Commun. Math. Phys. 103, 461–490 (1986)

Communications in
Mathematical
Physics

On the Existence of Eigenvalues of the Schrödinger Operator $H-\lambda W$ in a Gap of $\sigma(H)$

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Abstract. The authors prove a number of results asserting the existence of eigenvalues of the Schrödinger operator $H-\lambda W$ in a gap of $\sigma(H)$, where $H=-\Delta+V$ and V and W are bounded. The existence of these eigenvalues is an important element in the theory of the colour of crystals. The basic theorems are proved in $\mathbb{R}^\nu$; stronger results for $\nu=1$ are also presented.

Introduction

In the quantum theory of crystals one studies periodic Schrödinger operators

$$H=-\Delta+V, \qquad V(x+\underset{\sim}{a})=V(x), \qquad \underset{\sim}{a}\in\Lambda,$$

where Λ is a lattice in $\mathbb{R}^\nu$. The operator H is the basic ingredient in the so-called one-electron model (cf. Reed and Simon [14, Sect. XIII.16, p. 312], Kittel [8], and Ziman [20]) describing the energy levels of an electron in a pure crystal. The main feature of the model is that "allowed" energies for an electron moving in the crystal lattice lie in $\sigma(H)$, the spectrum of H, which consists of *bands*. In a typical insulator, there is a *gap* of some electron volts between the first and second bands, and absorption of photons from incoming electromagnetic radiation is possible *only* if the energy of the photon is greater than the gap (in this theory, all electron states in the first band are assumed to be filled, so that the electron has to "jump" over the gap by absorbing the photon energy). If impurities are present in the crystal, new energy levels may appear in the gaps of $\sigma(H)$ ("impurity levels"), which effectively reduce the width of the gaps and lead to a selective absorption of certain photon energies (cf. Stoneham [18] and Townsend and Kelly [19]). Perhaps the most appealing example in nature is the Al_2O_3 crystal (Corundum), which has a large gap between the first and second bands and is transparent and colourless. By

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1 Research partially supported by NSF Grant No. MCS-8301662
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3 Research supported by Deutsche Forschungsgemeinschaft

replacing some of the Al^{3+}-ions by Cr^{3+} (respectively Ti^{3+}), one obtains Ruby (respectively Sapphire). The impurity levels lead to absorption of green (respectively yellow) light, and the crystal appears in the familiar complementary colour red (respectively blue) (cf. e.g. Ludwig [10, pp. 372f.]).

The situation seems to be much the same for glasses, where one assumes that the "pure glass" is described by $H=-\Delta+V$, where V is no longer periodic, but has some "short range order," which produces gaps in $\sigma(H)$ (cf. Townsend and Kelly [19]). In general, we will not assume V to be periodic.

In this paper, we consider the following model problem for the one-electron theory of solids: let $H=-\Delta+V$ be the Schrödinger operator of the pure "crystal" and let the potential W describe the "impurity."

Question. *Given an energy $E\in\mathbb{R}\backslash\sigma(H)$, does there exist a (real) coupling constant λ so that $E\in\sigma(H-\lambda W)$?*

If $W(x)$ decays and

(i) $W(x)$ is of one sign,

or

(ii) $E<\inf\sigma(H)$,

the problem of the existence of real λ's reduces in a standard way (cf. Reed and Simon [14, p. 99]) to the classical existence theory for real, nonzero eigenvalues of the nonzero, compact, selfadjoint operators $|W|^{1/2}(H-E)^{-1}|W|^{1/2}$ and $(H-E)^{-1/2}W(H-E)^{-1/2}$, respectively (see also Klaus [9]). If E is in a gap, and W changes sign, then the above reduction leads to the *nonselfadjoint* operators $\operatorname{sgn}(W)|W|^{1/2}(H-E)^{-1}|W|^{1/2}$ and $\operatorname{sgn}(H-E)|H-E|^{-1/2}W|H-E|^{-1/2}$ respectively, and the existence of *real*, nonzero eigenvalues no longer follows on abstract grounds. Indeed, even in the case of matrices (cf. also Klaus [9]), for $H:=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $W:=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, the eigenvalues of $H-\lambda W$ are $E=\pm\sqrt{1+\lambda^2}$, which lie outside the gap $(-1,1)$ of $\sigma(H)$ for all real λ.

In the theorems that follow, we will *always* assume that V and W are real-valued, bounded and measurable. We normalize $V(x)\geqq 1$, so that $H\geqq 1$ and if E lies in a gap of $\sigma(H)$, then necessarily $E>1$. The assumption of boundedness for V and W is made mainly for convenience; many of the proofs that follow can be extended to include local singularities by using standard techniques from the theory of Schrödinger operators.

Definition. Let V and W be as above, $H:=-\Delta+V$, and let $S\subset\mathbb{R}$.

(a) We say that the triple (H,W,S) is *complete*, if for each $E\in\mathbb{R}\backslash\sigma(H)$ there exists $\lambda=\lambda(E)\in S$ such that $E\in\sigma(H-\lambda W)$.

(b) Let $\bigcup_{k=1}^{N}O_k$, $1\leqq N\leqq\infty$, be the decomposition of $\mathbb{R}\backslash\sigma(H)$ into a union of disjoint, open intervals (i.e. gaps). We say that the triple (H,W,S) is *essentially complete*, if, for each k, there exists at most one energy $E_k\in O_k$ such that $E_k\notin\bigcup_{\lambda\in S}\sigma(H-\lambda W)$. The E_k's, should they exist, are called *exceptional levels.*

The notion of essential completeness arises in a natural way. As we will see, at the technical level, the proofs of Theorems 1 and 3 below proceed by showing that $\bigcup_{\lambda>0} \sigma(H-\lambda W)$ is dense in $\mathbb{R}\backslash\sigma(H)$; this implies that the triple $(H, W, \mathbb{R}_+)$ is essentially complete, as first observed, in the case where W is relatively compact with respect to $-\Delta$, by Klaus [9]: for, if O_k is a gap in $\sigma(H)$ and $E_k < E_k'$ are two exceptional levels in O_k, then, by denseness, $\sigma(H-\lambda W)\cap(E_k, E_k') \neq \emptyset$, for some $\lambda > 0$. But $\sigma(H)\cap(E_k, E_k') = \emptyset$, so that by continuity of the spectrum, $\sigma(H-\lambda W)$ has to cross either E_k or E_k', as $\lambda\downarrow 0$, which is a contradiction. In other words, if $\bigcup_{\lambda>0} \sigma(H-\lambda W)$ is dense in $\mathbb{R}\backslash\sigma(H)$, there is at most *one* exceptional level per gap.

Exceptional levels can arise even in 1-dimension, if we allow W to be a δ-function. Let E be an energy in a gap of $\sigma(-d^2/dx^2 + V)$ and let $f_\pm$ be the (unique) solutions of $-f_\pm'' + (V-E)f_\pm = 0$ which are square integrable at $\pm\infty$, respectively. The existence of such solutions is a standard result in limit-point/limit-circle theory; note that f_+, f_- are linearly independent as $E \notin \sigma(H)$. Since $E > \inf\sigma(H)$, it is also clear that f_+ must have a zero at some point $x = x_0$. Set $W := \delta_{x_0}$. Any possible L_2-eigenfunction u of $(H-\lambda W)u = Eu$ is necessarily a multiple of f_+ for $x > x_0$, and a multiple of f_- for $x < x_0$. The matching conditions $a_+ f_+(x_0) = a_- f_-(x_0)$, $a_+ f_+'(x_0) = a_- f_-'(x_0) - \lambda a_+ f_+(x_0) = a_- f_-'(x_0)$ now imply that $a_+ = a_- = 0$.

Our results are as follows: In Sect. 1, we prove that $(H, W, \mathbb{R}_+)$ is essentially complete, provided $|x|^2 W(x) \to 0$, $|x| \to \infty$, and $W(x) \geqq h > 0$, for x in some ball B (Theorem 1). We also show that, fi W has compact support and E_0 is an exceptional level, then necessarily E_0 is an eigenvalue of the Dirichlet operator $H = -\Delta + V$ in $\mathbb{R}^\nu\backslash\operatorname{supp} W$ (Theorem 2).

If $W(x) = \text{const} \neq 0$, then clearly $(H, W, \mathbb{R})$ is trivially complete. In Sect. 2, we show that $(H, W, \mathbb{R})$ is essentially complete, provided W does not oscillate too much; see Theorem 3.

In Sect. 3, we consider complex E (but V and W still real) and show that for all but a discrete set of E's belonging to $\varrho(H)$, there exists a $\lambda \in \mathbb{C}$ such that $E \in \sigma(H-\lambda W)$, provided $W(-\Delta+1)^{-1} \in B_q(L_2(R^\nu))$, the q^{th} Schatten ideal, for some $q \in [1, \infty)$ (Theorem 4).

Finally, in Sect. 4, we show that for a variety of situations in 1-dimension, exceptional levels do not occur. In particular, this is true if

(i) $W(x)$ has compact support (Theorem 5),

or

(ii) W changes sign a finite number of times, $\operatorname{sgn} W(x) = \operatorname{sgn} W(-x)$ for x sufficiently large, and W is relatively compact with respect to $-\Delta$ (Theorem 6),

or

(iii) W changes sign precisely once and $|W(x)| \leqq c(1+|x|)^{-p}$, for some $p > 2$ (Theorem 7).

The proof of Theorem 7 involves the interesting question of estimating the growth rate of generalized eigenvalues λ_k of $(H-E)u_k = \lambda_k W u_k$ on $L_2(0,\infty)$, both as $\lambda_k \to +\infty$ and $\lambda_k \to -\infty$, in the case where W is positive and E lies in a gap of $\sigma(H)$. Standard techniques (Reed and Simon [14, Sect. XIII.15]; see also Fleckinger-Pelle [5] and Mingarelli [12]) do not apply as $H-E$ has *essential* spectrum above and below zero and Dirichlet-Neumann-bracketing is no longer useful. We obtain certain "phase space" bounds (see [14, loc. cit.]) for the λ_k's in the body of the text. In the appendix, using *Dirichlet-Dirichlet*-bracketing, we prove a Weyl's law for $\lambda_k \to +\infty$. The precise asymptotics for $\lambda_k \to -\infty$ remains open.

Remark. In all our results (except Theorem 3), W is relatively compact with respect to $-\Delta$ so that $\sigma_{\mathrm{ess}}(H-\lambda W)=\sigma_{\mathrm{ess}}(H)$, for all λ, by Weyl's theorem. This means that if E lies in a gap of $\sigma(H)$ and $E \in \sigma(H-\lambda W)$, then necessarily E is an *eigenvalue* of $H-\lambda W$.

1. W Decays an Infinity

The following Theorem 1 is our main result on the existence of eigenvalues of $H-\lambda W$ in a gap of $\sigma(H)$, where $H:=-\Delta+V$ acts in (real) $L_2(\mathbb{R}^\nu)$, $\nu \in \mathbb{N} = \{1,2,\ldots\}$.

Theorem 1. *Let $\nu \in \mathbb{N}$ and $V, W: \mathbb{R}^\nu \to \mathbb{R}$ be bounded. Suppose that $W(x) \geqq h > 0$, for x in some ball $B \subset \mathbb{R}^\nu$, and that $|x|^2 W(x) \to 0$ as $|x| \to \infty$. Then $(H, W, \mathbb{R}_+)$ is essentially complete.*

The idea of the proof is most easily described in the special case where V is periodic and W has compact support: let E be a fixed energy in a gap (a,b) of $\sigma(H)$ and let Π_n be the parallelepiped built of $(2n+1)^\nu$ lattice cells, centered at 0, and finally let $H_{n,p}:=-\Delta+V$, acting in $L_2(\Pi_n)$, with periodic boundary conditions. Note that any gap in $\sigma(H)$ is also a gap in $\sigma(H_{n,p})$, for any n. By a simple compactness argument [cf. Remark (c) following the proof of Lemma 3 below], there exist $\lambda_n > 0$ and $f_n \in D(H_{n,p})$, $\|f_n\|=1$, such that $(H_{n,p}-\lambda_n W-E) f_n = 0$. As $(a,b) \cap \sigma(H_{n,p}) = \emptyset$, f_n is concentrated near the support of W, and therefore we can find cutoff-functions $\psi_n \in C_0^\infty(\Pi_n)$ such that $(H_{n,p}-\lambda_n W - E)(\psi_n f_n) \to 0$ and $\|\psi_n f_n\| \to 1$ as $n \to \infty$. But $H_{n,p}(\psi_n f_n) = H(\psi_n f_n)$, and it follows that $H-\lambda_n W$ must have an eigenvalue close to E.

When V is not periodic, the ideas outlined above must be modified in a rather substantial way. As we will see, there are two main problems: first, we have to construct approximating operators $\hat{H}_n$ which have the *same* gap (a,b) as H, and second, we have to provide an estimate on the coupling constants λ_n which is complementary to the decay rate of W.

Proof of Theorem 1. We need only consider real $E \notin \sigma(H)$. Without restriction, we may assume $B = B_\varrho$, for some $\varrho > 0$, where $B_\varrho := \{x \in R^\nu; |x| < \varrho\}$. Let $a<b$, $[a,b] \cap \sigma(H) = \emptyset$, $E \in (a,b)$. We first construct an approximating operator $\hat{H}_n$: let $H_n := -\Delta + V$, acting in $L_2(B_n)$, with Dirichlet boundary condition; in other words, H_n is the Friedrichs extension of $-\Delta + V \upharpoonright C_0^\infty(B_n)$. Recall that H_n is self-adjoint and $C_0^\infty(B_n)$ is a form core.

