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Borel Summability of the $1/N$ Expansion for the N-Vector [$O(N)$ Non-Linear σ] Models

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Abstract. We construct an analytic interpolation in $1/N$ for the N-vector [$O(N)$ non-linear σ] models with N-component fields on a lattice. This interpolation, valid at sufficiently high temperatures, extends over a large domain in the complex plane containing the half plane $\mathrm{Re}(1/N) > 0$. We use this result to show that the $1/N$ expansion of the free energy density and of the correlation functions is Borel summable in the thermodynamic limit and at high temperature.

1. Introduction, Notations and Main Results

In this paper we continue a mathematically rigorous analysis of the $1/N$ expansion in the N-vector models, initiated by A. Kupiainen [1, 2]. Kupiainen has shown that the $1/N$ expansion is asymptotic for two families of models, the N-vector models on a simple, (hyper) cubic lattice $\mathbb{Z}^d$, $d = 2, 3, 4, \ldots$, at temperatures above the critical temperature of the spherical model ($N = \infty$), and a class of weakly coupled N-component $\lambda|\phi|^4$ models in two space-time dimensions. A careful analysis of the $1/N$ expansion for the three-dimensional $O(N)$ σ-models in the continuum limit has been carried out by I. Aref'eva [3] who, however, has not determined its nature. For a summary of the history of $1/N$ expansions and references to important, earlier work, see Kupiainen's papers [1, 2].

A natural problem is to study the analyticity properties in $1/N$ and to determine the summability properties of the $1/N$ expansion for the models mentioned above. Billionnet and Renouard have recently proven that the $1/N$ expansion for weakly coupled N-component $\lambda|\phi|^4$ models in two dimensions is Borel-summable [4]. In this paper we establish the same result for the $O(N)$ non-linear σ-models on a lattice of arbitrary dimension, at high temperature. The methods used in this paper are different from the ones in [4]. In [4] the main technical difficulty appears in the construction of the continuum (ultraviolet) limit. Here we do not construct

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the continuum limit. However, the $1/N$ expansion for the N-vector models is an expansion around a stationary point of a complex measure which has only "poor fall-off at infinity" and which is hard to control when $1/N$ is close to the imaginary axis. We resolve this difficulty by superimposing a second expansion related to a standard high temperature expansion. Each term in this double expansion can be calculated explicitly and turns out to be analytic in $1/N$ everywhere except on the interval $[-1/2, 0]$. Our double expansion can be interpreted as a random walk—or polymer representation of thermodynamic and correlation functions of the N-vector models. It is inspired by work of Symanzik [5] and a rigorous version thereof [6] which has recently been applied intensively to the study of the Ising ($N=1$) and classical rotor ($N=2$) models [6–8]. The representation described in [6] can be interpreted as a double expansion in powers of N and β. It is used in [9] to derive joint analyticity in N and β near $N=0$, $\beta=0$. As in [6], our double expansion represents the N-vector models as gases of random walks or polymer chains with soft core interaction. In contrast to the random walks used in [6], the random walks introduced in this paper make steps of arbitrary length, but with exponentially decaying probability. Our random walk representation is chosen in such a way that the spherical model limit, $N \to \infty$, is reached smoothly along rays in the half plane $\operatorname{Re} 1/N > 0$.

The convergence of our double expansion is controlled by cluster (or polymer) expansion methods described in [10]; see also [11, 12].

We establish convergence at high temperature in a large domain of the $1/N$ plane, uniformly in the volume cutoff. We also derive bounds on the k^{th} derivative in $1/N$ of thermodynamic and correlation functions. When combined with a theorem due to Nevanlinna and Sokal [13] they suffice to prove Borel summability of the $1/N$ expansion.

Our paper is organized as follows:

In the remainder of Sect. 1 we define the lattice N-vector models, introduce some notation and summarize our main results in the form of a theorem.

In Sect. 2 we derive the basic random walk representation of our models and express it in a compact form which makes the methods of [10, 12] accessible.

In Sect. 3 we recall the cluster expansion [10–12] and use it to control the convergence of our random walk representation. We construct the thermodynamic limit of our models and verify analyticity in $1/N$ in a large domain and prove Borel summability at $1/N=0$, for sufficiently high temperatures; (see Theorem A below).

In Sect. 4 we discuss our results and describe some open problems related to the main theme of this paper.

Some technical estimates needed in Sects. 2 and 3 are proven in an appendix.

Throughout this paper we follow quite closely the notation introduced in [1]. We adopt the conventions that

$$\text{an empty sum} = 0,$$

$$\text{an empty product} = 1.$$

With each site, j, of the simple (hyper) cubic lattice $\mathbb{Z}^d$, $d = 2, 3, 4, \ldots$, we associate a classical, N-component spin

$$\phi_j = (\phi_j^1, \ldots, \phi_j^N), \quad N = 1, 2, 3, \ldots \tag{1.1}$$

of length N, i.e. $|\phi_j|^2 = N$. The Hamilton function of the model in a finite sublattice Λ of $\mathbb{Z}^d$ is given by

$$H_\Lambda(\phi) = - \sum_{(jj') \subset \Lambda} \phi_j \cdot \phi_{j'}, \tag{1.2}$$

where the summation ranges over all pairs of nearest neighbors in Λ, and $\phi \cdot \phi'$ is the usual scalar product of ϕ with ϕ'. For reasons of technical simplicity we shall choose Λ to be a rectangle and impose periodic boundary conditions, but (in contrast to the methods used in [1]) we can accommodate arbitrary boundary conditions. The Gibbs state of the N-vector model ($=$ Euclidean vacuum functional of the $O(N)$ non-linear σ-model on the lattice) on the sublattice Λ is given by

$$d\mu_\beta^{(\Lambda)}(\phi) = (Z_\beta^{(\Lambda)})^{-1} e^{-\beta H_\Lambda(\phi)} \prod_{j \in \Lambda} \delta(|\phi_j|^2 - N) d^N \phi_j, \tag{1.3}$$

where β is the inverse temperature ($=$ inverse square coupling constant), and $Z_\beta^{(\Lambda)}$ is the usual partition function.

The correlation functions, $\langle \phi_{x_1}^{\alpha_1} \ldots \phi_{x_m}^{\alpha_m} \rangle_{\beta, z}$, $z = 1/\sqrt{N}$, are obtained from the characteristic functional

$$\begin{aligned} S(g) &= \int d\mu_\beta^{(\Lambda)}(\phi) \exp \phi(g), \\ \phi(g) &\equiv \sum_{\alpha=1}^{d} \sum_j \phi_j^\alpha g_j^\alpha \end{aligned} \tag{1.4}$$

by functional differentiation in g.

Following [7, 1, 6] we now introduce the dual representation of the N-vector model which displays N, or $1/N$, as a parameter. Let Δ be the usual finite difference Laplacian defined by

$$\Delta_{ij} = -2d\delta_{ij} + \sum_{|k|=1} \delta_{i,j+k},$$

with some boundary conditions (b.c.) (e.g. periodic) imposed at $\partial\Lambda$. We define

$$C \equiv C_\beta = (-\beta\Delta + m^2)^{-1}, \tag{1.5}$$

where $m \equiv m_j \geqq 0$ is some mass parameter which may, *a priori*, depend on the site $j \in \Lambda$ and will be specified below. We set

$$z = 1/\sqrt{N} \quad \text{and} \quad y = 2/N = 2z^2. \tag{1.6}$$

The first step in the derivation of the dual representation is to notice that in the definition of $d\mu_\beta^{(\Lambda)}$, Eq. (1.3), we may multiply each factor $\delta(|\phi_j|^2 - N)$ by $\exp[-(\beta d + m_j^2/2)|\phi_j|^2]$, which can be absorbed in a redefinition of $Z_\beta^{(\Lambda)}$. Second, we Fourier-decompose $\delta(|\phi_j|^2 - N)$,

$$\delta(|\phi_j|^2 - N) = (1/2\pi) \int d\alpha_j e^{i\alpha_j(|\phi_j|^2 - N)}$$

for each $j \in \Lambda$, and subsequently interchange integration over α and ϕ. The resulting ϕ-integral is Gaussian, and we obtain

$$S(g) = Z_1^{-1} \int \exp\{(1/2)(g, [-\beta\Delta + m^2 - 2i\alpha]^{-1} g)\}$$
$$\cdot \det(-\beta\Delta + m^2 - 2i\alpha)^{-N/2} \prod_{j\in\Lambda} e^{-i\alpha_j N} d\alpha_j, \tag{1.7}$$

where $Z_1 = \mathrm{const}^{|\Lambda|} Z_\beta^{(\Lambda)}$, and $|\Lambda|$ is the number of sites in Λ. Identity (1.7) is very convenient for analyzing the behaviour of $S(g)$ near $N = 0$; (one can prove joint analyticity in β and N inside some discs centered at $N = 0$ and $\beta = 0$. See [9]). In order to determine the behaviour of $S(g)$ near $N = \infty$, one changes variables,

