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$(\text{Higgs})_{2,3}$ Quantum Fields in a Finite Volume

I. A Lower Bound*

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Abstract. We consider a Euclidean model of interacting scalar and vector fields in two and three dimensions, and prove a lower bound for vacuum energy in a lattice approximation. The bound is independent of a lattice spacing; it is proved with the help of renormalization transformations in Wilson-Kadanoff form. It extends in principal also to generating functional for Schwinger functions.

1. Formulations of Results, Remarks on the Method, and Notations

The aim of this paper is to give some estimates on the partition function of a lattice approximation of two and three dimensional Euclidean models of interacting scalar and vector fields. These estimates are independent of the lattice spacing. The model is the so-called Proca model, and its (continuous) action is given by

$$S(A,\phi)=\int dx\left[\sum_{\mu=1}^{d}\frac{1}{2}|\partial_\mu\phi(x)+eqA_\mu(x)\phi(x)|^2+\frac{1}{2}m_0^2|\phi(x)|^2+|\lambda|\phi(x)|^4\right.$$
$$\left.+\sum_{\mu,\nu=1}^{d}\frac{1}{4}|F_{\mu\nu}(x)|^2+\frac{1}{2}\mu_0^2\sum_{\mu=1}^{d}|A_\mu(x)|^2\right],\tag{1.1}$$

where ϕ is a scalar field with values in R^N, q is an antisymmetric $N\times N$ matrix, A_μ are components of a vector field and $F_{\mu\nu}(x)=\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)$. This model was constructed in the two dimensional case by Brydges et al. [5–7] without any ultraviolet or space cutoffs, including the case $\mu_0^2=0$. Here we only prove the ultraviolet stability, however we consider both $d=2$ and 3, and we use a different method, based on a renormalization transformation. We take a lattice approximation for the model as our ultraviolet cutoff. Lattice approximations for gauge field models were introduced by Wilson in [29] and were studied by many authors [5, 6, 11, 16, 17, 22, 27, 28, 30]. The results of Brydges et al. [5, 6] are basic for our paper. They introduced several versions of lattice approximations and they verified their most important properties: physical positivity, diamagnetic in-

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equalities, gauge fixing properties, correlation inequalities etc. We make extensive use of some of the results of [5].

For the vector field action in (1.1) we approximate the action replacing derivatives with finite differences, yielding a quadratic form. Thus, from the technical point of view, the paper actually deals with scalar fields. In the future, we plan to consider the lattice approximation for the gauge field in the Wilson form, which is technically more difficult, but which serves as an introduction to gauge field models with non-abelian gauge groups. In this way we divide the technical problems of the full model into two parts; this is convenient and furthermore clarifies essential differences between the scalar field and the gauge field.

This and the next papers contain the main ideas of our proof, while many technical results are postponed for later papers.

We will consider the model on the d-dimensional torus T_ε identified with the subset of εZ^d:

$$T_\varepsilon=\{x\in\varepsilon Z^d : -L_\mu\leqq x_\mu<L_\mu,\ \mu=1,\dots,d\}, \tag{1.2}$$

where L_μ are such nonnegative numbers that $\varepsilon^{-1}L_\mu$ are integers.

More exactly we will assume that $\varepsilon^{-1}L_\mu=L^K M L'_\mu$, where K, L, M, L'_μ are some positive integers, $K=O(\log\varepsilon^{-1})$ and L, M will be described later.

We introduce a distance on T_ε by

$$|x-y|=\max_\mu \min\{|x_\mu-y_\mu|, 2L_\mu-|x_\mu-y_\mu|\}. \tag{1.3}$$

We consider two spaces of field configurations on T_ε: scalar fields and vector fields. The configurations of scalar field are functions $\phi: T_\varepsilon\to R^N$. Vector field configurations will be sometimes understood as the functions $A: T_\varepsilon\to R^d$, and sometimes as $A: T^*_\varepsilon\to R$, where T^*_ε is the set of all bonds from $T_\varepsilon: T^*_\varepsilon=\{\langle x,x'\rangle : x,x'\in T_\varepsilon,\ x,x'$ are the nearest neighbours$\}$. We will identify these two meanings by $A_{\langle x,x+\varepsilon e_\mu\rangle}=A_\mu(x)$, $\mu=1,\dots,d$. Bonds will be denoted by letters b, c also. Thus $b=\langle b_-,b_+\rangle$ where b_- is the initial point, b_+ is the final point. The points b_-, b_+ are of course the nearest neighbors. The bond b with the opposite orientation will be denoted by $-b=\langle b_+,b_-\rangle$. For a function g defined on bonds we assume $g_{-b}=-g_b$. The difference derivative is defined by the formula

