^{-1}),$$
$$B = \varphi(Z) X_{i+1} X_i^{-2} X_{i-1} \otimes \pi'(e_i q^{\varepsilon_i - \varepsilon_{i-1}})$$
$$+ \delta_{n3} \psi(Z) X_{i+1}^{-1} X_i^{-1} X_{i-1}^2 \otimes \pi'(e_{i+1} q^{\varepsilon_{i+1} - \varepsilon_{i-1}}) + O(x^{-1}),$$

where $\varphi(Z)$ and $\psi(Z)$ are some invertible polynomials in the Z_i. Specializing the Eqs. (A.2), (A.4) to $x = \infty$ one obtains

$$[\{aZ_i\} X_i X_{i+1}^{-1} \otimes \pi'(q^{-\varepsilon_i}), F] = 0, \tag{A.5a}$$

$$Z_i \otimes \pi'(q^{\varepsilon_i})F = q^{m_i} F Z_i \otimes \pi'(q^{\varepsilon_i}), \tag{A.5b}$$

$$[Z_{i+1} \otimes 1, F] = 0, \tag{A.5c}$$

$$[\varphi(Z) X_{i+1} X_i^{-2} X_{i-1} \otimes \pi'(e_i q^{\varepsilon_i - \varepsilon_{i-1}}), F] = 0, \tag{A.5d}$$

$$[\psi(Z) X_{i+1}^{-1} X_i^{-1} X_{i-1}^2 \otimes \pi'(e_{i+1} q^{\varepsilon_{i+1} - \varepsilon_{i-1}}), F] = 0 \quad \text{if} \quad n = 3. \tag{A.5e}$$

Here we have used the fact that $X_{i+1} X_i^{-2} X_{i-1}$ and $X_{i+1}^{-1} X_i^{-1} X_{i-1}^2$ are linearly independent. Equations (A.5a) through (A.5c) imply that F has the form $\prod_i Z_i^{k_i} \otimes F'$ with $k_i - k_{i+1} + m_i = 0$ and $\pi'(q^{\varepsilon_i})F' = q^{m_i} F' \pi'(q^{\varepsilon_i})$. From (A.5d) and (A.5e) one then concludes $[\pi'(e_i), F'] = 0$.

Next consider the case $n = 2$. Set

$$D_i = x^{-1}(\pi_x \otimes \pi')\Delta(e_i), \quad E = [D_0 D_1, D_1 D_0].$$

Noting that $q^{\varepsilon_0 + \varepsilon_1}$ is central, one has

$$D_i D_{i+1}(1 \otimes \pi'(q^{\varepsilon_0 + \varepsilon_1})) = \{aZ_i\}\{aqZ_{i+1}\} \otimes 1 + O(x^{-1}),$$

$$E(1 \otimes \pi'(q^{\varepsilon_0+\varepsilon_1}))/(x^{-1}a^{-1}(q+q^{-1}))$$
$$= \{aZ_0\}Z_0X_0X_1^{-1} \otimes \pi'(e_1q^{-\varepsilon_0})$$
$$- \{aZ_1\}Z_1X_1X_0^{-1} \otimes \pi'(e_0q^{-\varepsilon_1}) + O(x^{-1}).$$

Using D_i, E in place of $A_\pm, B$ and arguing similarly as above, one arrives at the same conclusion. □

A.2. Indecomposability of Tensor Products. Let (V, π) be a finite dimensional representation of $\tilde{U}_q$. It is said to be indecomposable if, for $F \in \mathrm{End}(V)$, $[F, \pi(g)] = 0$ for any $g \in \tilde{U}_q$ implies $F \in \mathbf{C}\mathrm{id}$.

For $p \geqq 1$ we set $\mathscr{S}_p = \bigcup_{\tilde{\gamma}} \overbrace{S_{\tilde{\gamma}} \times \cdots \times S_{\tilde{\gamma}}}^{p\text{-times}}$, where $S_{\tilde{\gamma}}$ denotes the variety defined in Sect. 2 and $\tilde{\gamma} = (\kappa_i, \lambda_i, \mu_i)_{0 \leqq i < n}$. Since $S_{\tilde{\gamma}}$ is irreducible if $\tilde{\gamma}$ is generic, $\mathscr{S}_p$ is also irreducible. Let $\Delta^{(p)} = (\Delta \otimes \cdots \otimes 1) \circ \Delta^{(p-1)}$, $\Delta^{(1)} = \Delta$. The following shows that the tensor products of the π_ξ are generically indecomposable.