$$\alpha_j = z a_j, \quad \text{for all } j \in \Lambda,$$

and applies the identity

$$\det(A) = \exp \operatorname{tr} \log(A).$$

This yields

$$S(g) = \int \exp\{(1/2)(g, [1 - 2izC_\beta a]^{-1} C_\beta g)\} d\mu_{\beta,z}(a), \tag{1.8}$$

where

$$d\mu_{\beta,z}(a) = Z_2^{-1} \exp\{-(1/y) \operatorname{tr} \log(1 - 2izC_\beta a)\} \cdot \prod_{j\in\Lambda} \exp(-ia_j/z) da_j, \tag{1.9}$$

and

$$Z_2 = \mathrm{const}^{|\Lambda|} Z_1.$$

We now choose $m^2 = m_j^2$ such that the terms linear in a in the exponents on the right side of (1.9) cancel. This is the case iff

$$C_{\beta,jj} = 1, \quad \text{for all } j \in \Lambda, \tag{1.10}$$

where

$$C_{\beta,ij} = (\delta_i, C_\beta \delta_j).$$

This condition ensures that the complex measure $d\mu_{\beta,z}$ is, in zeroth order in z, given by the positive Gaussian measure of the spherical model, the deviation from the Gaussian being of the form $\exp O(z)$. Condition (1.10) plays a crucial rôle in our subsequent analysis. It is a system of transcendental equations for the mass parameters $m_j = m_j(\beta, \Lambda)$. The simplifying feature of periodic b.c. is that condition (1.10) reduces to a single equation (because of translation invariance) which has a constant solution $m_j = m(\beta, \Lambda) > 0$, for all β, when $|\Lambda|$ is finite. This is the only instance in this paper where periodic b.c. are more convenient than other b.c. In [1], $m = m(\beta)$ is chosen so that (1.10) is only satisfied in the thermodynamic limit, $\Lambda \to \mathbb{Z}^d$. It then reduces to

$$\int_{[-\pi,\pi]^d} \frac{d^d p}{(2\pi)^d} \left[2\beta \sum_{\mu=1}^{d} (1 - \cos p_\mu) + m^2 \right]^{-1} = 1. \tag{1.11}$$

The inverse critical temperature β_S of the spherical model is the maximal value of β for which (1.11) has a solution, $m = m(\beta_S) = 0$. For $d \geqq 3, 0 < \beta_S < \infty$, while for $d = 2$, $\beta_S = \infty$. For $\beta < \beta_S$, (1.11) has a solution $m = m(\beta)$, and the solution, $m(\beta, \Lambda)$, of (1.10) (for periodic b.c.) converges to $m(\beta)$, as $\Lambda \to \mathbb{Z}^d$, for all $\beta < \beta_S$. Moreover

$$\lim_{\beta \to 0} m(\beta, \Lambda) = \sup_{\beta} m(\beta, \Lambda) = 1.$$

Note that on the right side of (1.8) and (1.9) $z = 1/\sqrt{N}$ appears as a parameter. It is not hard to see that $S(g)$ is a function of $y = 2z^2$ and that it is, in fact, analytic in y in a complex neighborhood of the *open* positive half axis when $|\Lambda|$ is finite. Moreover when $y = 2/N$ and N is an integer it agrees with the functional defined in (1.4). Our task is now to continue $S(g) = S^{(\Lambda)}_{\beta,y}(g)$ analytically in y (and β) to as large a domain as possible and to derive estimates on $|S^{(\Lambda)}_{\beta,y}(g)|$ in that domain which are uniform in Λ. Our main results can be summarized as follows.

Theorem A. *For $0 < \beta < \beta_\varepsilon$, for some positive constant $\beta_\varepsilon < \beta_S$, the thermodynamic limits of the free energy and the correlation functions exist and are analytic functions of $y = 2/N$ on the domain $\{y : |\arg y| < \pi - \varepsilon$,* or *$|y| > 1 + \varepsilon\}$. In that range of parameters connected correlation functions have exponential clustering. For $\beta < \beta_{\pi/2}$, the $1/N$ expansions of the free energy and the correlation functions are Borel summable.*

2. A New Random Walk Representation and the Polymer Method

We combine Eqs. (1.8) and (1.9) into the following formula for $S(g)$:

$$\begin{aligned} S(g) = Z_2^{-1} \int \exp\{(1/2)(g, [1 - 2izC_\beta a]^{-1} C_\beta g)\} \\ \cdot \exp\{-(1/y) \operatorname{tr} \log (1 - 2izC_\beta a)\} \prod_{j \in \Lambda} \exp(-ia_j/z) da_j, \end{aligned} \tag{2.1}$$

where Z_2 is chosen such that

$$S(g = 0) = 1. \tag{2.2}$$

The basic idea of this section is to split C_β into a diagonal part C^D_β and an off-diagonal part C^0_β and to expand perturbatively in C^0_β. By (1.10)

$$C^D_{\beta,ij} = C_{\beta,jj} \delta_{ij} = \delta_{ij}. \tag{2.3}$$

In order to understand the crucial significance of condition (2.3) in our scheme we first consider the case where

$$C_{\beta,jj} = \gamma. \tag{2.4}$$

for some parameter $\gamma > 0$ which will subsequently be set $= 1$.

We now insert the identity

$$\begin{aligned} 1 - 2izC_\beta a &= 1 - 2iz\gamma a - 2izC^0_\beta a \\ &= (1 - 2iz\gamma a)\left[1 - \gamma^{-1} C^0_\beta \left(\frac{1}{1 - 2iz\gamma a} - 1\right)\right] \end{aligned} \tag{2.5}$$

into (2.1) and expand in powers of $\gamma^{-1}C^0_\beta$. We make use of the identities

$$\log(1-2izC_\beta a)=\log(1-2iz\gamma a)\left\{-\sum_{k=1}^{\infty}\frac{1}{k}\left[\gamma^{-1}C^0_\beta\left(\frac{1}{1-2iz\gamma a}-1\right)\right]^k\right\}, \tag{2.6}$$

and

$$\begin{aligned}[1-2izC_\beta a]^{-1}C_\beta &=(1-2iz\gamma a)^{-1}\left[1-\gamma^{-1}C^0_\beta\left(\frac{1}{1-2iz\gamma a}-1\right)\right]^{-1}C_\beta\\ &=\gamma(1-2iz\gamma a)^{-1}\left\{1+\sum_{k=0}^{\infty}\left[\gamma^{-1}C^0_\beta\left(\frac{1}{1-2iz\gamma a}-1\right)\right]^k\right.\\ &\quad\left.\cdot\gamma^{-1}C^0_\beta(1-2iz\gamma a)^{-1}\right\}.\end{aligned} \tag{2.7}$$

Before we present the result of these expansions we introduce some notations and conventions:

An oriented random loop, denoted by ω, is a closed, directed random walk built of an arbitrary number, $p=p(\omega)$, of steps of arbitrary, but strictly positive length. Thus, an oriented random loop is a class of ordered sequences of jumps, from a site i_1 to a site i_2, from i_2 to i_3, and so on, the last jump being from i_p to $i_{p+1}=i_1$, but two sequences of jumps which only differ in the choice of the starting site correspond to the same random loop.

An open random walk, ω, from $x\in\Lambda$ to $u\in\Lambda$ consists of a sequence of $p=p(\omega)$ jumps, from $i_1=x$ to i_2, from i_2 to i_3, and so on, and finally from i_p to $i_{p+1}=u$; $i_k\neq i_{k+1}$, for all k. If $x\neq u$ we have $p(\omega)\geqq 1$. The circumstance that ω starts at x and ends at u is expressed, symbolically, as $\omega: x\to u$.

With every random walk, closed or open, we associate a weight, J_ω,

$$J_\omega=\prod_{k=1}^{p(\omega)}[\gamma^{-1}C_{\beta,i_ki_{k+1}}]. \tag{2.8}$$

Furthermore, we let $n_j(\omega)$ denote the number of visits of a random walk ω at site j. If $\omega: x\to u$ is open, we define $n_j(x)$ and $n_j(u)$ to be the total number of visits of ω at x, u, respectively, *not* counting the first visit of ω at x, the last visit of ω at u, respectively.

