Commun. Math. Phys. 136, 231–250 (1991)

Communications in
Mathematical
Physics
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Strong Asymptotic Abelianness for Entropic K-Systems

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Received July 29, 1989; in revised form July 20, 1990

Abstract. We prove that in entropic K-systems of type II_1 the automorphism is strongly asymptotically abelian.

1. Introduction

In classical statistical mechanics, approach to equilibrium is a well-known consequence of the mixing properties of the physical systems. These properties manifest themselves in the clustering behaviour of correlation functions. The strongest version of clustering occurs in those systems called K-systems [1]. Since nature is governed by noncommutative laws we are led to try to formulate concepts analogous to that of abelian ergodic theory also in a quantum frame.

It is clear that finite quantum systems cannot exhibit any mixing, because of the spectral character of the Hamiltonian governing their evolution. But in the thermodynamic limit of an infinite system we can have representations of the algebra of operators such that the generator of the dynamics does not belong to the algebra. Then we are in a good position to find an approach to equilibrium without invoking a coupling to an external reservoir that should drive the system to equilibrium.

Asymptotically abelian systems with ergodic or even mixing properties, as defined in the literature, are characterized by certain typical decays of the two point correlation functions which resemble the weak or strong mixing typical of the abelian case. Recently the extension of the Kolmogorov–Sinai entropy to the quantum realm [2] has been used to concretely formulate a noncommutative version of K-systems [3] as systems with "complete memory loss." This definition reduces to the classical one for abelian systems and there provides a certain "strong" clustering property.

Also for quantum systems we shall prove that entropic K-systems that are of type II_1 guarantee strong asymptotic abelianness and therefore also clustering.

★ Supported in part by "Fonds zur Förderung der wissenschaftlichen Forschung in Österreich," Project Nr. P7101-PHY

(Notice that there exists another definition of algebraic K-systems that is equivalent to that of entropic K-systems in the classical situation, but only leads to weak asymptotic abelianness [4].) Generalization to the general case of type III is under investigation.

2. Entropic K-Systems [3]

We describe a quantum dynamical system in terms of a C^* – or von Neumann algebra $\mathscr{M}$ and a faithful state ω over $\mathscr{M}$ invariant under the automorphism σ.

Definition (2.1) [2]. The following functional is called the n-subalgebra entropy:

$$H_\omega(\mathscr{A}_1,\ldots,\mathscr{A}_n) = \sup_{\omega=\sum_{(i)}\omega_{(i)}} H_{\{\omega_{(i)}\}}(\mathscr{A}_1,\ldots,\mathscr{A}_n),$$

$$H_{\{\omega_{(i)}\}}(\mathscr{A}_1,\ldots,\mathscr{A}_n) = \sum_{(i)}\eta(\omega_{(i)}(\mathbf{1})) - \sum_{k=1}^{n}\sum_{i_k}\eta(\omega^k_{i_k}(\mathbf{1})) + \sum_{k=1}^{n}\sum_{i_k}\omega^k_{i_k}(\mathbf{1})S(\omega\,|\,\hat{\omega}^k_{i_k})_{|\mathscr{A}_k},$$

where $\mathscr{A}_i$, $i=1,\ldots,n$, are finite dimensional subalgebras of $\mathscr{M}$, (i) is a multiindex $i_1, i_2, \ldots, i_n$, $\eta(x) = -x\log x$, and with $x_{(i)}\in\mathscr{M}$ such that

$$\omega_{(i)}(\cdot) = \omega(\sigma_\omega^{-i/2}(x_{(i)})(\cdot)), \quad 0 < x_{(i)} < \mathbf{1}, \quad \sum_{(i)} x_{(i)} = \mathbf{1},$$

σ_ω being the modular automorphism of ω,

$$\hat{\omega}^k_{i_k}(\cdot) = \omega\left(\sigma_\omega^{-i/2}\left(\sum_{(i),i_k\,\text{fixed}} x_{(i)}\right)\cdot\right)\Big/\omega^k_{i_k}(\mathbf{1}),$$

and, finally,

$$S(\omega\,|\,\hat{\omega}^k_{i_k})_{|\mathscr{A}_k} = \mathrm{Tr}\,\{\hat{\omega}^k_{i_k|\mathscr{A}_k}[\ln\hat{\omega}^k_{i_k|\mathscr{A}_k} - \ln\omega_{|\mathscr{A}_k}]\}$$

is the relative entropy of the states $\hat{\omega}^k_{i_k}$ and ω restricted to the finite dimensional subalgebra $\mathscr{A}_k$ and thus represented by suitable density matrices.

Properties (2.2)[2, 3].
1. $H_\omega(\mathscr{A}_1,\ldots,\mathscr{A}_n) \leqq H_\omega(\mathscr{B}_1,\ldots,\mathscr{B}_n)$ if $\mathscr{A}_i \subseteqq \mathscr{B}_i \forall i$,
$H_\omega(\mathscr{A}_1,\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_n) = H_\omega(\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_n)$
2. $H_\omega(\mathscr{A}_1,\ldots,\mathscr{A}_p,\mathscr{A}_{p+1},\ldots,\mathscr{A}_n) \leqq H_\omega(\mathscr{A}_1,\ldots,\mathscr{A}_p) + H_\omega(\mathscr{A}_{p+1},\ldots,\mathscr{A}_n)$.
3. $H_\omega(\tau(\mathscr{A}_1),\ldots,\tau(\mathscr{A}_n)) = H_\omega(\mathscr{A}_1,\ldots,\mathscr{A}_n)$ for τ an automorphism of $\mathscr{M}$ with $\omega\circ\tau = \omega$.
4. $H_\omega(\mathscr{A}_1,\mathscr{A}_2) - H_\omega(\mathscr{A}_2) \leqq H_\omega(\mathscr{A}_1|\mathscr{A}_2) \leqq H_\omega(\mathscr{A}_1)$, where

$$H_\omega(\mathscr{A}_1|\mathscr{A}_2) = \sup_{\omega=\sum_{(i)}\omega_i}\left\{\sum_i \omega_i(\mathbf{1})[S(\omega\,|\,\hat{\omega}_i)_{|\mathscr{A}_1} - S(\omega\,|\,\hat{\omega}_i)_{|\mathscr{A}_2}]\right\}.$$

5. $H_\omega(\mathscr{A}) > 0$ for ω faithful unless $\mathscr{A} = \mathbf{C1}$.

Remarks (2.3).
1. The supremum in the definition (2.1) is taken over all possible decompositions of the state ω and the n-subalgebra entropy represents the maximal information about the subalgebras achievable by refining the state over the whole algebra.

2. If $\mathscr{A}$ is abelian and ω is the tracial state then the 1-subalgebra entropy reduces to

$$S(\omega_{|\mathscr{A}}) = -\sum_{i=1}^{N_{\mathscr{A}}} \omega(P_i)\ln\omega(P_i),$$

$\{P_i\}_{i=1,\ldots,N_{\mathscr{A}}}$ being a set of minimal projectors.

Proof. $\sigma_\omega^{i/2} = \mathbf{1}$ due to the tracial property of ω. One has

$$H_\omega(\mathscr{A}) \geqq \sum_i \omega(x_i) S(\omega\,|\,\hat{\omega}(x_i))_{|\mathscr{A}} = S(\omega_{|\mathscr{A}}) - \sum_i \omega(x_i) S(\hat{\omega}(x_i)_{|\mathscr{A}}).$$

The supremum is attained for $\hat{\omega}(x_i\cdot)_{|\mathscr{A}}$ a pure state, thus for $x_i = P_i$. In this case we know the optimal decomposition exactly.

3. Observe that continuous decompositions of the reference state ω have been excluded from the above definition of the n-subalgebra entropy.

We make, indeed, the following important observation [2, p. 704, Lemma VI.1]: Given $H_\omega(A_1,\ldots,A_n)$, for any $\varepsilon > 0$ there exists a finite decomposition $\{x_{(i)}\}_{(i)\in I}$, card $I = \#(\varepsilon, d) < +\infty$, $d = \max_{1\leqq i\leqq n} \dim A_i$ such that

$$H_{\{x_{(i)}\}}(A_1,\ldots,A_n) \geqq H_\omega(A_1,\ldots,A_n) - \varepsilon_1(\varepsilon),$$

where $H_{\{x_{(i)}\}}(A_1,\ldots,A_n)$ is the value of the n-subalgebra functional attained at the given decomposition $\{x_{(i)}\}$. Moreover:

$$\varepsilon_1(\varepsilon) = 3\varepsilon(\tfrac{1}{2} + \log(1 + d\varepsilon^{-1})).$$

4. Remark 2.3.3 is the central point in the proof that the n-subalgebra entropy $H_\omega(A_1,\ldots,A_n)$ is continuous with respect to the strong topology induced on the von Neumann algebra M by the GNS construction relative to the reference state ω. Any finite dimensional C^*-algebra A_i can be embedded as a subalgebra within a full matrix algebra $M_d(\mathbf{C})$ with a conditional expectation $E_i: M_d(\mathbf{C}) \to A_i$. If $\{B_i\}_{i=1,\ldots,n}$ is another choice of finite dimensional subalgebras of M with $\bar{E}_i: M_d(\mathbf{C}) \to B_i$ the corresponding conditional expectations and $\forall x \in M_d(\mathbf{C})$,

$$\sup_{x\in M_d(\mathbf{C}),\|x\|\leqq 1} [\omega((E_i(x^+) - \bar{E}_i(x^+))(E_i(x) - \bar{E}_i(x))]^{1/2} < \varepsilon \quad \text{for some} \quad \varepsilon > 0,$$

then

$$|H_\omega(A_1,\ldots,A_n) - H_\omega(B_1,\ldots,B_n)| \leqq n\alpha(\varepsilon)$$

with $\alpha(\varepsilon) \xrightarrow{\varepsilon\to 0^+} 0$ [see 2, pp. 704–6].