Let E_{ni}, $i=1,\ldots,m(n)$, be the (repeated) eigenvalues of H_n in (a,b), with associated normalized eigenfunctions $u_{ni} \in D(H_n)$, $\int_{B_n} u_{ni}^2 = 1$. Now introduce

$$P_n := \sum_{i=1}^{m(n)} (u_{ni}, \cdot) u_{ni}$$

and

$$\hat{H}_n := H_n + (b-a) P_n .$$

Clearly,

$$\sigma(\hat{H}_n)\cap(a,b)=\emptyset\,. \tag{1.1}$$

By Lemma 3 below, there exists $0<\lambda_n\leqq cn^2$, such that $E\in\sigma(\hat{H}_n-\lambda_n W)$; let $f_n\in D(\hat{H}_n)=D(H_n)$ be a normalized eigenfunction, $\int\limits_{B_n} f_n^2=1$. Using (1.1), we will deduce that f_n is small near ∂B_n in the sense that, for any $0<\alpha<1$,

$$f_n\chi_{B_n\setminus B_{\alpha n}}\to 0\,,\qquad n\to\infty\,, \tag{1.2}$$

(χ_A denotes the characteristic function of a set $A\subset\mathbb{R}^\nu$). To prove (1.2), fix $\alpha\in(0,1)$ and let $\eta\in C^\infty(\mathbb{R}^\nu)$ satisfy $\eta\restriction B_{\alpha/2}=0$, $\eta|(\mathbb{R}^\nu\setminus B_\alpha)=1$, and $0\leqq\eta\leqq 1$ otherwise. Defining $\eta_n:=\eta(x/n)$, it is sufficient to show

$$\eta_n f_n\to 0\,,\qquad n\to\infty\,. \tag{1.3}$$

Let $\delta:=\min\{|E-a|,|E-b|\}$. As $\hat{H}_n$ is selfadjoint,

$$\delta\|\eta_n f_n\|\leqq\|(\hat{H}_n-E)(\eta_n f_n)\|\leqq\|(\hat{H}_n-\lambda_n W-E)(\eta_n f_n)\|+\|\lambda_n W\eta_n f_n\|\,;$$

the last term goes to 0, as $n\to\infty$, since $\|\lambda_n W\eta_n\|_\infty\to 0$, $n\to\infty$, by Lemma 3 and the decay rate of W. Further,

$$\begin{aligned}\|(\hat{H}_n-\lambda_n W-E)(\eta_n f_n)\|&\leqq\|\eta_n(\hat{H}_n-\lambda_n W-E)f_n\|+2\|\nabla\eta_n\nabla f_n\|\\&\quad+\|\Delta\eta_n f_n\|+(b-a)\|[P_n,\eta_n]f_n\|\,,\end{aligned}$$

where $[\cdot,\cdot]$ denotes the commutator. The first term on the right-hand side is zero. The terms $\|\nabla\eta_n\nabla f_n\|$ and $\|\Delta\eta_n f_n\|$ go to zero, as $\|\nabla\eta_n\|_\infty\leqq cn^{-1}$, $\|\Delta\eta_n\|_\infty\leqq cn^{-2}$, $\|f_n\|=1$, $\operatorname{supp}(\nabla\eta_n)\subset B_{\alpha n}\setminus B_{\alpha n/2}$, and $\|(\nabla f_n)|(B_{\alpha n}\setminus B_{\alpha n/2})\|\leqq$const, by Lemma 4 below. Finally, defining $\psi_n:=1-\eta_n$, we have

$$[P_n,\eta_n]f_n=\sum_{i=1}^{m(n)}(u_{ni},\eta_n f_n)\psi_n u_{ni}-\sum_{i=1}^{m(n)}(u_{ni},\psi_n f_n)\eta_n u_{ni}=-[P_n,\psi_n]f_n\,,$$

which goes to 0 since, by Lemma 6 below, $\|\psi_n u_{ni}\|\leqq e^{-\alpha n}$, and $m(n)\leqq cn^\nu$, by Lemma 5, and (1.3) follows. In particular, $\|\psi_n f_n\|\to 1$, $n\to\infty$, and as above

$$(\hat{H}_n-\lambda_n W-E)(\psi_n f_n)=-(\hat{H}_n-\lambda_n W-E)(\eta_n f_n)\to 0\,.$$

Again by Lemmas 5 and 6, it is clear that

$$P_n(\psi_n f_n)\to 0\,,\qquad n\to\infty\,.$$

Now fix $\varepsilon\in(0,\delta)$, $\varepsilon<1$. Then there is $n=n(\varepsilon)$ such that

$$\begin{aligned}&\|\psi_n f_n\|\geqq 1-\varepsilon/2\,,\\ &\|(H-\lambda_n W-E)(\psi_n f_n)\|\leqq\|(\hat{H}_n-\lambda_n W-E)(\psi_n f_n)\|+(b-a)\|P_n(\psi_n f_n)\|\\ &\qquad\leqq\varepsilon/2<\varepsilon\|\psi_n f_n\|\,,\end{aligned}$$

implying that $\sigma(H-\lambda_n W)\cap(E-\varepsilon,E+\varepsilon)\neq\emptyset$, which proves denseness. Essential completeness follows by the continuity argument given in the introduction. □

Remarks. (a) Since we have no uniform bound $\lambda_n \leqq \text{const}$, we cannot hope to obtain a nonzero solution f by defining $f := \text{w-lim} f_n$.

(b) It is clear from the proof of Lemma 3 that in fact there exist an infinite number of λ's such that $\sigma(H-\lambda W) \cap (E-\varepsilon, E+\varepsilon) \neq \emptyset$.

In the following lemmas, we always assume that the conditions of Theorem 1 hold, and we will use the notations of the proof of Theorem 1 without further comment.

The first step in finding $\lambda_n > 0$, $\lambda_n \leqq cn^2$, is an *upper* estimate on the eigenvalues of $\hat{H}_n^{-1/2}(W+\mu E)\hat{H}_n^{-1/2}$ which are greater than or equal to μ, for $\mu > 0$.

Lemma 1. *For $\mu \geqq 0$ and any n, let $\hat{\gamma}_{n1}(\mu) \geqq \hat{\gamma}_{n2}(\mu) \geqq \ldots$ denote the nonnegative eigenvalues of $\hat{H}_n^{-1/2}(W+\mu E)\hat{H}_n^{-1/2}$ ordered by the* min-max *principle. Let $p \in \mathbb{N}$, $p > \nu/2$. Then there exists a constant C such that for $\mu > 0$,*

$$\# \{\gamma_{ni}(\mu); \gamma_{ni}(\mu) \geqq \mu\} \leqq C \cdot n^{\nu}(\|W\|_{\infty} + \mu E)^p \cdot \mu^{-p}, \qquad n \in \mathbb{N}.$$

In particular, for $i \geqq Cn^{\nu}E^p + 1$, the eigenvalue branch $\gamma_{ni}(\mu)$, $\mu \geqq 0$, crosses the diagonal.

Proof. Writing $G_n(\mu) := \hat{H}_n^{-1/2}(W+\mu E)\hat{H}_n^{-1/2}$, we have

$$\mu^p \cdot \# \{\hat{\gamma}_{ni}(\mu); \hat{\gamma}_{ni}(\mu) \geqq \mu\} \leqq \sum_{i \in \mathbb{N}} \hat{\gamma}_{ni}(\mu)^p = \|(G_n(\mu)_+)^p\|_1,$$

where $G_n(\mu)_+$ is the positive part of $G_n(\mu)$ and $\|\cdot\|_1$ denotes the trace norm. Now we can estimate

$$\begin{aligned}\|(G_n(\mu)_+)^p\|_1 &\leqq (\|W\|_{\infty} + \mu E)^p \|\hat{H}_n^{-p}\|_1 \leqq (\|W\|_{\infty} + \mu E)^p \|H_n^{-p}\|_1 \\ &= (\|W\|_{\infty} + \mu E)^p \int_{B_n} G_n^{(p)}(x,x)\,dx,\end{aligned}$$

where $G_n^{(p)}(x,y)$ is the integral kernel of $(-\Delta+1)_n^{-p}$, and $(-\Delta+1)_n$ is $-\Delta+1$, acting in $L_2(B_n)$, with Dirichlet boundary condition. But, by the maximum principle, the integral kernel of $(-\Delta+1)_n^{-1}$ is dominated (pointwise a.e.) by the integral kernel of $(-\Delta+1)^{-1}$. Calling $G^{(p)}(x,y)$ the kernel of $(-\Delta+1)^{-p}$, we see that

$$\int_{B_n} G_n^{(p)}(x,x)\,dx \leqq \int_{B_n} G^{(p)}(x,x)\,dx \leqq cn^{\nu},$$

since

$$G^{(p)}(x,x) = c_p \int_0^{\infty} e^{-t\Delta}(x,x) e^{-t} t^{p-1}\,dt = c_p' \int_0^{\infty} t^{-\nu/2} e^{-t} t^{p-1}\,dt. \quad \square$$

By comparison with an eigenvalue problem on the ball B, we find a *lower* bound for the $\hat{\gamma}_{ni}(\mu)$, which is independent of both n and μ. This is the (only) place where we use that $W \restriction B \geqq h > 0$.

Lemma 2. *Assume that $W \restriction B_{\varrho} \geqq h > 0$. Then there is a constant $c > 0$ such that*

$$\hat{\gamma}_{ni}(\mu) \geqq ci^{-2/\nu}, \qquad n, i \in \mathbb{N}, \tag{1.4}$$

where the $\hat{\gamma}_{ni}(\mu)$'s are defined as in Lemma 1.

Proof. Let $H_\varrho := -\Delta + V$, acting in $L_2(B_\varrho)$, with Dirichlet boundary conditions [i.e., H_ϱ is the Friedrichs extension of $(-\Delta+V)\restriction C_0^\infty(B_\varrho)$], and let $\gamma_{\varrho 1} \geqq \gamma_{\varrho 2} \geqq \dots$ be the positive eigenvalues of H_ϱ^{-1}. We claim that

$$\hat{\gamma}_{ni}(\mu) \geqq \frac{h}{1+b-a}\gamma_{\varrho i}, \qquad n, i \in \mathbb{N}, \tag{1.5}$$

and that there exists a constant $c'>0$ such that

$$\gamma_{\varrho i} \geqq c' i^{-2/\nu}, \qquad i \in \mathbb{N}. \tag{1.6}$$

Clearly, (1.5) and (1.6) imply (1.4). We first prove (1.5): let O_j denote a j-dimensional subspace of a Hilbert space. As $C_0^\infty(B_n)$ is a core of $\hat{H}_n^{1/2}$, we have

$$\begin{aligned}
\hat{\gamma}_{ni}(\mu) &= \inf_{O_{i-1}\subset L_2(B_n)} \sup\left\{\frac{(\hat{H}_n^{-1/2}(W+\mu E)\hat{H}_n^{-1/2}u, u)}{u^2};\ 0 \neq u \in L_2(B_n),\ u \perp O_{i-1}\right\} \\
&= \inf_{O_{i-1} m C_0^\infty(B_n)} \sup\left\{\frac{((W+\mu E)v, v)}{(\hat{H}_n v, v)};\ 0 \neq v \in C_0^\infty(B_n),\ v \perp O_{i-1}\right\} \\
&\geqq \frac{h}{1+b-a} \inf_{O_{i-1}\subset C_0^\infty(B_n)} \sup\left\{\frac{(v,v)}{(H_n v, v)};\ 0 \neq v \in C_0^\infty(B_n),\ v \perp O_{i-1}\right\} \\
&\geqq \frac{h}{1+b-a} \inf_{O_{i-1}\subset C_0^\infty(B_n)} \sup\left\{\frac{(v,v)}{(H_\varrho v, v)};\ 0 \neq v \in C_0^\infty(B_\varrho),\ v \perp O_{i-1}\right\} \\
&= \frac{h}{1+b-a} \inf_{O_{i-1}\subset C_0^\infty(B_\varrho)} \sup\left\{\frac{(v,v)}{(H_\varrho v, v)};\ 0 \neq v \in C_0^\infty(B_\varrho),\ v \perp O_{i-1}\right\} \\
&= \frac{h}{1+b-a}\gamma_{\varrho i},
\end{aligned}$$

which proves (1.5).

By Weyl's law (cf. Reed and Simon [14, Theorem XIII.78]) and the inequality $-\Delta + V \leqq -\Delta + V_\infty$, there exist constants $c>0$ and $\lambda_0 > 0$ such that, for $\lambda \geqq \lambda_0$,

$$\#\{\xi_i;\ \xi_i \text{ eigenvalue of } H_\varrho, 0 < \xi_i \leqq \lambda\} \geqq c\lambda^{\nu/2}.$$

This means that the first $[c\lambda^{\nu/2}]$ eigenvalues $\gamma_{\varrho i}$ of H_ϱ^{-1} satisfy $\gamma_{\varrho i} \geqq \lambda^{-1}$, for $\lambda \geqq \lambda_0$. Hence, there exists an $i_0 \in \mathbb{N}$, such that $\gamma_{\varrho i} \geqq c' i^{-2/\nu}$, for $i \geqq i_0$, and (1.6) follows. □

The existence of a positive $\lambda_n \leqq cn^2$ is now an easy consequence of Lemmas 1 and 2, combined with an argument of Fleckinger-Pelle [5] and Fleckinger and Mingarelli [6] which connects our generalized eigenvalue problem with the spectrum of $\hat{H}_n^{-1/2}(W+\mu E)\hat{H}_n^{-1/2}$.

Lemma 3. *For any n, the indefinite generalized eigenvalue problem $(\hat{H}_n - E)u = \lambda W u$ has an eigenvalue λ_n satisfying the estimate $0 < \lambda_n \leqq cn^2$.*

Proof. For $\lambda > 0$, $(\hat{H}_n - E)u = \lambda W u$, for some $u \neq 0$, is equivalent to

$$\hat{H}_n u = (\lambda W + E)u = \lambda(W + \lambda^{-1}E)u,$$

which in turn is equivalent to

$$\mu \in \sigma(\hat{H}_n^{-1/2}(W+\mu E)\hat{H}_n^{-1/2}), \qquad \mu := \lambda^{-1},$$

i.e. $\mu=\hat{\gamma}_{ni}(\mu)$, for some $i\in\mathbb{N}$. But by Lemma 1, we know that, for any $i\geqq cn^{\nu}$, the (continuous) branch $\hat{\gamma}_{ni}(\mu)$, $\mu\geqq 0$, must cross the diagonal; hence, for $i\geqq cn^{\nu}$, there exists at least one $\mu_i>0$ such that $\hat{\gamma}_{ni}(\mu_i)=\mu_i$. Therefore, by Lemma 2,

$$\mu_i=\hat{\gamma}_{ni}(\mu_i)\geqq c'i^{-2/\nu}, \quad n\in\mathbb{N}, \quad i\geqq cn^{\nu}.$$

Inserting $i=i_n:=[cn^{\nu}]+1$, we get $\mu_{i_n}\geqq c''n^{-2}$, which proves the lemma. □

Remark (c). The fact that for each n there exists $0<\mu\in\sigma(\hat{H}_n^{-1/2}(W+\mu E)\hat{H}_n^{-1/2})$ follows immediately from compactness. Indeed, for $\bar{\mu}>0$, only a *finite* number of branches $\gamma_{ni}(\mu)$, $\mu\geqq\bar{\mu}$, can lie above the diagonal. On the other hand, there exists an *infinite* number of $\gamma_{ni}(0)>0$, since the quadratic form $(\hat{H}_n^{-1/2}W\hat{H}_n^{-1/2}\phi,\phi)$ is strictly positive on the ∞-dimensional space $\hat{H}_n^{1/2}(C_0^{\infty}(B_{\varrho}))$. The point of Lemmas 1–3 is to provide estimates which are "complementary" to the decay rate of W for the proof of Theorem 1. In the case where W has compact support, these estimates are not necessary [cf. Theorem 5 and Remark (c), following the proof of Theorem 5].

The following method for obtaining a gradient estimate is taken from Lemma C.2.1 in Simon [17].