The length, $l(\omega)$, of a random walk ω is defined by

$$l(\omega)=\sum_{k=1}^{p(\omega)}|i_k-i_{k+1}|, \tag{2.9}$$

where $|i-j|$ is the minimal number of nearest neighbor jumps necessary to get from i to j. Clearly

$$l(\omega)\geqq p(\omega). \tag{2.10}$$

Inserting (2.5), (2.6) into the equation defining Z_2 (see (2.2)) and making use of definition (2.8) we obtain

$$Z_2 = \sum_{m=0}^{\infty} \sum_{\omega_1,\ldots,\omega_m} \frac{1}{y^m m!} \prod_{k=1}^{m} J_{\omega_k} \int \prod_{j\in\Lambda} (1-2iz\gamma a_j)^{-1/y} \cdot \left(\frac{1}{1-2iz\gamma a_j}-1\right)^{n_j} e^{-ia_j/z} da_j, \tag{2.11}$$

where $\omega_1,\ldots,\omega_m$ are m not necessarily distinct oriented random loops and

$$n_j = \sum_{k=1}^{m} n_j(\omega_k). \tag{2.12}$$

If we insert identities (2.6) and (2.7) into the right side of Eq. (2.1) defining $S(g)$, differentiate twice with respect to g and make use of (2.8), we obtain the following expression for the two-point correlation, or Schwinger function

$$\langle \phi_x^1 \phi_u^1 \rangle_{\beta,z} = Z_2^{-1} \sum_{m=0}^{\infty} \sum_{\omega,\omega_1,\ldots,\omega_m} \frac{\gamma}{y^m m!} J_\omega \prod_{k=1}^{m} J_{\omega_k}$$

$$\int \prod_{j\in\Lambda} (1-2iz\gamma a_j)^{-(1/y)-\delta_{xj}-\delta_{uj}} \left(\frac{1}{1-2iz\gamma a_j}-1\right)^{n_j} e^{-ia_j/z} da_j, \tag{2.13}$$

where $\sum_{\omega,\omega_1,\ldots,\omega_m}$ ranges over all open random walks $\omega: x\to u$ and over m not necessarily distinct, oriented random loops $\omega_1,\ldots,\omega_m$, and

$$n_j = n_j(\omega) + \sum_{k=1}^{m} n_j(\omega_k). \tag{2.14}$$

If in (2.13) $x=u$, ω may be an empty random walk, in which case the exponent $-(1/y)-\delta_{xj}-\delta_{uj}$ on the right side of (2.13) must be replaced by $-(1/y)-\delta_{xj}$. Formulae analogous to (2.13) can be derived for an arbitrary $2n$ point correlation function, $\langle \phi_{x_1}^{\alpha_1} \ldots \phi_{x_{2n}}^{\alpha_{2n}} \rangle_{\beta,z}$, (with even numbers of α_j's equal to some $\alpha = 1,\ldots,N$). See also [6] for derivations of formulae quite similar to (2.11) and (2.13). For reasons of simplicity of our exposition we only study the two-point function, $\langle \phi_x^1 \phi_u^1 \rangle_{\beta,z}$, with $x \neq u$, henceforth, but our methods cover general correlations, as well. Now, note that the integrals over the parameters $a_j, j\in\Lambda$, in (2.11) and (2.13) factorize. They can be evaluated by using the simple identity

$$\int_{-\infty}^{\infty} da\, e^{-ia/z}(1-2iz\gamma a)^{-r} = \frac{1}{\Gamma(r)} \int_0^{\infty} dt\, t^{r-1} e^{-t} \int_{-\infty}^{\infty} e^{ia(2z\gamma t - z^{-1})} da$$

$$= \frac{2\pi z}{\Gamma(r)} (2z^2\gamma)^{-r} \exp[-1/2z^2\gamma]. \tag{2.15}$$

Let $\Omega = \{\omega_1,\ldots,\omega_m\}$ be an arbitrary, ordered m-tuple of oriented random loops,

$|\Omega| \equiv m, m = 1, 2, 3, \ldots,$ and

$$J_\Omega = \prod_{k=1}^{m} J_{\omega_k}. \tag{2.16}$$

We now insert the right side of (2.15) and (2.16) into the right side of (2.11) and of (2.13) and use repeatedly the identity

$$\Gamma(r) = (r-1)\Gamma(r-1).$$

This yields

$$Z_2 = \sum_\Omega \frac{1}{y^{|\Omega|}|\Omega|!} J_\Omega \prod_{j\in\Lambda} \Bigg[(\Gamma(1/y)^{-1} 2\pi z e^{-1/y}(1|y)^{1/y}) \cdot \left(\sum_{l=0}^{n_j} \binom{n_j}{l} (1/\gamma)^l (-1)^{n_j - l} \prod_{k=0}^{l-1} (1+ky)^{-1} \right) \Bigg], \tag{2.17}$$

and

$$Z_2 \langle \phi_x^1 \phi_u^1 \rangle_{\beta,z} = \sum_{\substack{\Omega \\ \omega: x \to u}} \frac{\gamma}{y^{|\Omega|}|\Omega|!} J_\Omega J_\omega \prod_{j\in\Lambda} \Bigg[(\Gamma(1/y)^{-1} 2\pi z e^{-1/y}(1/y)^{1/y}) \cdot \sum_{l=0}^{n_j} \binom{n_j}{l} (1/\gamma)^{l+\delta_{xj}+\delta_{uj}} (-1)^{n_j - l} \prod_{k=\delta_{xj}+\delta_{uj}}^{l-1+\delta_{xj}+\delta_{uj}} (1+ky)^{-1} \Bigg]. \tag{2.18}$$

The factor $[\Gamma(1/y)^{-1} 2\pi z e^{-1/y}(1/y)^{1/y}]^{|\Lambda|}$ can be absorbed in a redefinition of the partition function

$$Z_3 \equiv Z_2 [\Gamma(1/y)^{-1} 2\pi z e^{-1/y}(1/y)^{1/y}]^{-|\Lambda|}. \tag{2.19}$$

We notice that that factor is the total volume of the spheres over which the classical spins of the system may range, normalized so that the limit, as $y \to 0$, is finite.

If we define free energy densities

$$\left.\begin{aligned} \beta f'_\Lambda(\beta, z) &= \frac{1}{|\Lambda|} \log Z_2, \\ \beta f_\Lambda(\beta, y) &= \frac{1}{|\Lambda|} \log Z_3, \end{aligned}\right\} \tag{2.20}$$

then $\beta f'_\Lambda - \beta f_\Lambda$, expanded in powers of z, is given by Stirling's series which is Borel summable in the z variable [14] but not in y; (see also Sect. 4).

We now introduce the notation

$$R_n^q(\gamma; y) \equiv \frac{Q_n^q(\gamma; y)}{\prod_{k=q}^{n-1+q} (1+ky)} = \sum_{l=0}^{n} \binom{n}{l} \frac{\gamma^{-l-q}(-1)^{n-l}}{\prod_{k=q}^{l-1+q} (1+ky)}, \tag{2.21}$$

and

$$R_n^q(y) = R_n^q(1; y), \quad Q_n^q(y) = Q_n^q(1; y). \tag{2.22}$$

Clearly $Q_n^q(\gamma; y)$ is a polynomial in y, and $R_n^q(\gamma; y)$ is a rational function of y which is analytic in y in the entire complex plane, except in the interval $[-1, 0]$. In general $Q_n^q(\gamma; y=0) \neq 0$. The point is that for $\gamma = 1$ we have the following result.

Lemma 2.1. *For* $\gamma = 1$

$$\lim_{y \to 0} |y^{-E(n)} Q_n^q(1; y)| = \lim_{y \to 0} |y^{-E(n)} Q_n(y)|$$

is finite, where

$$E(n) = \text{integer part of } \frac{n+1}{2}. \qquad \square$$

The proof of Lemma 2.1 is given in the appendix.

From (2.17)–(2.19) and (2.21), (2.22) we obtain the representations

$$Z_3 = \sum_{\Omega} \frac{1}{y^{|\Omega|}|\Omega|!} J_{\Omega} \prod_{j \in \Lambda} R^0_{n_j(\Omega)}(y), \tag{2.23}$$

and

$$Z_3 \langle \phi_x^1 \phi_u^1 \rangle_{\beta,z} = \sum_{\substack{\Omega \\ \omega : x \to u}} \frac{1}{y^{|\Omega|}|\Omega|!} J_{\Omega} J_{\omega} \prod_{j \in \Lambda} R^{\delta_{xj} + \delta_{uj}}_{n_j(\omega,\Omega)}(y), \tag{2.24}$$

with $R_n^q(y)$ as in (2.21), (2.22), and

$$n_j(\Omega) = \sum_{\omega' \in \Omega} n_j(\omega'), \quad \text{see (2.12)},$$

$$n_j(\omega, \Omega) = n_j(\omega) + \sum_{\omega' \in \Omega} n_j(\omega'),$$

see (2.14).

It is easy to see that

$$\sum_{j \in \Lambda} E(n_j) > |\Omega|.$$

Therefore the singular factor $y^{-|\Omega|}$ on the right side of (2.23) and (2.24) is *cancelled* by

$$\prod_{j \in \Lambda} Q^q_{n_j}(\gamma; y)$$

provided $\gamma = 1$, i.e. $C_{\beta,jj} = 1$ (see (2.4)), as follows from Lemma 2.1. Thus, we now understand the crucial rôle of condition (1.10).

