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Classical and Thermodynamic Limits for Generalised Quantum Spin Systems

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Abstract. We prove that the rescaled upper and lower symbols for arbitrary generalised quantum spin systems converge in the classical limit. For a large class of models this enables us to derive the asymptotics of quantum free energies in the classical and in the thermodynamic limit.

1. Introduction

Several authors have studied the classical limit of quantum partition functions based on compact Lie groups. Particular cases were studied: first by Lieb [1] for $SU(2)$, and then by Fuller and Lenard [2] for $O(n)$, and Gilmore [3] for $SU(n)$. A unified treatment for general compact semi-simple Lie groups was given by Simon [4]. However, Simon's techniques were only successful for quantum systems built upon fully symmetric group representations and, as far as we are aware, this gap in the theory has not been filled. The contribution of this paper is to obtain classical and thermodynamic limits for quantum systems built upon arbitrary representations of compact semi-simple Lie groups. Furthermore, we are able to treat arbitrary polynomial Hamiltonians, rather than just the multiaffine Hamiltonians described in [4]. Our results rest on the proof of general limit theorems for the upper and lower symbols of polynomial Hamiltonians: we show that in all cases they coalesce in the classical limit. By a well-known procedure, this allows the computation of quantum-free energies in the classical limit. Furthermore, for a general class of mean-field models we are able to calculate the free-energies in the rather more interesting case of the thermodynamic limit.

In this introduction we will describe our framework and state our main technical result. We shall then summarize the contents of the paper.

Let G be a compact semi-simple Lie group with Lie algebra $\mathbf{g}, H$ a maximal abelian subgroup of G with Lie sub-algebra $\mathbf{h}$. Recall [5] that there exists a set of elements $\{\lambda_i : i \in E = \{1, 2, \ldots, \mathrm{rank}(G)\}\}$ of $\mathbf{h}^* = (i\mathbf{h})'$, the real dual of $i\mathbf{h}$, such that the

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irreducible representations of G are in one–one correspondence (up to unitary equivalence) with the elements of a subset $\mathscr{J}$ of $\mathbf{h}^*$ which is the intersection of some sublattice of the integer lattice generated by the $\{\lambda_i : i \in E\}$ with the positive cone

$$D = \left\{ \rho \in \mathbf{h}^* : \rho = \sum_{i \in E} l_i \lambda_i : l_i \geqq 0 \right\}.$$

The correspondence is that $\rho \in \mathscr{J}$ is the maximal weight for the representation, which we label Π_ρ. Representations Π_ρ for which $\rho = n\lambda_i$ for some $i \in E$ and some positive integer n are called fully symmetric.

The main tools in the application of our result will be the Berezin–Lieb inequalities for the upper and lower symbols of an operator [1, 4, 6]. We first fix our notation. Let Π_ρ act on the space V_ρ, and let π_ρ be its derivative acting on $\mathbf{g}$. Let e be the unit of G, and denote by A the adjoint representation of G on $\mathbf{g}$. For $x \in G, \rho \in \mathscr{J}$, define the projection $P_\rho(x) = \Pi_\rho(x) P_\rho(e) \Pi_\rho(x^{-1})$ in $\mathscr{L}(V_\rho)$, where $P_\rho(e)$ is the projection onto the vector ψ_ρ in V_ρ such that $\rho(X) = \langle \psi_\rho, \pi_\rho(X) \psi_\rho \rangle$ for all X in $\mathbf{g}$. Let $J_\rho = \{x \in G : \Pi_\rho(x)\psi_\rho = e^{i\alpha(x)}\psi_\rho : \alpha(x) \in \mathbb{R}\}$ denote the isotropy subgroup of ψ_ρ. We will write $x \sim y$ if x and y lie in the same element of G/H_ρ. Note that

$$x \sim y \Leftrightarrow P_\rho(x) = P_\rho(y). \tag{1.1}$$

We remark at this point that since we will be considering all possible representations, we will work directly with functions on the group rather than on the coadjoint orbits: the latter are representation dependent.

Let μ be the normalised Haar measure on G and write $\mu_\rho = \dim(V_\rho)\mu$. For each $\rho \in \mathscr{J}$ the $\{P_\rho(x) : x \in G\}$ are called a family of coherent projections and have the property that

$$\int_G P_\rho(x) d\mu_\rho(x) = \mathbf{1} \in \mathscr{L}(V_\rho). \tag{1.2}$$

Proposition 1. [1, 4, 6] *For each $B \in \mathscr{L}(V_\rho)$ there exists a function B^u in L^∞ (G, μ_ρ), not necessarily unique, called the upper symbol of B, such that*

$$B = U(B^u) := \int_G B^u(x) P_\rho(x) d\mu_\rho(x). \tag{1.3}$$

Defining the lower symbol $B^l \in L^\infty$ $(G, d\mu_\rho)$, of B by

$$B^l(x) = \operatorname{trace} P_\rho(x) B, \tag{1.4}$$

then for any convex function $f : \mathbb{R} \mapsto \mathbb{R}$ and self-adjoint B in $\mathscr{L}(V_\rho)$,

$$\int_G d\mu_\rho(x) f(B^l(x)) \leqq \operatorname{trace} f(B) \leqq \int_G d\mu_\rho(x) f(B^u(x)). \tag{1.5}$$

The main technical theorem is concerned with the limits of the upper and lower symbols as the dimension of the underlying representation becomes large. We formulate this as follows: let $\{\rho_L\}$ be a sequence in $\mathscr{J}$ with $\rho_L = L \sum_{i \in E} r_L^i \lambda_i$.