Proposition A.2. *For generic* $\tilde{\gamma}$ *and* $(\xi_i)_{1 \leqq i \leqq p} \in \overbrace{S_{\tilde{\gamma}} \times \cdots \times S_{\tilde{\gamma}}}^{p\text{-times}}$, *the only solution of the equation*

$$[(\pi_{\xi_1} \otimes \cdots \otimes \pi_{\xi_p}) \circ \Delta^{(p-1)}(e_i), F] = 0,$$
$$(\pi_{\xi_1} \otimes \cdots \otimes \pi_{\xi_p}) \circ \Delta^{(p-1)}(q^{\varepsilon_i})F$$
$$= q^{m_i} F(\pi_{\xi_1} \otimes \cdots \otimes \pi_{\xi_p}) \circ \Delta^{(p-1)}(q^{\varepsilon_i})$$

is

$$F = \text{scalar} \quad \text{if} \quad m = 0,$$
$$= 0 \qquad \text{otherwise.}$$

Proof. It is enough to show the assertion in the case where π_{ξ_i} are all trigonometric. Thanks to Lemma A.1 the proof is reduced to the case $p = 1$ by induction. But the case $p = 1$ can be shown easily. □

Remark. By decomposing F into joint eigenvectors of the $\mathrm{Ad}(q^{\varepsilon_i})$, it is clear from the proof that the indecomposability holds with respect to the subalgebra of $\tilde{U}_q$ generated by $e_i (0 \leqq i < n)$.

Corollary A.3. *For generic* $\tilde{\gamma}$ *and* $(\xi, \tilde{\xi})$ *the intertwiner (2.3) is unique up to scalar multiple.*

A.3. Yang–Baxter Equation. From the above results YBE follows by a general argument [15]. Let Q_L (respectively Q_R) denote the left-(respectively right-) hand side of (3.3). Since the $R(\xi, \eta)$ are intertwiners, $F = Q_L^{-1}Q_R$ commutes with $\pi_{\xi\eta\zeta} = (\pi_\xi \otimes \pi_\eta \otimes \pi_\zeta) \circ \Delta^{(2)}$:

$$[F, \pi_{\xi\eta\zeta}(g)] = 0 \quad \text{for any} \quad g \in \tilde{U}_q.$$

Proposition A.2 then shows for generic (ξ, η, ζ) that F is a scalar, namely

$$\rho Q_L = Q_R$$

with some scalar ρ. Comparing the determinant one finds that ρ is a root of unity, and hence is independent of the parameters (ξ, η, ζ). From the formula (3.1) it can be checked that $R(\xi, \xi)$ is a scalar. Hence setting $\xi = \eta = \zeta$ one obtains $\rho = 1$. This proves YBE.

Appendix B. Proof of Proposition 3.1.

Let

$$\begin{aligned}
\Omega_i &= (X_i X_{i+1}^{-1} \otimes X_i^{-1} X_{i+1})^{(1-N)/2},\\
C_i &= (Z_i^2 \otimes 1)(\Omega_i \Omega_{i-1})^{-1} = (\Omega_i \Omega_{i-1})^{-1}(Z_i^2 \otimes 1),\\
\bar{C}_i &= (1 \otimes Z_i^{-2})\Omega_i \Omega_{i-1} = \Omega_i \Omega_{i-1}(1 \otimes Z_i^{-2}),\\
Y_i &= X_i X_{i+1}^{-1} \otimes 1, \quad \bar{Y}_i = 1 \otimes X_i X_{i+1}^{-1}, \quad K_i = Z_i \otimes Z_i.
\end{aligned}$$