Lemma 4. *Let* $0<\beta'<\beta<1$, $A_n:=B_{\beta n}\backslash B_{\beta' n}$, *and* $f_n\in L_2(B_n)$ *be a solution of* $(\hat{H}_n-E)f_n=\lambda_n W f_n$, *with* $\|f_n\|=1$ *and* $|\lambda_n|\leqq cn^2$. *Then*

$$\|(\nabla f_n)\restriction A_n\|\leqq C=C(\beta,\beta').$$

Proof. Choose a cutoff-function ϕ satisfying $\phi(x)=1$, for $\beta'\leqq|x|\leqq\beta$, $\phi(x)=0$, for $|x|<\beta'/2$ and for $|x|>(\beta+1)/2$, $0\leqq\phi(x)\leqq 1$ else; let $\phi_n(x):=\phi(x/n)$. As in Lemma C.2.1 of Simon [17], the identity

$$\operatorname{div}(\phi_n f_n\nabla f_n-\tfrac{1}{2}\nabla\phi_n f_n^2)=\phi_n(\nabla\zeta_n)^2+\phi_n f_n\Delta f_n-\tfrac{1}{2}\Delta\phi_n f_n^2,$$

yields

$$\int_{B_n}\phi_n|\nabla f_n|^2\leqq\tfrac{1}{2}\int_{B_n}|\Delta\phi_n|f_n^2+\int_{B_n}\phi_n|f_n\Delta f_n|.$$

But $\|\Delta\phi_n\|_{\infty}\leqq c\cdot n^{-2}$, $\|f_n\|=1$, and

$$\begin{aligned}\|(\Delta f_n)|\operatorname{supp}\phi_n\|&\leqq\|\nabla f_n\|+\|\lambda_n W|\operatorname{supp}\phi_n\|_{\infty}\cdot\|f_n\|\\&\quad+E\|f_n\|+(b-a)\|P_n\|\cdot\|f_n\|\leqq\text{const},\end{aligned}$$

as W decays quadratically □

To control the projection operator P_n, we first give an estimate on $m(n)$ using Weyl's law, and then we show (in Lemma 6), that the eigenfunctions u_{ni} of H_n live close to ∂B_n, the boundary of B_n. More precisely, the exponential decay of the resolvent kernel of $(H-E)^{-1}$ implies that the u_{ni}'s are exponentially small on balls $B_{\beta n}$, $0<\beta<1$, as $n\to\infty$.

Lemma 5. *Let* $m(n)$ *be the number of (repeated) eigenvalues of* H_n *lying in* (a,b). *Then there exists a constant* c *such that* $m(n)\leqq cn^{\nu}$.

Proof. Let Δ_n be the Dirichlet operator on B_n. Since $V\geqq 1$, we have

$$\begin{aligned}m(n)&\leqq\#\{\lambda_i;\,\lambda_i\text{ eigenvalue of }-\Delta_n,\lambda_i\leqq b\}\\&=\#\{\lambda_i';\,\lambda_i'\text{ eigenvalue of }-\Delta_1,\lambda_i'\leqq bn^2\},\end{aligned}$$

by scaling. By Weyl's law (cf. Reed and Simon [14, Theorem XIII.78]), the last number is bounded by $c'(n^2b)^{\nu/2}$. □

Lemma 6. *Let u_{ni}, $i=1,\ldots,m(n)$, be the normalized eigenfunctions, $\int_{B_n} u_{ni}^2=1$, associated with the eigenvalues E_{ni} of H_n lying in (a,b), where $[a,b]\cap\sigma(H)=\emptyset$. Let $\beta\in(0,1)$. Then there exist $\alpha=\alpha(\beta)>0$ and $n_0=n_0(\beta)>0$ such that*

$$\|u_{ni}\chi_{B_{\beta n}}\|\leqq e^{-\alpha n}, \qquad n\geqq n_0, \qquad i=1,\ldots,m(n).$$

Proof. For $n\geqq 2$, let $\zeta_n\in C_0^\infty(B_n)$ be such that $\zeta_n\restriction B_{n-1}=1$, $0\leqq\zeta_n\leqq 1$ else, and $\|\nabla\zeta_n\|_\infty+\|\Delta\zeta_n\|_\infty\leqq C_0$, with C_0 independent of n. Then, for $i=1,\ldots,m(n)$,

$$(H_n-E_{ni})(\zeta_n u_{ni})=2\nabla\zeta_n\nabla u_{ni}+\Delta\zeta_n u_{ni}=:h_{ni}.$$

But $\|\nabla u_{ni}\|^2=\int_{B_n}(-V+E_{ni})u_{ni}^2\leqq\text{const}$, implying $\|h_{ni}\|\leqq c$, $n\geqq 2$, $i=1,\ldots,m(n)$. Clearly, $\zeta_n u_{ni}\in D(H)$, and $(H-E_{ni})(\zeta_n u_{ni})=(H_n-E_{ni})(\zeta_n u_{ni})$. As $E_{ni}\in\varrho(H)$, we get

$$\zeta_n u_{ni}=(H-E_{ni})^{-1}h_{ni}, \qquad n\geqq 2, \qquad i=1,\ldots,m(n) \tag{1.7}$$

with

$$\|h_{ni}\|\leqq c \quad\text{and}\quad \operatorname{supp} h_{ni}\subset B_n\backslash B_{n-1}.$$

Since we want an estimate for $\|u_{ni}\chi_{B_{\beta n}}\|$, and h_{ni} is supported in $B_n\backslash B_{n-1}$, the exponential decay in $|x-y|$ of the integral kernel $G(x,y;E_{ni})$ of $(H-E_{ni})^{-1}$ (which is *uniform* in $E\in[a,b]$) will give the desired result. For example, Eq. (4) of Simon [17, Theorem B.7.2] tells us that $|G(x,y;E)|\leqq Ce^{-\alpha|x-y|}$, for $|x-y|\geqq 1$. Now, for $n\geqq n_0$, we have $\zeta_n u_{ni}\restriction B_{\beta n}=u_{ni}\restriction B_{\beta n}$, and it follows from (1.7) that

$$|u_{ni}(x)|\leqq\int_{B_n\backslash B_{n-1}}|G(x,y;E_{ni})||h_{ni}(y)|dy\leqq Ce^{-\alpha v}, \qquad x\in B_{\beta n}.$$

Since the proof of Theorem B.7.2 in Simon [17] is rather lengthy and complicated, we include the following direct argument for the reader's convenience.

For $\varepsilon\in R^v$, we have $e^{-\varepsilon\cdot x}(H-E)e^{\varepsilon\cdot x}=H-E-2\varepsilon\cdot\nabla-\varepsilon^2$ [for simplicity of notation, we write $\varepsilon\cdot x$ instead of (ε,x)]. As $\|\nabla(H-E)^{-1}\|$ is uniformly bounded for $E\in[a,b]$, and

$$H-E-2\varepsilon\cdot\nabla-\varepsilon^2=(1-(2\varepsilon\cdot\nabla+\varepsilon^2)(H-E)^{-1})(H-E),$$

it follows that, for $|\varepsilon|\leqq\varepsilon_0$, we have $[a,b]\subset\varrho(e^{-\varepsilon\cdot x}He^{\varepsilon\cdot x})$, and

$$\|e^{-\varepsilon\cdot x}(H-E)^{-1}e^{\varepsilon\cdot x}\|=\|(e^{-\varepsilon\cdot x}He^{\varepsilon\cdot x}-E)^{-1}\|\leqq C, \qquad E\in[a,b], \qquad |\varepsilon|\leqq\varepsilon_0. \tag{1.8}$$

Clearly, (1.8) still holds true if we replace $e^{\pm\varepsilon\cdot x}$ by $e^{\pm\varepsilon\cdot(x-x_k)}$, and the constant will not depend on $x_k\in R^v$. Now we split $B_n\backslash B_{n-1}$ into sets D_k of diameter $\leqq 2$:

$$B_n\backslash B_{n-1}=\bigcup_{k=1}^{s(n)}D_k,$$

where $D_k\cap D_l=\emptyset$, $k\neq l$, $\operatorname{diam}D_k\leqq 2$ and $s(n)\leqq c\cdot n^{v-1}$; also let $y_k\in D_k$. Then

$$\begin{aligned}\|\chi_{B_{\beta n}}(H-E_{ni})^{-1}h_{ni}\|^2&\leqq cn^{v-1}\sum_{k=1}^{s(n)}\|\chi_{B_{\beta n}}(H-E_{ni})^{-1}(h_{ni}\chi_{D_k})\|^2\\&\leqq cn^{v-1}\sum_{k=1}^{s(n)}\|\chi_{B_{\beta n}}e^{\varepsilon_k\cdot(x-y_k)}e^{-\varepsilon_k\cdot(x-y_k)}(H-E_{ni})^{-1}\\&\quad\cdot e^{\varepsilon_k\cdot(x-y_k)}e^{-\varepsilon_k\cdot(x-y_k)}h_{ni}\chi_{D_k}\|^2=:D,\end{aligned}$$

where we have chosen $\varepsilon_k := \varepsilon_0 \|y_k\|^{-1} y_k$. But for $x \in B_{\beta n}$, $\varepsilon_k \cdot (x - y_k) \leqq -\varepsilon_0(n - \beta n - 1)$, and for $y \in D_k$, $e^{-\varepsilon_k \cdot (y - y_k)} \leqq e^{2\varepsilon_0}$. Now, using (1.8), we may estimate

$$\begin{aligned} D &\leqq C' n^{\nu-1} \sum_{k=1}^{s(n)} \|\chi_{B_{\beta n}}(x) e^{\varepsilon_k \cdot (x - y_k)}\|_\infty^2 \cdot \|e^{-\varepsilon_k \cdot (x - y_k)} (H - E_{ni})^{-1} e^{\varepsilon_k \cdot (x - y_k)}\|^2 \\ &\quad \cdot \|e^{-\varepsilon_k \cdot (x - y_k)} h_{ni} \chi_{D_k}\|^2 \\ &\leqq C' n^{\nu-1} e^{-2\varepsilon_0(n - n\beta - 1)} C^2 e^{4\varepsilon_0} \sum_{k=1}^{s(n)} \|h_{ni} \chi_{D_k}\|^2 . \quad \square \end{aligned}$$

In the case where W has compact support and is sufficiently regular, we can characterize exceptional levels as Dirichlet eigenvalues of the operator $-\Delta + V$ in $L_2(\mathbb{R}^\nu \backslash \operatorname{supp} W)$. The basic piece of information needed for the characterization is obtained by looking at eigenvalue branches emerging from the *lower* edge of a gap and close to the exceptional level E_0. Here one finds λ_n, E_n, and u_n such that $E_n \uparrow E_0$, $(H - \lambda_n W - E_n) u_n = 0$ *and* $(W u_n, u_n) \leqq 0$, so that $\|\nabla u_n\|^2 \leqq \text{const}$; in particular, we find a function u such that $u_n \to u$, weakly in $H_2^1(\mathbb{R}^\nu)$.

Theorem 2. *In addition to the assumptions of Theorem 1, suppose that W has compact support, $\mathbb{R}^\nu \backslash \operatorname{supp} W$ satisfies the segment condition and* $\operatorname{meas}\{x \in \operatorname{supp} W;\, W(x) = 0\} = 0$. *Let Ω be the unbounded component of $R^\nu \backslash \operatorname{supp} W$, and let $H_\Omega := -\Delta + V$, acting in $L_2(\Omega)$ with Dirichlet boundary conditions. Then we have: if E_0 is real and does not belong to $\bigcup_{\lambda \geqq 0} \sigma(H - \lambda W)$, then E_0 is an eigenvalue of H_Ω.*

Proof. (a) Let (a, b) be the gap in $\sigma(H)$ around E_0, let $E_n \uparrow E_0$, $E_n \in (a, E_0)$, and let λ_n be the smallest (positive) coupling constant such that $E_n \in \sigma(H - \lambda_n W)$; the existence of λ_n is guaranteed by Theorem 1. By regular perturbation theory, there exist $\varepsilon_n > 0$ and analytic functions $u_n(\lambda)$, $E_n(\lambda)$, defined for λ in $(\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n)$, such that

$$(H - \lambda W) u_n(\lambda) = E_n(\lambda) u_n(\lambda), \qquad \|u_n(\lambda)\| = 1, \tag{1.9}$$

and $E_n(\lambda_n) = E_n$. Now, $E_n'(\lambda_n) > 0$; otherwise, for some $0 < \bar{\lambda}_n < \lambda_n$, we would have $E_n(\bar{\lambda}_n) > E_n(\lambda_n) = E_n$. On the other hand, as λ decreases to 0, the branch $E_n(\cdot)$ can be continued until it is absorbed at the point a [note that $E_n(\cdot)$ cannot cross the level E_0]. It follows that there would exist $0 < \lambda_n' < \bar{\lambda}_n < \lambda_n$ such that $E_n(\lambda_n') = E_n(\lambda_n)$, contradicting the minimality assumption on λ_n.

Hence

$$(W u_n(\lambda_n), u_n(\lambda_n)) = -E_n'(\lambda_n) \leqq 0 .$$

But then, writing $u_n := u_n(\lambda_n)$,

$$\|\nabla u_n\|^2 + \left(\int V u_n^2\right) - E_n = \lambda_n (W u_n, u_n) \leqq 0 ,$$

which implies $\|\nabla u_n\|^2 \leqq \|V\|_\infty + b$. Hence there exists a function $u \in H_2^1(\mathbb{R}^\nu)$ such that $u_{n_j} \to u$ weakly in $H_2^1(\mathbb{R}^\nu)$, $u_{n_j} \to u$ in $L_{2,\mathrm{loc}}(\mathbb{R}^\nu)$, and $u_{n_j} \to u$, a.e., for a suitable subsequence $(u_{n_j}) \subset (u_n)$, which we will again call (u_n).

(b) Now first suppose that the λ_n's are bounded. Then it is clear from (1.9) that $\|\Delta u_n\| \leqq \text{const}$, and it follows that $u_n \to u$ weakly in $H_2^2(\mathbb{R}^\nu)$, and hence $(H - \lambda W) u$

$=E_0u$, for some $\lambda<\infty$. But $u\neq 0$; otherwise we would have $u_n\upharpoonright\operatorname{supp}W\to 0$ and $\lambda_n Wu_n\to 0$, in contradiction to

$$\|\lambda_n Wu_n\|=\|(H-E_n)u_n\|\geqq \operatorname{dist}(E_n,\sigma(H))\geqq\delta>0\,,\qquad n\geqq n_0\,.$$

So we see that u satisfies $(H-\lambda W)u=E_0u$, $u\neq 0$, contradicting the assumption about E_0.

