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On a Theorem of Deift and Hempel*

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Abstract. We provide an alternative proof of the main result of Deift and Hempel [1] on the existence of eigenvalues of ν-dimensional Schrödinger operators $H_\lambda = H_0 + \lambda W$ in spectral gaps of H_0.

In a beautiful paper, Deift and Hempel [1] proved the existence of eigenvalues of Schrödinger operators $H_\lambda = H_0 + \lambda W$ in spectral gaps of H_0. For the relevance of this result to the theory of the color of crystals, see [1] and the references therein. In this note, we present an alternative proof of their main Theorem 1. We present our proof because of its striking simplicity.

Our hypotheses read:

(H.1) $V \in L^\infty(\mathbb{R}^\nu)$ real-valued, $\nu \in \mathbb{N}$.

(H.2) $W \in L^\infty(\mathbb{R}^\nu)$ real-valued, $\mathrm{supp}(W)$ compact, $W_-(x) \geqq 1$ for

$$x \in B_{\varepsilon_0}(x_0) := \{x \in \mathbb{R}^\nu \,|\, |x - x_0| < \varepsilon_0\} \quad \text{for some} \quad x_0 \in \mathrm{supp}(W_-)$$

and some $\varepsilon_0 > 0$ (here $W_\pm(x) := [|W(x)| \pm W(x)]/2$).

Given (H.1) and (H.2) we define in $L^2(\mathbb{R}^\nu)$ the Schrödinger operators

$$H_0 = -\Delta + V, H_\lambda = H_0 + \lambda W, \lambda \geqq 0 \tag{1}$$

with Δ the Laplacian defined on the standard Sobolev space $H^{2,2}(\mathbb{R}^\nu)$. Without loss of generality, we next modify $W_\pm$ to $\tilde{W}_\pm$ so that

(α) $0 \leqq \tilde{W}_\pm \in L^\infty(\mathbb{R}^\nu)$, $\mathrm{supp}(\tilde{W}_\pm)$ compact,

(β) $W = W_+ - W_- = \tilde{W}_+ - \tilde{W}_-$,

(γ) $\mathrm{supp}(\tilde{W}_+) = \{x \in \mathbb{R}^\nu \,|\, \varepsilon \leqq |x - x_0| \leqq R\} := \Sigma$, where R is chosen so large that

$$\mathrm{supp}(W) \subset B_R(x_0),$$

and where $\varepsilon \leqq \varepsilon_0$ as well as R will be chosen later. Moreover $\tilde{W}_+ \geqq 1$ on Σ.

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Theorem. *Suppose hypotheses* (H.1) *and* (H.2). *Let* $(a,b)\subsetneqq\varrho(H_0)$ *be a spectral gap of* H_0 *and assume* $E_0\in(a,b)$. *Then there exists a sequence of positive numbers* $\lambda_n\uparrow\infty$ *such that* $E_0\in\sigma_p(H_{\lambda_n})$, $n\in\mathbb{N}$.

Proof. (i) In the first step we use the Birman-Schwinger principle (in the form of [2]), i.e.,

$$E_0\in\sigma_p(H_{\lambda_0})\Leftrightarrow\frac{1}{\lambda_0}\in\sigma(K_{\lambda_0}(E_0)) \tag{2}$$

with multiplicities preserved, where

$$K_\lambda(E):=\tilde{W}_-^{1/2}(H_0+\lambda\tilde{W}_+-E)^{-1}\tilde{W}_-^{1/2}. \tag{3}$$

(ii) Secondly, we recall that

$$K_\lambda(E_0)\xrightarrow[\lambda\uparrow\infty]{n}W_-^{1/2}|_{B_\varepsilon}(H_{0,D}^{B_\varepsilon}-E_0)^{-1}W_-^{1/2}|_{B_\varepsilon}:=K_\infty(E_0), \tag{4}$$

where $H_{0,D}^{B_\varepsilon}=-\Delta_D^{B_\varepsilon(x_0)}+V$ in $L^2(B_\varepsilon(x_0);d^\nu x)$. This follows e.g. from [3],

$$(H_0+\lambda\tilde{W}_+-z)^{-1}\xrightarrow[\lambda\uparrow\infty]{n}(-\Delta_D^{\mathbb{R}^\nu\setminus\Sigma}+V-z)^{-1}, z\in\mathbb{C}\setminus\sigma(-\Delta_D^{\mathbb{R}^\nu\setminus\Sigma}+V) \tag{5}$$

and the support properties of $\tilde{W}_-$. [Here Δ_D^Ω denotes the Dirichlet-Laplacian in $L^2(\Omega;d^\nu x)$, $\Omega\subseteq\mathbb{R}^\nu$ open.] By choosing ε and R appropriately, we may assume that $E_0\notin\sigma(-\Delta_D^{\mathbb{R}^\nu\setminus\Sigma}+V)$.

(iii) For $\varepsilon>0$ small enough, $H_{0,D}^{B_\varepsilon}\geqq E_0+1$. By commutation [4],

$$\sigma(K_\infty(E_0))=\sigma((H_{0,D}^{B_\varepsilon}-E_0)^{-1/2}W_-|_{B_\varepsilon}(H_{0,D}^{B_\varepsilon}-E_0)^{-1/2}). \tag{6}$$

By (H.2),

$$(H_{0,D}^{B_\varepsilon}-E_0)^{-1/2}W_-|_{B_\varepsilon}(H_{0,D}^{B_\varepsilon}-E_0)^{-1/2}\geqq(H_{0,D}^{B_\varepsilon}-E_0)^{-1}, \tag{7}$$

and hence by the min-max theorem [5], $K_\infty(E_0)\geqq0$ has (countably) infinitely many positive eigenvalues accumulating at zero.

(iv) Because of (ii), $K_\lambda(E_0)$ is analytic for $\lambda\geqq\Lambda_0$ for some $\Lambda_0>0$.

(v) $\Lambda_0K_{\Lambda_0}(E_0)$ has only finitely many eigenvalues above 1 by compactness. By (iii), as $\lambda\uparrow\infty$, $\lambda K_\lambda(E_0)$ has arbitrarily many eigenvalues above 1. It follows that there are infinitely many $\lambda>\Lambda_0$ to which $1\in\sigma(\lambda K_\lambda(E_0))$.

Remarks. (i) $V,W\in L^\infty(\mathbb{R}^\nu)$ are inessential assumptions. Local singularities can be handled in a standard manner [5].

(ii) While our assumption supp(W) compact has not been used in [1], our result is stronger than their Theorem 1 in the sense that we do not have to worry about "exceptional levels." In the meantime, however, these exceptional levels have also beem removed in [6]. Using involved arguments, entirely different from ours, the author in [6] was also able to dispense with our condition supp(W_-) compact. After completing this work, we were informed by P. Deift that prior to our work he, S. Alama and R. Hempel [7], replaced the condition supp(W) compact by an appropriate falloff of W at infinity. The methods in [7] are generalizations of those in [1] and [6] and substantially different from ours.

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