5. If $A \subset M$ is a finite dimensional subalgebra invariant under the modular automorphisms of ω, $\forall t: \sigma_\omega^t(A) \subset A$, then

$$H_\omega(\mathscr{A}) = S(\omega_{|\mathscr{A}}) = S(\omega_{|\mathscr{B}})$$

for any maximal abelian subalgebra $\mathscr{B}$ in the centralizer of $\mathscr{A}$ [2, p. 711, VIII.6], e.g. when ω is tracial the result holds for any subalgebra $\mathscr{A}$ and, if $\mathscr{A}$ is abelian and generated by the set $\{P_i\}_{i=1,\ldots,N<+\infty}$ of minimal projectors, then:

$$H_\omega(\mathscr{A}) = -\sum_{i=1}^{N} \omega(P_i)\log\omega(P_i).$$

Definition (*2.4*) [2]. The quantity

$$h_\omega(\mathscr{A},\sigma)=\lim_n \frac{1}{n} H_\omega(\mathscr{A},\sigma(\mathscr{A}),\ldots,\sigma^{n-1}(\mathscr{A}))$$

is called the dynamical entropy of the automorphism σ with respect to the finite dimensional subalgebra $\mathscr{A}\subset\mathscr{M}$.

Let us now consider the quantum dynamical system described by the triple $(\mathscr{M},\omega,\sigma)$, where $\mathscr{M}$ is a von Neumann algebra with a state ω invariant under the dynamics represented by the automorphism σ of $\mathscr{M}$.

Definition (*2.5*) [*3*]. $(\mathscr{M},\sigma,\omega)$ is called an entropic K-system if

$$\lim_n h_\omega(\mathscr{A},\sigma^n)=H_\omega(\mathscr{A}) \quad \forall \text{ finite dimensional subalgebras } \mathscr{A} \text{ of } \mathscr{M}.$$

Remarks (*2.6*).
1. Due to Property 2.2.1 and using the subadditivity 2.2.2 of the n-subalgebra entropy it can be proved that for entropic K-systems the dynamical entropy is strictly positive, $h_\omega(\sigma,\mathscr{A})>0$, for any subalgebra $\mathscr{A}\neq \mathbf{C1}$. For abelian algebras this complete positivity is equivalent to the requirement in the above Definition, whereas it is not proved to be the same in the quantum case [3].
2. Intuitively speaking $h_\omega(\sigma,\mathscr{A})$ is the long run averaged information about $\mathscr{A}$ obtained by letting $\mathscr{A}$ evolve through unit time intervals. Definition (2.5) would indicate that in the limit of increasing time steps the information about $\mathscr{A}$ is not retained any more and the system develops a complete memory loss [3].

3 Strong Clustering for a Type II_1 Entropic K-System

3.1.
Let $(\mathscr{M},\omega,\sigma)$ be a quantum dynamical system as specified above.

Definition (*3.1.1*). Set $b_n=\sigma^n(b)\ \forall b\in\mathscr{M}$; then $(\mathscr{M},\omega,\sigma)$ is

1. weakly clustering if $\lim_n \omega(ab_nc)=\omega(ac)\omega(b)$,
2. strongly clustering if $\lim_n \omega(ab_ncd_ne)=\omega(ace)\omega(bd)$

$\forall\ a,b,c,d,e\in\mathscr{M}$ [5, 6, 7].

Remarks (*3.1.2*).
1. Besides the obvious observation that strong clustering implies weak clustering we easily deduce that the former implies

$$\lim_n \omega(a^\dagger[b^\dagger,c_n^\dagger][c_n,b]a)=0 \quad \forall\, a,b,c\in\mathscr{M},$$

and the latter

$$\lim_n \omega(a[c_n,b]d)=0 \quad \forall\, a,b,c,d\in\mathscr{M}.$$

Accordingly we have that in the GNS representation based on ω, strong (respectively weak) clustering implies strong (respectively weak) asymptotic abelianness.

2. If the state ω is assumed to be faithful, e.g. equilibrium states over simple C^*-algebras, then, by means of the KMS conditions we can replace:

$$\mathbf{1} \to \mathbf{1}': \lim_n \omega(ab_n) = \omega(a)\omega(b)\, \forall\, a, b \in \mathcal{M},$$

$$\mathbf{2} \to \mathbf{2}': \lim_n \omega(ab_n cd_n) = \omega(ac)\omega(bd)\, \forall\, a, b, c, d \in \mathcal{M}.$$

3. Physically one is much more used to weak-clustering which implies mixing, that is every observable tends weakly in time to its expectation with respect to the invariant state ω:

$$\pi_\omega(\sigma^n(a)) \xrightarrow{n} \omega(a)\mathbf{1} \quad \forall\, a \in \mathcal{M}.$$

This in turn guarantees the ergodicity of the dynamical triple $(\mathcal{M}, \omega, \sigma)$: that is ω is extremal invariant. Strong clustering is more related to its consequence, namely strong asymptotic abelianness which holds true for the free evolution of the Boson algebra and the even Fermi algebra and which we would like to extend to a more general class of evolutions.

We now specialize the triple $(\mathcal{M}, \omega, \sigma)$ to be a type II_1 factor with ω its unique trace, invariant under any automorphism σ of $\mathcal{M}$ and state the main result of this work:

Theorem (3.1.3). *If the triple $(\mathcal{M}, \omega, \sigma)$ is an entropic K-system then it is strongly clustering and hence*:

1. *weakly clustering*:

$$\lim_n \langle \Omega_\omega | \pi_\omega(a)\pi_\omega(b_n)\pi_\omega(c) | \Omega_\omega \rangle = \langle \Omega_\omega | \pi_\omega(a)\pi_\omega(c) | \Omega_\omega \rangle \langle \Omega_\omega | \pi_\omega(b) | \Omega_\omega \rangle.$$

2. *strongly asymptotic abelian*:

$$\lim_n \| [\pi_\omega(c), \pi_\omega(b_n)]\pi_\omega(a) | \Omega_\omega \rangle \| = 0 \quad \forall\, a, b, c \in \mathcal{M},$$

where $(|\Omega_\omega\rangle, \pi_\omega(\cdot), \mathcal{H}_\omega)$ is the GNS triple associated to ω.

The proof of the theorem requires several steps.

Lemma (3.1.4). *If $(\mathcal{M}, \omega, \sigma)$ is weakly clustering and strongly asymptotically abelian then it is strongly clustering.*

Proof.

$$\omega(ab_n cd_n) = \omega(ac(bd)_n) + \omega(a[b_n, c]d_n)$$

and

$$|\omega(a[b_n, c]d_n)|^2 \leqq \omega(d_n^\dagger d_n a a^\dagger)\omega([b_n, c]^\dagger [b_n, c]).$$

Lemma (3.1.5). *If for general projectors we have weak clustering, and if for commuting projectors P and Q it happens that*

$$\lim_n \omega(PQ_n PQ_n) = \omega(P)\omega(Q),$$

then strong clustering holds.

Proof. Since we are dealing with a von Neumann algebra, according to Lemma (3.1.4), strong clustering holds if $\lim_n \omega([R,S_n][R,S_n])=0$ for R,S general projectors in $\mathscr{M}$. Since

$$\omega([R,S_n][R,S_n])=2[\omega(RS_nRS_n)-\omega(RS_n)],$$

the lemma is proved if $\lim_n \omega(RS_nRS_n)=\omega(R)\omega(S)$. Now, we can write

$$\begin{aligned}\omega(RS_nRS_n)&=\omega((R+S)S_nRS_n)-\omega(SS_nRS_n)\\&=\omega((R+S)(R+S)_nRS_n)-\omega((R+S)R_nRS_n)-\omega(S_nSS_nR)\\&=\omega(S_n(R+S)_n(R+S)R)-\omega((R+S)R(RS)_n)-\omega(S_nSR)\\&\quad+\omega([(R+S),(R+S)_n]RS_n)-\omega((R+S)[R_n,R]S_n)-\omega(S_n[S,S_n]R).\end{aligned}$$

Hence

$$\begin{aligned}\lim_n \omega(RS_nRS_n)&=\omega(S(R+S))\omega(R(R+S))-\omega((R+S)R)\omega(RS)-\omega(S)\omega(SR)\\&\quad+\lim_n\{\omega([R+S),(R+S)_n]RS_n)-\omega((R+S)[R_n,R]S_n)-\omega(S_n[S,S_n]R)\}.\end{aligned}$$

Since the constant part gives $\omega(R)\omega(S)$ it is sufficient to prove that

$$\lim_n \omega([A,A_n][A,A_n])=0$$

for A selfadjoint. This again holds if $\lim_{n\to\infty}\omega([P,Q_n][P,Q_n])=0$ for all P,Q with $[P,Q]=0$, or, following the above strategy, if $\lim_n \omega(PQ_nPQ_n)=\omega(P)\omega(Q)$ for all P,Q with $[P,Q]=0$.

We wish now to elucidate how $H_\omega(\mathscr{A}_1,\dots,\mathscr{A}_n)$ behaves asymptotically if $(\mathscr{M},\omega,\sigma)$ displays a complete memory loss.

We prove the following result which is actually valid for all entropic K-systems without restrictions on the type of $\mathscr{M}$ and the invariant state ω.