$$(\partial^\varepsilon f)(b)=\varepsilon^{-1}(f(b_+)-f(b_-)). \tag{1.4}$$

It is an operator on functions defined on a lattice into functions defined on bonds of this lattice. The difference derivative in a given direction μ can be defined in an obvious way also. The adjoint operators with respect to the scalar product

$$\langle f,g\rangle=\sum_{x\in T_\varepsilon}\varepsilon^d f(x)\cdot g(x) \tag{1.5}$$

can be easily written up. We define also the derivative for a function on bonds:

$$(\partial^\varepsilon A)(P)=\varepsilon^{-1}\sum_{b\subset\partial P}A_b=(\partial^\varepsilon_\mu A_\nu)(x)-(\partial^\varepsilon_\nu A_\mu)(x), \tag{1.6}$$

where P is a plaquette, i.e. an elementary square of the lattice,

$$P=\langle x, x+\varepsilon e_\mu, x+\varepsilon e_\mu+\varepsilon e_\nu, x+\varepsilon e_\nu\rangle \quad \text{for} \quad x\in T_\varepsilon, \mu<\nu.$$

To describe the interaction of scalar and vector fields we introduce a representation of the additive group of real numbers R in unitary operators on R^N: $U(A)=\exp(q\varepsilon eA)$, $A\in R$, where e is a coupling constant and q is an antisymmetric $N\times N$ matrix. We will assume only that $\|q\|\leqq 1$. Antisymmetry of q implies $U(A)^*=U(-A)=U(-A)^{-1}$. A covariant derivative for scalar fields is defined by the formula

$$(D_A^\varepsilon\phi)(b)=\varepsilon^{-1}(U(A_b)\phi(b_+)-\phi(b_-)), \qquad b=\langle b_-,b_+\rangle. \tag{1.7}$$

Now we can define the fundamental action for the lattice approximation of our model

$$S'^\varepsilon(A,\phi)=\frac{1}{2}\sum_{b\subset T_\varepsilon}\varepsilon^d|(D_A^\varepsilon\phi)(b)|^2+\sum_{X\in T_\varepsilon}\varepsilon^d\left(\frac{1}{2}m_0^2|\phi(x)|^2+\lambda|\phi(x)|^4\right)$$
$$+\frac{1}{2}\sum_{P\subset T_\varepsilon}\varepsilon^d|(\partial^\varepsilon A)(P)|^2+\frac{1}{2}\sum_{b\subset T_\varepsilon}\varepsilon^d\mu_0^2|A_b|^2, \tag{1.8}$$

and the corresponding partition function

$$Z'^\varepsilon=\int dA\int d\phi\exp(-S'^\varepsilon(A,\phi)). \tag{1.9}$$

In the above integral the natural Lebesgue measure is used on the spaces of configurations of scalar and vector fields. We will now use some results of the paper [5]. In Sect. 5.1 the authors have shown how to introduce the gauge fixing terms in the integral (1.9), using the properties of (1.8) under the gauge transformations. The same gauge fixing procedure can be applied also to gauge-invariant Schwinger functions. We introduce here the Feynman gauge and we will consider the integral

$$Z^\varepsilon=\int dA\int d\phi\exp(-S^\varepsilon(A,\phi)) \tag{1.10}$$

instead of (1.9), where the action S^ε is now defined by the formula

$$S(A,\phi)=\tfrac{1}{2}\langle\phi,(-\Delta_A^\varepsilon)\phi\rangle+\sum_{x\in T_\varepsilon}\varepsilon^d(\tfrac{1}{2}m_0^2|\phi(x)|^2+\lambda|\phi(x)|^4)$$
$$+\tfrac{1}{2}\langle A,(-\Delta^\varepsilon+\mu_0^2)A\rangle+E. \tag{1.11}$$