(c) Assume $\lambda_n\to\infty$. As $((H-E)\phi,u_n)=\lambda_n(W\phi,u_n)$, $\phi\in C_0^\infty(\mathbb{R}^\nu)$, we see that $\int W\phi u=0$, for all $\phi\in C_0^\infty(\mathbb{R}^\nu)$ and hence $u\upharpoonright\operatorname{supp}W=0$, a.e. (here we use the assumption $\operatorname{meas}\{x\in\operatorname{supp}W;\,W(x)=0\}=0$). We claim that

$$u\upharpoonright\Omega\in\mathring{H}_2^1(\Omega)\,,\tag{1.10}$$

$$u\upharpoonright\Omega\in H_2^2(\Omega)\,,\tag{1.11}$$

$$u\neq 0\,.\tag{1.12}$$

Then, clearly, $u\in D(H_\Omega)=\mathring{H}_2^1\cap H_2^2(\mathbb{R}^\nu)$ and, for $\phi\in C_0^\infty(\Omega)$,

$$(u,(H_\Omega-E_0)\phi)=\lim(u_n,(H-E_n)\phi)=\lim(u_n,\lambda_n W\phi)=0\,,$$

implying $H_\Omega u=E_0u$, $u\neq 0$. It remains to show (1.10)–(1.12). Clearly, (1.11) follows from the inequality

$$\|\Delta u_n\upharpoonright\Omega\|=\|(E_n-V)u_n\upharpoonright\Omega\|\leqq E_0+\|V\|_\infty\,.$$

To show (1.12), note that $u=0$ would imply $u_n\to 0$ in $L_{2,\mathrm{loc}}(\mathbb{R}^\nu)$. Now let $\eta_n\in C^\infty(\mathbb{R}^\nu)$ be the cutoff-function used in the proof of Theorem 1. Then, with $\delta>0$ as above,

$$\delta\|\eta_m u_n\|\leqq\|(H-E_n)(\eta_m u_n)\|\leqq 2\|\nabla\eta_m\|_\infty\|\nabla u_n\|+\|\Delta\eta_m\|_\infty\|u_n\|\leqq c\cdot m^{-1}\,,$$
$$n\geqq n_0\,,\qquad m\geqq m_0\,.$$

But, as $(1-\eta_m)u_n\to 0$, $n\to\infty$, the above inequality is incompatible with $\|u_n\|=1$.

Finally, (1.10) is clear from Lemma 7 below, which we apply to $G:=\Omega$ and $v\upharpoonright\Omega:=u$, $v\upharpoonright(\mathbb{R}^\nu\backslash\Omega):=0$; as Ω satisfies the segment condition, it is easy to check that $v\in H_2^1(\mathbb{R}^\nu)$. □

Lemma 7. *Let $G=\mathring{G}\subset\mathbb{R}^\nu$ satisfy the segment condition, and let $v\in H_2^1(\mathbb{R}^\nu)$, $v\upharpoonright\mathbb{R}^\nu\backslash G\equiv 0$. Then $v\upharpoonright G\in\mathring{H}_2^1(G)$.*

The proof of this lemma is standard (cf. Adams [1, Proof of Theorem 3.18, p. 54f.]) and omitted.

2. The Oscillation of W is Small

The following Theorem 3 describes a different mechanism to produce spectrum of $H-\lambda W$ in a gap of $\sigma(H)$. As mentioned in the introduction, $(H,W,\mathbb{R})$ is trivially complete if W is a nonzero constant. As we will see, $(H,W,\mathbb{R})$ is essentially complete if W does not oscillate too much, in a sense to be made precise below. The main idea of the proof is to construct approximate eigenfunctions which live in regions where W is nearly constant.

Recall that, if $\Omega \subset \mathbb{R}^\nu$, $W: \mathbb{R}^\nu \to \mathbb{R}$, then

$$\operatorname*{osc}_{\Omega} W := \sup_{\Omega} W - \inf_{\Omega} W.$$

Theorem 3. *Let $V, W: \mathbb{R}^\nu \to \mathbb{R}$ be bounded and assume that, for any $n \in \mathbb{N}$,*

$$\operatorname*{osc}_{A_{ni}} W \Big/ \inf_{A_{ni}} W \to 0, \qquad i \to \infty, \tag{2.1}$$

where $A_{ni} := \{x \in \mathbb{R}^\nu; (i-1)n \leqq |x| \leqq (i+2)n\}$. Then $(H, W, \mathbb{R})$ is essentially complete.

Remarks. (a) It is an easy exercise to check that if (2.1) holds, then W is eventually of one sign.

(b) Condition (2.1) is satisfied, for example, if $W(x) = c_1(c_2 + |x|)^{-\alpha}$ for $|x| \geqq R_0$ and some $\alpha > 0$.

(c) If V is periodic, condition (2.1) can be replaced by the weaker condition

$$\operatorname*{osc}_{B_n(x_n)} W \Big/ \inf_{B_n(x_n)} |W| \to 0, \qquad n \to \infty, \tag{2.2}$$

where $x_n \to \infty$ and $B_n(x_n) = \{x \in \mathbb{R}^\nu; |x - x_n| < n\}$, and where W has one sign in each ball $B_n(x_n)$; see also Remark (d) below.

The proof of Theorem 3 is based on the following Lemma 8, which constructs sequences, similar to Weyl (singular) sequences, with special support properties. We defer the proof of this lemma to the end of the section.

Lemma 8. *Let $V: \mathbb{R}^\nu \to \mathbb{R}$ be bounded, $H := -\Delta + V$ acting in $L_2(\mathbb{R}^\nu)$. Then, for each $n = 1, 2, \ldots$, there exist sequences $(i_{nk})_{k \in \mathbb{N}} \subset \mathbb{N}$, $i_{nk} \to \infty$ as $k \to \infty$, $(\lambda_{nk})_{k \in \mathbb{N}} \subset \sigma(H)$, $|\lambda_{nk}| \leqq C_1$, and $(u_{nk})_{k \in \mathbb{N}} \subset D(H)$, $\|u_{nk}\| = 1$, such that $\|(H - \lambda_{nk})u_{nk}\| \leqq C_0 n^{-1}$, and $\operatorname{supp} u_{nk} \subset A_{n, i_{nk}}$, with c_0, c_1 independent of n, k, and A_{ni} being defined as in Theorem 3.*

Remark (d). If V is periodic and W satisfies condition (2.2), then it is rather easy to construct a singular sequence (u_n) such that $\|(H - \lambda)u_n\| \to 0$, for some $\lambda \in \sigma_{\mathrm{ess}}(H)$, $\|u_n\| = 1$, and $\operatorname{supp} u_n \subset B_n(x_n)$, where $x_n \to \infty$ as $n \to \infty$ (cf. e.g. Hempel [7, proof of Theorem 3.1]). This construction would then replace (the rather complicated) Lemma 8.

Proof of Theorem 3. Let $(a, b) \cap \sigma(H) = \emptyset$, and assume, without loss of generality, that $W(x) > 0$ for $|x| > R_0$. With the notation of Lemma 8 and the definition

$$\mu_{nk} := (\lambda_{nk} - E)\Big(\inf_{A_{n, i_{nk}}} W\Big)^{-1},$$

we have, for $n = 1, 2, \ldots$,

$$\begin{aligned}
\|(H - \mu_{nk}W - E)u_{nk}\| &\leqq \|(H - \lambda_{nk})u_{nk}\| + \|(\lambda_{nk} - \mu_{nk}W - E)u_{nk}\| \\
&\leqq c_0 n^{-1} + |\lambda_{nk} - E| \cdot \Big\|1 - W\Big(\inf_{A_{n, i_{nk}}} W\Big)^{-1}\Big\|_\infty \cdot \|u_{nk}\| \\
&\leqq c_0 n^{-1} + c_1 \operatorname*{osc}_{A_{n, i_{nk}}} W \Big/ \inf_{A_{n, i_{nk}}} W \to C_0 n^{-1}, \qquad k \to \infty,
\end{aligned}$$

by (2.1) and Lemma 8. As in the proof of Theorem 1, this implies that $(a, b) \cap \sigma(H - \mu_{nk}W) \neq \emptyset$, for n, k sufficiently large. Finally, denseness together with continuity of the spectrum gives the result (cf. the remarks in the introduction). □

The idea behind Lemma 8 is simply to take a generalized eigenfunction of $-\Delta+V$ and use a cutoff-procedure to get the right support properties. In order to control the error introduced by cutting off the eigenfucntion, we first show, following ideas of Sch'nol [16] (see also Simon [17, p. 501]) that there exist "enough" regions A_{nk} such that cutoff is possible with a function $\psi_{nk}\in C_0^\infty(\mathbb{R}^\nu)$, $\psi_{nk}\upharpoonright A_{nk}=1$.

Proof of Lemma 8. Let $n\in\mathbb{N}$ be fixed, and define

$$C_i:=\{x\in\mathbb{R}^\nu; ni\leqq|x|\leqq n(i+1)\}.$$

By the generalized eigenfunction expansion form of the spectral theorem (cf. e.g. Simon [17, Corollary C.5.5]), there exist polynomially bounded solutions $\phi(x,\lambda)$ of $(-\Delta+V)\phi=\lambda\phi$, for $\lambda\in\sigma(-\Delta+V)$ (spectrally a.e.). We have to consider two main cases:

Case A. Suppose, that there exists at least one such $\phi\neq 0$ which is *not* an $L_2(\mathbb{R}^\nu)$-function.

Consider the sequence (α_i), given by

$$\alpha_i:=\alpha_i(\phi):=\int_{C_i}|\phi(x,\lambda)|^2dx. \tag{2.3}$$

Note, that by unique continuation, $\alpha_i>0$. We will now show that there is a sequence of integers (i_k), $i_k\to\infty$ as $k\to\infty$, such that

$$\alpha_{i_k}\geqq\tfrac{1}{2}\max\{\alpha_{i_k-1},\alpha_{i_k+1}\},\qquad k\in\mathbb{N}. \tag{2.4}$$

The sequence (α_{i_k}) is *either* eventually monotonic (nonincreasing or nondecreasing) *or* has an infinite number of local maxima $\neq 0$. In the latter case, (2.4) follows immediately.

Assume (α_i) is eventually nondecreasing. Then, for an infinite number of indices i_k we must have $\alpha_{i_k+1}\leqq 2\alpha_{i_k}$; otherwise $\int_{i_0\leqq|x|\leqq i}|\phi|^2\geqq c\cdot 2^{i-i_0}$, $i\geqq i_0$, which contradicts the polynomial bound on ϕ.

Finally, if (α_i) is eventually nonincreasing, then, for an infinite number of i_k, we must have

$$\alpha_{i_k-1}\leqq 2\alpha_{i_k};$$

otherwise we would have $\phi\in L_2(\mathbb{R}^\nu)$, contradiction! Hence (2.4) is proven.

Now let $\chi_i:=\chi_{in}$ be the characteristic function of the set

$$\{x\in\mathbb{R}^\nu;(i-1/3)n\leqq|x|\leqq(i+4/3)n\}$$

and $\psi_i:=\psi_{in}:=\chi_i * j_{n/3}$, where j_ε is the usual Friedrichs mollifier. In particular, $\psi_i\upharpoonright C_i=1$ and $\|\nabla\psi_i\|_\infty\leqq cn^{-1}$, $\|\Delta\psi_i\|_\infty\leqq cn^{-1}$, c independent of n,i. We define

$$u_{nk}:=\psi_{i_k}\phi(\cdot,\lambda)\cdot\|\psi_{i_k}\phi\|^{-1},\qquad \lambda_{nk}:=\lambda,\qquad i_{nk}:=i_k,$$

with (i_k) as in (2.4). Then, clearly, $\operatorname{supp}u_{nk}\subset\operatorname{supp}\psi_{i_k}\subset A_{n,i_k}$, and

$$\begin{aligned}\|(H-\lambda_{nk})u_{nk}\|&\leqq[2\|\nabla\psi_{i_k}\|_\infty\|\nabla\phi\upharpoonright\operatorname{supp}\nabla\psi_{i_k}\|+\|\Delta\psi_{i_k}\|_\infty\|\phi\upharpoonright\operatorname{supp}\psi_{i_k}\|]\cdot\|\psi_{i_k}\phi\|^{-1}\\&\leqq c'\cdot n^{-1}[\|\nabla\phi\upharpoonright\operatorname{supp}\nabla\psi_{i_k}\|+\sqrt{5\alpha_{i_k}}]\cdot\alpha_{i_k}^{-1/2}.\end{aligned}$$

By the arguments in the proof of Lemma 4, Sect. 1, we can estimate

$$\|\nabla\phi \upharpoonright \operatorname{supp} \nabla\psi_{i_k}\| \leqq C\|\phi \upharpoonright A_{n,i_k}\| \leqq C\sqrt{5\alpha_{i_k}}\,,$$

where C can be chosen independent of k and n. Hence $\|(H-\lambda_{nk})u_{nk}\| \leqq c'' n^{-1}$, with C'' independent of k, n.

Case B. Suppose that *all* $\phi(x,\lambda)$ which occur in the eigenfunction expansion theorem are L_2-functions. In this case necessarily H has pure point spectrum and all the ϕ's are ordinary eigenfunctions; in particular, $(\phi(\cdot,\lambda),\phi(\cdot,\lambda'))=0$, for $\lambda \neq \lambda'$. As V is bounded, $\sigma_{\mathrm{ess}}(-\Delta)\neq\emptyset$ implies $\sigma_{\mathrm{ess}}(H)\neq\emptyset$, by min-max, and there exists a sequence $\phi_j := \phi(\cdot,\lambda_j)$, with $\lambda_j \in \sigma(H)$, $\lambda_j \neq \lambda_{j'}$ $(j\neq j')$, and $|\lambda_j| \leqq$ const. Without restriction, we may assume $(\phi_j,\phi_{j'})=\delta_{jj'}$, so that $\phi_j \to 0$ weakly, as $j\to\infty$. As the λ_j's are bounded, $\|\nabla\phi_j\| \leqq$ const, and by compactness it follows that

$$\phi_j \to 0 \quad \text{in} \quad L_{2,\mathrm{loc}}(\mathbb{R}^\nu)\,, \qquad j\to\infty\,. \tag{2.5}$$

Let C_i be as above and define

$$\alpha_{ji} := \int_{C_i} |\phi_j|^2 dx\,.$$

As in Case A, we have to consider several possibilities:

Case B1. For each $k\in\mathbb{N}$ there exists a j_k such that $(\alpha_{j_k i})_{i\geqq k}$ is not monotonic. This means, that for every $k\in\mathbb{N}$, we can find $j_k\in\mathbb{N}$ and $i_k \geqq k$ such that

$$\alpha_{j_k i_k} \geqq \tfrac{1}{2}\max\{\alpha_{j_k, i_k-1}, \alpha_{j_k, i_k+1}\}\,. \tag{2.6}$$

Case B2. There exists $k_0\in\mathbb{N}$ such that $(\alpha_{ji})_{i\geqq k_0}$ is nondecreasing or nonincreasing for all j. As $\phi_j \in L_2(\mathbb{R}^\nu)$, only the latter possibility occurs, and we have the following (final) two subcases (of Case B2):

Case B2a. $\exists k_1 \geqq k_0$, $\forall j\in\mathbb{N}$, $\forall i \geqq k_1 : \alpha_{ji} \leqq \tfrac{1}{2}\alpha_{j,i-1}$;

Case B2b. $\forall k \geqq k_0$, $\exists j_k \in\mathbb{N}$, $\exists i_k \geqq k : \alpha_{j_k i_k} \geqq \tfrac{1}{2}\alpha_{j_k, i_k-1}$.