Lemma (3.1.6). *If $(\mathscr{M},\omega,\sigma)$ is an entropic K-system, the n-subalgebra entropy $H_\omega(\mathscr{A},\sigma^n(\mathscr{A}),\dots,\sigma^{n(k-1)}(\mathscr{A}))$ becomes additive in the limit of large steps for any finite dimensional subalgebra $\mathscr{A}\subset\mathscr{M}$:*

$$\lim_n H_\omega(\mathscr{A},\sigma^n(\mathscr{A}),\dots,\sigma^{n(k-1)}(\mathscr{A}))=kH_\omega(\mathscr{A})\quad\forall\, k\geqq 2.$$

Proof. Let $\mathscr{A}_n$ indicate $\sigma^n(\mathscr{A})$. From Definitions (2.4, 5) we get:

$$\lim_n h_\omega(\mathscr{A},\sigma^n)=\lim_n\lim_p\frac{1}{p}H_\omega(\mathscr{A},\mathscr{A}_n,\dots,\mathscr{A}_{n(p-1)})=H_\omega(\mathscr{A}). \tag{1}$$

For p large enough the left-hand side of (1) can be controlled, by virtue of properties 2.2.2, 3, in the following way: Fix $k\in\mathbf{N}$ and write $p=mk+q$, $1\leqq q<k$; then:

$$\begin{aligned}&\frac{1}{p}H_\omega(\mathscr{A},\mathscr{A}_n,\dots,\mathscr{A}_{n(p-1)})\\&\overset{\text{subadditivity}}{\leqq}\frac{1}{mk}[H_\omega(\mathscr{A},\dots,\mathscr{A}_{n(k-1)})+H_\omega(\mathscr{A}_{nk},\dots,\mathscr{A}_{n(2k-1)})+\cdots\\&\quad+H_\omega(\mathscr{A}_{nmk},\dots,\mathscr{A}_{nmk+n(q-1)})]\end{aligned}$$

$$\overset{\text{covariance}}{=} \frac{1}{k} H_\omega(\mathscr{A}, \ldots, \mathscr{A}_{n(k-1)}) + \frac{1}{mk} H_\omega(\mathscr{A}, \ldots, \mathscr{A}_{n(q-1)}).$$

Since $H_\omega(\mathscr{A}, \ldots, \mathscr{A}_{n(k-1)}) \leqq k H_\omega(\mathscr{A})$ it eventually follows:

$$H_\omega(\mathscr{A}) = \lim_n h_\omega(\mathscr{A}, \sigma^n) \leqq \frac{1}{k} \lim_n H_\omega(\mathscr{A}, \ldots, \mathscr{A}_{n(k-1)}) \leqq H_\omega(\mathscr{A}).$$

Remark (*3.1.7*). When $k = 2$ the above result reads: $\lim_n H_\omega(\mathscr{A}, \mathscr{A}_n) = 2H_\omega(\mathscr{A})$. Let us write down $H_\omega(\mathscr{A}, \mathscr{A}_n)$ explicitly:

$$H_\omega(\mathscr{A}, \mathscr{A}_n) = \sup_{\omega = \sum_{ij} \omega_{ij}} \left\{ \sum_{ij} \eta(\omega_{ij}(1)) - \sum_i \eta(\omega_i^1(1)) - \sum_j \eta(\omega_j^2(1)) \right.$$
$$\left. + \sum_i \omega_i^1(1) S(\omega, \hat{\omega}_i^1)_{|\mathscr{A}} + \sum_j \omega_j^2(1) S(\omega, \hat{\omega}_j^2)_{|\mathscr{A}_n} \right\}. \tag{2}$$

Note that:

1.
$$(2) = \sum_{ij} \omega_{ij}(1) \log \frac{\omega_i^1(1)\omega_j^2(1)}{\omega_{ij}(1)} \leqq 0,$$

since $\omega_i^1(1) = \sum_j \omega_{ij}(1), \omega_j^2(1) = \sum_i \omega_{ij}(1)$, and $x(\ln x - \ln y) \geqq x - y, x, y \geqq 0$. The equality holds if and only if $\omega_i^1(1)\omega_j^2(1) = \omega_{ij}(1)$.

2.
$$\sum_i \omega_i^1(1) S(\omega, \hat{\omega}_i^1)_{|\mathscr{A}} \leqq H_\omega(\mathscr{A}), \quad \sum_j \omega_j^2(1) S(\omega, \hat{\omega}_j^2)_{|\mathscr{A}_n} \leqq H_\omega(\mathscr{A}), \tag{3}$$

owing to Definition (2.1) with $n = 1$ and to the covariance of the n-subalgebra entropy.

If we denote by $\{y_{ij}(n, \varepsilon)\}$ the set of positive operators which, according to Definition (2.1), give the optimal decomposition for $H_\omega(\mathscr{A}, \mathscr{A}_n)$ within ε and additivity holds in the limit of larger and larger steps then, from 1. above, we expect

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} [\omega(y_{ij}(n, \varepsilon)) - \omega(y_i^1(n, \varepsilon))\omega(y_j^2(n, \varepsilon))] = 0.$$

In order to give a precise meaning to this limit we should be able to control the decompositions when n is large.

In the next paragraph we use inequalities (3) and Remark (2.3.5) to achieve this control.

3.2.

We recall that ω is the unique trace on the type II_1 factor $\mathscr{M}$. If $\{P_i\}_{i=1,\ldots,N}$, $\sum_{i=1}^N P_i = \mathbf{1}$, $P_i P_j = \delta_{ij} P_j$ are the minimal projectors which generate the abelian finite dimensional subalgebra $\mathscr{A}$, then $H_\omega(\mathscr{A}) = -\sum_{i=1}^N \omega(P_i) \log \omega(P_i)$ (see Remark 2.3.5).

We prove the following

Theorem (3.2.1). *Let $\mathscr{A}$ be a finite dimensional abelian subalgebra of $\mathscr{M}$, generated* *by the set* $\{P_i\}_{i=1,\ldots,N}$ *of minimal projectors. Correspondingly* $\mathcal{A}_n=\sigma^n(\mathcal{A})$ *is generated by the set* $\{P_i(n)\}_{i=1,\ldots,N}$, *after setting* $P_i(n)\equiv\sigma^n(P_i)$.

Since ω *is the trace on* $\mathcal{M}$, *we consider the canonical conditional expectations* $E:\mathcal{M}\to\mathcal{A}$ *and* $E_n:\mathcal{M}\to\mathcal{A}_n$. *For any* $\varepsilon>0$ *there exists an integer* M *and some functions* $\delta_1(\varepsilon),\delta_2(\varepsilon)$ *with* $\lim_{\varepsilon\to 0}\delta_i(\varepsilon)=0$ *such that for any* $n\geqq M$ *we can construct a decomposition of* ω *given by the finite set of positive operators* $\{y_{ij}(n)\}_{i,j=1,\ldots,N+1}$, $0<y_{ij}(n)<1$, *for which:*

1.
$$H_{\{y_{ij}(n)\}}(\mathcal{A},\mathcal{A}_n)\geqq H_\omega(\mathcal{A},\mathcal{A}_n)-\delta_1(\varepsilon),\quad \delta_1(\varepsilon)\xrightarrow{\varepsilon\to 0^+}0^+.$$

Moreover:

2.
$$\|E(y_k^1(n))-P_k\|\leqq\delta_2(\varepsilon)\quad k=1,\ldots,N,$$
3.
$$\|E(y_k^2(n))-P_k(n)\|\leqq\delta_2(\varepsilon)\quad k=1,\ldots,N,$$

$$y_k^1(n)=\sum_{j=1}^{N+1}y_{kj}(n);\quad y_k^2(n)=\sum_{i=1}^{N+1}y_{ik}(n);\quad \delta_2(\varepsilon)\xrightarrow{\varepsilon\to 0^+}0^+;\quad k=1,\ldots,N.$$

Proof. Part I: *Construction of* $\{y_{ij}(n)\}$.

$$H_\omega(\mathcal{A},\mathcal{A}_n)\leqq 2H_\omega(\mathcal{A})\quad \text{owing to subadditivity and covariance;}$$
$$\lim_n H_\omega(\mathcal{A},\mathcal{A}_n)=2H_\omega(\mathcal{A})\quad \text{by assumption (complete memory loss).}$$

Then

$$\forall\varepsilon_1>0\,\exists M\in\mathbf{N}:\forall n\geqq M\quad 2H_\omega(\mathcal{A})-H_\omega(\mathcal{A},\mathcal{A}_n)\leqq\varepsilon_1.$$

Remark (2.3.3) tells us that $\forall\varepsilon_2>0\,\exists$ a decomposition of ω given by $\{x_{\alpha\beta}(n)\}_{(\alpha,\beta)\in I\times J}$ with finite cardinality $\#(\varepsilon_2,N)$, such that:

$$H_\omega(\mathcal{A},\mathcal{A}_n)-H_{\{x_{\alpha\beta}(n)\}}(\mathcal{A},\mathcal{A}_n)\leqq 2\eta_2(\varepsilon_2)=6\varepsilon_2\left(\frac{1}{2}+\log\left(1+\frac{N}{\varepsilon_2}\right)\right).\tag{4}$$