The proof of analyticity of $\beta f(\beta, y)$ and of $\langle \phi_x^1 \phi_u^1 \rangle_{\beta,z}$ in y in the domain

$$\{y : |\arg y| < \pi - \varepsilon \quad \text{or} \quad |y| > 1 + \varepsilon\}$$

for β sufficiently small (depending on ε), and of Borel summability in y at $y = 0$ is now reduced to proving convergence of the expansions (2.23) and (2.24) and of their Taylor remainders (at $y = 0$) in the above domains of y and β, uniformly in $|\Lambda|$.

We shall establish convergence by applying the polymer method [10–12] to the right side of (2.23) and (2.24). We first notice that a term on the right side of

(2.23) or (2.24) indexed by a family $\Omega=\{\omega_1,\dots,\omega_m\}$ of oriented random loops does not depend on the ordering of $\{\omega_1,\dots,\omega_m\}$. We may thus resum over all families of oriented random loops which only differ in their ordering. This yields expansions of Z_3 and $\langle\phi_x^1\phi_u^1\rangle_{\beta,z}$ with terms labelled by what are called multi-indices in [10], i.e. by functions from the set Γ of all random walks to $\mathbb{N}$. In the following, these labels, the multi-indices are called g-sets (for "generalized sets") of random walks. A g-set may contain an arbitrary number of copies of the same random walk. We shall use set-theoretical language (whenever it is unambiguous), speaking of unions of g-sets = addition of multi-indices, of elements of g-sets = particular copies of particular random walks in a g-set, etc. Let Ω be some g-set. Given some random walk $\omega\in\Gamma$, let $\nu(\Omega,\omega)$ be the number of copies of ω appearing in Ω. We define

$$[\Omega]!=\prod_{\omega\in\Gamma}\nu(\Omega,\omega)! \tag{2.25}$$

with the usual convention that $0!=1$.

We may now rewrite expansions (2.23) and (2.24) by summing over all distinct g-sets Ω of oriented random loops (rather than over arbitrary, ordered collections of random loops) and replacing $|\Omega|!$ by $[\Omega]!$.

Next, we observe that each term in these modified expansions labelled by a disconnected g-set can be factorized in a product of terms labelled by connected g-sets, also called polymers, and denoted by Ω^c. (A g-set Ω is called connected, i.e. a polymer, if it is given by oriented random loops, or walks $\omega_1,\dots,\omega_m$, not necessarily distinct, with the property that $\omega_k\cap\omega_{k+1}\neq\emptyset$, $k=1,\dots,m-1$.) Two polymers are said to be compatible if their union is not a polymer. (Otherwise they are said to be incompatible.) If $\tilde\Omega$ is a g-set of oriented random loops with the property that $\tilde\Omega\cup\{\omega\}$ is a polymer, where $\omega: x\to u$ is an open random walk, then $\Omega^c=\tilde\Omega\cup\{\omega\}$ is called an $x\to u$ polymer.

Given some polymer, Ω^c, we let $|\Omega^c|$ denote the total number of oriented random loops in Ω^c, counting multiplicities,

$$p(\Omega^c)=\sum_{\omega\in\Omega^c}p(\omega), \tag{2.26}$$

(we recall that $p(\omega)$ is the number of steps made by the random walk ω),

$$n_j(\Omega^c)=\sum_{\omega\in\Omega^c}n_j(\omega),\quad j\in\Lambda, \tag{2.27}$$

$$l(\Omega^c)=\sum_{\omega\in\Omega^c}l(\omega). \tag{2.28}$$

Clearly

$$2|\Omega^c|\leqq p(\Omega^c)=\sum_{j\in\Lambda}n_j(\Omega^c)\leqq l(\Omega^c). \tag{2.29}$$

We also define an activity, z, of each polymer,

$$z(\Omega^c)=\frac{1}{y^{|\Omega^c|}[\Omega^c]!}J_{\Omega^c}\prod_{j\in\Lambda}R^{q_j}_{n_j(\Omega^c)}(y), \tag{2.30}$$

where $q_j = \delta_{xj} + \delta_{uj}$ if Ω^c is an $x \to u$ polymer, and $q_j = 0$, otherwise. (Note that $|\Omega^c|$ is the total number of random loops in Ω^c, not counting open random walks.)

The expansions (2.23) and (2.24) can now be written as sums of products of activities associated with compatible polymers. The constraint of compatibility is equivalent to a hard core exclusion between polymers. Let

$$g(\Omega^c, \Omega^{c'}) = \begin{cases} 0 & \text{if } \Omega^c \text{ and } \Omega^{c'} \text{ are} \\ & \text{compatible polymers} \\ -1, & \text{otherwise.} \end{cases} \tag{2.31}$$

Then the expansions (2.23) and (2.24) take the form

$$Z_3 = \sum_{r=0}^{\infty} \sum_{\{\Omega_1^c, \dots, \Omega_r^c\}} \prod_{k=1}^{r} z(\Omega_k^c) \prod_{1 \leqq k < k' \leqq r} [1 + g(\Omega_k^c, \Omega_{k'}^c)], \tag{2.32}$$

$$\langle \phi_x^1 \phi_u^1 \rangle_{\beta,z} = Z_3^{-1} \sum_{r=0}^{\infty} \sum_{\{\Omega_1^c, \dots, \Omega_r^c\}} \prod_{k=1}^{r} z(\Omega_k^c) \cdot \prod_{1 \leqq k < k' \leqq r} [1 + g(\Omega_k^c, \Omega_{k'}^c)], \tag{2.33}$$

where in (2.32) summation extends over all sets of polymers made of oriented random loops while in (2.33) exactly one polymer is an $x \to u$ polymer. The factor

$$\prod_{1 \leqq k < k' \leqq r} [1 + g(\Omega_k^c, \Omega_{k'}^c)]$$

is the Boltzmann factor of the hard core interaction between polymers. The activities $z(\Omega^c) = z(\Omega^c; y)$ are functions of $y = 2/N$.

Criteria for convergence of the expansions (2.32) and (2.33) are well known [11, 12]. See, in particular, the clear presentation in [10]. In the appendix we prove the following bound on the activity $z(\Omega^c)$ of a polymer, Ω^c:

Lemma 2.2. *If y is such that $|\arg y| < \pi - \varepsilon$ or $|y| > 1 + \varepsilon$, then there exists a finite constant K_ε such that*

$$|z(\Omega^c)| \leqq [2d\beta K_\varepsilon (m^2(\beta, \Lambda))^{-1}]^{l(\Omega^C)}, \tag{2.34}$$

where $m^2(\beta, \Lambda)$ is the solution of Eq. (1.10).

Remarks. 1) It follows from [10–12] and (2.34) that the expansions (2.32) and (2.33) can be exponentiated and converge uniformly in $|\Lambda|$ if β is sufficiently small, ($\beta < \text{const}\, K_\varepsilon^{-1}$). In the next section we recall some of the machinery developed in [10–12], following [10], which can be used to prove convergence.

2) The main analytical facts needed to prove (2.34) are the following estimates:

a) For $k \leqq E(n)$,

$$|y^{-k} Q_n^q(y)| \leqq \text{const}_q{}^n k! \prod_{k=q}^{n-1+q} (1 + k|y|), \tag{2.35}$$

where Q_n^q has been defined in (2.21), (2.22).

b) $$|J_{\Omega^c}| \leqq \left[\frac{2d\beta}{m^2(\beta,\Lambda)}\right]^{l(\Omega^c)}, \tag{2.36}$$

with $m(\beta,\Lambda)\searrow 1$, as $\beta\searrow 0$; (see (1.10), (1.11)).

Fact a) follows from the definition of Q_n^q and Lemma 2.1 which asserts that, for $k \leqq E(n), y^{-k}Q_n^q(y)$ is analytic near $y=0$, so that one can use the Cauchy estimate. Fact b) is standard (see e.g. [6]). Details are provided in the appendix. Lemma 2.2 follows from facts a) and b) combined with some fairly subtle combinatorial arguments contained in the appendix.

3. Convergence of the Expansions (2.32) and (2.33), and Proof of Theorem A

The form of expansions (2.32) and (2.33) and estimate (2.34) on the activity of polymers permit us to use the polymer method described in [10–12] in order to prove convergence. (We follow [10].) We now briefly recall the main features of that method and some combinatorial estimates which we also need in our proof of Lemma 2.2, contained in the appendix.

Our aim is to prove existence of the thermodynamic limit, analyticity properties in y and Borel summability of the $1/N$ expansion of

$$\beta f_\Lambda(\beta,y) = |\Lambda|^{-1}\log Z_3, \tag{3.1}$$

and

$$\langle \phi_x^1\phi_u^1\rangle_{\beta,z} = \frac{d}{dt}\log[Z_3 + tZ_3\langle\phi_x^1\phi_u^1\rangle_{\beta,z}]|_{t=0}. \tag{3.2}$$

By (2.33), the unnormalized two point function $tZ_3\langle\phi_x^1\phi_u^1\rangle_{\beta,z}$, has a polymer expansion in which the activities of the $x\to u$ polymer are multiplied by a factor t. In order to control the logarithms of polymer expansions we require two basic properties (A and B, below) of polymers; (see e.g. [10]).