Let $\rho\in D$ with $\rho=\sum_{i\in E(\rho)} r^i\lambda_i$, for some subset $E(\rho)$ of E, and let $D(\rho)=\left\{\tilde{\rho}\in D:\tilde{\rho}=\sum_{i\in E(\rho)}\tilde{r}^i\lambda_i\right\}$.

Definition. *We say that the sequence $\{\rho_L\}$ satisfies convergence condition C with respect to ρ if*

$$\lim_{L\to\infty} L^{-1}\rho_L=\rho \tag{1.6}$$

and

$$\rho_L\in D(\rho)\qquad \forall L. \tag{1.7}$$

For arbitrary $\rho=\sum_E r^i\lambda_i$ in $D, X\in g, x\in G$ define

$$Q_\rho(X,x)=\sum_{i\in E} r_i\langle\psi_{\lambda_i},\Pi_{\lambda_i}(x)\pi_{\lambda_i}(X)\Pi_{\lambda_i}(x^{-1})\psi_{\lambda_i}\rangle. \tag{1.8}$$

Note, [7] that when $\rho\in\mathscr{J}$, $Q_\rho(X,\cdot)$ is just the lower symbol of X in the representation ρ.

Let $\mathbf{g}^n=\mathbf{g}\otimes\cdots\otimes\mathbf{g}$ (n terms) and define $\pi^n_\rho:\mathbf{g}^n\to\mathscr{L}(V_\rho)$ by

$$\pi^n_\rho(X^1\otimes\cdots\otimes X^n)=\pi_\rho(X^1)\cdots\pi_\rho(X^n). \tag{1.9}$$

Gilmore proves the following result about the lower symbols:

Theorem 2A. [7] *Let $\underline{X}=X^1\otimes\cdots\otimes X^n\in\mathbf{g}^n$, and suppose that $\{\rho_L\}$ satisfies convergence condition C with respect to ρ. Then*

$$\lim_{L\to\infty} L^{-n}(\pi^n_{\rho_L}(\underline{X}))^l(x)=\prod_{m=1}^n Q_\rho(X^m,x) \tag{1.10}$$

the limit existing uniformly in x and norm-bounded subsets of $\mathbf{g}^n$.

Remark. The uniformity, although not stated in [7], is straightforward to obtain. Our main technical result is the following:

Theorem 2B. *Let $\{\rho_L\}$ satisfy convergence condition C with respect to ρ. Then*

$$\lim_{L\to\infty} L^{-n}(\pi^n_{\rho_L}(\underline{X}))^u(x)=\lim_{L\to\infty} L^{-n}(\pi^n_{\rho_L}(\underline{X}))^l(x) \tag{1.11}$$

for any $\underline{X}$ in $\mathbf{g}^n$, the limits existing uniformly as in Theorem 2A, and the right-hand side being obtained by linearity from Theorem 2A.

Theorem 2A for lower symbols is well known. The upper symbols have been far more problematic. Simon [4] gives explicit formulae for the upper symbol of an operator B in $\mathscr{L}(V_\rho)$ under the conditions that (i) $B=\pi_\rho(X)$ for some $X\in\mathbf{g}$, and (ii) Π_ρ is fully symmetric (see above).

Our main limit theorem is proved in Sect. 2. We do not obtain explicit formulae for the upper symbols, but in Proposition 3 and Lemma 5 we prove that they are bounded and equicontinuous. This turns out to be sufficient. In Proposition 4, we formalise an argument often given (like the main result) as an assumption (see e.g. [8]), namely, that since $\langle \psi_{\rho L}, \Pi_{\rho L}(x)\psi_{\rho L}\rangle$ behaves like $\exp Lf(x)$ for some $f(x)$ whose real part takes its maximum value, zero, at e, only the diagonal parts in the coherent state representation of an operator are important in the classical limit. In fact we work with the modulus of this matrix element: the complex oscillations of the matrix element itself make it untreatable as $L \nearrow \infty$. This result, along with the equicontinuity mentioned above, allows us to use a compactness argument to prove Proposition 2B: the only possible limit point of sequence of upper symbols is precisely the limit of the lower symbol. In Sect. 3 we apply our result to obtain classical limits of the free energy for a wide class of Hamiltonians. In Sect. 4 we combine the result with one from [9] concerning the limiting distributions of multiplicities in the decomposition of tensor products of representations into irreducible components. By use of the theory of large deviations [10, 11], we obtain a variational expression for thermodynamic limit of the free energy for a large class of mean-field models. This result can be seen as an extension of the work of [12], where a similar result was proved for the $G = SU(2)$. Finally, we note that simultaneously with our own work, the authors of [13] have obtained variational expressions for the free energy in the thermodynamic limit of a general class of mean-field models. It remains to be seen which scheme is more accessible for computations.