In Case B2b, we immediately have (2.6), since, by monotonicity, $\alpha_{j_k, i_k+1} \leqq \alpha_{j_k i_k}$. But Case B2a implies that for any $\varepsilon>0$, there exists $R>0$ such that $\int_{|x|\geqq R} |\phi_j|^2 \leqq \varepsilon$, for all j [note that $\alpha_{jk_1} \leqq 1$ and $\alpha_{j,k_1+m} \leqq (\tfrac{1}{2})^m \alpha_{jk_1}$, in Case B2a]. Together with (2.5) this would imply $\phi_j \to 0$, in contradiction to $\|\phi_j\|=1$.

Hence we have found (j_k), (i_k) such that $(\alpha_{j_k i_k})_{k\in\mathbb{N}}$ satisfies (2.6), and as in Case A, we can now define

$$u_{nk} := \psi_{i_k}\phi(\cdot,\lambda_{j_k}) \cdot \|\psi_{i_k}\phi(\cdot,\lambda_{j_k})\|^{-1}\,, \qquad \lambda_{nk} := \lambda_{j_k}\,, \qquad i_{nk} = i_k\,.$$

The proof is finished as in Case A; when estimating $\|\nabla\phi(\cdot,\lambda_{j_k}) \upharpoonright \operatorname{supp}\nabla\psi_{i_k}\|$ one has to use the fact that the λ_{jk}'s belong to a bounded set. □

3. Complex E and λ

The main result of this section is Theorem 4 below.

Lemma 9. *Let $\underline{H}$ be a Hilbert space, $D\subset\mathbb{C}$ a domain and $q\in[1,\infty)$. Suppose that $B: D\to \underline{B}_q(\underline{H})$ is holomorphic. Then the set $M := \{z\in D;\ \sigma(B(z))=\{0\}\}$ is either a discrete subset of D, or $M=D$.*

Proof. Assume there exists a sequence $(z_n) \subset M$, $z_n \to z_0 \in D$. Choose $\nu_0 \in \mathbb{N}$ so large that $B(z)^{\nu_0} \in \underline{B}_1(\underline{H})$, and let $A(z) := B(z)^{\nu_0}$, $z \in D$. Then $D \ni z \mapsto A(z)$ is a holomorphic map $D \to \underline{B}_1(\underline{H})$, and, by the spectral mapping theorem,

$$\sigma(A(z_n)^\nu) = \{0\}, \quad \text{for} \quad n, \nu = 1, 2, \ldots.$$

By a theorem of Lidskij (cf. Reed, Simon [14; p. 328]), this implies

$$\operatorname{tr}(A(z_n)^\nu) = 0, \qquad n, \nu = 1, 2, \ldots,$$

and hence $\operatorname{tr}(A(z)^\nu) = 0$, for all $z \in D$, since $z \mapsto \operatorname{tr}(A(z)^\nu)$ is analytic. By the Plemelj-Smithies theorem ([14, Theorem XIII.108], e.g.), it follows that

$$\det(1 + \mu A(z)) = 1, \qquad \mu \in \mathbb{C}, \qquad z \in D,$$

implying $\sigma(A(z)) = \{0\}$, $z \in D$, by [14, Theorem XIII.105]. Hence, by the spectral mapping theorem, $\sigma(B(z)) = \{0\}$, for $z \in D$.

Theorem 4. *In the complex Hilbert space* $\underline{H} := L_2(\mathbb{R}^\nu)$, *let* $H := -\Delta + V$, *with* $V: \mathbb{R}^\nu \to \mathbb{R}$ *bounded. Assume that* $W: \mathbb{R}^\nu \to \mathbb{R}$, $W \not\equiv 0$, *is such that* $W(-\Delta + 1)^{-1} \in \underline{B}_q(\underline{H})$, *for some* $q \in [1, \infty)$. *Then* $\varrho(H) \setminus \bigcup_{\lambda \in \mathbb{C}} \sigma(H - \lambda W)$ *is a discrete subset of* $\varrho(H)$.

Proof. It is enough to show that the set of E such that $\sigma(W(H-E)^{-1}) = \{0\}$ is discrete. By variational principles, there exists $\tilde{E} < \inf \sigma(H)$ which belongs to $\bigcup_{\lambda \in \mathbb{R}} \sigma(H - \lambda W)$. Hence $\sigma(W(H - \tilde{E})^{-1}) \neq \{0\}$. For $E \in \varrho(H)$ (which is connected), we write

$$B(E) := W(H-E)^{-1} = W(-\Delta + 1)^{-1}(-\Delta + 1)(H-E)^{-1},$$

and see that $B(E)$ is analytic from $\varrho(H)$ to $\underline{B}(\underline{H})$, and the result follows from Lemma 9. $\square$

Remark. Conditions on W which guarantee that $W(-\Delta + 1)^{-1} \in \underline{B}_q(\underline{H})$, may be found in Reed and Simon [13, p. 47]; e.g. $|W(x)| \leqq C(1 + |x|)^{-\alpha}$, for some $\alpha > 0$, will do.

4. Completeness in 1-Dimension

In this section, we use o.d.e. techniques to show that, under suitable assumptions on V and W, exceptional levels do not occur. Our first result is

Theorem 5. *Let* $V, W: \mathbb{R} \to \mathbb{R}$ *be bounded,* W *of compact support and* $W(x) \geqq h > 0$ *for* $-\eta < x < \eta$. *Let* $H := -d^2/dx^2 + V$, *acting in* $L_2(\mathbb{R})$. *Then* $(H, W, \mathbb{R}^+)$ *is complete.*

Proof. Once again, consider real $E \notin \sigma(H)$. By limit-point/limit-circle theory (cf. Coddington, Levinson [2]), there exist solutions $f_\pm$ (unique up to scalar multiples) of $-f_\pm'' + Vf_\pm = Ef_\pm$, with $f_\pm$ square integrable at $\pm\infty$, respectively. Fix $R > 0$ such that $\operatorname{supp} W \subset (-R, R)$ and $f_\pm(\pm R) \neq 0$. Let

$$\alpha_\pm := f_\pm'(\pm R)/f_\pm(\pm R),$$

and $H_R := -d^2/dx^2 + V$, acting in $L_2(-R, R)$, with boundary conditions $u'(\pm R) = \alpha_\pm u(\pm R)$, $u \in D(H_R)$.

By Remark (c) following the proof of Lemma 3 in Sect. 1, there exist $\lambda_R > 0$ and $0 \neq u_R \in D(H_R)$ such that $(H_R - \lambda_R W)u_R = Eu_R$ [here one has to find

$$0 < \mu \in \sigma((H_R + c)^{-1/2}(W + \mu(E+c)(H_R+c)^{-1/2}),$$

where we have chosen c such that $H_R + c \geqq 1$]. Now, defining

$$w(x) := \begin{cases} (u_R(-R)/f_-(-R))f_-(x), & x \leqq -R, \\ u_R(x), & -R \leqq x \leqq R, \\ (u_R(R)/f_+(R))f_+(x), & x \geqq R, \end{cases}$$

it is easy to see that $w \in D(H)$ and $(H - \lambda_R W - E)w = 0$. □

Remarks. (a) Unfortunately, this proof does not generalize to $\mathbb{R}^\nu$.

(b) This theorem also shows that one cannot "smoothe out" the δ-function in the example given in the introduction.

(c) By using a more detailed analysis following Lemmas 1–3 in Sect. 1, it is possible to give an estimate on the *smallest* coupling constant λ (for a given E), under the additional assumption that V is periodic. In particular one finds that λ can be chosen less than $\frac{\pi^2}{4h\eta^2}(E(R+\Pi)+1)^2$, where Π is the period of V.

(d) Theorem 5 is nearly contained in Theorem 6 below, but its proof is so simple that we have chosen to present it as a separate result.

In the remaining theorems of this section, we will assume that W is continuous and has a finite number k of changes of sign. More precisely, we assume that there exist k points $x_1 < x_2 < \ldots < x_k$, for which $W(x_i) = 0$, $i = 1, \ldots, k$, and $W(x) \neq 0$ for $x \notin \{x_1, \ldots, x_k\}$. The results below extend to the case $W(x) \geqq 0$ or $W(x) \leqq 0$ in each interval (x_{i-1}, x_i), but involve additional technicalities.

The following Theorem 6 considers the case where $\operatorname{sgn} W(x) = \operatorname{sgn} W(-x)$ for x sufficiently large. The proof uses solutions $u_\pm = (u_\pm(x, \lambda)$ of $-u''_\pm + (V - \lambda W)u_\pm = Eu_\pm$, $u_\pm$ square integrable at $\pm\infty$, respectively, and shows that, as λ increases, a zero of u_+ has to meet a zero of u_-, giving rise to a solution of our problem. The intricate estimates needed to control the behaviour of the zeros of $u_\pm(x, \lambda)$, are based on results of Richardson [15].

Theorem 6. *Let $V: \mathbb{R} \to \mathbb{R}$ be bounded and $\geqq 1$, $H := -d^2/dx^2 + V$. Suppose that $W \in C(\mathbb{R})$ is relatively compact with respect to $-\Delta$ and has a finite number k of changes of sign, k even. Then $(H, W, \mathbb{R})$ is complete.*

An essential ingredient in the proof of Theorem 6 is the following extremely interesting lemma due to Richardson [15]; we give a complete proof as some of the details are omitted in Richardson's original text.

Lemma 10. *Let V and W be as above, $E > 1$. Suppose $-\infty < a < b < \infty$ and $W(x) \leqq -h < 0$, for $a \leqq x \leqq b$. Then there exists $\lambda_0 > 0$ such that $\lambda \geqq \lambda_0$ implies*

$$\int_a^b u'^2 + (V - E)u^2 > 0,$$

for all solutions u of $-u'' + (V - E) = \lambda Wu$ in (a, b).

Proof. Choose $\lambda_1 > 0$ such that $\lambda_1 h + V - E \geqq E$, for $x \in [a,b]$. It follows that any solution u of the differential equation as above, with $\lambda \geqq \lambda_1$, has at most one zero in $[a,b]$. Indeed, if $u(x) > 0$ for $a \leqq \alpha < x < \beta \leqq b$, $u(\alpha) = u(\beta) = 0$, then $u'' = (-\lambda W + V - E)u \geqq Eu > 0$ in (α, β), which is a contradiction. Hence there are two cases:

(i) $u(\eta) = 0$, for precisely one $\eta \in [a,b]$,

(ii) $u(x) > 0$, for $a \leqq x \leqq b$.

Case (i). Assume first, that $a < \eta < b$; the cases where $\eta = a$ or $\eta = b$ are similar. Without loss, suppose $u(x) > 0$ for $\eta < x \leqq b$, $u(x) < 0$ for $a \leqq x < \eta$. For $\eta < x \leqq b$, we have $u'' > 0$ so that $u'(x) > 0$ and hence

$$\frac{1}{2} u'(x)^2 \geqq \int_\eta^x u'u'' \geqq E \int_\eta^x u'u = \frac{E}{2} u(x)^2 ;$$

thus

$$\int_\eta^b u'^2 + (V-E)u^2 \geqq \int_\eta^b (E + V - E)u^2 > 0 ,$$

as $V \geqq 1$. A similar argument shows that $\int_a^\eta u'^2 + (V-E)u^2 > 0$, and we are done.

Case (ii). Let $\varepsilon_0 := (b-a)/4$; choose $\bar{\lambda} \geqq \lambda_1$ such that $\bar{\lambda} h + 1 - E > 0$ and $(\bar{\lambda} h + 1 - E)\varepsilon_0^2/2 = 1$. Now consider $\lambda \geqq \bar{\lambda}$ and let $\varepsilon = \varepsilon(\lambda)$ solve

$$(\lambda h + 1 - E)\varepsilon^2/2 = 1 . \tag{4.1}$$

Clearly, $\varepsilon \leqq \varepsilon_0 = (b-a)/4$, so that $a < a + \varepsilon < b - \varepsilon < b$. Let $\eta \in [a,b]$ be a point where the solution u above obtains its minimum, so that $u(x) \geqq u(\eta) > 0$, $x \in [a,b]$. Assume first that $a + \varepsilon \leqq \eta \leqq b - \varepsilon$, and set $M = M(\lambda) := \lambda h + 1 - E$. Then $u'' \geqq Mu$, and integrating once, one obtains

$$u'(x) \geqq M u(\eta)(x - \eta) , \qquad \eta \leqq x \leqq b , \tag{4.2}$$

as $u'(\eta) = 0$, and integrating again, $u(x) \geqq u(\eta)(1 + M(x-\eta)^2/2)$, $\eta \leqq x \leqq b$. Moreover, as $M\varepsilon^2/2 = 1$, we have

$$u(x) \geqq 2u(\eta) , \qquad \eta + \varepsilon \leqq x \leqq b . \tag{4.3}$$

From (4.2), we see that $u'(x)$ is positive in $[\eta, b]$, so that

$$u'(x)^2 \geqq 2 \int_\eta^x u'u'' \geqq 2M \int_\eta^x u'u \geqq M(u(x)^2 - u(\eta)^2) \geqq \tfrac{3}{4} M u(x)^2 , \tag{4.4}$$

by (4.3). Thus

$$\int_{\eta+\varepsilon}^b u'^2 + (V-E)u^2 \geqq \int_{\eta+\varepsilon}^b (\tfrac{3}{4}M + V - E)u^2 .$$

Combining this result with a similar calculation in (a, η), one obtains that

$$\int_a^{\eta-\varepsilon} + \int_{\eta+\varepsilon}^b u'^2 + (V-E)u^2 \geqq \int_a^{\eta-\varepsilon} + \int_{\eta+\varepsilon}^b (\tfrac{3}{4}M + V - E)u^2 . \tag{4.5}$$

On the other hand, $u'' \leqq ku$, where $k := \lambda \|W\|_\infty + \|V\|_\infty$, and hence, for $\eta - \varepsilon \leqq x \leqq \eta + \varepsilon$, $u(x) \leqq u(\eta) \exp \sqrt{k\varepsilon^2}$. But, by (4.1), $\lambda\varepsilon^2 = h^{-1}(2+(E-1)\varepsilon^2) \leqq h^{-1}(2+(E-1)\varepsilon_0^2)$; thus $u(x) \leqq \gamma u(\eta)$, $\eta - \varepsilon \leqq x \leqq \eta + \varepsilon$, where

$$\gamma := \exp[(h^{-1}(2+(E-1)\varepsilon_0^2)\|W\|_\infty + \varepsilon_0^2 \|V\|_\infty)^{1/2}].$$

Thus

$$-\int_{\eta-\varepsilon}^{\eta+\varepsilon} u'^2 + (V-E)u^2 \leqq (E-1)2\varepsilon\gamma^2 u(\eta)^2. \tag{4.6}$$

Now choose $\lambda_0 \geqq \bar{\lambda}$ such that

$$\tfrac{3}{4}M(\lambda_0) + 1 - E > 0 \quad \text{and} \quad 8(\tfrac{3}{4}M(\lambda_0) + 1 - E)\varepsilon_0 \geqq \sqrt{2/M(\lambda_0)} \cdot 2\gamma^2(E-1).$$

Then, as $u(x) \geqq 2u(\eta)$ for $x \notin (\eta - \varepsilon, \eta + \varepsilon)$, one verifies, using (4.5), (4.6) and $M(\lambda)\varepsilon(\lambda)^2 = 2$, that

$$\int_a^b u'^2 + (V-E)u^2 > 0.$$

The case where $\eta \in (b-\varepsilon, b)$ or $\eta \in (a, a+\varepsilon)$ is similar and left to the reader. Finally, when $\eta = a$ or $\eta = b$, then $u'(\eta)$ is not necessarily zero. However, in the case $\eta = a$, say u' is positive in $[a,b]$, so that (4.4) holds true for $x \in [a,b]$, and the estimates

$$\int_{a+\varepsilon}^{b} u'^2 + (V-E)u^2 \geqq (\tfrac{3}{4}M + 1 - E)(b-a-\varepsilon)u(a+\varepsilon)^2,$$

$$-\int_a^{a+\varepsilon} u'^2 + (V-E)u^2 \leqq E\varepsilon u(a+\varepsilon)^2,$$

are clearly sufficient, provided $M = M(\lambda) > \tfrac{4}{3}(2E-1)$. □

Proof of Theorem 6. Suppose W changes sign at the points $x_1, \ldots, x_k$. Without loss of generality, we assume that $W(x) > 0$ for $x < x_1$ and $x > x_k$. For each real $E \notin \sigma(H)$, we will produce a positive λ for which $E \in \sigma(H - \lambda W)$.