Then $C_i, \bar{C}_i, \Omega_i, Y_i, \bar{Y}_i$ and K_i satisfy the following relations:

$$\begin{aligned}
&[Y_i, Y_j] = [\bar{Y}_i, \bar{Y}_j] = [Y_i, \bar{Y}_j] = [C_i, \bar{C}_j] = [C_i, \bar{Y}_j] = [\bar{C}_i, Y_j] = 0,\\
&[\Omega_i, \Omega_j] = [K_i, \Omega_j] = [K_i, C_j] = [K_i, \bar{C}_j] = 0,\\
&\quad \Omega_i^2 = Y_i(\bar{Y}_i)^{-1}, \quad C_i \Omega_{i-1}^2 = K_i^2 \bar{C}_i \Omega_i^{-2},\\
&C_i C_{i+1} = q^{-2} C_{i+1} C_i, \quad [C_i, C_j] = 0 && (j \not\equiv i \pm 1 \bmod n),\\
&\bar{C}_i \bar{C}_{i+1} = q^2 \bar{C}_{i+1} \bar{C}_i, \quad [\bar{C}_i, \bar{C}_j] = 0 && (j \not\equiv i \pm 1 \bmod n),\\
&\quad C_i \Omega_i = q \Omega_i C_i, \quad C_i \Omega_{i-1} = q^{-1} \Omega_{i-1} C_i, \quad [C_i, \Omega_j] = 0 && (j \not\equiv i, i-1 \bmod n),\\
&\quad \bar{C}_i \Omega_i = q \Omega_i \bar{C}_i, \quad \bar{C}_i \Omega_{i-1} = q^{-1} \Omega_{i-1} \bar{C}_i, \quad [\bar{C}_i, \Omega_j] = 0 && (j \not\equiv i, i-1 \bmod n).
\end{aligned}$$

In terms of these operators, we have

$$\begin{aligned}
\eta_i \pi_{\xi\bar{\xi}}(e_i) &= v'_{i-1}\tilde{v}_{i-1} K_i^{-1} C_i (u_i \tilde{v}'_i \Omega_i - \tilde{u}'_i v_i \Omega_i^{-1}) \Omega_{i-1} Y_i\\
&\quad + v'_{i-1}\tilde{v}'_{i-1} \tilde{u}_i v_i K_i \bar{Y}_i - v_{i-1}\tilde{v}_{i-1} u'_i \tilde{v}'_i K_i^{-1} Y_i.
\end{aligned}$$

Using (3.1a), we have

$$\begin{aligned}
\delta_i S_{r\bar{r}'}(\Omega) \eta_i \pi_{\xi\bar{\xi}}(e_i) S_{r\bar{r}'}(\Omega)^{-1} &= v'_{i-1}\tilde{v}'_{i-1} K_i (q\delta_{i-1} u_{i-1} \tilde{v}_{i-1} \bar{C}_i + \delta_i \tilde{u}_i v_i) \bar{Y}_i\\
&\quad - v_{i-1}\tilde{v}_{i-1} K_i^{-1}(q^{-1}\delta_{i-1} \tilde{u}'_{i-1} v'_{i-1} C_i + \delta_i u'_i \tilde{v}'_i) Y_i.
\end{aligned}$$

Using (3.1b, c), we have

$$\begin{aligned}
&\delta_i T_{r'\bar{r}'}(C) \bar{T}_{r\bar{r}}(\bar{C}) S_{r\bar{r}'}(\Omega) \eta_i \pi_{\xi\bar{\xi}}(e_i) S_{r\bar{r}'}(\Omega)^{-1} \bar{T}_{r\bar{r}}(\bar{C})^{-1} T_{r'\bar{r}'}(C)^{-1}\\
&= \tilde{v}'_{i-1} v_{i-1} K_i^{-1} \delta_{i-1} (\tilde{u}_{i-1} v'_{i-1} \Omega_{i-1} - u'_{i-1} \tilde{v}_{i-1} \Omega_{i-1}^{-1}) C_i \Omega_{i-1} Y_i\\
&\quad + \delta_i (v'_{i-1} \tilde{v}'_{i-1} u_i \tilde{v}_i K_i \bar{Y}_i - v_{i-1} \tilde{v}_{i-1} \tilde{u}'_i v'_i K_i^{-1} Y_i).
\end{aligned}$$

Finally using (3.1a) again, we get (2.3) for $g = e_i$.

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