Here $-\Delta_A^\varepsilon=D_A^{\varepsilon*}D_A^\varepsilon$ is the covariant Laplace operator and $-\Delta^\varepsilon=\partial^{\varepsilon*}\partial^\varepsilon$ is the Laplace operator on the torus T_ε. We will assume that $\mu_0^2>0$ and $\lambda>0$, and these assumptions are basic for this paper. The coefficient m_0^2 is more complicated. There are (linearly) divergent self-energy diagrams for scalar field in the theory, so a mass-renormalization counterterm is needed. As usual, we will take $m_0^2=m^2+\delta m^2$, where $m^2>0$ and δm^2 is the counterterm given by a perturbation expansion in e and λ. This expansion, as all the other expansions in the paper, will be taken up to some order $\bar{n}$, and $\bar{n}\geqq 6$ is necessary for renormalization of the theory. It can be written down explicitly but we will not do it because we will not use it in this paper. Let us only notice that δm^2 is a function of ε, e, λ, m^2, μ_0^2, q. Without any further assumptions on q, δm^2 is a $N\times N$-matrix depending on q^2. For simplicity of notations we will treat it as a number. It is a number if q^2 is a number and it is so in the simplest and most important case when $N=2$ and $q^2=-1$. The constant E in (1.12) is a sum $E=E_0+E_1$, and E_0 is a normalization constant obtained from

(1.10) by taking there $e=\lambda=0$, $E=0$ and $m_0^2=m^2$; therefore we have

$$\int dA \exp(-\tfrac{1}{2}\langle A,(-\Delta^\varepsilon+\mu_0^2)A\rangle)\int d\phi \exp(-\tfrac{1}{2}\langle\phi,(-\Delta_0^\varepsilon+\mathrm{m}^2)\phi\rangle)$$
$$=\exp(E_{0,v})\exp(E_{0,s})=\exp(E_0). \tag{1.12}$$

The constant E_1 is given again by a perturbation expansion which can be written as

$$E_1=\sum_{1\leqq\alpha+\beta\leqq\bar{n}}\frac{1}{\alpha!\beta!}e^\alpha\lambda^\beta\left(\frac{\partial^{\alpha+\beta}}{\partial e^\alpha\partial\lambda^\beta}\log\int dA\int d\phi\exp(-S^\varepsilon(A,\phi)+E)\right)\Bigg|_{e=\lambda=0}. \tag{1.13}$$

Now the fundamental result of this paper can be formulated.

Theorem. *For the dimensions* $d=2,3$ *there exist the constants* E_-, E_+ *independent of* ε, T_ε *and such that*

$$\exp(-E_-|T_\varepsilon|)\leqq Z^\varepsilon\leqq\exp(E_+|T_\varepsilon|). \tag{1.14}$$

Let us make some comments on this Theorem.

First, we would like to stress that the assumption $\lambda>0$ is essential for our method.

The second remark concerns the estimates of a generating functional. Using the same method of proof we can extend the Theorem in several ways. One of the simplest extensions is

$$\exp(-E_-(\|J\|,\|f\|)|T_\varepsilon|)\leqq Z^\varepsilon(J,f)=\int dA\int d\phi\exp(-S^\varepsilon(A,\phi)+\langle J,A\rangle+\langle f,\phi\rangle)$$
$$\leqq\exp(E_+(\|J\|,\|f\|)|T_\varepsilon|), \tag{1.15}$$

where the "sources" J, f are arbitrary, $\|\cdot\|$ is some norm, e.g. we can take the norm of L^∞, and the functions $E_\pm(\cdot,\cdot)$ are continuous and independent of ε, T_ε. Using the techniques developed in [11, 16, 25] it is possible to prove more refined estimates.

The fundamental aim of the paper is a presentation of some method. This method is a modification of the method of Gallavotti et al. [2–4, 13]. As is well known, every proof of ultraviolet stability is based on some "slicing" of momentum range and successive analysis of the effective actions in the slices, i.e. on the so-called phase-space cell analysis. This slicing can be done in several ways, see the papers [2, 8, 10, 15, 21–23, 29, 30]. Here we will obtain it applying the renormalization group transformations in the form proposed by Wilson and Kadanoff in the papers [17–19, 32, 34]. These transformations, together with the method of Gallavotti et al. are the basic ingredients of the method applied here.

Let us mention that a method similar to the one described in this paper was applied by Gawędzki and Kupiainen in [14] to another problem.