(1) We first note that there exists $\varepsilon_0 > 0$ such that for any $a \in \mathbb{R}$ and all $0 < \varepsilon \leqq \varepsilon_0$

$$\int_a^{a+\varepsilon} v'^2 + (V-E)v^2 > 0, \tag{4.7}$$

for all $v \in \mathrm{AC}[a, a+\varepsilon]$ such that $v(a) = 0$; here AC denotes the space of absolutely continuous functions. Inequality (4.7) follows immediately from the inequality $\int_0^\varepsilon w^2 \leqq \varepsilon^2 \int_0^\varepsilon w'^2$, valid for $w \in \mathrm{AC}[0,\varepsilon]$, $w(0) = 0$.

(2) Let $y_1 < y_2 < y_3$ be fixed points strictly to the right of x_k, and let $\varepsilon_1 > 0$ be a number such that $2\varepsilon_1$ is smaller than any of the numbers ε_0, $y_3 - y_2$, $y_2 - y_1$, $y_1 - x_k$, and $x_j - x_{j-1}$, $j = 1, \ldots, k-1$. Choose $\lambda_0 > 0$ such that for $\lambda \geqq \lambda_0$, any solution u of $-u'' + (V-E) = \lambda W u$ has at least one zero in each of the intervals $(x_1 - \varepsilon_1, x_1 - \varepsilon_1/2)$, $(x_2 + \varepsilon_1/2, x_2 + \varepsilon_1)$, $(x_3 - \varepsilon_1, x_3 - \varepsilon_1/2)$, $(x_4 + \varepsilon_1/2, x_4 + \varepsilon_1), \ldots$, $(x_k + \varepsilon_1/2, x_k + \varepsilon_1)$, and in (y_1, y_2), (y_2, y_3). [This is possible by Sturm oscillation theory: choose λ_0 such that $\lambda_0 h + E - \|V\|_\infty > \lambda$, where $\lambda = 4\pi^2/\varepsilon_1^2$ is the lowest

Dirichlet eigenvalue of $-d^2/dx^2$ in $L_2(0, \varepsilon_1/2)$, and h is the minimum of W over all above intervals. h is the minimum of W over all above intervals.]

(3) Now, since WH^{-1} is compact, we have $\sigma_{ess}(H-\lambda W)=\sigma_{ess}(H)$, for any λ, by Weyl's theorem. Again by limit-point/limit-circle theory (see e.g. Coddington and Levinson [2]), there exist solutions $u_{\pm}(x,\lambda)$ of $-u''_{\pm}+(V-E)u_{\pm}=\lambda W u_{\pm}$, with $u_{\pm}$ square integrable at $\pm\infty$, respectively. Moreover, $u_{\pm}(x,\lambda)$ are unique up to scalar multiples, and for each $\bar{\lambda}$ there is a neighborhood $N(\bar{\lambda})$ where $u_{\pm}(x,\lambda)$ can be chosen to depend analytically on λ.

Let $\gamma_j^{\pm}(\lambda)$ be zeros of $u_{\pm}(x,\lambda)$ lying in the intervals $(x_j-\varepsilon_1, x_j-\varepsilon_1/2)$, j odd, and in $(x_j+\varepsilon_1/2, x_j+\varepsilon_1)$, j even $(1\leqq j\leqq k)$, and let $\gamma^+=\gamma^+(\lambda)$ be the last zero of $u_+(x,\lambda)$ in $(y_1, y_2]$, $\gamma^-=\gamma^-(\lambda)$ the first zero of $u_-(x,\lambda)$ in $[y_2, y_3)$.

Our next aim is to show that there exists $\lambda_1\geqq\lambda_0$ such that $\lambda>\lambda_1$ implies

$$\int_{-\infty}^{\gamma^-} u_-'^2+(V-E)u_-^2=\int_{-\infty}^{\gamma^-} Wu_-^2>0\,, \tag{4.8}$$

and

$$\int_{\gamma^+}^{\infty} u_+'^2+(V-E)u_+^2=\int_{\gamma^+}^{\infty} Wu_+^2>0\,. \tag{4.9}$$

As $u_-(\gamma^-)=0$, and $W\restriction(\gamma^-,\infty)>0$, Eq. (4.9) is immediate by partial integration; partial integration also proves the equality in (4.8). Now we split up the integral on the left-hand side of (4.8), and obtain, again by partial integration,

$$\int_{-\infty}^{\gamma_1^-} u_-'^2+(V-E)u_-^2>0\,,\qquad \int_{\gamma_j^-}^{\gamma_{j+1}^-} u_-'^2+(V-E)u_-^2>0\,,\qquad j\ even\,,$$

$$\int_{\gamma_k}^{\gamma^-} u_-'^2+(V-E)u_-^2>0\,.$$

Further, by (4.7),

$$\int_{\gamma_j^-}^{x_j+\varepsilon_1} u_-'^2+(V-E)u_-^2>0\,,\qquad j\ odd\,,\qquad \int_{x_j-\varepsilon_1}^{\gamma_j} u_-'^2+(V-E)u_-^2>0\,,\qquad j\ even\,.$$

Finally, by Lemma 10 above,

$$\int_{x_j+\varepsilon_1}^{x_{j+1}-\varepsilon_1} u_-'^2+(V-E)u_-^2>0\,,\qquad j\ odd\,,$$

and (4.8) follows.

(4) Differentiating the equation $-u''_{\pm}+(V-E)u_{\pm}=\lambda Wu_{\pm}$ with respect to λ, one obtains $(\partial_\lambda:=\partial/\partial\lambda)$

$$-(\partial_\lambda u_{\pm})''+(V-E-\lambda W)\partial_\lambda u_{\pm}=Wu_{\pm}\,,$$

so that

$$\frac{d}{dx}((\partial_\lambda u_{\pm})'u_{\pm}-(\partial_\lambda u_{\pm})u'_{\pm})=-Wu_{\pm}^2\,.$$

Integrating, we find

$$\int_{-\infty}^{\gamma^-} Wu_-^2 = \partial_\lambda u_-(\gamma^-,\lambda)\cdot u_-'(\gamma^-,\lambda) - (\partial_\lambda u_-)'(\gamma^-,\lambda)\cdot u_-(\gamma^-,\lambda),$$

and

$$\int_{\gamma^+}^{\infty} Wu_+^2 = (\partial_\lambda u_+)'(\gamma^+,\lambda)\cdot u_+(\gamma^+,\lambda) - \partial_\lambda u_+(\gamma^+,\lambda)\cdot u_+'(\gamma^+,\lambda).$$

As $u_-(\gamma^-,\lambda)=u_+(\gamma^+,\lambda)=0$, it follows from (4.8), (4.9) that

$$\partial_\lambda u_-(\gamma^-,\lambda)\cdot u_-'(\gamma^-,\lambda)>0, \qquad \partial_\lambda u_+(\gamma^+,\lambda)\cdot u_+'(\gamma^+,\lambda)<0. \tag{4.10}$$

(5) We are now able to draw our basic conclusion that for any $\lambda \geqq \lambda_1$, there are solutions $u_\pm(x,\lambda)$ which are square integrable at $\pm\infty$, respectively, and which have zeros $\gamma^+=\gamma^+(\lambda)\in(y_1,y_2]$ and $\gamma^-=\gamma^-(\lambda)\in[y_2,y_3)$, respectively, such that (4.10) holds true. Differentiating the equations $u_\pm(\gamma^\pm(\lambda),\lambda)=0$ with respect to λ, we see that the above inequality imply that γ^+ moves to the right and γ^- moves to the left, as λ increases. A standard argument in Sturm-Liouville theory shows that for some $\tilde{\lambda}\geqq\lambda$, the continuations of $\gamma^+(\lambda_1)$ and $\gamma^-(\lambda_1)$ must coincide at some point $\tilde{\gamma}$, say. Then

$$u(x):=\begin{cases} u_-(x,\tilde{\lambda})\cdot u_+'(\tilde{\gamma},\tilde{\lambda}), & x\leqq\tilde{\gamma},\\ u_+(x,\tilde{\lambda})\cdot u_-'(\tilde{\gamma},\tilde{\lambda}), & x\geqq\tilde{\gamma},\end{cases}$$

is an L_2-eigenfunction of $H-\lambda W-E$. □

The above proof breaks down if W has opposite signs in $(-\infty,x_1)$ and (x_k,∞). To treat this case, one needs different methods and our results are less general; in fact, we consider only the case where W has one zero [but see Remark (h)]. The method employed here is entirely different from our other approaches. The essential idea is that by introducing a Dirichlet boundary condition at the zero of W, the Birman-Schwinger kernel $\operatorname{sgn}(W)|W|^{1/2}(H-E)^{-1}|W|^{1/2}$ can be written as a direct sum of two *selfadjoint* operators plus a rank-one perturbation. We have the following theorem:

Theorem 7. *Let $V:\mathbb{R}\to\mathbb{R}$ be periodic and bounded, $H:=-\frac{d^2}{dx^2}+V$ acting in $L_2(\mathbb{R})$, and suppose that $W:\mathbb{R}\to\mathbb{R}$ is continuous, $W(x)>0$ for $x<0$, and $W(x)<0$ for $x>0$. Furthermore, assume that there exists $p>2$ such that $|x|^pW(x)$ is bounded. Then $(H,W,\mathbb{R})$ is complete.*

Proof. For simplicity, we will again assume $V\geqq1$ and consider $E\notin\sigma(H)$, $E>\inf\sigma(H)$. We write $W=\sigma q^2$, with $q\geqq0$, $\sigma(x)=\operatorname{sgn}(W(x)$. Clearly, to find a solution of the problem $(H-E)u=\lambda Wu$, it is enough to find v and $\mu=\lambda^{-1}$ such that $\mu v=\sigma q(H-E)^{-1}qy$.

Again by limit-point/limit-circle theory (Coddington and Levinson [2]), there exist solutions $f_\pm$ of $-f_\pm''+(V-E)f_\pm=0$, unique up to scalar multiples, which are square integrable at $\pm\infty$, respectively. Now let H_L and H_R be $-\frac{d^2}{dx^2}+V$, acting in $L_2(-\infty,0)$ and in $L_2(0,\infty)$, respectively, with Dirichlet boundary condition at 0, provided $f_+(0)f_-(0)\neq 0$. If $f_+(0)f_-(0)=0$, choose some selfadjoint boundary condition $u(0)+au'(0)=0$ such that $f_\pm(0)+f'_\pm(0)\neq 0$, and define H_L, H_R accordingly. Let $H_D=H_L\oplus H_R$; by construction, $E\notin\sigma(H)\cup\sigma(H_D)$. Our first goal is to determine $(H-E)^{-1}-(H_D-E)^{-1}$: for any $g\in L_2(\mathbb{R})$, the function $h=(H-E)^{-1}g-(H_D-E)^{-1}g$ satisfies $h\in L_2(\mathbb{R})$, $-h''(x)+(V-E)h(x)=0$ for $x\neq 0$, and $h(x)+ah'(x)$ continuous (in the case of Dirichlet-condition at 0, take $a=0$). It follows that $h(x)=a_-(g)f_-(x)$, $x<0$, and $h(x)=a_+(g)f_+(x)$, $x>0$, where $a_\pm(g)\in\mathbb{R}$. Since $h+ah'$ is continuous, there exists a constant $\gamma\in\mathbb{R}$, independent of g, such that $a_-(g)=\gamma a_+(g)$, and we see that

$$(H-E)^{-1}g-(H_D-E)^{-1}g=a_+(g)\,(f_+(x)\chi_{(0,\infty)}(x)+\gamma f_-(x)\chi_{(-\infty,0)}(x))\,.$$

By linearity, there exists $w\in L_2(\mathbb{R})$ such that $a_+(g)=(w,g)$. Letting

$$f:=(f_-(0)+af'_-(0))f_+(x)\chi_{(0,\infty)}(x)+(f_+(0)+af'_+(0))f_-(x)\chi_{(-\infty,0)}(x)\,,$$

it is clear by selfadjointness that, for some $\alpha\in\mathbb{R}$, $\alpha\neq 0$,

$$(H-E)^{-1}-(H_D-E)^{-1}=\alpha(f,\cdot)f\,. \tag{4.11}$$

For later applications in Lemma 12 and Proposition A, we do the explicit calculations for the case of Dirichlet boundary conditions: $h(x)$ being continuous at 0, we have

$$f=f_-(0)f_+(x)\chi_{(0,\infty)}(x)+f_+(0)f_-(x)\chi_{(-\infty,0)}(x)\,.$$

To determine α in (4.11), take any nonzero $g\in C_0^\infty(0,\infty)$. Then $((H_D-E)^{-1}g)(x)=0$, $x<0$, implying

$$((H-E)^{-1}g)(x)=\alpha\left(f_-(0)\int_0^\infty f_+(y)g(y)dy\right)f_+(0)f_-(x)\,,\qquad x<0\,.$$

On the other hand, by the standard form of the Green's function of $(H-E)^{-1}$,

$$((H-E)^{-1}g)(x)=f_-(x)\int_0^\infty f_+(y)g(y)dy/[f_-,f_+]\,,$$

and therefore

$$\alpha^{-1}=f_-(0)f_+(0)\,[f_-,f_+]\,, \tag{4.12}$$

with $[f_-,f_+]=f'_-f_+-f_-f'_+$ denoting the Wronskian of f_- and f_+.

