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A Note and Erratum Concerning "Min–Max Theory for the Yang-Mills–Higgs Equations"*

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Abstract. An error in an argument which was used to prove the existence of non-minimal solutions to the $SU(2)$ Yang-Mills–Higgs equations has been shown to the author. A revised proof is presented here to establish the existence of infinitely many non-minimal solutions to the afore-mentioned equations.

1. Introduction

In [T1], I described the topology of the configuration space of finite action pairs (A, Φ) of connection on the principal bundle $\mathbb{R}^3 \times SU(2)$ and section of the associated vector bundle $\mathbb{R}^3 \times$ Lie Alg $SU(2)$. The action functional which defines the topology is the Yang-Mills–Higgs action in the Prasad Sommerfield limit. The space of all finite action pairs, modulo the action of the gauge group $C^\infty(\mathbb{R}^3; SU(2))$, was denoted by B; and I proved that B was homotopy equivalent to the $\Omega^2(S^2)/S^1$, where $\Omega^2(S^2)$ is the space of smooth maps from S^2 to S^2 which take the north pole to itself and the group S^1 acts by rotating the image S^2 about the equator.

Associated to each configuration (A, Φ) is a Dirac operator coupled to the vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$, and I showed that the assignment to each (A, Φ) of this Dirac operator defines a continuous map, δ, from B into the space of Fredholm operators. Proposition C3.1 of [T1] asserts that this map δ is homotopically non-trivial and pulls back non-zero cohomology of arbitrarily high degree from the space of Fredholm operators. Ralph Cohen has shown me an error in the proof of Proposition C3.1, and in fact, he has thrown considerable doubt onto its veracity.

Proposition C3.1 now sits unproved because Lemma C3.2 is erroneous. This lemma claims to construct an embedding of the configuration space C_n of unordered n-tuples of distinct points of $\mathbb{R}^3$ into the monopole number n component, B_n, of B. In fact, the construction in Definition C4.2 provides only an embedding of a fiber bundle over C_n, the fiber being $(x_n S^1)/S^1$, where S^1 acts on the n-torus diagonally. This fiber bundle has no continuous sections—the proof of Lemma C2.1 errs in assuming the existence of a section. This bundle is described for the interested reader at the end of this note.

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With Proposition C3.1 unproved, Theorem A1.2 stands unproved, and thus so does Theorem A1.3 in [T2] which asserts that the full Yang-Mills–Higgs equations in the Prasad–Sommerfeld limit have an infinite set of gauge inequivalent solutions in each component B_n of B which have action greater than any fixed value. The main purpose of this note is to give a proof of Theorem A1.2 of [T1], and thus of Theorem A1.3 of [T2].

2. The Hessian Proof

To prove Theorem A1.3 of [T2], the Dirac operator in the erroneous proof of the theorem will be replaced by another elliptic operator, namely, the hessian operator for the Yang-Mills–Higgs functional which was introduced in [T3]. Using the hessian operator, one can make an argument which is similar, at least in outline, as the failed argument which used the Dirac operator.

For this new argument, introduce from [T1] the space B' which admits an S^1 action whose orbit space is B. The space B' is homotopy equivalent to $\Omega^2(S^2)$. On the one hand, we know from Sect. C.2 of [T1] and from [T3] that the value of the Yang-Mills–Higgs functional on B' determines a bound on the number of eigenvectors of the hessian with small eigenvalue. On the other hand, we have, from topology, that the cohomology of a path component of $\Omega^2(S^2)$ and hence of B' is non-trivial in infinitely many dimensions [M]. Morse theory relates the hessian of a function to the topology of the space, and once a suitable Morse theory has been established for the Yang-Mills–Higgs functional, the preceding two facts give Theorem A1.3.

The point of [T2] was to establish a min–max theory for the Yang-Mills–Higgs functional. The step from min–max theory to Morse theory is a technical one which involves, mostly, the establishment of an appropriate manifold structure on B'.