Recalling that in our case

$$H_\omega(\mathcal{A})=H_\omega(\mathcal{A}_n)=S(\omega_{|\mathcal{A}})$$

and

$$\hat\omega(x_\alpha^1(n)\cdot)=\omega(x_\alpha)^{-1}\sum_{\beta\in J}\omega(x_{\alpha\beta}(n)\cdot),\quad \hat\omega(x_\beta^2(n)\cdot)=\omega(x_\beta)^{-1}\sum_{\alpha\in I}\omega(x_{\alpha\beta}(n)\cdot)$$

are normalized states for which

$$\omega=\sum_{\alpha\in I}\omega(x_\alpha^1(n))\hat\omega(x_\alpha^1(n)\cdot)=\sum_{\beta\in J}\omega(x_\beta^2(n))\hat\omega(x_\beta^2(n)\cdot),$$

we get:

$$\sum_{\alpha\in I}S(\omega,\hat\omega(x_\alpha^1(n)\cdot))_{|\mathcal{A}}\omega(x_\alpha^1(n))=S(\omega_{|\mathcal{A}})-\sum_{\alpha\in I}\omega(x_\alpha^1(n))S(\hat\omega(x_\alpha^1(n)\cdot)_{|\mathcal{A}}),$$
$$\sum_{\beta\in J}S(\omega,\hat\omega(x_\beta^2(n)\cdot))_{|\mathcal{A}_n}\omega(x_\beta^2(n))=S(\omega_{|\mathcal{A}_n})-\sum_{\beta\in J}\omega(x_\beta^2(n))S(\hat\omega(x_\beta^2(n)\cdot)_{|\mathcal{A}_n}).$$

Together with the explicit expression of $H_\omega(\mathcal{A},\mathcal{A}_n)$ given in Remark (3.1.7), we eventually obtain:

$$2H_{\omega}(\mathscr{A}) - H_{\{x_{\alpha\beta}(n)\}}(\mathscr{A}, \mathscr{A}_n) = \sum_{\alpha,\beta \in I \times J} \omega(x_{\alpha\beta}(n)) \log \frac{\omega(x_{\alpha\beta}(n))}{\omega(x_{\alpha}^1(n))\omega(x_{\beta}^2(n))}$$
$$+ \sum_{\alpha \in I} \omega(x_{\alpha}^1(n)) S(\hat{\omega}(x_{\alpha}^1(n)\cdot)_{|\mathscr{A}})$$
$$+ \sum_{\beta \in J} \omega(x_{\beta}^2(n)) S(\hat{\omega}(x_{\beta}^2(n)\cdot)_{|\mathscr{A}_n}) \leqq \varepsilon_1 + \eta_2(\varepsilon_2) =: \varepsilon \quad (5)$$

for $n \geqq M$.

All the summands in (5) are positive, and therefore

$$\sum_{\alpha \in I} \omega(x_{\alpha}^1(n)) S(\hat{\omega}(x_{\alpha}^1(n)\cdot)_{|\mathscr{A}}) \leqq \varepsilon, \quad (6)$$

$$\sum_{\beta \in J} \omega(x_{\beta}^2(n)) S(\hat{\omega}(x_{\beta}^2(n)\cdot)_{|\mathscr{A}_n}) \leqq \varepsilon. \quad (7)$$

We now consider $I = I^1 \cup I^2$ and $J = J^1 \cup J^2$ with:

$$\begin{cases} \alpha \in I^1 \Rightarrow S(\hat{\omega}(x_{\alpha}^1(n)\cdot)_{|\mathscr{A}}) \leqq \sqrt{\varepsilon} \\ \beta \in J^1 \Rightarrow S(\hat{\omega}(x_{\beta}^2(n)\cdot)_{|\mathscr{A}_n}) \leqq \sqrt{\varepsilon} \end{cases} \quad (8)$$

and I^2, J^2 the complements of I^1 and J^1 in I and J, respectively. From (6), (7) and (8) it follows that

$$\begin{cases} \sum_{\alpha \in I^2} \omega(x_{\alpha}^1(n)) \leqq \sqrt{\varepsilon} \\ \sum_{\beta \in J^2} \omega(x_{\beta}^2(n)) = \sqrt{\varepsilon}. \end{cases} \quad (9)$$

Since $\mathscr{A}$ and $\mathscr{A}_n$ are abelian:

$$S(\hat{\omega}(x_{\alpha}^1(n)\cdot)_{|\mathscr{A}}) = -\sum_{i=1}^{N} \hat{\omega}(x_{\alpha}^1(n)P_i) \log \hat{\omega}(x_{\alpha}^1(n)P_i),$$

$$S(\hat{\omega}(x_{\beta}^2(n)\cdot)_{|\mathscr{A}_n}) = -\sum_{i=1}^{N} \hat{\omega}(x_{\beta}^2(n)P_i(n)) \log \hat{\omega}(x_{\beta}^2(n)P_i(n)).$$

Therefore:

$$\begin{cases} -\hat{\omega}(x_{\alpha}^1(n)P_i) \log \hat{\omega}(x_{\alpha}^1(n)P_i) \leqq \sqrt{\varepsilon} \\ -\hat{\omega}(x_{\beta}^2(n)P_i(n)) \log \hat{\omega}(x_{\beta}^2(n)P_i(n)) \leqq \sqrt{\varepsilon} \end{cases} \quad (10)$$

$\forall i = 1, \ldots, N$; $\alpha \in I^1$; $\beta \in J^1$.

The function $-x \ln x$ is understood to be zero at $x = 0$; this continuity makes us consider the right neighbourhood of $x = 0$ and the left neighbourhood of $x = 1$ determined by $\eta_1(\varepsilon)$ according to:

$$-\eta_1(\varepsilon) \log \eta_1(\varepsilon) = \sqrt{\varepsilon},$$

and the corresponding two possibilities for $\alpha \in I^1$ and $\beta \in J^1$:

$$\hat{\omega}(x_{\alpha}^1(n)P_i) \begin{cases} \leqq \eta_1(\varepsilon) \\ \geqq 1 - \eta_1(\varepsilon) \end{cases} \quad i = 1, \ldots, N,$$

$$\hat{\omega}(x_{\beta}^2(n)P_i) \begin{cases} \leqq \eta_1(\varepsilon) \\ \geqq 1 - \eta_1(\varepsilon) \end{cases} \quad i = 1, \ldots, N. \quad (11)$$

Since $\hat{\omega}(x_\alpha^1(n)) = \hat{\omega}(x_\beta^2(n)) = 1\ \forall\, \alpha, \beta$ and $\sum_{i=1}^N P_i = \sum_{i=1}^N P_i(n) = \mathbf{1}$, it turns out that, for ε_2 small enough, there can be only one $P_i(P_j(n))$ for a given $\alpha \in I^1$ ($\beta \in J^1$) such that the second possibilities in (11) show up.

We are thus led, by making ε small enough, to the following partitions of the sets I and J and the corresponding coarse graining of $\{x_{\alpha\beta}(n)\}_{(\alpha,\beta)\in I\times J}$:

$$\left.\begin{array}{ll}\alpha \in I_i^1: & \hat{\omega}(x_\alpha^1(n)P_i) \geqq 1 - \eta_1(\varepsilon), \quad \hat{\omega}(x_\alpha^1(n)P_r) \leqq \eta_1(\varepsilon) \quad \text{if} \quad r \neq i \\ & \sum_{\alpha\in I^2} \omega(x_\alpha^1(n)) \leqq \sqrt{\varepsilon}, \\ \beta \in J_j^2: & \hat{\omega}(x_\beta^2(n)P_j(n)) \geqq 1 - \eta(\varepsilon), \quad \hat{\omega}(x_\beta^2(n)P_s(n)) \leqq \eta(\varepsilon) \quad \text{if} \quad s \neq j \\ & \sum_{\beta\in J^2} \omega(x_\beta^2(n)) \leqq \sqrt{\varepsilon}\end{array}\right\} \tag{12}$$

$$\left.\begin{array}{ll} y_{ij}(n) = \sum_{\alpha\in I_i}\sum_{\beta\in J_j} x_{\alpha\beta}(n), & i, j = 1, \ldots, N \\ y_{N+1,j}(n) = \sum_{\alpha\in I^2}\sum_{\beta\in J_j} x_{\alpha\beta}(n), & j = 1, \ldots, N \\ y_{i,N+1}(n) = \sum_{\beta\in J^2}\sum_{\alpha\in I_i} x_{\alpha\beta}(n), & i = 1, \ldots, N \\ y_{N+1,N+1}(n) = \sum_{\beta\in J^2}\sum_{\alpha\in I^2} x_{\alpha\beta}(n). & \end{array}\right\} \tag{13}$$

If then follows:

$$y_i^1(n) = \sum_{j=1}^{N+1} y_{ij}(n) = \begin{cases} \sum_{\alpha\in I_i} x_\alpha^1(n) & i \leqq N \\ \sum_{\alpha\in I^2} x_\alpha^1(n) & i = N+1 \end{cases},$$

$$y_j^2(n) = \sum_{i=1}^{N+1} y_{ij}(n) = \begin{cases} \sum_{\beta\in J_j} x_\beta^2(n) & j \leqq N \\ \sum_{\beta\in J^2} x_\beta^2(n) & j = N+1 \end{cases},$$

$$\sum_{i=1}^{N+1}\sum_{j=1}^{N+1} y_{ij}(n) = \mathbf{1}, \quad \sum_{i=1}^{N+1} y_i^1(n) = \mathbf{1}, \quad \sum_{j=1}^{N+1} y_j^2(n) = \mathbf{1}.$$

Part II: Evaluation of the Estimates. We have to control the quantity:

$$|H_{\{y_{ij}(n)\}}(\mathscr{A}, \mathscr{A}_n) - H_{\{x_{\alpha\beta}(n)\}}(\mathscr{A}, \mathscr{A}_n)| \tag{14}$$

which is smaller than (see Remark (3.1.7))

$$\left|\sum_{i,j=1}^{N+1} \omega(y_{ij}(n)) \log \frac{\omega(y_i^1(n))\omega(y_j^2(n))}{\omega(y_{ij}(n))} - \sum_{\alpha,\beta\in I\times J} \omega(x_{\alpha\beta}(n)) \log \frac{\omega(x_\alpha^1(n))\omega(x_\beta^2(n))}{\omega(x_{\alpha\beta}(n))}\right|, \tag{15}$$

$$+$$

$$\left|\sum_{i=1}^{N+1} \omega(y_i^1(n)) S(\omega, \hat{\omega}(y_i^1(n)\cdot))_{|\mathscr{A}} - \sum_{\alpha\in I} \omega(x_\alpha^1(n)) S(\omega, \hat{\omega}(x_\alpha^1(n)\cdot))_{|\mathscr{A}}\right|, \tag{16}$$

$$+$$

$$\left|\sum_{j=1}^{N+1} \omega(y_j^2(n)) S(\omega, \hat{\omega}(y_j^2(n)\cdot))_{|\mathscr{A}_n} - \sum_{\beta\in J} \omega(x_\beta^2(n)) S(\omega, \hat{\omega}(x_\beta^2(n)\cdot))_{|\mathscr{A}_n}\right|. \tag{17}$$

The properties of the function $x(\log x - \log y)$ imply that the second term in (15) always decreases under a coarse graining (see [2], Lemma VI.1).