Now let us gather together the definitions and notations used in the paper. For a lattice of an arbitrary scale η, and also for its arbitrary subset $\Lambda\subset\eta Z^d$, we define the "prime" operation

$$\Lambda'=\Lambda\cap L\eta Z^d. \tag{1.16}$$

A block $B(y)$ of the lattice ηZ^d, $y\in L\eta Z^d$, is defined by

$$B(y)=\{x\in\eta Z^d : y_\mu\leqq x_\mu<y_\mu+L\eta,\ \mu=1,\dots,d\}. \tag{1.17}$$

Together with the operation $'$, we introduce also a dual operation B:

$$\text{for} \quad \Lambda \subset \eta Z^d, \qquad B(\Lambda) = \bigcup_{x \in \Lambda} B(x) \subset L^{-1}\eta Z^d. \tag{1.18}$$

The operations' and B can be iterated and we define

$$T^{(k+1)}_{L^{k+1}\varepsilon} = (T^{(k)}_{L^k\varepsilon})', \qquad k = 0, 1, 2, \ldots, T^{(0)}_\varepsilon = T_\varepsilon. \tag{1.19}$$

From the definition of T_ε, and more exactly, the definition of the numbers L_μ, it follows that all the above sets are the sums of the corresponding blocks, i.e.

$$T^{(k)}_{L^k\varepsilon} = B(T^{(k+1)}_{L^{k+1}\varepsilon}), \qquad k = 0, 1, \ldots, \text{ thus } T_\varepsilon = B^k(T^{(k)}_{L^k\varepsilon}). \tag{1.20}$$

Further we will use the partitions into large blocks besides the partitions into blocks described above. They are defined in the same way, only L is replaced by ML, where M is a sufficiently large natural number. This number will be determined later. The sets $T^{(k)}_{L^k\varepsilon}$ are the sums of large blocks also. For the subsets Λ of the lattice ηZ^d a measure is defined by the formula

$$|\Lambda| = \sum_{x \in \Lambda} \eta^d = \eta^d \text{ (a number of points in } \Lambda). \tag{1.21}$$

For the functions defined on the points of the lattice ηZ^d or on the bonds of this lattice, a scalar product is defined as usual by the formula (1.5) with η instead of ε. Generally, we will denote a scale of a lattice by a sub- or super-script at an operator or a set, except the unit scale.

An important role is played in the paper by the operations of rescaling done on the fields and operators. Let us consider the rescalings of the fields, because they determine the others. We use only the canonical rescalings, so we have

$$\phi(x) = \left(\frac{\eta}{\delta}\right)^{-\frac{d-2}{2}} \phi'\left(\frac{\delta}{\eta}x\right), \quad x \in \eta Z^d, \text{ or } \phi\left(\frac{\eta}{\delta}x\right) = \left(\frac{\eta}{\delta}\right)^{-\frac{d-2}{2}} \phi'(x), \; x \in \delta Z^d, \tag{1.22}$$

and the same formulas for vector fields. They imply the formulas for the rescalings of the functions of fields, e.g. we have for a derivative

$$(\partial^\eta_\mu \phi)(x) = \left(\frac{\eta}{\delta}\right)^{-d/2} (\partial^\delta_\mu \phi')\left(\frac{\delta}{\eta}x\right) \quad \text{for} \quad x \in \eta Z^d. \tag{1.23}$$

Finally, it is very important to know how the constants in the action (1.11) are changed after a rescaling. If we rescale (1.11) from the ε-lattice to an $s\varepsilon$-lattice, then from (1.22), (1.23) it follows that we get the action again of the form (1.11), but with ε replaced by $s\varepsilon$, T_ε by $T_{s\varepsilon}$, the constants m_0^2, μ_0^2 are replaced by $m_0^2 s^{-2}$, $\mu_0^2 s^{-2}$, and the coupling constants e, λ are replaced by $\lambda_s = \lambda s^{-(4-d)}$, $e_s = e s^{-\frac{4-d}{2}}$. We can use this to remove one of the coupling constants, for dimensions $d = 2, 3$. It is convenient to rescale in such a way that $\lambda_s = 1$, so $s = \lambda^{\frac{1}{4-d}}$ and then $e_s = e\lambda^{-1/2}$. We will assume that this rescaling is done, but we will leave λ in all the formulas because it is convenient in the perturbation expansions, only in the estimations λ will be replaced by 1.

We will identify a set Λ with the characteristic function defined by the set, thus we will write Λf and it means the product of the characteristic function Λ and the function f. Other notations will be introduced in the places where the corresponding objects appear for the first time.