2. Proof of the Limit Theorem

Proposition 3. *Let $\mathbf{g}^n = \mathbf{g} \otimes \cdots \otimes \mathbf{g}$ (n terms). For all $x \in G$ define the map $A^n : \mathbf{g}^n \to \mathbf{g}^n$ by*

$$A^n(X^1 \otimes \cdots \otimes X^n) = A(x)X^1 \otimes \cdots \otimes A(x)X^n$$

and extend by linearity to the whole of $\mathbf{g}^n$. Let $B \in \mathscr{L}(V_\rho)$ be equal to $\pi^n_\rho(\underline{X})$ for some $\underline{X} \in G^n$. Then

$$B^u(x) = M_{n,\rho}(A^n(x^{-1})\underline{X}) \tag{2.1}$$

for some $M_{n,\rho} \in (\mathbf{g}^n)^*$, the dual of $\mathbf{g}^n$. Furthermore,

$$M_{n,\rho} = (A^n)^*(x) M_{n,\rho} \tag{2.2}$$

for any x in J_ρ.

Proof. For $n = 1$, the proof of (2.1) is just Theorem A.2.3 of [4]. For $n \geqq 2$ the proof turns out to be similar. Fix a basis $\{X_i : i \in E\}$ in $\mathbf{g}$, and hence a basis $X_{(i)} = X_{i_1} \otimes \cdots \otimes X_{i_n}$ in $\mathbf{g}^n$, where (i) stands for $(i_1, \ldots, i_n)$. The action of $A(x)$ in $\mathbf{g}$ is written

$$A(x)X_i = A(x)_{ji} X_j \tag{2.3}$$

(summation convention assumed) and so

$$\Pi_\rho(x)\pi^n_\rho(X_{(i)})\Pi_\rho(x^{-1}) = A^n(x)_{(j),(i)}\pi_\rho(X_{(j)}), \tag{2.4}$$

where

$$A^n(x)_{(j),(i)} = A(x)_{j_1 i_1} \cdots A(x)_{j_n, i_n}. \tag{2.5}$$

By Proposition 1 we know that $\pi_\rho^n(X_{(i)})$ has an upper symbol $b_{(i)}$. Since the $\{X_{(i)}\}$ form a basis in $\mathbf{g}^n$ we can use the $b_{(i)}$ to define a linear function $b:\mathbf{g}^n \to L^\infty(G,\mu_\rho)$ so that

$$\pi_\rho^n(\underline{X}) = U(b(\underline{X})). \tag{2.6}$$

For any function h on G define $(L_y h)(x) = h(y^{-1}x)$. Then [4],

$$U(L_y h) = \Pi_\rho(y) U(h) \Pi_\rho(y^{-1}), \tag{2.7}$$

and so

$$U(L_y b(X_{(i)})) = \Pi_\rho(y)\pi_\rho^n(X_{(i)})\Pi_\rho(y^{-1}) = A^n(y)_{(j),(i)}\pi_\rho(X_{(j)})). \tag{2.8}$$

Multiplying by $A^n(y^{-1})_{(i),(k)}$, summing over (i) and integrating over G with μ,

$$\tilde{b}(X_{(i)}) = \int_G d\mu(y) A^n(y^{-1})_{(j),(i)} L_y b(X_{(j)}) \tag{2.9}$$

is also an upper symbol for $\pi^n(X_{(i)})$. A brief calculation shows that

$$\tilde{b}(X_{(i)})(y) = (L_y \tilde{b}_{(i)})(e) = A^n(y^{-1})_{(k),(i)} \tilde{b}(X_{(k)})(e). \tag{2.10}$$

Since $\{X_{(i)}\}$ are a basis and $\tilde{b}(\cdot)(e)$ defines a linear map from the basis to $\mathbb{C}$, we can write

$$\tilde{b}(X_{(i)})(y) = A^n(y^{-1})_{(k),(i)} \tilde{M}_{n,\rho}(X_{(k)}) = \tilde{M}_{n,\rho}(A^n(y^{-1})X_{(i)}) \tag{2.11}$$

for some $\tilde{M}_{n,\rho} \in (\mathbf{g}^n)^*$. Now extend (2.11) by linearity to obtain (2.1). For (2.2), note that by (1.1) and (1.3),

$$\begin{aligned}(\pi_\rho^n(X_{(i)}))^u(x) &= \int_G d\mu_\rho(x) P_\rho(x) M_{n,\rho}(A^n(x^{-1})X_{(i)}) \\ &= \int_G d\mu_\rho(x) P_\rho(x) M_{n,\rho}(A^n(yx^{-1})X_{(i)})\end{aligned} \tag{2.12}$$

for $y \in J_\rho$, so that we can replace $\tilde{M}_{n,\rho}$ by

$$M_{n,\rho} = \int_{J_\rho} d\hat{\mu}_\rho(x)(A^n)^*(x)\tilde{M}_{n,\rho} \tag{2.13}$$

in (2.11), where $\hat{\mu}_\rho$ is the measure on J_ρ with mass 1 inherited from μ. $\square$

For $\beta = \sum_E b^i \lambda_i$ in D, define the quantity

$$F_\beta(x) = \prod_{i \in E} \langle \psi_{\lambda_i}, \Pi_{\lambda_i}(x)\psi_{\lambda_i}\rangle^{b_i}. \tag{2.14}$$

Clearly $F_{\beta+\beta'} = F_\beta F_{\beta'}$. In fact, [4], it can be shown when $\rho = \sum_E r^i \lambda_i \in \mathscr{J}$ that

$$F_\rho(x) = \langle \psi_\rho, \Pi_\rho(x)\psi_\rho \rangle. \tag{2.15}$$

By (1.1) $x \in J_\rho$ if and only if $|F_\rho(x)| = 1$. Since all representations are unitary, this occurs if and only if $|F_{\lambda_i}(x)| = 1$ for all i such that $r^i \neq 0$, i.e. if and only if $x \in J_{\lambda_i}$ for all such i. The point of (1.7) is that if $\{\rho_L\}$ satisfies the convergence condition

C with respect to ρ, then all ρ_L have the same isotropy subgroup. We shall call this $\tilde{J}_\rho$.