Together with (5) we get $(15) \leqq 2\varepsilon$.

We can rewrite

$$(16) = \left| \sum_{i=1}^{N+1} \omega(y_i^1(n)) S(\hat{\omega}(y_i^1(n)\cdot))_{|\mathscr{A}} - \sum_{\alpha\in I} \omega(x_\alpha^1(n)) S(\hat{\omega}(x_\alpha^1(n)\cdot))_{|\mathscr{A}} \right|$$

$$\leqq \left| \omega(y_{N+1}^1(n)) S(\hat{\omega}(y_{N+1}^1(n)\cdot))_{|\mathscr{A}} - \sum_{\alpha\in I^2} \omega(x_\alpha^1(n)) S(\hat{\omega}(x_\alpha^1(n)\cdot))_{|\mathscr{A}} \right|$$

$$+ \left| \sum_{k=1}^{N} \left\{ \omega(y_k^1(n)) S(\hat{\omega}(y_k^1(n)\cdot))_{|\mathscr{A}} - \sum_{\alpha\in I_k^1} \omega(x_\alpha^1(n)) S(\hat{\omega}(x_\alpha^1(n))\cdot))_{|\mathscr{A}} \right\} \right|.$$

Since $\dim A = N$ and $S(\omega_{|\mathscr{A}}) \leqq \ln N$ for any state ω, along with (12) we get:

$$(16) \leqq 2\sqrt{\varepsilon} \ln N + \sum_{k=1}^{N} \sum_{\alpha\in I_k^1} \omega(x_\alpha^1(n)) | S(\hat{\omega}(y_k^1(n)\cdot))_{|\mathscr{A}} - S(\hat{\omega}(x_\alpha^1(n)\cdot))_{|\mathscr{A}} |.$$

(Recall: $y_k^1(n) = \sum_{\alpha\in I_k^1} x_\alpha^1(n)$)

Let us concentrate on $\hat{\omega}(x_\alpha^1(n)\cdot)_{|\mathscr{A}}$ for $\alpha\in I_k^1$,

$$\hat{\omega}(x_\alpha^1(n)P_i) - \hat{\omega}(P_k P_i) = \omega\left[\left(\frac{x_\alpha^1(n)}{\omega(x_\alpha^1(n))} - \frac{P_k}{\omega(P_k)} \right) P_i \right]$$

$$= \begin{cases} \hat{\omega}(x_\alpha^1(n)P_k) - 1 \geqq \eta_1(\varepsilon)\cdots & i = k \\ \hat{\omega}(x_\alpha^1(n)P_i) \leqq \eta_1(\varepsilon)\cdots & i \neq k \end{cases}$$

due to (12).

If $E: \mathscr{M} \to \mathscr{A}$ is the conditional expectation into $\mathscr{A}$ which respects the trace ω, then:

$$|\hat{\omega}(x_\alpha^1(n)P_i) - \hat{\omega}(P_k P_i)| = \left| \omega\left[\left(\frac{E[x_\alpha^1(n)]}{\omega(x_\alpha^1(n))} - \frac{P_k}{\omega(P_k)} \right) P_i \right] \right| \leqq \eta_1(\varepsilon) \quad \forall i = 1, \dots, N. \quad (18)$$

Set

$$E(x_\alpha^1(n)) = \sum_{r=1}^{N} \mu_\alpha^{1,r}(n) P_r. \quad (19)$$

As $0 < x_\alpha^1(n) < \mathbf{1} \, \forall \alpha\in I$, $0 < \mu_\alpha^{1,r}(n) < 1$ and, for $\alpha\in I_k^1$, we get:

$$\mu_\alpha^{1,k}(n)\omega(P_k) = \omega[E(x_\alpha^1(n))P_k] = \hat{\omega}(x_\alpha^1(n)P_k)\omega(x_\alpha^1(n)) \geqq \omega(x_\alpha^1(n))(1 - \eta_1(\varepsilon))$$

$$\mu_\alpha^{1,r}(n)\omega(P_r) = \omega[E(x_\alpha^1(n))P_r] = \hat{\omega}(x_\alpha^1(n)P_r)\omega(x_\alpha^1(n)) \leqq \omega(x_\alpha^1(n))\eta_1(\varepsilon) \qquad r \neq k. \quad (20)$$

Therefore: $\forall \alpha\in I_k^1$, $\forall i = 1, \dots, N$,

$$|\omega[(E(x_\alpha^1(n)) - \mu_\alpha^{1,k}(n)P_k)P_i]| \leqq \eta_1(\varepsilon)\omega(x_\alpha^1(n)),$$

and, after summing over $\alpha\in I_k^1$ and setting $c_k^1(n) = \sum_{\alpha\in I_k^1} \mu_\alpha^{1,k}(n)$,

$$|\omega[(E(y_k^1(n)) - c_k^1(n)P_k)P_i]| \leqq \eta_1(\varepsilon). \quad (21)$$

Let ω_* indicate $\min_{1 \leqq i \leqq N} \omega(P_i)$, then

$$\| E(y_k^1(n)) - c_k^1(n) P_k \| \leqq \eta_1(\varepsilon) \omega_*^{-1}. \tag{22}$$

From (19) it follows:

$$\sum_{\alpha \in I_k^1} \mu_\alpha^{1,k}(n) \omega(P_k) = \sum_{\alpha \in I_k^1} \omega(E(x_\alpha^1(n)) P_k) = \omega(P_k y_k^1(n) P_k) \leqq \omega(P_k),$$

and thus $0 \leqq c_k^1(n) \leqq 1$.

Furthermore, using (22) in the last inequality:

$$\begin{aligned} 1 &= \sum_{k=1}^{N+1} \omega(y_k^1(n)) \\ &= \omega(y_{N+1}^1(n)) + \sum_{k=1}^{N} [\omega(y_k^1(n)) - \omega(P_k) c_k^1(n)] + \sum_{k=1}^{N} c_k^1(n) \omega(P_k) \\ &\leqq \sqrt{\varepsilon} + N \omega_*^{-1} \eta_1(\varepsilon) + \sum_{k=1}^{N} c_k^1(n) \omega(P_k). \end{aligned}$$

As $\sum_{k=1}^{N} \omega(P_k) = 1$ and $c_k^1(n) \leqq 1$ for any $k = 1, \dots, N$ it turns out:

$$1 - c_k^1(n) \leqq \omega_*^{-1}(\sqrt{\varepsilon} + N \omega_*^{-1} \eta_1(\varepsilon)). \tag{23}$$

Putting together the preceding estimates (22), (23) we eventually obtain:

$$\begin{aligned} \| E(y_k^1(n)) - P_k \| &\leqq \| E(y_k^1(n)) - c_k^1(n) P_k \| + (1 - c_k^1(n)) \\ &\leqq \omega_*^{-1} [\sqrt{\varepsilon} + \eta_1(\varepsilon)(1 + N \omega_*^{-1})] =: \delta_2(\varepsilon), \quad k = 1, \dots, N. \end{aligned} \tag{24}$$

The second estimate in the statement of the theorem is therefore obtained and the proof of the third one follows by exactly the same argument with the replacement of $\mathscr{A}$, $E: \mathscr{M} \to \mathscr{A}$, $\{P_i\}_{i=1,\dots,N}$ by $\mathscr{A}_n$, $E_n: \mathscr{M} \to \mathscr{A}_n$, $\{P_i(n)\}_{i=1,\dots,N}$, respectively.