To begin, let B'_k be the component of B' with monopole number k. We know from [G] that the action functional, a maps B'_k into $[4\pi\cdot|k|, \infty)$, and we know from [T4] that $M'_k \equiv a^{-1}(4\pi|k|)\cap B'_k$ is non-empty. Fix $E>0$ and let B'_{kE} denote the subspace of B'_k on which the a has value no more than E. From [T2], we can deduce

Proposition 1. (1) *The set of critical points of a with critical values in $(4\pi|k|, E]$ is a compact set.* (2) *There exists $E>4\pi|k|$ and a smooth, a-nonincreasing retraction of B'_{kE} onto M'_k.*

Here, the term a-nonincreasing means that the retraction $r:[0,1]\times B'_{kE}\to B'_{kE}$ is such that for fixed c in B'_{kE}, $a(r(\cdot,c))$ is a non-increasing function on $[0,1]$.

Now, recall that the main theorem in [T2] asserts that min–max converges for the functional a. Indeed, if we let F denote a homotopy invariant family of compact subsets of B'_k, and if we let E denote the infimum over the subsets of F of the supremum of a on the subset, then we can deduce from [T2]

Theorem 2. *Given an open set N which contains the set of critical points of a in B'_{kE}, there exists $\delta>0$ and a set U in the family F with the property that $U\subset B'_{kE+\delta}$ and that $U\cap(B'_{kE+\delta}-B'_{kE-\delta})$ lies in N.*

By definition, the gradient of a vanishes at a critical point, so that a is then approximated in a neighborhood by the higher order terms of its Taylor's expansion about the critical point. The second order term is its hessian, so the hessian, if not too degenerate, can be used to study the topology of $B'_{kE+\delta} - B'_{kE-\delta}$. This is, of course, the heart of Morse theory.

To study this hessian, we require a nice local coordinate chart around each critical point; and such a chart is provided in the next proposition. In this proposition, $V \to \mathbb{R}^3$ denotes the vector bundle $(T^*\mathbb{R}^3 \oplus \mathbb{I}) \times su(2)$ and, for $c \in B'$, $H_c(V)$ (with the inner product $\langle \cdot \rangle_c$) is the Hilbert space of sections of V which is defined in Sect. B6 of [T1]. Let $T_c(V)$ denote the (closed) subspace of $H_c(V)$ which consists of pairs (a, φ) which obey the generalized divergence condition $*d_A * a + [\Phi, \varphi] = 0$.

Proposition 3. *Let $c = [A, \Phi] \subset B'_k$ be a critical point of the Yang-Mills–Higgs functional. There exists $\varepsilon > 0$ and a neighborhood N of c and a homeomorphism of N onto the open ball $\{\psi \equiv (a, \varphi) \in C^\infty(V) \cap T_c(V) : \|\psi\|_c < \varepsilon\}$.*

A generalization of this proposition is proved by Floer in [F], but with the a priori estimates of Sect. C of [T2], the proof is much easier than Floer's generalization. Mostly, the proof is a fairly straightforward modification of the local slice argument that is used in Yang-Mills theories on compact manifolds (cf. [F–U]). However, to start, one must know an obscure fact: When an L^2-function f on $\mathbb{R}^3$ is given, there exists a unique, continuous function u on $\mathbb{R}^3$ which obeys $u(0) = 0$, $du \in L^6(T^*\mathbb{R}^3)$, $\nabla du \in L^2$ and the Laplace equation $d^*du = f$.

Proposition 3 allows us to use the results of [T3] to analyze the hessian, *ha*, of a as a bilinear form on $T_c(V)$. When $c \in B'_k$ is a critical point of a, say that $\psi \in T_c(V)$ is an eigenvector of the hessian if there exists a real number λ (the eigenvalue) such that $ha\,(\psi, \eta) = \lambda \cdot \langle \psi, \eta \rangle_c$ for all $\eta \in T_c(V)$. For $\lambda < 1$, elliptic regularity insures that these eigenvectors are a priori smooth. One can deduce from [T3] that:

Proposition 4. *Suppose that $c \in B'_k$ is a critical point of a and that $\lambda < 1$. The number of eigenvectors of $ha|_c$ with eigenvalue in $(-\infty, \lambda]$ is bounded a priori from $a(c)$. Furthermore, if $\psi \in T_c(V)$ is orthogonal to the span of these eigenvectors, then $ha(\psi, \psi) > \lambda \cdot \langle \psi, \psi \rangle_c$.*

The next proposition is also an immediate corollary of [T3] (it is also proved in [F]):

Proposition 5. *The space M_k is a manifold of dimension $4 \cdot |k|$.*

With Proposition A.4.3. of [T2], we now have enough data to establish Theorem A.1.3 of [T2]:

Proof of Theorem A.1.3. Fix E; then using the coordinate system in Proposition 3, it is not hard to perturb a on B'_{kE} to a functional a' which agrees with a in a neighbourhood of the minimal manifold M_k, but whose non-minimal critical set is a finite set of discrete, non-degenerate critical points, all with Morse index bounded a priori by E. (This follows from Proposition 4). Indeed, given an open set which contains the non-minimal critical points of a, one can require that a' agree with a on the compliment of this set.