2. Renormalization Transformations, Their Properties and Action on the Gaussian Densities

Now we will introduce a renormalization transformation for scalar fields. Its form will be imposed by Gaussian densities defined by a covariant Laplace operator in an external vector field. The renormalization transformations for vector fields will be obtained by taking $N=d$ and an external vector field $A=0$, so we will not consider them separately.

At first let us define a family of contours on the ε-lattice T_ε. For $y\in T^{(k+1)}_{L^{k+1}\varepsilon}$, $x\in B(y)$, we define a contour $\Gamma_{y,x}$:

$$\begin{aligned}\Gamma_{y,x}=&\langle y,(y_1,\dots,y_{d-1},x_d)\rangle\\ &\cup\langle(y_1,\dots,y_{d-1},x_d),(y_1,\dots,y_{d-2},x_{d-1},x_d)\rangle\\ &\dots\cup\langle(y_1,x_2,\dots,x_d),x\rangle,\end{aligned}\tag{2.1}$$

where $\langle(y_1,\dots,y_i,x_{i+1},\dots,x_d),(y_1,\dots,y_{i-1},x_i,x_{i+1},\dots,x_d)\rangle$ is a segment on T_ε connecting the corresponding points. We consider $\Gamma_{y,x}$ as an oriented contour, with y as an initial point and x as a final point. Let us notice that this contour depends on the lattice $T^{(k)}_{L^k\varepsilon}$ to which both points y, x belong. Each such contour is composed of the bonds of ε-lattice, so the number of these bonds in $\Gamma_{y,x}$ grows with k. Next we will define the contours $\Gamma^{(k)}_{y,x}$ for $y\in T^{(k)}_{L^k\varepsilon}$, $x\in B^k(y)$. Let us denote by x_j a point of torus $T^{(j)}_{L^j\varepsilon}$, such that $x\in B^j(x_j)$. Of course $x_k=y$ and $x_j\in B(x_{j+1})$. We define

$$\begin{gathered}\Gamma^{(k)}_{y,x}=\Gamma_{y,x_{k-1}}\cup\Gamma_{x_{k-1},x_{k-2}}\cup\dots\cup\Gamma_{x_1,x},\\ y\in T^{(k)}_{L^k\varepsilon},\quad x\in B^k(y).\end{gathered}\tag{2.2}$$

The contour $\Gamma^{(k)}_{y,x}$ is considered as an oriented contour. We will denote the rescaled contours by the same symbol. For an arbitrary oriented contour Γ, let us put

$$A(\Gamma)=\sum_{b\subset\Gamma}A_b,\tag{2.3}$$

where the bonds b are taken with the orientations according to the orientation of Γ.

Now we can define the renormalization transformations for scalar fields on a subset Ω of the $L^k\varepsilon$-lattice $T^{(k)}_{L^k\varepsilon}$ satisfying the condition $B(\Omega')=\Omega$.

$$\varrho'(A,\psi)=T^{L^k\varepsilon}_{a,L,A}[\Omega,\varrho]=\int d\phi\, t^{L^k\varepsilon}_{a,L,A}(\Omega;\psi,\phi)\varrho(A,\phi),\tag{2.4}$$

$$t^{L^k\varepsilon}_{a,L,A}(\Omega;\psi,\phi)=\prod_{y\in\Omega'}t^{L^k\varepsilon}_{a,L,A}(\psi(y),\phi\restriction_{B(y)}),\tag{2.5}$$

$$t^{L^k\varepsilon}_{a,L,A}(\psi(y),\phi\restriction_{B(y)})=\left(\frac{a(L^{k+1}\varepsilon)^{d-2}}{2\pi}\right)^{N/2}\exp\left(-\tfrac{1}{2}a(L^{k+1}\varepsilon)^{d-2}|\psi(y)-(Q(A)\phi)(y)|^2\right),\tag{2.6}$$

$$(Q(A)\phi)(y)=L^{-d}\sum_{x\in B(y)}U(A(\Gamma_{y,x}))\phi(x).\tag{2.7}$$

We have the normalization properties

$$\int d\psi(y) t^{L^k\varepsilon}_{a,L,A}(\psi(y), \phi\restriction_{B(y)}) = 1\,, \tag{2.8}$$

$$\int d\psi \varrho'(A,\psi) = \int d\phi \varrho(A,\phi)\,. \tag{2.9}$$

Although formally the same for all k, the above definitions are in fact different because the contours $\Gamma_{y,x}$ depend on the scale $L^k\varepsilon$, as it was noted above. Here the external vector field A defined on T_ε is arbitrary.