We go back now to the estimate:

$$(16) \leqq 2\sqrt{\varepsilon} \ln N + \sum_{k=1}^{N} \sum_{\alpha \in I_k^1} \omega(x_\alpha^1(n)) | S(\hat{\omega}(y_k^1(n) \cdot))_{|\mathscr{A}} - S(\hat{\omega}(x_\alpha^1(n) \cdot))_{|\mathscr{A}} |,$$

and observe that

$$\begin{aligned} \| \hat{\omega}(y_k^1(n) \cdot)_{|\mathscr{A}} - \hat{\omega}(x_\alpha^1(n) \cdot)_{|\mathscr{A}} \| &= \sup_{a \in A, \|a\| \leqq 1} | \hat{\omega}(y_k^1(n) a) - \hat{\omega}(x_\alpha^1(n) a) | \\ &\leqq \sup_{a \in A, \|a\| \leqq 1} \left| \hat{\omega}(y_k^1(n) a) - \frac{\omega(P_k a)}{\omega(P_k)} \right| + \sup_{a \in A, \|a\| \leqq 1} \left| \frac{\omega(P_k a)}{\omega(P_k)} - \hat{\omega}(x_\alpha^1(n) a) \right|. \end{aligned} \tag{25}$$

From (18) we have:

$$\sup_{a \in A, \|a\| \leqq 1} | \hat{\omega}(x_\alpha^1(n) a) - \hat{\omega}(P_k a) | \leqq N \eta_1(\varepsilon). \tag{26}$$

On the other hand:

$$\begin{aligned} | \hat{\omega}(y_k^1(n) a) - \hat{\omega}(P_k a) | &\leqq \left| \frac{\omega(y_k^1(n) a)}{\omega(y_k^1(n))} - \frac{\omega(y_k^1(n) a)}{\omega(P_k)} \right| + \left| \frac{\omega(E(y_k^1(n)) a) - \omega(P_k a)}{\omega(P_k)} \right| \\ &\leqq \omega(y_k^1(n)) \left| \frac{1}{\omega(y_k^1(n))} - \frac{1}{\omega(P_k)} \right| + \frac{\delta_2(\varepsilon)}{\omega(P_k)} \leqq 2 \delta_2(\varepsilon) \omega_*^{-1}, \end{aligned} \tag{27}$$

where (24) has been used and the fact that $0 \leqq y_k^1(n) \leqq 1$. (26) and (27) yield:

$$\| \hat{\omega}(y_k^1(n)\cdot)_{|\mathscr{A}} - \hat{\omega}(x_\alpha^1(n)\cdot)_{|\mathscr{A}} \| \leqq N\eta_1(\varepsilon) + 2\delta_2(\varepsilon)\omega_*^{-1} =: \gamma_1(\varepsilon). \tag{28}$$

Inserting (28) in the estimate

$$|S(\psi_{|\mathscr{A}}) - S(\phi_{|\mathscr{A}}| \leqq 3 \| \psi_{|\mathscr{A}} - \phi_{|\mathscr{A}} \| \left[\frac{1}{2} + \log\left(1 + \frac{\dim A}{\| \psi_{|\mathscr{A}} - \phi_{|\mathscr{A}} \|} \right) \right]$$

given in [2, Lemma IV.1], we get

$$(16) \leqq 2\sqrt{\varepsilon} \ln N + 3\gamma_1(\varepsilon) \left[\frac{1}{2} + \log\left(1 + \frac{N}{\gamma_1(\varepsilon)} \right) \right] =: \gamma_2(\varepsilon).$$

The estimate $(17) \leqq \gamma_2(\varepsilon)$ can be performed following the very same argument, so that:

$$|H_{\{y_{ij}(n)\}}(\mathscr{A}, \mathscr{A}_n) - H_{\{x_{\alpha\beta}(n)\}}(\mathscr{A}, \mathscr{A}_n)| \leqq 2\varepsilon + 2\gamma_2(\varepsilon).$$

Remembering the inequality (4) at the beginning of Part I together with the definition $\varepsilon = \varepsilon_1 + \eta_2(\varepsilon_2)$ we get:

$$H_{\{y_{ij}(n)\}}(\mathscr{A}, \mathscr{A}_n) \geqq H_\omega(\mathscr{A}, \mathscr{A}_n) - \delta_1(\varepsilon) \quad \forall n \geqq M$$

with $\delta_1(\varepsilon) = 4\varepsilon + 2\gamma_2(\varepsilon)$ the generic ε of the statement of the theorem being obtained by choosing ε_2 small enough and $M \in \mathbf{N}$ large enough.

Remarks (3.2.2).
1. In the first part of the proof of the theorem the main point was selecting those decomposing states whose weight was not too small in comparison with ε. Indeed we required the corresponding entropies to be smaller than $\sqrt{\varepsilon}$: in other words we have eliminated, via the coarse graining, "the points with too small measure." As

$$\sum_{\alpha \in I} \omega(x_\alpha^1(n)) = \sum_{\beta \in J} \omega(x_\beta^2(n)) = 1,$$

we see that for any ε there will be α and β for which $S(\hat{\omega}(x_\alpha^1(n)\cdot)_{|\mathscr{A}}) \leqq \sqrt{\varepsilon}$, $S(\hat{\omega}(x_\beta^2(n)\cdot)_{|\mathscr{A}_n}) \leqq \sqrt{\varepsilon}$, otherwise the above two sums would result smaller than $\sqrt{\varepsilon}$.
2. We have indicated by $\{x_{\alpha\beta}(n)\}_{(\alpha,\beta)\in I \times J}$ the decomposition of cardinality $N(\varepsilon_2, N)$ which gives $H_\omega(\mathscr{A}, \mathscr{A}_n)$ within the infinitesimal function, $\eta(\varepsilon_2)$, of ε_2. As the coarse graining has been performed using the operators $x_{\alpha\beta}(n)$ which should carry another cumbersome parameter ε_2, also the resulting $y_{ij}(n)$'s depend on ε_2. By the very construction of the coarse grained decomposition we see that the result is stable with respect to ε_2. If we let ε_2 go to zero the corresponding $\varepsilon = \varepsilon_1 + \eta_2(\varepsilon_2)$ turns into $\varepsilon = \varepsilon_1$ and all the estimates hold with this new ε. On the other hand, $\varepsilon_2 = 0$ means that $\{y_{ij}(n)\}$ is, in this case, a coarse graining of an optimal decomposition. This coarse graining depends now only on $\varepsilon_1 = \varepsilon$ and we are thus in the position of performing the limit $n \to \infty$, with respect to the selected sequence of decompositions $\{y_{ij}(n)\}_{i,j=1,\ldots,N+1}$.

This is the content of:

Corollary (3.2.3). *There exists a sequence of decompositions* $\{y_{ij}(n)\}_{i,j=1,\ldots,N+1}$ *such that*
1.

$$\lim_n [H_{\{y_{ij}(n)\}}(\mathscr{A}, \mathscr{A}_n) - H_\omega(\mathscr{A}, \mathscr{A}_n)] = 0,$$

2.
$$\lim_n \| E(y_i^1(n)) - P_i \| = 0, \quad i = 1, \ldots, N,$$
3.
$$\lim_n \| E_n(y_j^2(n) - P_j(n) \| = 0, \quad j = 1, \ldots, N,$$
4.
$$\lim_n \omega(y_{N+1}^1(n)) = \lim_n \omega(y_{N+1}^2(n)) = 0,$$

E, E_n, $y_i^1(n)$, $y_j^2(n)$ have been introduced and explained during the proof of Theorem (3.2.1).

4. The Clustering Properties

We now apply the results of Sect. 3.2 to prove Theorem (3.1.3). We recall that, according to Lemma (3.1.5), we should show:

1.
$$\lim_n \omega(PQ_n) = \omega(P)\omega(Q),$$
2.
$$\lim_n \omega(PQ_nPQ_n) = \omega(P)\omega(Q) \quad \text{if} \quad [P, Q] = 0.$$

$(Q_n = \sigma^n(Q)\cdot)$

We start from Corollary (3.2.3) and prove:

Lemma (4.1).
$$\lim_n \| E[y_{ij}(n)] - P_i E[(P_j)_n] P_i \| = \lim_n \| E_n[y_{ij}(n)] - (P_j)_n E_n[P_i](P_j)_n \| = 0.$$

Proof. From $\lim_n \| E(y_k^1(n)) - P_k \| = 0$ together with $0 < y_k^1(n) < \mathbf{1}$ we get
$$\lim_n \omega([y_k^1(n) - P_k]^2) \leqq \lim_n \omega(E(y_k^1(n)) - 2P_k E(y_k^1(n)) + P_k) = 0.$$

In complete analogy, $\lim_n \omega([y_k^2(n) - (P_k)_n]^2) = 0$, so that $y_{ij}(n)$ tends strongly to both refinements of P_i and $(P_j)_n$, respectively:
$$\text{s-}\lim_n [y_{ij}(n) - P_i y_{ij}(n) P_i] = \text{s-}\lim_n [y_{ij}(n) - (P_j)_n y_{ij}(n)(P_j)_n] = 0. \tag{29}$$

This is a consequence of the following observations: We write
$$y_{ij}(n) = \sqrt{y_i^1(n)}\, z_{ij}(n) \sqrt{y_i^1(n)}$$
and use
$$\text{st-}\lim_n \sqrt{y_i^1(n)}\,(1 - P_i) = 0,$$
so that
$$0 = \text{st-}\lim_n (y_{ij}(n) - \sqrt{y_i^1(n)}\, z_{ij}(n) \sqrt{y_i^1(n)}\, P_i)\Omega$$
$$= \text{st-}\lim_n J(y_{ij}(n) - \sqrt{y_i^1(n)}\, z_{ij}(n) \sqrt{y_i^1(n)}\, P_i)\Omega$$

$$= \text{st-}\lim_n (y_{ij}(n) - P_i\sqrt{y_i^1(n)}\, z_{ij}(n)\sqrt{y_i^1(n)})\Omega$$

$$= \text{st-}\lim_n (y_{ij}(n) - P_i y_{ij}(n) P_i)\Omega.$$

We now consider

$$\lim_n \omega([E(y_{ij}(n)) - P_i E[(P_j)_n] P_i] P_k)$$

$$= \begin{cases} \lim_n \omega(y_{ij}(n)P_k) & \text{for } \quad i \neq k, \\ \lim_n \omega(E(y_{ij}(n))P_i - P_i E[(P_j)_n]P_i) = \lim_n \omega(y_{ij}(n) - P_i(P_j)_n P_i) & \text{for } \quad i = k, \end{cases}$$

where $E: M \to A$ is the canonical conditional expectation already used. From above we know:

$$\text{s-lim}\,[y_k^2(n) - (P_k)_n] = \text{s-lim}\left(\sum_{i=1}^{N+1} y_{ik}(n) - (P_k)_n\right) = 0, \quad k = 1, \ldots, N, \tag{30}$$

$$\text{s-lim}\,[y_k^1(n) - P_k] = \text{s-lim}\left(\sum_{i=1}^{N+1} y_{ki}(n) - P_k\right) = 0, \quad k = 1, \ldots, N. \tag{31}$$