Property A. Let Ω_0^c be a polymer of length l_0. The total number of polymers, Ω^c, of length l which are incompatible with Ω_c^0 is bounded by $l_0K_1^l$, where K_1 is a constant proportional to d, and d is the dimension of the lattice.

To verify Property A in our case, we first note that we may associate with each polymer Ω^c a walk $\overline{\Omega^c}$, and two subsets $S(\Omega^c)$ and $E(\Omega^c)$ made of jumps in $\overline{\Omega^c}$. We construct $\overline{\Omega^c}$ first. We start at some site $j_1\in\Omega^c$ ($j_1=x$ if Ω^c is an $x\to u$ polymer). All walks visiting j_1 form a g-subset H_1. We choose one of them and follow its jumps till we arrive at some site j_2 visited by a non-empty g-subset H_2 of walks in $\Omega^c\sim H_1$. We choose one of the walks in H_2 and follow its jumps till we arrive at a site j_3 visited by some walk in $\Omega^c\sim H_1\sim H_2$, etc. Sometimes the walk in $H_k, k\geqq 1$, that we were following terminates. This can only happen at the site j_k. We then choose another walk in H_k whose jumps we have not yet followed, or—if no such walk exists anymore—we follow further the jumps of the walk in H_{k-1} along which we arrived at j_k the first time. In this way we must finally get back to j_1 (or land at u if Ω^c is an $x\to u$ polymer), having completely followed all the paths in Ω^c (provided, if Ω^c is an $x \to u$ polymer, we choose the $x \to u$ path in H_1 as late as possible...). The walk which we have followed in this fashion is $\overline{\Omega^c}$. Our definition of $\overline{\Omega^c}$ depends most of the time on many arbitrary choices, but this need not concern us. The subsets $S(\Omega^c)$ and $E(\Omega^c)$ are defined by the following condition: a jump j in $\overline{\Omega^c}$ belongs to $S(\Omega^c)$ (respectively: to $E(\Omega^c)$) if and only if there exists a walk ω in Ω^c such that the jump j is the first one (respectively: the last one) that we followed on the walk ω in the process described above.

We claim that given a random walk ω and two subsets S and E of its jumps, there exists at most one polymer Ω^c such that $\overline{\Omega^c} = \omega$, $S(\Omega^c) = S$ and $E(\Omega^c) = E$. Indeed, if such a polymer Ω^c exists, by the way $\overline{\Omega^c}$ is constructed, there has to be a subloop ω_1 of ω, i.e. a closed sequence of consecutive jumps in ω, which start (respectively: end) by a jump in S (respectively: in E) and is minimal with this property. If $\omega^1 = \omega \sim \omega_1$ is not empty, we can find a subloop ω_2 of ω^1 with the same property, replacing S and E by $S_1 = S \cap \omega^1$ and $E_1 = E \cap \omega^1$. Finally we end up with ω^{k+1} empty, and Ω^c must be $\{\omega_1, \omega_2, \dots, \omega_k\}$. There is no arbitrariness in fact in this construction of Ω^c and this proves the claim. Since there are no more than $4^{p(\omega)}$ possible subsets S and E for a given ω, and since there are no more than $l_0(2d)^l$ random walks ω of length l incompatible with Ω_0^c, the number of polymers, Ω^c, of length l incompatible with Ω_0^c is bounded by

$$l_0 4^{p(\Omega^c)}(2d)^l \leqq l_0(8d)^l, \tag{3.3}$$

which proves Property A.

Similar arguments show that the total number of polymers $\Omega_1^c, \Omega_2^c, \dots$, with $\Omega_k^c \cap \Omega_{k+1}^c \neq \emptyset, \Omega_1^c \cup \Omega_2^c \cup \dots$ contains 0, and $l = \sum_{k=1,2,\dots} l(\Omega_k^c)$ fixed is bounded by $(16\mathrm{d})^l$.

Property B. The activity $z(\Omega^c)$ of each polymer Ω^c is bounded by $K_2^{l(\Omega^c)}$, where K_2 is some geometric constant depending on d.

Obviously, Property B follows from Lemma 2.2: For $\beta < \beta_\varepsilon < \beta_S$,

$$K_\varepsilon \beta (2d\beta + m^2(\beta, \Lambda))^{-1} < K_2. \tag{3.4}$$

We now perform the cluster expansion, following closely [10].

In what follows, g-sets of polymers are denoted by X. If a g-set of polymers, X, contains precisely one $x \to u$ polymer we write $X : x \to u$. With each g-set of polymers, X, we associate a graph $G(x)$ whose vertices are the polymers in X. Two vertices of $G(X)$ are joined by a line iff the corresponding polymers are incompatible. The total number of lines in a graph, C, is denoted by $L(C)$. Given a g-set X of polymers and a polymer Ω^c, let $v(X, \Omega^c)$ be the number of copies of Ω^c in X, and let

$$[X]! = \prod_{\Omega^c} v(X, \Omega^c)!. \tag{3.5}$$

Finally

$$z(X) = \prod_{\Omega^c} z(\Omega^c)^{v(X, \Omega^c)} \tag{3.6}$$

We now have

Lemma 3.1. [10, 11].

$$Bf_\Lambda(\beta, y) = |\Lambda|^{-1} \sum_X \frac{1}{[X]!} \phi^T(X) z(X), \tag{3.7}$$

$$\langle \phi_X^1 \phi_u^1 \rangle_{\beta,z} = \sum_{X : x \to u} \frac{1}{[X]!} \phi^T(X) z(X), \tag{3.8}$$

where

$$\phi^T(X) = \sum_{C \subset G(X)} (-1)^{L(C)},$$

and the sum ranges over all connected subgraphs C of $G(X)$ containing all the vertices of $G(X)$.

Note that $\phi^T(X) = 0$, unless $G(X)$ is connected (i.e. $X = \{\Omega_1^c, \ldots, \Omega_r^c\}, r = 1, 2, 3, \ldots$, with Ω_k^c and Ω_{k+1}^c incompatible, for all $k = 1, \ldots, r-1$).

Lemma 3.2. [10, 12].

$$|\phi^T(X)| \leqq \prod_{\Omega^c \in X} I_X(\Omega^c), \tag{3.9}$$

where $I_X(\Omega^c)$ is the number of polymers in X which are incompatible with Ω^c, [*i.e. $I_X(\Omega^c)$ is the number of lines of $G(X)$ leaving the vertex corresponding to Ω^c*].

The key combinatorial lemma based on Property A is the following □

Lemma 3.3. (Malyshev's theorem [12], [10])

$$\prod_{\Omega \in X} I_X(\Omega^c) \leqq [X]! K_3^{l(X)}, \tag{3.10}$$

where

$$l(X) = \sum_{\Omega^c \in X} l(\Omega^c),$$

and K_3 is a geometric constant depending on d.

We do not recall the proofs of Lemmas 3.2, 3.3; see [10, 12]. We have stated them here, because they are basic in our proof of convergence of the expansions (3.7), (3.8), see Lemma 3.1, and because we also need them to prove the required estimates on $z(\Omega^c)$, Lemma 2.2; (for this purpose X is replaced by Ω^c and Ω^c by ω, see Lemmas A.1, A.2, Appendix).

Combining (3.6) with Lemmas 3.1 through 3.3 we obtain

Theorem 3.1. *Let y be such that $|\arg y| < \pi - \varepsilon$ or $|y| > 1 + \varepsilon$. Let β be so small that*

$$K_\varepsilon \beta (2d\beta + m^2(\beta))^{-1} < K_2, \tag{3.11}$$

where $K_2 \propto d^{-1}$ is the constant introduced in Property B. Then the cluster expansions (3.7), (3.8) for the free energy and the correlation functions converge uniformly in $|\Lambda|$. The limits, as $\Lambda \to \mathbb{Z}^d$, exist and are analytic in y in the domain defined above. The limiting connected correlation functions have exponential cluster properties.

Remarks. 1) The last part of Theorem 3.1 is a standard consequence of convergence of the cluster expansion.

2) Condition (3.11), can be satisfied for sufficiently small β, because $m(\beta) \to 1$, as $\beta \to 0$; see Conditions (1.10), (1.11). Moreover, if (3.11) holds then it also holds for sufficiently large, finite Λ, because

$$\lim_{\Lambda \to \mathbb{Z}^d} m(\beta, \Lambda) = m(\beta).$$

3) Although we have tacitly assumed that periodic b.c. were imposed at $\partial\Lambda$, our techniques extend to a general class of b.c. and can be used to establish independence of the thermodynamic limit on b.c. This requires solving condition (1.10) [see also (2.3)] for Dirichlet b.c. and introducing random walks terminating in sites in the boundary of Λ which arise from contractions with boundary fields. Although we shall not discuss these generalizations any further we think they show some nice features of our techniques.