Next let us define a renormalization transformation of k^{th} order $T^\varepsilon_{a_k, L^k, A}$ by the formulas (2.4), (2.5), but instead of (2.6) we take

$$t^\varepsilon_{a_k, L^k, A}(\psi(y), \phi\restriction_{B^k(y)}) = \left(\frac{a_k(L^k\varepsilon)^{d-2}}{2\pi}\right)^{N/2} \exp\left(-\tfrac{1}{2} a_k (L^k\varepsilon)^{d-2} |\psi(y) - (Q_k(A)\phi)(y)|^2\right), \quad y \in T^{(k)}_{L^k\varepsilon}, \tag{2.10}$$

where $Q_k(A)$ is an operator transforming functions on the ε-lattice T_ε into functions on the $L^k\varepsilon$-lattice $T^{(k)}_{L^k\varepsilon}$ and given by the formula

$$(Q_k(A)f)(y) = L^{-kd} \sum_{x\in B^k(y)} U(A(\Gamma^{(k)}_{y,x})) f(x)\,, \quad y \in T^{(k)}_{L^k\varepsilon}\,. \tag{2.11}$$

An easy Gaussian integration gives the formula

$$\int \prod_{y\in B(z)} d\theta(y) t^{L^k\varepsilon}_{a,L,A}(\psi(z), \theta\restriction_{B(z)}) \prod_{y\in B(z)} t^\varepsilon_{a_k, L^k, A}(\theta(y), \phi\restriction_{B^k(y)}) = t^\varepsilon_{a_{k+1}, L^{k+1}, A}(\psi(z), \phi\restriction_{B^{k+1}(z)})\,, \tag{2.12}$$

where a, a_k, a_{k+1} satisfy the relation

$$a_{k+1} = \frac{a a_k}{aL^{-2} + a_k}\,. \tag{2.13}$$

From this formula we obtain

$$T^{L^k\varepsilon}_{a,L,A} T^\varepsilon_{a_k, L^k, A} = T^\varepsilon_{a_{k+1}, L^{k+1}, A}\,. \tag{2.14}$$

A sequence of numbers a_k satisfying the above recurrent equation and an initial condition $a_1 = a$ is uniquely determined and it is equal to

$$a_k = a\frac{1-L^{-2}}{1-L^{-2k}}\,, \quad a_k \searrow a_\infty = a(1-L^{-2}) \quad \text{as} \quad k\to\infty\,. \tag{2.15}$$

Thus we get the following formula for the composition of k successive renormalization transformations

$$T^{L^{k-1}\varepsilon}_{a,L,A} \dots T^{L\varepsilon}_{a,L,A} T^\varepsilon_{a,L,A} = T^\varepsilon_{a_k, L^k, A}\,. \tag{2.16}$$

Now we will define sequences of operators and covariances which are fundamental for the rest of the paper. They are obtained by application of renormalization transformations to the basic Gaussian density $\exp(-\frac{1}{2}\langle\phi, (-\Delta^{\varepsilon,N}_{A,\Omega} + m^2)\phi\rangle)$, where $-\Delta^{\varepsilon,N}_{A,\Omega}$ is a covariant Laplace operator on the set Ω with Neumann boundary conditions. A set Ω is a subset of T_ε and we assume that it satisfies the condition (1.22), i.e. $\Omega = B^k(\Omega^{(k)})$ for the suitable indices k. In this paper we will use the case $\Omega = T_\varepsilon$ only, but in the second part the necessity of the considerations of more general Ω will arise. We define inductively

$$\Delta^{(0),\varepsilon}(\Omega, A) = -\Delta^{\varepsilon,N}_{A,\Omega} + m^2, \tag{2.17}$$

$$\begin{aligned} &Z^{(k),L^k\varepsilon}(\Omega, A)\exp(-\tfrac{1}{2}\langle\psi, \Delta^{(k+1),L^{k+1}\varepsilon}(\Omega, A)\psi\rangle) \\ &= T^{L^k\varepsilon}_{a,L,A}[\Omega^{(k)}, \exp(-\tfrac{1}{2}\langle\phi, \Delta^{(k),L^k\varepsilon}(\Omega, A)\phi\rangle)]. \end{aligned} \tag{2.18}$$