Proposition 4. *For all* $\rho = \sum_E r^i \lambda_i \in D$ *and* $\rho \in \mathscr{J}$ *define the measure* γ_ρ *on* G *by*

$$d\gamma_\rho(x) = d\mu_\rho(x) |F_\rho(x)|^2. \tag{2.16}$$

Then

(i) γ_ρ *is a probability measure.*
(ii) *If* $\{\rho_L\}$ *satisfies convergence condition C with respect to* ρ, *then for all* $g \in \mathscr{C}(G, \mathbb{C})$

$$\lim_{L \to \infty} \int_G d\gamma_{\rho_L}(x) g(x) = \hat{g}(e), \tag{2.17}$$

where

$$\hat{g}(x) = \int_{\tilde{J}_\rho} d\tilde{\mu}_\rho(y) g(xy), \tag{2.18}$$

convergence being uniform on uniformly bounded and equicontinuous subsets of $\mathscr{C}(G, \mathbb{C})$

Proof. (i) follows from (1.2) and the fact that the $P_\rho(x)$ are one dimensional. For (ii), first note that by (1.1), $|F_\rho(x)| = |F_\rho(y)|$ if and only if $x \sim y$, so

$$\int_G d\gamma_\rho(x) g(x) = \int_G d\gamma_\rho(x) \hat{g}(x). \tag{2.19}$$

Let

$$N_L(\varepsilon) = \{x \in G : 2\log|F_{\rho_L}(x)| \geqq -\varepsilon L\}. \tag{2.20}$$

Let C be a uniformly bounded and equicontinuous subset of $\mathscr{C}(G, \mathbb{C})$. Then for all g in C,

$$\hat{g}(e) - \int_G d\gamma_{\rho_L}(x) g(x) = \int_{N_L(\varepsilon)} d\gamma_{\rho_L}(x)(\hat{g}(e) - \hat{g}(x)) + \int_{N_L^c(\varepsilon)} d\gamma_{\rho_L}(x)(\hat{g}(\rho) - \hat{g}(x)), \tag{2.21}$$

and so

$$\left| \hat{g}(e) - \int_G d\gamma_{\rho_L}(x) g(x) \right| \leqq \sup_{g \in C} \sup_{x \in N_L(\varepsilon)} |\hat{g}(e) - \hat{g}(x)| + 2c\mu_{\rho_L}(G) e^{-\varepsilon L}, \tag{2.22}$$

where $c = \sup_{g \in C} \|g\|$. Since $r_L^i \to r^i$ for all $i \in E$, we can find L_0 such that $r_L^i \geqq r^i/2$ for all i when $L \geqq L_0$. Define

$$N(\varepsilon) = \{x \in G : 2\log|F_\rho(x)| \geqq -\varepsilon\}. \tag{2.23}$$

Since $\log|F_{\rho_L}(x)| = \sum_E r_L^i \log|F_{\lambda_i}(x)|$ we see that for $L \geqq L_0$, $N_L(\varepsilon) \subset N(2\varepsilon)$. Consequently the upper bound of (2.22) is itself bounded by

$$\sup_{g \in C} \sup_{x \in N(2\varepsilon)} |\hat{g}(e) - \hat{g}(x)| + 2c\mu_{\rho_L}(G) e^{-\varepsilon L}. \tag{2.24}$$

By the argument preceding the proposition, if $|F_\rho(x)| = 1$ then $x \in \tilde{J}_\rho$. Since the Π_{λ_i}

are continuous and unitary, $|F_\rho(x)|$ is continuous and bounded above by $1{:}\log|F_\rho|$ is upper semicontinuous. Hence $N(\varepsilon)$ is a neighbourhood of $\tilde{J}_\rho$ for ε sufficiently small and $\bigcap_{\varepsilon\geqq 0} N(\varepsilon)=\tilde{J}_\rho$. By Ascoli's theorem C is compact so that first supremum in (2.24) is $|\hat{g}_\varepsilon(x_\varepsilon)-\hat{g}(e)|$ for some $g_\varepsilon\in C$ and x_ε such that the limit points of x_ε as $\varepsilon\to 0^+$ lie in $\tilde{J}_\rho$. Note furthermore that $\hat{g}(x)=\hat{g}(e)$ if $x\in\tilde{J}_\rho$. For the second supremum just note that $\mu_{\rho_L}(G)$ is just $(\dim V_{\rho_L})$ which is bounded (see e.g. [9, Lemma 2.2]) by a polynomial in L. The result follows by a standard argument. □

Lemma 5. *Let $\{\rho_L\}$ satisfy convergence condition C with respect to ρ. For any fixed n, the sequence $\{L^{-n}M_{n,\rho_L}\}$ is bounded in norm in $(\mathbf{g}^n)^*$ as $L\to\infty$.*