By estimating Taylor remainders in y of the activities $z(\Omega^c)$ (which are analytic functions of y outside $[-1,0]$ and applying Lemmas 3.1–3.3 and [13] we obtain

Theorem 3.2. *For β small enough, the $1/N$ expansions for the free energy and the correlation functions are Borel summable at $1/N = 0$.*

Remarks. *1*) The required estimates on the Taylor remainders in y of the activities $z(\Omega^c)$ are proven in the Appendix (Lemma A.5).

2) Our main result, Theorem A, is equivalent to Theorems 3.1 and 3.2.

(3) We used Nevanlinna–Sokal's theorem on Borel summability for simplicity, but with a little additional work one can modify Lemma A.5 to verify that the stronger hypotheses of Watson's theorem on Borel summability [15] also hold in this case, provided the temperature is high enough.

4. Discussion of Results and Open Problems

1) We wish to add a comment on the difference between the two free energies $f'(\beta, z)$ and $f(\beta, y)$ introduced in (2.19), (2.20): In Theorem 3.2 we have shown that the expansion of $f(\beta, y)$ in powers of y is Borel summable. However, $\beta[f'(\beta, z) - f(\beta, y)]$ has no expansion in powers of y. This quantity is the normalized volume of a sphere over which the classical spin may range. It is thus the free energy for a 0-dimensional model consisting of a single spin (the "toy integral"). Its expansion in powers of z (not of y) is Stirling's series which is Borel summable. Its Borel transform is given by

$$\begin{aligned} \frac{1}{z}\beta[f'(\beta, z) - f(\beta, y)] &= \sum_{r=0}^{\infty} (-1)^r \frac{B_{r+1}}{[2(r+1)]!}(z')^{2r} \\ &= \frac{i}{2z'}\cot\left(\frac{iz'}{2}\right) - \left(\frac{1}{z'}\right)^2, \end{aligned} \tag{4.1}$$

where z' is the "Borel variable dual to z," and the B_r are the Bernoulli numbers. It has singularities on the imaginary axis, at $z' = k\pi i$, $k = \pm 1, \pm 2, \pm 3, \ldots$.

It is an open problem to locate the singularities of the Borel transform of the $1/N$ expansion and to give such singularities a physical interpretation, as expected of the singularities in the Borel transform of the coupling constant expansion in field theories like $\lambda\phi_4^4$ [16].

2) The techniques we have used to establish our main results [Theorems 3.1 and 3.2] have the following feature which is both remarkable and annoying: The cluster expansion of Lemma 3.1 converges for all y with $\operatorname{Re} y > 0$, and our estimates on the activities $z(\Omega^c)$ are uniform in y, for $\operatorname{Re} y > 0$. Thus the constant $\beta_{\pi/2}$ (see Theorem A, Theorem 3.1) is bounded by

$$\min_{0<N<\infty} \beta_{\text{crit}}(N)$$

which is well below β_S.
In contrast, Kupiainen [1] has established estimates which are valid for $0 \leqq y < y_0$ and $\beta < \beta_{y_0}$, for small values of y_0, with $\lim_{y_0 \to 0} \beta_{y_0} = \beta_S$.

It is conceivable that we could improve our estimates in this direction, i.e. that we could prove convergence of our cluster expansions in domains $0 < \beta < \beta_{y_0}$, $|\arg y| < \frac{\pi}{2}$, $|y| < y_0$, with $\lim_{y_0 \to 0} \beta_{y_0} \sim \beta_S$. [The $y \to 0$ (spherical model) limit exists term by term in our expansion.] The main obstruction preventing us from establishing such a result is that it is not easy to establish optimal bounds on

$$|y^{-k} Q_n^q(y)|, \tag{4.2}$$

where $Q_n^q(y)$ is one of the polynomials defined in (2.21), (2.22) and $k \leqq E(n)$.

We believe that with more work one might be able to improve the estimates on $|y^{-k} Q_n^q(y)|$ [see Lemmas A.1–A.4, in the appendix] and eventually prove the above conjecture.

3) *It might be worthwhile to try to extend our methods to other models with* a good $1/N$-expansion, like the $\mathbb{C}P^{N-1}$ models. The first step in this program consists of choosing the right lattice approximation permitting one to use double [high temperature, $1/N$] expansions. For the $\mathbb{C}P^{N-1}$ models, it is known that certain lattice actions lead to first order transitions in β at $1/N = 0$ [17, 18]. Such actions must be rejected in our scheme. We have reasons to believe that there are lattice actions for the $\mathbb{C}P^{N-1}$ models which are not plagued with such pathologies.

4) It would be interesting to extend our methods to analyze other expansions in statistical physics, e.g. the $1/d$ expansion. Random walk representation should be particularly convenient to analyze the $1/d$ expansion [since random walks on $\mathbb{Z}^d$ become mutually avoiding and self-avoiding, as $d \to \infty$, thus identifying this limit with mean field theory. Systematic corrections in $1/d$ appear to be accessible].

Appendix. Proofs of Lemmas 2.1 and 2.2 and of Theorem 3.2

In this appendix we supply some technical details which we have used in the main text without proof. The key estimates are Lemmas 2.1 and 2.2 and (2.35). In the proof of Lemma 2.2, Lemma 3.3 (Malyshev's theorem [12, 10]) plays an important rôle.

*A.*1. *Proof of Lemma* 2.1. Let

$$Q^q_{n,l}(y)=\prod_{k=l+q}^{n+q-1}(1+ky),\quad Q^q_{n,n}(y)=1$$

By (2.21) and (2.22),

$$Q^q_n(y)=\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}Q^q_{n,l}.$$

We propose to prove

$$Q^q_n(y)=a_k y^k+O(y^{k+1}),\tag{A.1}$$

where $k\geqq E(n)\equiv$ integer part of $\frac{n+1}{2}$, and $a_k<\infty$. Clearly, (A.1) yields Lemma 2.1.

Proof of (A.1.). Our proof is inductive and makes use of Pascal's triangular formula

$$\binom{n+1}{l}=\binom{n}{l}+\binom{n}{l-1}.$$

To start the induction we note that

$$Q^q_0(y)=1,\ \text{ and }\ Q^q_1(y)=-qy,\quad \forall\ q.\tag{A.2}$$

In order to do the induction step we use the identity

$$\begin{aligned}Q^q_{n+1}&=(n+q)y\left[\sum_{l=0}^{n}\binom{n+1}{l}(-1)^{n+1-l}Q^q_{n,l}\right]\\&\quad+\sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n+1-l}[Q^q_{n,l}-Q^q_{n,l+1}].\end{aligned}\tag{A.3}$$

Now, notice that

$$\binom{n}{l}[Q^q_{n,l}-Q^q_{n,l+1}]=ny\binom{n-1}{l-1}Q^q_{n,l+1}+qy\binom{n}{l}Q^q_{n,l+1}.$$

Inserting this identity into (A.3) we get

$$\begin{aligned}Q^q_{n+1}&=ny\left[\sum_{l=0}^{n}\binom{n+1}{l}(-1)^{n+1-l}Q^q_{n,l}-\sum_{l=2}^{n}\binom{n-1}{l-2}(-1)^{n+1-l}Q^q_{n,l}\right]\\&\quad+qy\left[\sum_{l=0}^{n}\binom{n+1}{l}(-1)^{n+1-l}Q^q_{n,l}-\sum_{l=1}^{n}\binom{n}{l-1}(-1)^{n+1-l}Q^q_{n,l}\right]\\&=ny\left[2\sum_{l=1}^{n}\binom{n-1}{l-1}(-1)^{n+1-l}Q^q_{n,l}+\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^{n-1-l}Q^q_{n,l}\right]\\&\quad+qy\left[\sum_{l=0}^{n}\binom{n}{l}(-1)^{n+1-l}Q^q_{n,l}\right]\\&=ny[(1+(n-1)y)Q^q_{n-1}-2Q^{q+1}_{n-1}]-qyQ^q_n.\end{aligned}\tag{A.4}$$

Formulas (A.4) and (A.2) yield (A.1) by induction in n. This completes the proof of Lemma 2.1.

A.2. Proof of Lemma 2.2. Let $\Omega^c = \{\omega_1, \ldots, \omega_m\}$ be some polymer. (Different elements of Ω^c may correspond to different copies of the same random walk.) Our purpose is to estimate the activity $z(\Omega^c)$ of Ω^c by proving a bound on $|y^{-k} Q_n^q(y)|$, $k \leqq E(n)$, and straightening out the combinatorial properties of polymers. We start with the latter.