When $i \neq k$ then we can use the first equality in (29) and thus arrive at:

$$\lim_n \omega(y_{ij}(n)P_k) = \lim_n \omega(P_i y_{ij}(n) P_i P_k) = 0.$$

If $i = k$ then (30) can be exploited first to show:

$$\lim_n \omega(P_i(P_j)_n P_i) = \lim_n \omega(P_i y_j^2(n) P_i)$$

$$= \lim_n \sum_{k=1}^{N+1} \omega(P_i y_{kj}(n) P_i) = \lim_n \left[\sum_{k=1}^{N} \omega(P_i y_{kj}(n) P_i) + \omega(P_i y_{N+1,j}(n) P_i)\right].$$

Again the first equality in (29) serves to get:

$$\lim_n \omega(P_i y_{kj}(n) P_i) = \lim_n \omega(P_i P_k y_{kj}(n) P_k P_i) = \delta_{ik} \lim_n \omega(P_i y_{ij}(n) P_i),$$

and thus

$$\lim_n \omega(P_i(P_j)_n P_i) = \lim_n [\omega(P_i y_{ij}(n) P_i) + \omega(P_i y_{N+1,j}(n) P_i)].$$

The first limit on the right-hand side of the above equality is equal to $\lim_n \omega(y_{ij}(n))$ owing to (29), whereas the second one is zero according to (3.2.3, 4).

Hence

$$\lim_n \omega([E[y_{ij}(n)] - P_i E[(P_j)_n] P_i] P_k) = 0 \forall i, j, k = 1, \ldots, N.$$

Owing to the finite dimensionality of the subalgebra $\mathscr{A}$, from the above it follows that

$$\lim_n \| E[y_{ij}(n) - P_i(P_j)_n P_i] \| = 0.$$

On considering the canonical conditional expectation $E_n: M \to \mathscr{A}_n$, the very same argument, now using (29) and (31), leads to

$$\lim_n \| E_n[y_{ij}(n) - (P_j)_n P_i (P_j)_n] \| = 0.$$

In Property (2.2.4) we have introduced the quantity $H_\omega(\mathscr{A}_1 | \mathscr{A}_2)$ which is analogous to the classical conditional entropy [3] and can be fruitfully used at this point.

Lemma (4.2).
1. If $(\mathscr{M}, \omega, \sigma)$ is an entropic K-system and $\mathscr{A}$ is a finite dimensional subalgebra of $\mathscr{M}$, then:

$$\lim_n H_\omega(\mathscr{A} | \mathscr{A}_n) = H_\omega(\mathscr{A}) \quad (\mathscr{A}_n \text{ being } \sigma^n(\mathscr{A})).$$

2. If $\mathscr{M}$ is in addition a type II_1 factor and ω the trace, then:

$$\lim_n H_\omega(\mathscr{B} | \mathscr{A}_n) = H_\omega(\mathscr{B})$$

if $\mathscr{B}$ is a maximal abelian subalgebra of $\mathscr{A}$.

Proof. From covariance (Property (2.2.3)) we know: $H_\omega(\mathscr{A}_n) = H_\omega(\mathscr{A})\ \forall n$, and from the complete memory loss assumption (Definition (2.4)) we have: $2H_\omega(\mathscr{A}) = \lim_n H_\omega(\mathscr{A}, \mathscr{A}_n)$. Using Property (2.2.4), we have:

$$\begin{aligned} 2H_\omega(\mathscr{A}) &= \lim_n \{H_\omega(\mathscr{A}, \mathscr{A}_n) - H_\omega(\mathscr{A}_n) + H_\omega(\mathscr{A}_n)\} \\ &\leqq \lim_n H_\omega(\mathscr{A} | \mathscr{A}_n) + H_\omega(\mathscr{A}) \leqq 2H_\omega(\mathscr{A}). \end{aligned}$$

From covariance and Remark (2.3.5) we get:

$$H_\omega(\mathscr{A}) = H_\omega(\mathscr{B}) = H_\omega(A_n) = H_\omega(\mathscr{B}_n).$$

$\mathscr{B} \subset \mathscr{A}$ and monotonicity (Property (2.2.1)) imply

$$H_\omega(\mathscr{B}, \mathscr{B}_n) \leqq H_\omega(\mathscr{B}, \mathscr{A}_n).$$

Together with $\lim_n H_\omega(\mathscr{B}, \mathscr{B}_n) = 2H_\omega(\mathscr{B})$ we obtain:

$$\begin{aligned} H_\omega(\mathscr{B}) &= \lim_n [H_\omega(\mathscr{B}, \mathscr{B}_n) - H_\omega(\mathscr{B}_n)] \leqq \lim_n [H_\omega(\mathscr{B}, \mathscr{A}_n) - H_\omega(\mathscr{A}_n)] \\ &= \lim_n H_\omega(\mathscr{B} | \mathscr{A}_n) \leqq H_\omega(\mathscr{B}). \end{aligned}$$

Lemma (4.3). *Assume that $(\mathscr{M}, \omega, \sigma)$ is the dynamical system of Theorem (3.1.3) and that the projectors $P, Q \in \mathscr{M}$ generate a finite dimensional subalgebra $\mathscr{A} = (P \vee Q)''$ of $\mathscr{M}$ (such operator pairs are dense in $\mathscr{M}$). Then*

$$\lim_n \omega(PQ_n) = \omega(P)\omega(Q).$$

Proof. Let us consider a maximal abelian subalgebra $\mathscr{B}$ of $\mathscr{A}$ to which P belongs and let $\{P_i\}_{i=1,\ldots,N}$ be the generating set of minimal projectors. From Remark (2.3.2) we know:

$$H_\omega(\mathscr{B}) = -\sum_{i=1}^{N} \omega(P_i) \log \omega(P_i) = S(\omega_{|\mathscr{B}}).$$

From Lemma (4.2) and the definition of $H_\omega(\mathscr{A}_1|\mathscr{A}_2)$ in Property (2.2.4):

$$\lim_n H_\omega(\mathscr{B}|\mathscr{A}_n) = \lim_n \sup_{\omega=\sum_i \omega(x_i\cdot)} \left[\sum_i \omega(x_i)[S(\omega|\hat{\omega}(x_i\cdot))_{|\mathscr{B}} - S(\omega|\hat{\omega}(x_i\cdot))_{|\mathscr{A}_n}]\right]$$

$$= -\sum_j \omega(P_j)\log\omega(P_j).$$

As $H_\omega(\mathscr{B}|\mathscr{A}_n) \leqq H_\omega(\mathscr{B})$ we have that $\forall \varepsilon_1 > 0 \exists M \in \mathbf{N}$:

$$\forall n \geqq M \quad H_\omega(\mathscr{B}) - H_\omega(\mathscr{B}|\mathscr{A}_n) \leqq \varepsilon_1. \tag{32}$$

Introducing in (32) the expression for $H_\omega(\mathscr{A}_1|\mathscr{A}_2)$ given in (2.2.4) we get:

$$H_\omega(\mathscr{B}) - \sup_{\omega=\sum_i \omega(x_i\cdot)} \left\{\sum_i \omega(x_i)[S(\omega,\hat{\omega}(x_i\cdot))_{|\mathscr{B}} - S(\omega,\hat{\omega}(x_i\cdot))_{|\mathscr{A}_n}]\right\} \leqq \varepsilon_1. \tag{33}$$

To the second term of the left-hand side of (33) we can apply the same argument of [2, Lemma VI.1] to construct $\forall \varepsilon > 0$ a finite decomposition $\{x_\alpha(n)\}$ whose cardinality depends on ε_2 such that:

$$H_\omega(\mathscr{B}|\mathscr{A}_n) \leqq \sum_\alpha \omega(x_\alpha(n))[S(\omega,\hat{\omega}(x_\alpha(n)\cdot))_{|\mathscr{B}} - S(\omega,\hat{\omega}(x_\alpha(n)\cdot))_{|\mathscr{A}_n}] + \eta_2(\varepsilon_2),$$

$$\eta_2(\varepsilon_2) \xrightarrow{\varepsilon_2 \to 0^+} 0^+. \tag{34}$$

Using (34), (33) turns into

$$H_\omega(\mathscr{B}) - \sum_\alpha \omega(x_\alpha(n))[S(\omega,\hat{\omega}(x_\alpha(n)\cdot))_{|\mathscr{B}} - S(\omega,\hat{\omega}(x_\alpha(n)\cdot))_{|\mathscr{A}_n}] \leqq \varepsilon_1 + \eta_2(\varepsilon_2) =: \varepsilon.$$