Proof. Suppose the opposite is true. We can find a sequence $\{\underline{X}_L\}$ in $\mathbf{g}^n$ with $\|\underline{X}_L\|=1$ such that

$$M_{n,\rho_L}(\underline{X}_L)=\|M_{n,\rho_L}\|. \tag{2.26}$$

From (1.4)

$$\begin{aligned}\mathscr{R}(\pi^n_{\rho_L}(\underline{X}_L))^l(e)&=\mathscr{R}\operatorname{trace}\pi^n_{\rho_L}(\underline{X}_L)P_{\rho_L}(e)\\&=\mathscr{R}\int_G d\gamma_{\rho_L}(x)M_{n,\rho_L}(A^n(x^{-1})\underline{X}_L)\\&=\int_G d\gamma_{\rho_L}(x)\mathscr{R}M_{n,\rho_L}(A^n(x^{-1})\underline{X}_L),\end{aligned} \tag{2.27}$$

where $\mathscr{R}$ denotes the real part, the last equality following because γ_{ρ_L} is a real measure. Now,

$$|\mathscr{R}(M_{n,\rho_L}(\underline{X}_L-A^n(x^{-1})\underline{X}_L))|\leqq|M_{n,\rho_L}(\underline{X}_L-A^n(x^{-1})\underline{X}_L)|, \tag{2.28}$$

which by (2.2)

$$=\left|M_{n,\rho_L}\left(\int_{J_\rho} d\hat{\mu}_\rho(y)A^n(y)(\underline{X}_L-A^n(x^{-1})\underline{X}_L)\right)\right|\leqq\|M_{n,\rho_L}\|f_L(x), \tag{2.29}$$

where

$$f_L(x)=\left\|\int_{J_\rho} d\hat{\mu}_\rho(y)A^n(y)(\underline{X}_L\dot{-}A^n(x^{-1})\underline{X}_L)\right\|. \tag{2.30}$$

By (2.6)

$$\mathscr{R}M_{n,\rho_L}(A^n(x^{-1})\underline{X}_L)\geqq\|M_{n,\rho_L}\|(1-f_L(x)) \tag{2.31}$$

and by the hypothesis that the norm diverges, then for any positive K

$$\mathscr{R}L^{-n}(\pi^n_{\rho_L}(\underline{X}_L)^l(e)\geqq K\int_G d\gamma_{\rho_L}(x)(1-f_L(x)) \tag{2.32}$$

for L sufficiently large. Note that $f_L(x)=0$ when $x\in J_\rho$, and that since $\|\underline{X}_L\|$ is bounded in L, $\{f_L\}$ is a sequence of uniformly bounded and equicontinuous functions on G. Taking the limit as $L\to\infty$ along a suitable subsequence we find from Proposition 4 that the limit of the right-hand side of (2.32) is simply K. But K is arbitrary, which contradicts Theorem 2A. □

Proof of Theorem 2B. Fix $X_{(i)}$ in $\mathbf{g}^n$. Combining (1.3) and (1.4) and (2.15),

$$L^{-n}(\pi^n_{\rho_L}(X_{(i)}))^l(x) = L^{-n}\int_G d\gamma_{\rho_L}(y)(\pi^n_{\rho_L}(X_{(i)}))^u(xy). \tag{2.33}$$

Note that by (1.1) (respectively (2.2)), the lower (respectively upper) symbols are equal to their $\hat{\mu}$-averages. (Remark: This is just saying that the symbols can be defined as functions on the coadjoint orbit $\Gamma_\rho = \{A^*(x)\rho : x\in G\}$, which is isomorphic with G/J_ρ.) Since, by the previous lemma, the norm of $L^{-n}M_{n,\rho_L}$ is uniformly bounded in L, the sequence of upper symbols is uniformly bounded and equicontinuous on G for all (i) and L. Thus some subsequence converges uniformly to some limit. By application of Proposition 4 to (2.33) and the note above, this limit is simply the limit of the lower symbol. The above argument holds for any convergent subsequence so the theorem is proved. In terms of the M_{n,ρ_L} we have that

$$\lim_{L\nearrow\infty} L^{-n}M_{n,\rho_L} = \underbrace{\rho\otimes\cdots\otimes\rho}_{n}. \quad \square$$

3. The Classical Limit for Free Energies

For our first application, we combine Theorem 2 with the Berezin–Lieb inequalities of Proposition 1 to obtain classical limits of quantum free energies. This procedure is, of course, well understood from other cases where the upper and lower symbols coalesce (see for example [1, 4, 7]).

Fix a maximum weight vector ρ and let $\rho_L = L\rho$. Trivially, the sequence $\{\rho_L\}$ satisfies the convergence condition C with respect to ρ. By the note in the proof of Theorem 2B, we can write the upper and lower symbols as functions on the coadjoint orbit Γ_ρ. Correspondingly we define the measure ν_ρ on Γ_ρ by

$$\nu_\rho(Q\subset\Gamma_\rho) = \mu_\rho(\{x\in G: A^*(x)\rho\in Q\}). \tag{3.1}$$

Let Λ be a finite set. Attach a copy $\mathbf{g}_\alpha$ of $\mathbf{g}$ to each α in Λ and for X_α in $\mathbf{g}_\alpha$ define

$$\tilde{\pi}_\rho(X_\alpha) = \pi_\rho(X_\alpha)\otimes 1_{\Lambda\setminus\alpha}. \tag{3.2}$$