From (2.16) we have

$$Z^\varepsilon_k(\Omega, A)\exp(-\tfrac{1}{2}\langle\psi, \Delta^{(k),L^k\varepsilon}(\Omega, A)\psi\rangle) = T^\varepsilon_{a_k,L^k,A}[\Omega, \exp(-\tfrac{1}{2}\langle\phi, (-\Delta^{\varepsilon,N}_{A,\Omega} + m^2)\phi\rangle)]. \tag{2.19}$$

Defining the propagator

$$G^\varepsilon_k(\Omega, A) = (-\Delta^{\varepsilon,N}_{A,\Omega} + m^2 + a_k(L^k\varepsilon)^{-2}P_k(A))^{-1}, \quad P_k(A) = Q^*_k(A)Q_k(A), \tag{2.20}$$

and calculating the integral in (2.19), we obtain

$$\langle\psi, \Delta^{(k),L^k\varepsilon}(\Omega, A)\psi\rangle = a_k(L^k\varepsilon)^{-2}\langle\psi,\psi\rangle - a_k^2(L^k\varepsilon)^{-4}\langle\psi, Q_k(A)G^\varepsilon_k(\Omega, A)Q^*_k(A)\psi\rangle. \tag{2.21}$$

In the sequel the properties of the propagator $G^\varepsilon_k(\Omega, A)$ rescaled to the η-lattice, $\eta = L^{-k}$, will be very important. Let us notice that the rescaled propagator is given by

$$G_k(\Omega, A) = (-\Delta^{\eta,N}_{A,\Omega} + m^2(L^k\varepsilon)^2 + a_kP_k(A))^{-1}. \tag{2.22}$$

The following proposition gathers the most basic properties of $G_k(\Omega, A)$:

Proposition 2.1. *Let a set Ω satisfies $\Omega = B^k(\Omega^{(k)})$ and let $\Omega^{(k)} \subset T^{(k)}_1$ be a sum of big blocks with M sufficiently large. Further, let a configuration A be regular on Ω in the sense that*

$$|(\partial^\eta_\mu A)(x)| \leqq c(e(L^k\varepsilon))^{\beta-1}, \quad x \in \Omega, \quad \mu = 1, \ldots, d, \tag{2.23}$$

where $e(L^k\varepsilon) = e(L^k\varepsilon)^{\frac{4-d}{2}}$, $\eta = L^{-k}$, $\beta > 0$ and c is some universal constant. For an arbitrary pair of points $x, x' \in T_\eta$ let us denote yb $\Gamma_{x,x'}$ a shortest contour connecting these points. Then for $e(L^k\varepsilon)$ sufficiently small and $\alpha < 1$ there exist positive constants δ_0, c_0, R_0 independent of A, k, Ω and depending on d, a, M only, c_0 on α also, such that for an arbitrary function $f: \Omega \to R^N$ we have

$$\begin{aligned} &\frac{1}{|x - x'|^\alpha}|U(A(\Gamma_{x,x'}))(D^\eta_{A,\mu}G_k(\Omega, A)f)(x') - (D^\eta_{A,\mu}G_k(\Omega, A)f)(x)| \\ &\leqq c_0\exp(-\delta_0\,\mathrm{dist}(\{x, x'\}, \mathrm{supp} f))\|f\|_\infty \end{aligned} \tag{2.24}$$

for $x, x' \in \Omega$ and satisfying the condition $\mathrm{dist}(\{x, x'\}, \Omega^c) \geqq R_0$. *Similarly we have*

$$|(D^\eta_{A,\mu}G_k(\Omega, A)f)(x)|, |(G_k(\Omega, A)f)(x)| \leqq c_0\exp(-\delta_0\,\mathrm{dist}(x, \mathrm{supp} f))\|f\|_\infty \tag{2.25}$$

for $x \in \Omega$, $\mathrm{dist}(x, \Omega^c) \geqq R_0$. *If $\Omega \subset \Omega_0$, then for $\delta G_k(\Omega, \Omega_0, A)$ defined by the equality*

$$\delta G_k(\Omega, \Omega_0, A) = G_k(\Omega, A) - G_k(\Omega_0, A), \tag{2.26}$$

we have the inequalities (2.24), (2.25) *with the additional factor*

$$\exp(-\delta_0\,\mathrm{dist}(\mathrm{supp} f, \Omega^c) - \delta_0\,\mathrm{dist}(\{x, x'\}, \Omega^c))$$

on the right sides. For some simple sets Ω, e.g. *for rectangular parallelepipeds, the inequalities hold without any restrictions on the points* x, x', i.e. *for all* $x, x' \in \Omega$.