Lemma A.1. *Consider some g-set Ω consisting of oriented (not necessarily distinct) random loops. With each site $j \in \Lambda \subseteq \mathbb{Z}^d$ one can associate a (possibly empty) g-subset $I_j = I_j(\Omega)$ of Ω, with the properties that*

i) *each $\omega \in \Omega$ belongs to precisely one I_j;*
ii) *each $\omega \in I_j$ visits j;*
iii) *the total number of walks in I_j, $|I_j|$, satisfies*

$$|I_j| \leqq E(n_j(\Omega)), \quad \text{for all } j \in \Lambda.$$

Remark. Formula (2.30) for $z(\Omega^c)$, property (A.1) and Lemma A.1 (with $\Omega = \Omega^c$, or $\Omega = \Omega^c \sim \{\omega\}$, where $\omega : x \to u$) show that $z(\Omega^c)$ is regular at $y = 2/N = 0$.

Proof. Our proof proceeds by induction in $|\Omega|$. For $|\Omega| = 1$, Lemma A.1 is trivial. We may thus assume it is true for $|\Omega| \leqq m$. We then show that it holds for $|\Omega| = m+1$. Since any closed random loop makes at least two steps, we may choose, for any ω_l in Ω, $l = 1, \ldots, m+1$, two distinct sites j_l^1 and j_l^2; we then call ω_l' the closed random loop made of only two steps which visits j_l^1 and j_l^2, and $\Omega' = \{\omega_1', \ldots, \omega_{m+1}'\}$. It is now clear that if Lemma A.1 holds for Ω', it holds for Ω; indeed there is a one to one correspondence between Ω' and Ω which induces a natural correspondence between the g-subsets $I_j(\Omega')$ and $I_j(\Omega)$; and obviously $n_j(\Omega') \leqq n_j(\Omega)$ implies $E(n_j(\Omega')) \leqq E(n_j(\Omega)) \forall j \in \Lambda$. Since it also suffices to prove Lemma A.1 for connected g-sets (otherwise we may apply the induction hypothesis to each connected component) we are left with the case of a connected g-set $\Omega = \{\omega_1, \ldots, \omega_{m+1}\}$ of random loops, each of them made of two steps. Then:

either there exists $j \in \Lambda$ such that $n_j(\Omega)$ is odd. Then j is visited by a walk $\omega^{(j)} \in \Omega$, and

$$E(n_j(\Omega)) \geqq E(n_j(\Omega \sim \{\omega^j\})) + 1.$$

We may then assign $\omega^{(j)}$ to $I_j(\Omega)$ and Lemma A.1 follows from the induction hypothesis for $\Omega \sim \{\omega^{(j)}\}$,

or

$$n_j(\Omega) \text{ is even for all } j \in \Lambda. \tag{A.5}$$

Let $\omega_1 \in \Omega$. Choose a site $j_1 \in \omega_1$, and assign ω_1 to $I_{j_1}(\Omega)$. Let j_2 be the other site visited by ω_1, and $\omega_2 \neq \omega_1$ a loop of Ω visiting j_2. We assign ω_2 to $I_{j_2}(\Omega)$, and define j_3 as the other site visited by ω_2 (possibly $j_3 = j_1$). If j_3 is visited by a path ω_3 in Ω, not equal to ω_1 or ω_3, we assign ω_3 to $I_{j_3}(\Omega)$, and so on. This construction may stop after having chosen j_k and ω_k, $k \geqq 2$. This can only be the case if the other site visited by ω_k is $j_{k+1} = j_1$. Indeed from A.5 and the construction we made,

$n_{j_{k+1}}(\Omega \sim \{\omega_1, \ldots, \omega_k\})$ should be odd, hence not zero, except if $j_{k+1} = j_1$. In this way we therefore obtain partial g-subsets $K_j = \{\omega_i \in I_j, i \leqq k\}$. We complete the construction if $k < m+1$, by applying the induction hypothesis to $\Omega^k = \Omega \sim \{\omega_1, \ldots, \omega_k\}$; this constructs g-subsets $I_j(\Omega^k)$, and finally we define

$$I_j(\Omega) = K_j \cup I_j(\Omega^k).$$

It is easy to verify then all items of Lemma A.1, because by our construction:

$$E(n_j(\Omega)) = E(n_j(\Omega^k)) - |K_j|. \qquad \square$$

The next (and last) combinatorial result is an easy consequence of Lemma 3.3.

Lemma A.2. *There is a constant K_4 such that for any closed polymer Ω^c*

$$\prod_{j \in \Lambda} |I_j(\Omega^c)|! \leqq [\Omega^c]! K_4^{l(\Omega)}, \tag{A.9}$$

where $I_j(\Omega^c)$ is defined as in Lemma A.1, unless Ω^c is an $x \to u$ polymer containing $\omega : x \to u$, in which case $I_j(\Omega^c) = I_j(\Omega^c \sim \{\omega\})$.

Proof. We apply Lemma 3.3, with X replaced by Ω^c and $\Omega^c \in X$ replaced by $\omega \in \Omega^c$. Clearly, Property A holds if we identify Ω_0^c with a random walk ω_0 of length l_0 and count the total number of random walks ω of length $l(\omega) = l$ incompatible with ω_0. This guarantees that the hypotheses of Lemma 3.3, with $X \to \Omega^c$, $\Omega^c \to \omega$, are fulfilled. Thus, by (3.10),

$$\prod_{\omega \in \Omega^c} I_{\Omega^c}(\omega) \leqq [\Omega^c]! K_3^{l(\Omega^c)}. \tag{A.10}$$

Now, for $\omega \in I_j(\Omega^c)$, $I_{\Omega^c}(\omega) \geqq |I_j|$, with $I_j = I_j(\Omega^c)$. Thus

$$\prod_{\omega \in I_j} I_{\Omega^c}(\omega) \geqq |I_j|^{|I_j|} \geqq c^{|I_j|} |I_j|!. \tag{A.11}$$

Since

$$\prod_{j \in \Lambda} \prod_{\omega \in I_j} I_{\Omega^c}(\omega) = \prod_{\omega \in \Omega^c} I_{\Omega^c}(\omega),$$

(A.9) follows from (A.10) and (A.11).
End of proof.

We now come to the more analytical part of the proof of Lemma 2.2.

Lemma A.3.

$$0 \leqq C_{\beta,ij} \leqq \left(\frac{2d\beta}{m(\beta,\Lambda)^2} \right)^{|i-j|},$$

$$0 \leqq J_{\Omega^c} \leqq \left(\frac{2d\beta}{m(\beta,\Lambda)^2} \right)^{l(\Omega^c)}.$$

Proof. When $i = j$, $C_{\beta,ij} = 1$, by condition (1.10), so the lemma holds trivially. For $i \neq j$, we apply the random walk expansion of C_β; see [7].

$$C_{\beta,ij} = \sum_{\tilde{\omega}: i \to j} \prod_{k \in \Lambda} \left(\frac{\beta}{2d\beta + m(\beta,\Lambda)^2} \right)^{\tilde{n}_k(\tilde{\omega})} > 0,$$

where $\tilde{\omega}$ is a nearest neighbor random walk, and $\tilde{n}_k(\tilde{\omega})$ the number of visits of $\tilde{\omega}$ at k. There are at most $(2d)^l$ such walks of length l. Thus

$$\begin{aligned}
C_{\beta,ij} &\leqq \frac{\beta}{2d\beta+m(\beta,\Lambda)^2}\sum_{l=|i-j|}^{\infty}\left(\frac{2d\beta}{2d\beta+m(\beta,\Lambda)^2}\right)^l \\
&\leqq \left(\frac{\beta}{2d\beta+\mathrm{m}^2(\beta,\Lambda)}\right)\left(\frac{2d\beta}{2d\beta+m(\beta,\Lambda)^2}\right)^{|i-j|}\left(1-\frac{2d\beta}{2d\beta+m(\beta,\Lambda)^2}\right)^{-1} \\
&\leqq \left(\frac{2d\beta}{m(\beta,\Lambda)^2}\right)^{|i-j|}.
\end{aligned}$$

The bound on J_{Ω^c} follows from the bound on $C_{\beta,ij}$ and the definition of J_{Ω^c}.

Next, we prove our crucial bounds on the polynomials $Q_n^q(y)$; see (2.35).

Lemma A.4. *For $k \leqq E(n)$,*

$$|y^{-k}Q_n^q(y)| \leqq e^{2(n+q-1)}2^n k!\prod_{k=q}^{n+q-1}(1+k|y|).$$

Proof. If $|y| \geqq (2\rho)^{-1}$ then, by (2.21) and (2.22)

$$\begin{aligned}
|y^{-k}Q_n^q(y)| &\leqq (2\rho)^k\left|\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}\prod_{k=l+q}^{n+q-1}(1+ky)\right| \\
&\leqq e^{2\rho}k!2^n\prod_{k=q}^{n+q-1}(1+k|y|).
\end{aligned}$$

If $|y| \leqq (2\rho)^{-1}$ we may apply Cauchy's estimate: We choose Γ to be the circle of radius ρ^{-1} centered at the origin. Let

$$f(y)=y^{-k}Q_n^q(y).$$

By (A.1), $f(y)$ is analytic in $y\in\mathbb{C}\sim\{\infty\}$. For $|y|<(2\rho)^{-1}$, we may therefore apply the Cauchy formula

$$f(y)=\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(\zeta)}{\zeta-y}d\zeta,$$

which yields

$$\begin{aligned}
|f(y)| &\leqq 2\rho^k\max_{\zeta\in\Gamma}\left|\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}\prod_{k=l+q}^{n+q-1}(1+k\zeta)\right| \\
&\leqq 2e^{\rho}k!2^n\prod_{k=q}^{n+q-1}(1+k/\rho).
\end{aligned}$$

Lemma A.4 now follows by choosing $\rho=n+q-1$.