In a Hamiltonian which is polynomial in these operators, the classical limit consists of rescaling the Lie algebras as the dimension of the underlying representation increases, i.e. replacing (3.2) by $\tilde{\pi}_{\rho_L}(L^{-1}X_\alpha)$ and taking the limit as $L\to\infty$. We obtain the class of models with polynomial local interactions. Define the Hamiltonian

$$H^L_\Lambda = \sum_{r=1}^{n}\sum_{\alpha_1\in\Lambda}\sum_{(\alpha_2,\ldots,\alpha_r)\in\Lambda^r_{\alpha_1}} \tilde{\pi}_{\rho_L}(L^{-1}X^r_{\alpha_1})\cdots\tilde{\pi}_{\rho_L}(L^{-1}X^r_{\alpha_n}), \tag{3.3}$$

where for each $\alpha\in\Lambda$, Λ^r_α is a subset of $\Lambda\times\cdots\times\Lambda$ (r − 1 terms) with the property that $|\Lambda^r_\alpha|$ is a constant for all α if $|\Lambda|$ is greater than some Λ_0. We can define both upper and lower symbols on $(\Gamma_\rho)^{\times\Lambda}$ with respect to the measure $d\nu^\Lambda(\underline{\omega}) = \prod_{\alpha\in\Lambda} d\nu_\rho(\omega_\alpha)$. By Theorem 2 we see

$$|(H^L_\Lambda)^l(\underline{\omega}) - (H^L_\Lambda)^u(\underline{\omega})| \leqq |\Lambda|\, p^n_L(\varepsilon_L)(\underline{\omega}), \tag{3.4}$$

where $\{\varepsilon_L\}$ is some sequence of positive numbers with $\lim_{L\to\infty} \varepsilon_L = 0$ and $p^{n,L}_\Lambda(\underline{\omega})$ is some polynomial of order n whose coefficients are functions on $\Gamma^{\times\Lambda}_\rho$ which are uniformly bounded in L. Define free-energies

$$f^\Lambda_L(\beta) = -\frac{1}{\beta|\Lambda|}\log((\dim V_{\rho L})^{-|\Lambda|}\operatorname{trace}\exp - \beta H^\Lambda_L) \tag{3.5}$$

and

$$f^{\Lambda,\#}_L(\beta) = -\frac{1}{\beta|\Lambda|}\log \int_{G^{|\Lambda|}} d\tilde{\nu}^\Lambda(\underline{\omega})\exp - \beta(H^\Lambda_L)^\#(\underline{\omega}), \tag{3.6}$$

where $\tilde{\nu}^\Lambda$ is ν^Λ normalised to mass 1, and # stands for u or l, denoting upper or lower symbol. Combing (3.4) with the Berezin–Lieb inequalities

$$f^{\Lambda,l}_L(\beta) \geqq f^\Lambda_L(\beta) \geqq f^{\Lambda,u}_L(\beta) \geqq f^{\Lambda,l}_L(\beta) + \delta_L \tag{3.7}$$

with $\lim_{L\to\infty} \delta_L = 0$. Let

$$f^\Lambda(\beta) = -\frac{1}{\beta|\Lambda|}\log \int_{G^{|\Lambda|}} d\mu^\Lambda \exp - \beta H^\Lambda(\underline{\omega}), \tag{3.8}$$

where H^Λ is the polynomial in $\underline{\omega}$ obtained by replacing in H^Λ_L each $\tilde{\pi}_{\rho_L}(L^{-1}X^{(r,s)}_\alpha)$ by $\tilde{\pi}_\rho(X^{(r,s)}_\alpha)^l(\omega_\alpha)$. By Theorem 2A,

$$|f^\Lambda_L(\beta) - f^\Lambda(\beta)| \leqq \delta'_L \tag{3.9}$$

for some sequence $\{\delta'_L\}$ coverging to zero. Thus

$$\lim_{L\to\infty} f^\Lambda_L(\beta) = f^\Lambda(\beta). \tag{3.10}$$

Note that since δ_L and δ'_L are independent of Λ, if the existence of a limit of for $f^\Lambda(\beta)$ as $|\Lambda|\to\infty$ implies that the limits $|\Lambda|\to\infty$ and $L\to\infty$ are interchangable for the quantum free energy f^Λ_L.

4. The Thermodynamic Limit for Mean Field Generalised Spin Systems

As a second application of Theorem 2, we obtain a variational expression for the free energy in the thermodynamic limit for mean-field generalised spin systems. We rely heavily on the theory of large deviations [10, 11] and in a particular employ a result proved in [9] about the limiting distribution of multiplicities occurring in the decomposition of tensor product representations into their irreducible components.

We consider the following class of models: fix a maximal weight λ and let H^λ_N be a self-adjoint operator on $(V_{\rho L})^{\otimes N}$ of the form

$$H^\lambda_N = N\sum_{r=1}^{n}\prod_{s=1}^{r}\sum_{\alpha=1}^{N}\tilde{\pi}_\lambda(X^{(r,s)}_\alpha/N) \tag{4.1}$$

with $X^{(r,s)}_\alpha$ identical for fixed (r,s) and equal to $X^{(r,s)}$. Define $h: D\times G\to\mathbb{R}$ by letting

$h(\rho, x)$ be the number obtained by dividing H_N^λ by N and replacing $\tilde{\pi}_\lambda(X_\alpha^{(r,s)}/N)$ by $Q_\rho(X^{(r,s)}, x)$. This function turns out to be the limiting classical form for the Hamiltonian density.