Returning to the proof of Theorem 7, it follows from (4.11) that

$$\sigma q(H-E)^{-1}q=\sigma q(H_D-E)^{-1}q+\alpha(qf,\cdot)\sigma qf;$$

we write $B:=\sigma q(H-E)^{-1}q$, $A:=\sigma q(H_D-E)^{-1}q$, and note that $A,B\in\underline{B}_1(L_2(\mathbb{R}))$ [as $q(-d^2/dx^2+1)^{-1/2}$ is Hilbert-Schmidt]. It follows that

$$\begin{aligned}\det\frac{1-B/z}{1-A/z}&=\det\frac{B-z}{A-z}=\det(1+[\alpha(qf,\cdot)\sigma qf](A-z)^{-1})\\&=1+\alpha(qf,(A-z)^{-1}\sigma qf)\,.\end{aligned}$$

Decomposing A into its left and right parts A_L and A_R, we get

$$A = A_L \oplus A_R, \quad (A-z)^{-1} = (A_L - z)^{-1} \oplus (A_R - z)^{-1},$$

with A_L, A_R selfadjoint. Furthermore,

$$\frac{1}{\alpha}\det\frac{B-z}{A-z} = \frac{1}{\alpha} + (qf, (A_L - z)^{-1} qf)_L - (qf, (A_R - z)^{-1} qf)_R, \tag{4.13}$$

with $(\cdot,\cdot)_L$ and $(\cdot,\cdot)_R$ denoting the scalar product in $L_2(-\infty, 0)$ and $L_2(0,\infty)$, respectively. Let ν_{Li} (ν_{Ri}) denote the negative eigenvalues of A_L and A_R, respectively, with (normalized) eigenfunctions n_{Li} and n_{Ri}, and let μ_{Li} and μ_{Ri} denote the positive eigenvalues of A_L and A_R, respectively, with (normalized) eigenfunctions m_{Li} and m_{Ri}; without restriction, let us assume $\nu_{Li} < \nu_{L,i+1}$, $\nu_{Ri} < \nu_{R,i+1}$, $\mu_{Li} > \mu_{L,i+1}$, and $\mu_{Ri} > \mu_{R,i+1}$.

Expanding the right-hand side of (4.13), we obtain

$$\frac{1}{\alpha}\det\frac{B-z}{A-z} = \frac{1}{\alpha} + \sum_i \frac{(qf, m_{Li})_L^2}{\mu_{Li} - z} + \sum_i \frac{(qf, n_{Li})_L^2}{\nu_{Li} - z} - \sum_i (qf, m_{Ri})_R^2 \mu_{Ri} - z - \sum_i \frac{(qf, n_{Ri})_R^2}{\nu_{Ri} - z}. \tag{4.14}$$

We are looking for real zeros $z \neq 0$ of $\alpha^{-1}\det((B-z)/(A-z))$. Consider $z > 0$ henceforth. The second and fourth sum in the right-hand side of (4.14) are continuous functions for $z > 0$; between any two roots $\mu_{Li} > \mu_{L,i+1}$, the first sum varies monotonically from $-\infty$ to $+\infty$ as z increases from $\mu_{L,i+1}$ to μ_{Li}. If there are *no* roots μ_{Ri} in the interval $(\mu_{L,i+1}, \mu_{Li})$, then the third sum is also continuous, for $\mu_{L,i+1} < z < \mu_{Li}$, and it is then clear that $\alpha^{-1}\det((B-z)/(A-z))$ must have a real zero in this interval. But it follows from Lemmas 11 and 12 below, that in fact "most" intervals $(\mu_{L,i+1}, \mu_{Li})$ are free of roots μ_{Ri}, and we are done. □

The following lemma establishes a lower bound for the positive eigenvalues of $-u_i'' + (V-E)u_i = \lambda_i W u_i$, on $(-\infty, 0)$.

Lemma 11. *Let* $V: (-\infty, 0) \to \mathbb{R}$ *be bounded,* $V \geqq 1$, *and let* $\tilde{H} := -d^2/dx^2 + V$, *acting in* $L_2(-\infty, 0)$, *with some fixed selfadjoint boundary condition at* 0. *Let* $W: (-\infty, 0] \to \mathbb{R}$ *be continuous,* $W(x) > 0$, *for* $-\infty < x < 0$, *and suppose that* $\int_{-\infty}^{0} \sqrt{W} < \infty$. *Fix* $E \in \mathbb{R}$, *and let*

$$N_+(\lambda) := \#\{\lambda_i; (\tilde{H} - E)u_i = \lambda_i W u_i, 0 < \lambda_i \leqq \lambda\},$$

for $\lambda > 0$. *Then*

$$\liminf_{\lambda\to\infty} N_+(\lambda)/\lambda^{1/2} \geqq \frac{1}{\pi}\int_{-\infty}^{0} \sqrt{W}.$$

Proof. For $n \in \mathbb{N}$, let

$$W_n(x) := \begin{cases} W(x), & -n \leqq x \leqq 0, \\ 0, & x < -n, \end{cases}$$

and let $\lambda_i^{(n)}$ denote the positive eigenvalues of $(\tilde{H}-E)u_i^{(n)}=\lambda_i^{(n)}W_n u_i^{(n)}$; we also define

$$N_+^{(n)}(\lambda):=\#\{\lambda_i^{(n)};0<\lambda_i^{(n)}\leqq\lambda\},$$

for $\lambda>0$. We first remark that $\lambda_i^{(n)}\geqq\lambda_i$, $i=1,2,\ldots$, for all n. To prove this, let $W(t):=(1-t)W+tW_n$, and consider $(\tilde{H}-E)u_i(t)=\lambda_i(t)W(t)u_i(t)$; as the eigenvalues $\lambda_i(\cdot)$ are clearly simple, they are differentiable, and it follows that

$$(\tilde{H}-E-\lambda_i(t)W(t))\dot{u}_i=\dot{\lambda}_i(t)W(t)u_i(t)+\lambda_i(t)\dot{W}(t)u_i(t),$$

where the dot means $\partial/\partial t$; multiplying by $u(t)$ and using $(\tilde{H}-E-\lambda_i(t)W(t))u(t)=0$, we get

$$0=\dot{\lambda}_i(t)\,(u(t),W(t)u(t))+\lambda_i(t)\,(u(t),\dot{W}(t)u(t)),$$

and therefore $\dot{\lambda}_i(t)/\lambda_i(t)>0$. Hence it is enough to consider the asymptotics of the $\lambda_i^{(n)}$'s.

If $u_i=u_i^{(n)}$ satisfies $(\tilde{H}-E)u_i=\lambda_i^{(n)}W_n u_i$, then necessarily $((\tilde{H}-E)u_i)(x)=0$, for $x<-n$, and hence $u_i(x)=c_i f_-(x)$, for $x<-n$, where f_- solves $-f_-''+(V-E)f_-=0$, f_- square integrable at $-\infty$; the existence of f_- is again guaranteed by limit-point/limit-circle theory. This implies that $u_i=u_i^{(n)}$ satisfies the selfadjoint boundary condition

$$f_-(-n)u_i'(-n)-f_-'(-n)u_i(-n)=0,$$

independently of $i=1,2,\ldots$. Conversely, it is clear that each eigenfunction u_i of $-u_i''+(V-E)u_i=\lambda_i u_i$ on $(-n,0)$, satisfying the above boundary conditions, extends to an eigenfunction on $(-\infty,0)$. It follows then by standard min-max-arguments using Dirichlet-Neumann bracketing (cf. e.g. Reed and Simon [14, Sect. XIII.15] and Courant and Hilbert [3]) that

$$\lim_{\lambda\to\infty}N_+^{(n)}(\lambda)/\lambda^{1/2}=\frac{1}{\pi}\int_{-n}^{0}\sqrt{W},$$

and we see that, for any $n\in\mathbb{N}$,

$$\liminf_{\lambda\to\infty}N_+(\lambda)/\lambda^{1/2}\geqq\lim_{\lambda\to\infty}N_+^{(n)}(\lambda)/\lambda^{1/2}=\frac{1}{\pi}\int_{-n}^{0}\sqrt{W},$$

and the result follows. □

To obtain an *upper* bound on the *negative* eigenvalues λ_i [for the interval $(0,\infty)$, where $W>0$], we first show that we can insert an infinite number of equidistant Dirichlet points x_k in such a way, that all eigenvalues λ_i go up. The decay of W then gives the bound on the number of negative eigenvalues contributed by each interval (x_k,x_{k+1}).

Lemma 12. *Let* $1\leqq V:\mathbb{R}\to\mathbb{R}$ *be bounded and periodic, and let* $W\in C([0,\infty))$ *be such that* $0\leqq W(x)\leqq c(1+|x|)^{-p}$, $0\leqq x<\infty$, *for some* $p>2$. *Let* $\tilde{H}=-d^2/dx^2+V$, *acting* *in* $L_2(0,\infty)$, *with boundary condition* $\cos\theta u(0)+\sin\theta u'(0)=0$, *for some fixed* $\theta\in R$. *Then, for any* $E\in R\backslash\sigma(\tilde{H})$, *there exists a constant* C *such that for all* $\lambda>0$

$$N_-(\lambda):=\#\{\lambda_i:(\tilde{H}-E)u_i=\lambda_i W u_i,\ -\lambda\leqq\lambda_i<0\}\leqq C\lambda^{1/p}.$$

Remarks. (e) Clearly, $N_-(\lambda)=0$ if $E<\inf\sigma(\tilde{H})$. (f) A slight modification of the proof shows that the result is also true if $E\in\sigma(\tilde{H})$.

Proof. For simplicity of notation, we will assume that V has period 1. Again, let $f_\pm$ be the (unique) solutions of $-f_\pm''+(V-E)f_\pm=0$, $f_\pm$ square integrable at $\pm\infty$, respectively. Without loss, assume that the Wronskian $[f,f_+]$ is *positive*.

(1) In this the first step, we wish to insert an infinite number of Dirichlet points $x_k, x_k\to\infty$, in such a way that the λ_i's go up. So let us first analyze what happens if we add a Dirichlet condition at some point $x'>0$:

Let $a,b\in\mathbb{R}$ be such that $\phi:=af_++bf_-$ satisfies the boundary condition

$$\cos\theta\phi(0)+\sin\theta\phi'(0)=0.$$

As in the proof of Theorem 7 [see Eqs. (4.11) and (4.12)], it follows that

$$(\tilde{H}-E)^{-1}=(H'-E)^{-1}+(\phi(x')f_+(x')[\phi,f_+])^{-1}(v,\cdot)v,$$

where

$$v(x):=f_+(x')\phi(x)\chi_{(0,x')}(x)+\phi(x')f_+(x)\chi_{(x',\infty)}(x),$$

and $H':=-d^2/dx^2+V$ in $L_2(0,x')\oplus L_2(x',\infty)$ with the boundary conditions $\cos\theta u(0)+\sin\theta u'(0)=0$ and $u(x')=0$, for all $u\in D(H')$. Without restriction, we may again assume that $[\phi,f_+]>0$.

(2) Now let us first consider the case where $\tilde{H}$ is Dirichlet at 0: as $E\notin\sigma(\tilde{H})$, we necessarily have $f_+(0)\neq 0$. We distinguish between the following two cases:

(2a): $f_+(0)f_-(0)>0$: Let $H_k:=-d^2/dx^2+V$ in $L_2(k-1,k)$, $k=1,2,\ldots$, with Dirichlet boundary condition at $k-1$ and at k. We claim that

$$E\notin\sigma(H_k)\quad\text{and}\quad(\tilde{H}-E)^{-1}\geqq\bigoplus_{k=1}^{\infty}(H_k-E)^{-1}.\tag{4.15}$$

To see this, let $H_{(k)}:=-d^2/dx^2+V$ in $L_2(k,\infty)$ with Dirichlet boundary condition at k. Then, by step (1),

$$(\tilde{H}-E)^{-1}=(H_1-E)^{-1}\oplus(H_{(1)}-E)^{-1}+(\phi(1)f_+(1)[\phi,f_+])^{-1}(v,\cdot)v,$$

and we only have to check that $\phi(1)f_+(1)>0$. Note that $E\notin\sigma(\tilde{H})$ implies $E\notin\sigma(H_{(1)})$, since $H_{(1)}$ is a translate of $\tilde{H}$, and $E\notin\sigma(H_1)$, since $\phi(1)\neq 0$, as we will now see: By definition, $\phi=f_+(0)f_-(x)-f_+(x)f_-(0)$, and hence

$$\begin{aligned}\phi(1)f_+(1)[\phi,f_+]&=(f_+(0)f_-(0)(z_--z_+)z_+f_+^2(0)[f_-,f_+]\\&=f_+^2(0)[f_-,f_+]f_+(0)f_-(0)(1-z_+^2)>0,\end{aligned}$$

where $z_\pm$, $|z_+|<1$, $|z_-|>1$, are the multipliers of $f_\pm$, respectively (cf. Eastham [4]).

Hence $(\tilde{H}-E)^{-1} \geqq (H_1-E)^{-1} \oplus (H_{(1)}-E)^{-1}$. By periodicity,

$$(\tilde{H}-E)^{-1} \simeq (H_{(1)}-E)^{-1},$$

and (4.15) follows. Writing $D := \bigoplus_{k=1}^{\infty} q(H_k-E)^{-1}q$, $q=\sqrt{W}$, we see that

$$q(\tilde{H}-E)^{-1}q \geqq D. \tag{4.16}$$

By a now familiar calculation ($\mu=\lambda^{-1}>0$),

$$\begin{aligned} N_-(\lambda) &= \#\{\mu_i;\ \mu_i \text{ eigenvalue of } q(\tilde{H}-E)^{-1}q,\ \mu_i \leqq -\mu<0\} \\ &\leqq \#\{\gamma_i;\ \gamma_i \text{ eigenvalue of } D,\ \gamma_i \leqq -\mu<0\} \\ &= \sum_{k=1}^{\infty} N_k(\mu), \end{aligned} \tag{4.17}$$

where we have used (4.16) in the second step, and

$$N_k(\mu) := \#\{\gamma_{ki};\ \gamma_{ki} \text{ eigenvalue of } q(H_k-E)^{-1}q,\ \gamma_{ki} \leqq -\mu<0\}. \tag{4.18}$$

We will show that there exist constants m and c_0 such that

$$N_k(\mu) \leqq m, \qquad k=1,2,\ldots, \mu>0, \tag{4.19}$$

and

$$N_k(\mu)=0, \qquad k \geqq c_0 \mu^{-1/p}. \tag{4.20}$$

From (4.17)–(4.20) it is clear that

$$N_-(\lambda) \leqq m c_0 \mu^{-1/p} = m c_0 \lambda^{1/p}.$$

We claim that $N_k(\mu) \leqq m$, where m is the number of negative eigenvalues of H_k-E (= order of the gap containing E = number of zeros of f_+ or f_- in $[0,1)$); see also Mingarelli [11].

Suppose, there would exist $m+1$ eigenvalues $\lambda_{ki}<0$ of $(H_k-E)u_{ki}=\lambda_{ki}u_{ki}$. Then we can find $u := \sum b_i u_{ki}$, $u \neq 0$, such that $(u,(H_k-E)u)>0$. But

$$\begin{aligned} (u,(H_k-E)u) &= \sum_{i,j} b_i b_j (u_{ki},(H_k-E)u_{k_j}) \\ &= \sum_{i,j} b_i^2 (u_{ki},(H_k-E)u_{ki}) = \sum_i b_i^2 \lambda_{ki} (u_{ki}, W u_{ki}) < 0, \end{aligned}$$

where we have used the fact that $(u_i,(H_k-E)u_j)=0$, $i \neq j$, as all eigenvalues are simple. Hence (4.19) is proven.