Now, with this understood, and with Theorem 2, the usual Morse theory arguments can be applied to a' to show that B'_{kE} has the homotopy type of a cell complex with an E-dependent, apriori bound on the dimension of the cells. Hence

$H^p(B'_{kE})$ vanishes for p sufficiently large, and so there can be no retraction of B'_k onto B'_{kE}. Now, Theorem A.1.2 of [T2] yields immediately Theorem A.1.3.

3. The Parameter Space

I will end this note with a description of the parameter space of approximately self-dual monopole configurations. To understand this space, introduce the space T_k of *distinct* k-tuples of points in $\mathbb{R}^3$; $T_k \equiv \otimes_k \mathbb{R}^3 - \Delta$, where Δ is the set of k-tuples of points which are not distinct. For example, $T_2 = \mathbb{R}^3 \times (\mathbb{R}^3 - \{0\})$, with the identification sending (x_1, x_2) to $(x_1 + x_2, x_1 - x_2)$. Remark that the symmetric group on k-letters, Σ_k, acts freely on T_k.

Note that $H^2(T_2) = \mathbb{Z}$ as T_2 retracts onto a 2-sphere. Fix a complex line bundle $L \to T_2$ whose first Chern class generates $H^2(T_2)$. The generator of $\Sigma_2 \approx \mathbb{Z}/2\cdot\mathbb{Z}$ pulls L back to L^{-1}.

One can readily check that $H^2(T_k) \approx \mathbb{Z}^{k(k-1)/2}$. Generators are had by introducing, for each pair of indices (i, j), the map $t_{ij}: T_k \to T_2$ which sends $(x_1, \dots, x_k)$ to (x_i, x_j); then $H^2(T_k)$ is generated freely by $\{t_{ij}{}^* c_1(L)\}_{i<j}$. For $i \neq j$, introduce the line bundle $L_{ij} \equiv t_{ij}{}^* L \to T_k$ and note that $L_{ji} = L_{ij}^{-1}$.

For each index i, define the line bundle $L_i \to T_k$ to be $L_i \equiv \otimes_{j \neq i} L_{ij}$ and introduce the square, L_i^2. Next, introduce the unit circle bundle, $S_i \subset L_i^2$, an S^1 principle bundle over T_k. Observe that the transposition, $x_i \leftrightarrow x_j$ in the symmetric group acts by pull back to interchange S_i and S_j. With this understood, let $\mathbb{T} \equiv \oplus_i S_i \to T_k$ denote the $\otimes_s S^1$ bundle. Pull back defines a lifting of the action of Σ_k on T_k to an action of Σ_k on $\mathbb{T}$. This action commutes with the diagonal action of S^1 on $\mathbb{T}$ and so one can consider the quotients $\mathbb{N}'_k \equiv \mathbb{T}/\Sigma_k$ and $\mathbb{N}_k \equiv \mathbb{T}/(\Sigma_k \times S^1)$.

Careful consideration of Definition C4.2 in [T1] provides

Proposition 6. *There is a generalization of Definition C4.2 which defines a smooth, S^1 equivariant embedding of $\mathbb{N}'_k$ into B'_k. There is a smooth, isotopy of this embedding to an embedding of $\mathbb{N}_k$ into an open subset of M_k.*

The extra circle parameters are not explicit in Definition C4.2. To see them, note that the definition required the choice of a polar coordinate system, (r_i, θ_i, χ_i) centered at each component point x_i of the k-tuple $(x_1, \dots, x_k) \in T_k$. There is a circle's worth of choices for the ray $(\theta_i, \chi_i) = (\pi/2, 0)$ and this ambiguity constitutes the circle in the fiber of S_i—the circle fiber in S_i can be identified with the circle $\chi_i \to \chi_i + \alpha$ for $\alpha \in [0, 2\pi]$.

The isotopy in the proposition refers to the deformation to the self-dual space which is constructed by the author in [T4].

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