Let j be the natural bijection from $i\mathbf{h}$ its dual $\mathbf{h}^*$ furnished by the Killing form on $\mathbf{h}$. (See e.g. [5].) Define $I^\lambda: \mathbf{h}^* \to [0, \infty]$:

$$I^\lambda(\rho) = \sup_{z \in j^{-1} D \subset ih} \left\{ \rho(z) - \log\left(\frac{\operatorname{trace}(e^z)}{d^\lambda} \right) \right\} \tag{4.2}$$

with $d^\lambda = \dim(V_\lambda)$. Then:

Theorem 6. *Define the free energy* $f_N(\beta) = -(1/N) \log \operatorname{trace} \exp(-\beta H_N^\lambda)$,

$$\lim_{N \to \infty} f_N(\beta) = f(\beta) := \inf_{\rho \in \mathbf{h}^*, x \in G} \{ h(\rho, x) - \tilde{I}^\lambda(\rho) \}, \tag{4.3}$$

where $\tilde{I}^\lambda = \log d^\lambda - I^\lambda$.

Remarks. (1) The effective domain of I^λ is [9] $\mathscr{D} = D \cap \{\lambda - \bar{D}\}$, where $\bar{D}$ is the positive cone generated by the dual basis of the $\{\lambda_i : i \in E\}$. Consequently the ρ-supremum can be restricted to $\mathscr{D}$. (2) $h(\rho, \cdot)$ is constant within each element of G/J, where $J = \bigcap_{i \in E} J_{\lambda_i}$. Thus $h(\rho, \cdot)$ can be written as a function on the coset space ($=$ coadjoint orbit).

For the proof of the theorem we will need the technology of the theory of large deviations.

Definition. Let $\{\mathbb{K}_n : n = 1, 2, \ldots\}$ be a sequence of probability measures on the Borel subsets of a complete separable metric space E and $\{V_n\}$ a divergent sequence of positive numbers. We say that $\{\mathbb{K}_n\}$ satisfies a **Large Deviation Principle** with constants $\{V_n\}$ and rate function $I: E \to [0, \infty]$ if the following conditions hold:

(LD1) I is lower semicontinuous.
(LD2) For each $m < \infty$, $\{x : I(x) \leqq m\}$ is compact.
(LD3) For each closed subset C of E

$$\limsup_{n \to \infty} \frac{1}{V_n} \log \mathbb{K}_n(C) \leqq -\inf_{x \in C} I(x).$$

(LD4) For each open subset G of E

$$\liminf_{n \to \infty} \frac{1}{V_n} \log \mathbb{K}_n(G) \geqq -\inf_{x \in G} I(x).$$

Varadhan's Theorem (1). *Suppose that the sequence of probability measures* $\{\mathbb{K}_n\}$ *on* E *satisfies a large deviation principle with constants* $\{V_n\}$ *and rate function* I. *Let* $\{h_n\}$ *be a sequence of functions* $h_n : E \to \mathbb{R}$ *which are uniformly bounded above, with the property that if* $x_n \to x$ *in* E *then* $\limsup_{n \to \infty} h_n(x_n) \leqq h(x)$ *for some function* h *on* E. *Then*

$$\limsup_{n \to \infty} \frac{1}{V_n} \log \int_E \exp(V_n h_n(x)) \mathbb{K}_n(dx) \leqq \sup_E \{ h(x) - I(x) \}.$$

Varadhan's Theorem (2). *In the above theorem, suppose that in addition the h_n are continuous and converge to $h:E\to\mathbb{R}$ uniformly on bounded sets. Then*

$$\lim_{n\to\infty}\frac{1}{V_n}\log\int_E \exp(V_n h_n(x))\mathbb{K}_n(dx)=\sup_E\{h(x)-I(x)\}.$$

Proof of Theorem 6. Since H_N^λ depends only on the total generalised spin operators:

$$\sum_{\alpha=1}^N \tilde{\pi}_\lambda(X_\alpha^{(r,s)}),$$

we can decompose it in terms of the irreducible representations occurring in the decomposition of the tensor product $\Pi_\lambda^N=\Pi_\lambda\otimes\cdots\otimes\Pi_\lambda$ (N terms). Thus

$$H_N^\lambda=\bigoplus_{\rho\in D_N^\lambda}\bigoplus_{k=1}^{b^\lambda(N,\rho)}\bar{H}_N^{\rho,k},\tag{4.4}$$

where $D_N^\lambda\subset\mathscr{J}$ is the set of maximal weight vectors for the irreducible representations occurring in the decomposition, $b^\lambda(N,\rho)$ is the multiplicity of Π_ρ in the decomposition, and $\bar{H}_N^{\rho,k}$ is a copy of $\bar{H}_N^\rho$, the operator obtained by replacing $\sum_\alpha \pi_\lambda(X_\alpha^{(r,s)}/N)$ with $\pi_\rho(X^{(r,s)}/N)$ in (4.1). Consequently

$$\operatorname{trace} e^{-\beta H_N^\lambda}=\sum_{\rho\in D_N^\lambda} b^\lambda(N,\rho)\operatorname{trace} e^{-\beta\bar{H}_N^\rho}.\tag{4.5}$$