To show (4.20), we first remark that there exists a positive constant c' such that

$$\|(H_k-E)u\| \geqq c'\|u\|, \qquad u \in D(H_k), \qquad k=1,2,\ldots, \tag{4.21}$$

since $E \notin \sigma(H_k)$ and all the H_k are the same. Now considering the eigenvalue equation $(H_k-E)u_{ki}=\lambda_{ki}Wu_{ki}$, $\|u_{ki}\|=1$, in $L_2(k-1,k)$, it is clear from (4.21) that $c' \leqq |\lambda_{ki}|\,\|Wu_{ki}\| \leqq |\lambda_{ki}| c k^{-p}$, by the decay property of W. Hence all λ_{ki} satisfy $|\lambda_{ki}| \geqq c'/c \cdot k^p$, which is equivalent to saying that [in the notation of (4.18)] $|\gamma_{ki}| \leqq c/c' \cdot k^{-p}$, or $N_k(\mu)=0$, provided $\mu > c/c' \cdot k^{-p}$.

This concludes our proof in the case (2a).

(2b): $f_+(0)f_-(0) \leqq 0$. Let again $\phi = f_+(0)f_-(x) - f_+(x)f_-(0)$. As E lies in a gap, f_+ and f_- have zeros in $(0,1)$; in particular, we can find $x_0 \in (0,1)$ such that $f_-(x_0)f_+(x_0) > 0$. By construction, E will not belong to $\sigma(H_0) \cup \sigma(H_{(0)})$, where $H_0 := -d^2/dx^2 + V$ in $(0, x_0)$ and $H_{(0)} := -d^2/dx^2 + V$ in (x_0, ∞), with Dirichlet boundary conditions at 0 and x_0. As in (2a), to get $\tilde{H} \leqq H_0 \oplus H_{(0)}$, we have to control the sign of $\phi(x_0)f_+(x_0)[\phi, f_+]$. But $[\phi, f_+] = f_+(0)[f_-, f_+]$, and hence

$$\phi(x_0)f_+(x_0)[\phi, f_+] = \{f_+^2(0)f_-(x_0)f_+(x_0) - f_+^2(x_0)f_-(0)f_+(0)\}[f_-, f_+],$$

which is positive since $f_-(x_0)f_+(x_0) > 0$, $f_-(0)f_+(0) \leqq 0$ and $[f_-, f_+] > 0$. Now, as $f_+(x_0)f_-(x_0) > 0$, we can proceed as in (2a) and insert Dirichlet boundary conditions at the points $x_0 + k$, $k = 1, 2, \ldots$ [note that the function ϕ in (2a) is now replaced by $f_+(x_0)f_-(x) - f_-(x_0)f_+(x)$], obtaining finally an inequality analogous to (4.16). The eigenvalues contributed by the intervals $(x_0 + k - 1, x_0 + k)$ are estimated as in (2a), and the contribution by $(0, x_0)$ is bounded by the constant m, as in the proof of Eq. (4.19).

(3) Finally, let us consider the case where $\tilde{H}$ does not have Dirichlet boundary condition at 0. Let $\tilde{\phi} : \tilde{\alpha} f_+ + \tilde{\beta} f_-$ satisfy the boundary condition of $\tilde{H}$ at 0, and note that $\tilde{\beta} \neq 0$ as $E \notin \sigma(\tilde{H})$. In particular, $\tilde{\phi}$ and f_+ are not proportional and, as above, we can find $\tilde{x} \in (0,1)$ such that $\tilde{\phi}(\tilde{x})f_+(\tilde{x})[\tilde{\phi}, f_+] > 0$, $f_+(\tilde{x}) \neq 0$. Again by construction, E is not in the spectrum of $-d^2/dx^2 + V$ on $(0, \tilde{x})$ (with the boundary condition of $\tilde{H}$ at 0, and Dirichlet boundary condition at $\tilde{x}$), and not in the spectrum of $-d^2/dx^2 + V$ on $(\tilde{x}, \infty)$ (with Dirichlet boundary condition at $\tilde{x}$). But on $(\tilde{x}, \infty)$ we are now in the case (2a)/(2b), and the result follows, as the interval $(0, \tilde{x})$ can contribute at most m eigenvalues. □

Remarks. (g) In general, the folk theorem is that the asymptotic distribution of eigenvalues is related to volumes in phase space (cf. the discussion in Reed and Simon [14, p. 261f.]). Applied to our situation, one would expect that [assuming $W(x) > 0$, $x > 0$] $N_+(\lambda)$ = the number of *positive* eigenvalues λ_i of $(H_D - E)u_i = \lambda_i W u_i$ which are less than or equal to λ, is given asymptotically as $\lambda \to \infty$ by

$$\frac{1}{2\pi} \int_{\{(x,p);\, 0 \leqq p^2 + V - E \leqq \lambda W\}} dx\, dp = \frac{1}{2\pi} \int_{\{(x,p);\, E - V \leqq p^2 \leqq E - V + \lambda W\}} dx\, dp$$
$$\sim \frac{1}{\pi} \int_0^\infty (\sqrt{\lambda W + E - V} - \sqrt{E - V})\, dx \sim \frac{1}{\pi} \int_0^\infty \sqrt{\lambda W}\, dx$$

(cf. the shaded area in Fig. 1 below), and indeed we will verify this in the appendix.

Similarly, one would like to prove that the number of negative eigenvalues λ_i, $-\lambda \leqq \lambda_i < 0$, is approximately given by

$$\frac{1}{2\pi} \int_{\{(x,p);\, E - V - \lambda W \leqq p^2 \leqq E - V\}} dx\, dp,$$

which behaves like const $\lambda^{1/p}$, if $W(x) \sim c|x|^{-p'}$ for x large (cf. the shaded area in Fig. 2 below). In this case, Lemma 12 provides only an *upper* bound $\leqq c\lambda^{1/p'}$, and the question of the precise asymptotic distribution remains open.

(h) With considerable effort it is possible to extend Theorem 7 to the case where W has more than one change of sign. Even the case where W has only two changes of sign requires a proof which is far longer than the proof of Theorem 6. Amongst the many additional difficulties, perhaps the most cumbersome is that on inserting Dirichlet-points inductively at the zeros of W, the $(k+1)^{\text{th}}$ operator is no longer a sum of two selfadjoint operators (+ a rank-one perturbation).

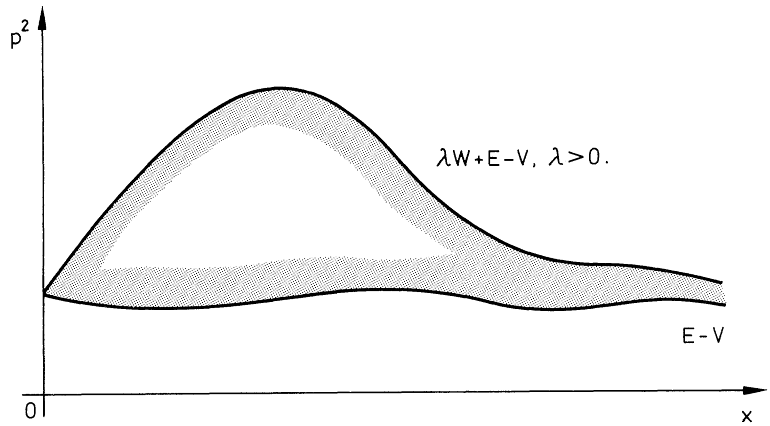

Fig. 1

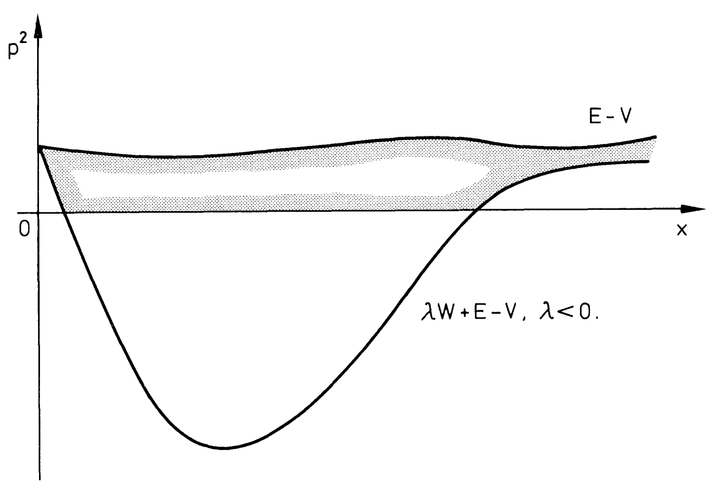

Fig. 2

Appendix

Here we combine the method of Lemma 12 of inserting Dirichlet points with the exact asymptotics for finite intervals, to obtain a sharp upper bound on the number of positive eigenvalues (for W positive). Together with the lower bound of Lemma 11, this leads to the following result:

Proposition A. *Let* $V\colon \mathbb{R}\to\mathbb{R}$ *be bounded and periodic and let* $H := -d^2/dx^2 + V$, *acting in* $L_2(0,\infty)$, *with Dirichlet boundary condition at* 0. *Suppose that*

$W \in C([0,\infty))$ satisfies $0 < W(x) \leqq c(1+|x|)^{-p}$, $0 < x < \infty$, for some $p > 2$. Defining

$$N(\lambda) := \#\{\lambda_i : (H-E)u_i = \lambda_i W u_i, 0 < \lambda_i \leqq \lambda\},$$

for $\lambda > 0$ and $E \in R \cap \varrho(H)$, $E > \inf\sigma(H)$, we have

$$\lim_{\lambda\to\infty} N(\lambda)/\lambda^{1/2} = \frac{1}{\pi}\int_0^\infty \sqrt{W}\,.$$

Proof. We know already

$$\liminf N(\lambda)/\lambda^{1/2} \geqq \frac{1}{\pi}\int_0^\infty \sqrt{W}\,,$$

by Lemma 12. In order to show the complementary inequality

$$\limsup N(\lambda)/\lambda^{1/2} \leqq \frac{1}{\pi}\int_0^\infty \sqrt{W},$$

we wish to insert Dirichlet points, as in the proof of Lemma 12; for simplicity, we shall again assume that V has period 1. So let $f_\pm$ be as in the proof of Lemma 12, and assume that $f_+(0)f_-(0) \neq 0$ [if $f_+(0)f_-(0) = 0$, replace E by $E+\varepsilon$ for some $\varepsilon > 0$, chosen small enough so that no λ_i crosses 0; then the eigenvalues $\lambda_i = \lambda_i(\varepsilon)$ will decrease and we will have $N(\lambda; E+\varepsilon) \geqq N(\lambda; E)$]. Proceeding as in the proof of Lemma 12, but with reversed inequalities, we insert Dirichlet points

$$x_k := k\,, \qquad k = 1, 2, \ldots, \quad \text{if} \quad f_+(0)f_-(0) < 0\,,$$

or

$$\left.\begin{array}{ll} x_0 \in (0,1)\,, & f_+(x_0)f_-(x_0) < 0\,, \\ x_k := x_0 + k\,, & k = 0, 1, 2, \ldots\,, \end{array}\right\} \quad \text{if} \quad f_+(0)f_-(0) \geqq 0\,,$$

and denote by H_k the Dirichlet operators on the k^{th} interval (x_{k-1}, x_k) (with $x_{-1} = 0$, in the second case). We obtain $q(H-E)^{-1}q \leqq \bigoplus_k q(H_k - E)^{-1}q$, so that $N(\lambda) \leqq \sum_k N_k(\lambda)$, where

$$N_k(\lambda) := \#\{\lambda_{ki};\ (H_k - E)u_{ki} = \lambda_{ki} W u_{ki},\ 0 < \lambda_{ki} \leqq \lambda\}\,.$$

We will show below that there exist $a_k > 0$ such that

$$N_k(\lambda)/\lambda^{1/2} \leqq a_k\,, \qquad \sum_k a_k < \infty\,. \tag{4.22}$$

Using (4.22), dominated convergence implies

$$\limsup_{\lambda\to\infty} N(\lambda)/\lambda^{1/2} \leqq \sum_k \limsup_{\lambda\to\infty} N_k(\lambda)/\lambda^{1/2} = \sum_k \frac{1}{\pi}\int_{x_{k-1}}^{x_k} \sqrt{W} = \frac{1}{\pi}\int_0^\infty \sqrt{W},$$

where we have again used the standard asymptotics on the finite intervals (x_{k-1}, x_k).

It remains to prove (4.22): in the k^{th} interval, consider the equations $-u_i'' + (V-E)u_i = \lambda_i W u_i$ and $-\tilde{u}_i'' + (V-E)\tilde{u}_i = \tilde{\lambda}_i c(1+k)^{-p}\tilde{u}_i$, with Dirichlet boundary conditions at x_{k-1} and x_k. Then it is easy to see that $\tilde{\lambda}_i \leqq \lambda_i$ [first increase

$W\uparrow c(1+k)^{-p}$ and then decrease $V\downarrow 1$; note that in the second step some of the λ_i's may cross zero], so that

$$N_k(\lambda) \leqq \tilde{N}_k(\lambda) := \#\{\tilde{\lambda}_{ki}; \tilde{\lambda}_{ki} \leqq \lambda\}.$$

But we can explicitly calculate the $\tilde{\lambda}_i$'s; in fact,

$$\tilde{\lambda}_i c(1+k)^{-p} + E - 1 = i^2\pi, \qquad k, i = 1, 2, \ldots$$

(and similarly for the interval $(0, x_0)$ in the second case above). It follows that

$$N_k(\lambda) \leqq \tilde{N}_k(\lambda) \leqq \frac{1}{\pi}\sqrt{c\lambda(1+k)^{-p} + E - 1}. \tag{4.23}$$

On the other hand, by the same arguments as in the proof of Lemma 12, there exists a constant $c' > 0$ such that

$$c'\|u\| \leqq \|(H_k - E)u\|, \qquad u \in D(H_k), \qquad k = 0, 1, 2, \ldots,$$

which implies again that on (x_{k-1}, x_k), all λ_{ki} satisfy

$$\lambda_{ki} \geqq (c'/c)k^p,$$

and hence

$$N_k(\lambda) = 0, \qquad k > c''\lambda^{1/p}. \tag{4.24}$$

From (4.23), (4.24), it is now easy to see that there exists a constant $\tilde{c}$ such that

$$N_k(\lambda) \leqq \tilde{c}\lambda^{1/2}k^{-p/2}, \qquad \lambda > 0, \qquad k = 0, 1, 2, \ldots,$$

which proves (4.22), as $\sum k^{-p/2} < \infty$. $\square$

Acknowledgements. The second author (R.H.) wishes to thank the Courant Institute for its hospitality, and the Deutsche Forschungsgemeinschaft for financial support. He also would like to thank Dr. A. Hinz (Munich) for drawing his attention to problems of the type considered here.

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Communicated by B. Simon

Received May 20, 1985; in revised form July 29, 1985