Since

$$\sum_\alpha \omega(x_\alpha(n))S(\omega,\hat{\omega}(x_\alpha(n)\cdot))_{|\mathscr{B}} = S(\omega_{|\mathscr{B}}) - \sum_\alpha \omega(x_\alpha(n))S(\hat{\omega}(x_\alpha(n)\cdot))_{|\mathscr{B}}$$

and

$$S(\omega_{|\mathscr{B}}) = H_\omega(\mathscr{B}) \quad (\mathscr{B} \text{ is abelian})$$

we obtain:

$$\sum_\alpha \omega(x_\alpha(n))[S(\hat{\omega}(x_\alpha(n)\cdot))_{|\mathscr{B}} + S(\omega,\hat{\omega}(x_\alpha(n)\cdot))_{|\mathscr{A}_n}] \leqq \varepsilon. \tag{35}$$

We can thus follow the proof of Theorem (3.2.1), Part I, and Corollary (3.2.3) to construct, via the coarse graining, a sequence of decompositions $\{y_i(n)\}_{i=1,\ldots,N+1}$ such that:

$$\lim_n \| E(y_i(n)) - P_i \| = 0, \quad i = 1,\ldots,N, \tag{36}$$

$$\lim_n \omega(y_{N+1}(n)) = 0, \tag{37}$$

$$\lim_n S(\omega,\hat{\omega}(y_i(n)\cdot))_{|\mathscr{A}_n} = 0, \quad i = 1,\ldots,N. \tag{38}$$

Using the proof of Lemma (4.1) we have from (36),

$$\lim_n \omega((y_i(n) - P_i)^2) = 0, \quad i = 1,\ldots,N. \tag{39}$$

Since the relative entropy $S(\omega, \hat{\omega}(y_i(n)\cdot))_{|\mathscr{A}_n}$ is strictly positive unless $\hat{\omega}(y_i(n)\cdot)_{|\mathscr{A}_n} = \omega_{|\mathscr{A}_n}$, using (39) we obtain:

$$\lim_n \hat{\omega}(y_i(n)Q_n) = \lim_n \frac{\omega(P_i Q_n)}{\omega(P_i)} = \omega(Q_n) = \omega(Q), \quad i = 1, \dots, N;$$

therefore

$$\lim_n \omega(PQ_n) = \omega(P)\omega(Q)$$

when $P \in \mathscr{B}$.

P and Q are in general not commuting, and therefore $(P \vee Q)'' =: \mathscr{A}$ might be infinite dimensional. In order to exploit the behaviour of the entropic functionals we need finite dimensionality, but we can assume this without loss of generality due to the following Lemma (4.4):

Lemma (4.4). *Let P and Q be two projectors, then there exists a sequence of projectors Q_r such that $\lim_{r\to\infty} \| Q_r - Q \| = 0$ and $(P \vee Q_r)''$ is finite dimensional.*

Proof. Restricting P and Q to the range of

$$E \equiv 1 - \{P \wedge Q + (1-P) \wedge Q + P \wedge (1-Q) + (1-P) \wedge (1-Q)\},$$

we can assume that the partial isometry u obtained by the polar decomposition of $PQ(1-P)$ satisfies $uu^\dagger = P, u^\dagger u = 1 - P$. Since $R = 1 - (P-Q)^2$ commutes with P and Q, it follows (e.g. [9]) that the von Neumann algebra built by P and Q is isomorphic to $I_2 \otimes \{R\}''$ when restricted to the range of E, and is abelian with at most minimal projections when restricted to the range of $1-E$. In the former, P and Q can be written as:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} R & \sqrt{R(1-R)} \\ \sqrt{R(1-R)} & 1-R \end{pmatrix}. \tag{40}$$

R can be approximated by step functions R_τ so $\{R_\tau\}''$ is finite dimensional and Q_r (with R replaced by R_τ) satisfies the condition of Lemma (4.5).

Now we estimate

$$\lim_{n\to\infty} |\omega(PQ_n) - \omega(P)\omega(Q)| \leqq \lim_{\tau\to\infty} \Big[\lim_{n\to\infty} |\omega(PQ_{n,r}) - \omega(P)\omega(Q_{n,r})| + 2\| Q - Q_\tau \| \Big] = 0. \tag{41}$$

Hence the weak clustering can be extended beyond the restrictions needed in the proof of Lemma (4.3).

Lemma (4.5). *Let P and Q be two commuting projectors in $\mathscr{M}$ and denote by $\mathscr{A}$ the four dimensional abelian subalgebra they generate, $\{P_i\}_{i=1,\dots,4}$ being the set of minimal projectors constructed with P and Q. Let $E_n: \mathscr{M} \to \mathscr{A}_n := \sigma^n(\mathscr{A})$ be the conditional expectation from $\mathscr{M}$ onto $\sigma^n(\mathscr{A})$; then*

$$\lim_n \| E_n(P_j) - \omega(P_j) \| = 0, \quad j = 1, \dots, 4.$$

Proof. From Lemma (4.3), results (36) and (38) we draw the same conclusions as before:

$$\lim_n \frac{\omega(y_i(n)P_j(n))}{\omega(y_i(n))} = \lim_n \frac{\omega(P_iP_j(n))}{\omega(P_i)} = \lim_n \frac{\omega(E_n(P_i)P_j(n))}{\omega(P_i)} = \omega(P_j)$$

for $i,j = 1,\ldots,4$ and $P_j(n) = \sigma^n(P_j)$, and exploit the finite dimensionality of $\mathscr{A}$.

Lemma (4.6). *With the same hypothesis as in Lemma (4.5), let $\{y_{ij}(n)\}_{i,j=1,\ldots,5}$ be the sequence of decompositions for $H_\omega(\mathscr{A},\mathscr{A}_n)$ which behave according to Theorem (3.2.2) and Corollary (3.2.3); then:*

$$\lim_n \| E_n(y_{ij}(n)) - \omega(P_i)P_j(n) \| = 0, \quad i,j = 1,\ldots,4.$$

Proof. From Lemma (4.1) we get:

$$\lim_n \| E_n(y_{ij}(n)) - P_j(n)E_n(P_i)P_j(n) \| = 0, \quad i,j = 1,\ldots,4,$$

and the result thus follows from Lemma (4.5).

Now we can conclude with:

Lemma (4.7). *With the hypothesis of Lemma (4.6):*

$$\lim_n \omega(PQ_nPQ_n) = \omega(P)\omega(Q).$$

Proof. Let $P_1 := PQ$, $P_2 := PQ^c := P(1-Q)$, $P_3 := P^cQ := (1-P)Q$ and $P_4 := P^cQ^c$ be the four minimal projectors which generate the abelian subalgebra $\mathscr{A}$ of the lemma.

Let again $P_i(n)$ indicate $\sigma^n(P_i) \in \sigma^n(\mathscr{A})$, $i = 1,\ldots,4$. We want to control $\lim_n \omega(P_iP_j(n)P_kP_l(n))$. From Lemma (4.1) we have, using the tracial property of ω,

$$\lim_n \omega(P_iP_j(n)P_kP_l(n)) = \lim_n \omega(y_j^2(n)P_kP_l(n)P_i) = \sum_{r=1}^{5} \lim_n \omega(y_{rj}(n)P_kP_l(n)P_i).$$

From Corollary (3.2.3), result (4), and Lemma (4.1), formula (29), we get:

$$\begin{aligned}\lim_n \omega(P_iP_j(n)P_kP_l(n)) &= \sum_{r=1}^{4} \lim_n \omega(P_ry_{rj}(n)P_rP_kP_l(n)P_i) \\ &= \delta_{ik}\lim_n \omega(P_iy_{ij}(n)P_iP_l(n)).\end{aligned}$$

Again Lemma (4.1), formula (29), yields:

$$\lim_n \omega(P_iP_j(n)P_kP_l(n)) = \delta_{ik}\lim_n \omega(y_{ij}(n)P_l(n)) = \delta_{ik}\lim_n \omega(E_n[y_{ij}(n)]P_l(n)).$$

The last step uses Lemma (4.6):

$$\lim_n \omega(P_iP_j(n)P_kP_l(n)) = \delta_{ik}\delta_{jl}\omega(P_i)\omega(P_j).$$

Since $P = P_1 + P_2$ and $Q = P_3 + P_4$, this result can be applied to obtain:

$$\begin{aligned}\lim_n \omega(PQ_nPQ_n) &= \omega(P_1)\omega(P_3) + \omega(P_1)\omega(P_4) + \omega(P_2)\omega(P_3) + \omega(P_2)\omega(P_4) \\ &= \omega(P)\omega(Q).\end{aligned}$$

5. Conclusion

In abelian ergodic theory K-systems are clustering where no distinction between weak and strong clustering has to be made. As a consequence $\mathscr{A}$ and $\sigma^n\mathscr{A}$ from approximately a tensor product and also the state has tensor product structure. In the nonabelian situation weak clustering is sufficient to guarantee convergence to equilibrium. It holds if the system is an algebraic K-system [4], but such a system still allows nontrivial relations between $\mathscr{A}$ and $\sigma^n\mathscr{A}$ as we know from the odd elements of the Fermi algebra. For entropic K-systems $\mathscr{A}$ and $\sigma^n\mathscr{A}$ become completely independent from one another in the sense that they form approximately a tensor product. Therefore, the n-subalgebra entropy is the relevant quantity to measure to which extent different subalgebras are independent from one another. Furthermore, strong asymptotic abelianness is usually expected to hold for the time evolution of the observable algebra (the even part of the Fermi algebra), and is used, for example, to show that dynamically stable states are KMS states [8]. But it suffices that chaotic properties are satisfied by the observable algebra, whereas it does not matter if they are violated for the field algebra.

Acknowledgement. One of the authors (F. B.) would like to warmly thank the Italian Foundation "Angelo Della Riccia" for financial support.

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Communicated by H. Araki