We can bound each term in the sum (4.5) by using the Berezin–Lieb inequalities (1.5), and so

$$d^\rho\int_G d\mu(x)e^{-\beta(\bar{H}_N^\rho)^l(x)}\leqq\operatorname{trace} e^{-\beta\bar{H}_N^\rho}\leqq d^\rho\int_G d\mu(x)e^{-\beta(\bar{H}_N^\rho)^u(x)}.\tag{4.6}$$

We now rewrite the sum (4.5). Firstly define the measures $\{\mathbb{P}_N^\lambda:N=1,2,\ldots\}$ by

$$\mathbb{P}_N^\lambda[A]=\frac{1}{(d^\lambda)^N}\sum_{\rho\in D_N^\lambda:\rho/N\in A} b^\lambda(N,\rho)d^\rho\tag{4.7}$$

for A a Borel subset of D. Secondly, define $h_N^l:D\times G\to\mathbb{R}$ by

$$h_N^l(\rho,x)=\frac{1}{N}(\bar{H}_N^{N\rho})^l(x)\tag{4.8}$$

if $N_\rho\in D_N^\lambda$, and by interpolation elsewhere. Lastly, given $\rho\in D$, let ρ_N to be any element ρ' of $D_N^\lambda\cap D(\rho)$ which minimises $\|N\rho-\rho'\|$, and define $h_N^u:D\times G\to\mathbb{R}$ by

$$h_N^u(\rho,x)=\frac{1}{N}(\bar{H}_N^{(N\rho)_N})^u(x).\tag{4.9}$$

Then

$$(d^\lambda)^N\int_D d\mathbb{P}_N^\lambda(\rho)\int_G d\mu(x)e^{-\beta h_N^l(\rho,x)}\leqq\operatorname{trace} e^{-\beta H_N^\lambda}$$

$$\leqq(d^\lambda)^N\int_D d\mathbb{P}_N^\lambda(\rho)\int_G d\mu(x)e^{-\beta h_N^u(\rho,x)}.\tag{4.10}$$

The point of making this rearrangement is given by the following

Proposition 7. *The family of measures* $\{\mathbb{P}_N^\lambda : N = 1, 2, \ldots\}$ *satisfies a large deviation principle with constants* $\{N\}$ *and rate function* I^λ.

Proof. Define the family of measures $\{\mathbb{L}_N^\lambda : N = 1, 2, \ldots\}$ by

$$\mathbb{L}_N^\lambda = \frac{1}{b^\lambda(N)} \sum_{\rho \in D_N^\lambda : \rho/N \in A} b^\lambda(N, \rho), \tag{4.11}$$

where

$$b^\lambda(N) = \sum_{\rho \in D_N^\lambda} b^\lambda(N, \rho). \tag{4.12}$$

Then [9, Theorem 2.1], the sequence $\{\mathbb{L}_N^\lambda : N = 1, 2, \ldots\}$ satisfies a large deviation principle with constants $\{N\}$ and rate function I^λ. For any Borel subset A of D we have that

$$\frac{b^\lambda(N)}{(d^\lambda)^N} \mathbb{L}_N^\lambda[A] \leqq \mathbb{P}_N^\lambda[A] \leqq \frac{d^{N\lambda} b^\lambda(N)}{(d^\lambda)^N} \mathbb{L}_N^\lambda[A]. \tag{4.13}$$

By [9, Lemma 2.2], both $(d^\lambda)^N / b^\lambda(N)$ and $d^{N\lambda}$ are bounded below by 1 and above by a polynomial in N, and so (LD3) and (LD4) are satisfied by $\{\mathbb{P}_N^\lambda\}$ with constants $\{N\}$ and rate function I^λ. (LD1) and (LD2) carry over trivially, so the proposition is proved. □

Remark. Roughly speaking, Proposition 7 says that $d\mathbb{P}_N^\lambda(\rho) \sim \exp(-NI^\lambda(\rho))d\rho$. $\tilde{I}^\lambda$ can be viewed as a specific entropy density for the limiting distribution of multiplicities, and so (4.3) just expresses energy–entropy balance.

The constant sequence of measures $\{\mu\}$ on G trivially satisfies a large deviation principle on G with constants $\{N\}$ and rate function zero. Using the theorem in [12] for product measures, we see that the sequence $\{P_N^\lambda \otimes \mu\}$ also satisfies a large deviation principle on $D \times G$ with constants $\{N\}$ and rate function I^λ. By Theorem 2A, h_N^l converges uniformly on $D \times G$ to h. Given a sequence $\{(\rho^N, x^N)\}$ in $D \times G$ converging to (ρ, x), then clearly the sequence $\{(N\rho_N)_N\}_{N \geqq N_0}$ satisfies the convergence condition C with respect to ρ for some N_0, and so by Theorem 2B, $h_N^u(\rho_N, x_N)$ converges to $h(\rho, x)$. Using the first (respectively second) version of Varadhan's Theorem to treat the upper (respectively lower) symbols in (4.10):

$$f(\beta) \geqq \limsup_{N \to \infty} f_N(\beta) \geqq \liminf_{N \to \infty} f_N(\beta) \geqq f(\beta), \tag{4.14}$$

and so the theorem is proved. □

Remark. By comparison with [14], it can be seen that this result can be generalised in two directions. Firstly, the strict mean-field nature of the Hamiltonian can be relaxed to include heterogeneous interactions (but still with the mean-field scaling). Secondly, by calculating the gradients of the free-energy, variational expressions can be obtained for the thermodynamic limit of the expectation values of intensive observables.

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