Corollary A.5. *For $k \leqq E(n)$ and y in the domain*

$$D_\varepsilon=\{y:|\arg y|<\pi-\varepsilon,\text{ or }|y|>1+\varepsilon\} \tag{A.12}$$

$$|y^{-k}R_n^q(y)| \leqq C_\varepsilon^{n+q}k!, \tag{A.13}$$

where R_n^q is the rational function defined in (2.21) and (2.22), and C_ε is a finite constant, for each $\varepsilon > 0$. If $\operatorname{Re} y > 0$

$$|y^{-k} R_n^q(y)| \leqq (\sqrt{2}2)^n e^{2(n+q-1)} k!. \tag{A.14}$$

Proof. For $y \in D_\varepsilon$,

$$\prod_{k=q}^{n+q-1} (1+k|y|)|1+ky|^{-1} \leqq c_\varepsilon^{n+q-1}, \tag{A.15}$$

for some constant c_ε which is finite for all $\varepsilon > 0$. If Re $y \geqq 0$, $c_\varepsilon \leqq \sqrt{2}$ in (A.15). Thus Corollary A.5 follows from the definition of R_n^q and Lemma A.1. □

Proof of Lemma 2.2. By formula (2.30)

$$\begin{aligned} z(\Omega^c) &= \frac{1}{[\Omega^c]!} J_{\Omega^c} y^{-|\Omega^c|} \prod_{j \in \Lambda} R^{q_j}_{n_j(\Omega^c)}(y) \\ &= \frac{1}{[\Omega^c]!} J_{\Omega^c} \prod_{j \in \Lambda} y^{-|I_j(\Omega^c)|} R^{q_j}_{n_j(\Omega^c)}(y), \end{aligned} \tag{A.16}$$

where $I_j(\Omega^c)$ is as in Lemma A.2, and the last equation follows from Lemma A.1. By Lemma A.1, iii)

$$|I_j(\Omega^c)| \leqq E(n_j(\Omega^c)), \quad \text{for all } j.$$

We may therefore apply Corollary A.5 to estimate $|y^{-|I_j(\Omega^c)|} R^{q_j}_{n_j(\Omega^c)}(y)|$ which yields:

$$|z(\Omega^c)| \leqq \frac{1}{[\Omega^c]!} J_{\Omega^c} \prod_{j \in \Lambda} |I_j(\Omega^c)|! \, C_\varepsilon^{n_j(\Omega^c)+q_j} \tag{A.17}$$

for $y \in D_\varepsilon$ (defined in (A.12)). If we now use inequality (A.9), Lemma A.2, and the inequality

$$\sum_j n_j(\Omega^c) + q_j \leqq l(\Omega^c) + 1, \tag{A.18}$$

we find

$$|z(\Omega^c)| \leqq J_{\Omega^c} (C_\varepsilon K_4)^{l(\Omega^c)} \tag{A.19}$$

which, together with Lemma A.3, completes the proof of Lemma 2.2. [If Re $y \geqq 0$ we may use (A.14) instead of (A.13).]

A.3. Proof of Theorem 3.2. It suffices to prove uniform estimates on the r^{th} derivative in y of $z(\Omega^c; y) \equiv z(\Omega^c)$, for Re $y \geqq 0$ and arbitrary polymers Ω^c. One then uses Lemmas 3.1 through 3.3 to transfer such estimates to $\beta f(\beta, y)$ and $\langle \phi_x^1 \phi_u^1 \rangle_{\beta,z}$. Therefore we only need one further lemma, namely:

Lemma A.5. *If* $\operatorname{Re} 1/y > 1$

$$\left| \frac{\partial^r}{\partial y^r} z(\Omega^c; y) \right| \leqq c\, 2^r (r!)^2 J_{\Omega^c} K_5^{l(\Omega^c)} \tag{A.20}$$

for some constants c and K_5.

Proof. According to (A.16) and Leibniz' rule:

$$\frac{\partial^r}{\partial y^r} z(\Omega^c; y) = \frac{1}{[\Omega^c]!} J_{\Omega^c} \sum_{\sum_j s_j = r} \prod_j \frac{\partial^{s_j}}{\partial y^{s_j}} f_j, \tag{A.21}$$

where f_j stands for $y^{-|I_j(\Omega^c)|} R^{q_j}_{n_j(\Omega^c)}(y)$.

We estimate the derivatives on the right side by using the Cauchy formula, as in the proof of Lemma A.4. Let C_1 be the disk $\{y; \mathrm{Re}\, 1/y > 1\}$ and Γ_ρ the circle surrounding it at distance $(2\rho)^{-1}$, namely the set of points y with $\mathrm{dist}\{y, C_1\} = (2\rho)^{-1}$.

For f analytic inside Γ_ρ and y in C_1 we have

$$\frac{\partial^s}{\partial y^s} f(y) = \frac{s!}{2\pi i} \oint_{\Gamma_\rho} \frac{f(\zeta)}{(\zeta - y)^{s+1}} d\zeta. \tag{A.22}$$

We can apply this Cauchy formula to the derivative of f_j in the right side of (A.21), provided we choose $\rho = n_j + q_j$; indeed by (A.1), f_j is analytic inside $\Gamma_{n_j + q_j}$.

We need the following easy generalization of (A.14):

$$\left| \prod_{k=q}^{n+q-1} [(1 + k|y|)(1 + ky)^{-1}] \right| \leqq 4^{n+q}, \quad \text{if } y \in \Gamma_{n+q}. \tag{A.23}$$

Using Lemma A.4, it yields for $y \in \Gamma_{n_j + q_j}$:

$$|f_j(y)| \leqq 8^{n_j + q_j} e^{2(n_j + q_j - 1)} |I_j(\Omega^c)|!. \tag{A.24}$$

We insert this bound in (A.22) and obtain, for y in C_1

$$\left| \frac{\partial^{s_j}}{\partial y^{s_j}} f_j(y) \right| \leqq s_j! (8e^2)^{n_j + q_j} [2(n_j + q_j)]^{s_j + 1} |I_j(\Omega^c)|! \tag{A.25}$$

We use the estimate

$$[2(n_j + q_j)]^{s_j + 1} \leqq 2(n_j + q_j) s_j! e^{2(n_j + q_j)} \leqq s_j! (2e^2)^{n_j + q_j}. \tag{A.26}$$

We can now bound the sum in the right side of (A.21), using (A.25), (A.26) and (A.18). Since there are at most $2^{r + l(\Omega^c)}$ possible choices of sequences $(s_j)_{j \in \Lambda}$ such that $\sum_j s_j = r$ and $s_j = 0$ if $n_j(\Omega^c) = 0$, and since $\prod_j s_j! \leqq r!$ we get the bound

$$(r!)^2 2^{r + l(\Omega^c)} (16e^4)^{l(\Omega^c) + 1} \prod_j |I_j(\Omega^c)|!.$$

Lemma A.5 follows from this bound and Lemma A.2, with $K_5 = 32e^4 K_4$. Of course we do not try to find best possible constants. Theorem 3.2 now follows by appealing to the result of Nevanlinna and Sokal [13]:

Lemma A.6. *Let f be analytic in the circle $C_R = \{y; \mathrm{Re}\, 1/y > 1/R\}$. Suppose f admits an asymptotic expansion such that*

$$f(y) = \sum_{k=0}^{r-1} a_k y^k + R_r(y), \tag{A.27}$$

with

$$|R_r(y)| \leqq c\sigma^r r!|y|^r \tag{A.28}$$

uniformly in r and $y \in C_R$, *for some constants* σ *and c. Then f is Borel summable, which means that the power series* $\sum_k a_k t^k/k!$ *converges for* $|t| < 1/\sigma$, *that it defines a function* $B(t)$ *which has an analytic continuation in the strip*

$$S_\sigma = \{t; \operatorname{dist}\{t, R_+\} < 1/\sigma\},$$

and that this function satisfies the bound

$$|B(t)| \leqq K \exp t/R \quad \text{for } t \in R_+ .$$

Finally f is represented by the following absolutely convergent integral:

$$f(y) = \frac{1}{y} \int_0^\infty \exp(-t/y) B(t) dt, \quad y \in C_R .$$

There is also a reciprocal theorem which we do not use here. We apply this theorem in our case, with $R = 1$, by simply noticing that (A.27) and (A.28) follow from Lemma A.5 by Taylor's formula.

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