Let us notice that Ω^c means a complement in T_η, so in the case $\Omega = T_\eta$ the condition $\mathrm{dist}(\{x, x'\}, \Omega^c) \geqq R_0$ is meaningless and is omitted.

Let us formulate now some consequences of the above theorem. The first concerns the operator $\Delta^{(k), L^k\varepsilon}(\Omega, A)$ rescaled to the unit lattice.

Proposition 2.2. *If a configuration A is regular in the same sense as in Proposition 2.1 then there exist constants* $\delta_0 > 0$ *and* c_0, *depending on the same quantities as in Proposition 2.1, such that*

$$|\Delta^{(k)}(\Omega, A; x, x')| \leqq c_0 \exp(-\delta_0 |x - x'|), \qquad x, x' \in \Omega^{(k)}. \tag{2.27}$$

Putting for $\Omega \subset \Omega_0$

$$\delta\Delta^{(k)}(\Omega, \Omega_0, A) = \Delta^{(k)}(\Omega, A) - \Delta^{(k)}(\Omega_0, A), \tag{2.28}$$

the following inequality holds

$$|\delta\Delta^{(k)}(\Omega, \Omega_0, A; x, x')| \leqq c_0 \exp(-\delta_0(|x - x'| + \mathrm{dist}(x, \Omega^{(k)c}) + \mathrm{dist}(x', \Omega^{(k)c}))). \tag{2.29}$$

Another important class of covariances is the one connected with the effective integration in (2.18). We define a convariance $C^{(k), L^k\varepsilon}(\Omega, A)$ by means of the quadratic form in ϕ in this integral:

$$C^{(k), L^k\varepsilon}(\Omega, A) = (a(L^{k+1}\varepsilon)^{-2} P(A) + \Delta^{(k), L^k\varepsilon}(\Omega, A))^{-1}. \tag{2.30}$$

In the sequel we will use the covariance rescaled to the unit lattice and it is of the form

$$C^{(k)}(\Omega, A) = (aL^{-2} P(A) + \Delta^{(k)}(\Omega, A))^{-1}. \tag{2.31}$$

It is not clear from the formulas (2.30), (2.31) that the covariances are well defined. It is so, and it is one of the assertions of Proposition 2.3. Beside the covariances (2.31), we will need the covariances which are obtained by conditioning with respect to some set $\Lambda \subset \Omega^{(k)}$. For an operator A defined on configurations $\phi : \Omega^{(k)} \to R^N$ we define $A \upharpoonright_\Lambda$ as an operator on configurations $\phi : \Lambda \to R^N$ by the formula $A \upharpoonright_\Lambda \phi = \Lambda A \Lambda \phi$. We will use the following covariances also

$$C_\Lambda^{(k)}(\Omega, A) = ((aL^{-2} P(A) + \Delta^{(k)}(\Omega, A)) \upharpoonright_\Lambda)^{-1}. \tag{2.32}$$

Here we will assume that the set Λ is a union of big blocks of $T_1^{(k)}$.

Proposition 2.3. *If a configuration A is regular on* Ω *in the sense defined in Proposition 2.1, then there exist positive constants* δ_0, c_0, γ_0, γ_1, *dependent on d and a, and independent of A, k,* Ω *and* Λ, *such that*

$$\gamma_0 I \leqq aL^{-2} P(A) + \Delta^{(k)}(\Omega, A) \leqq \gamma_1 I, \tag{2.33}$$

$$|C_\Lambda^{(k)}(\Omega, A; x, x')| \leqq c_0 \exp(-\delta_0 |x - x'|), \qquad x, x' \in \Lambda. \tag{2.34}$$

In particular the above inequality holds for $C^{(k)}(\Omega, A)$. *Putting*

$$\delta C_\Lambda^{(k)}(\Omega, A) = C_\Lambda^{(k)}(\Omega, A) - C^{(k)}(\Omega, A), \tag{2.35}$$

we have

$$|\delta C_{\Lambda}^{(k)}(\Omega, A; x, x')|$$
$$\leqq c_0 \exp(-\delta_0(|x-x'| + \operatorname{dist}(x, \Lambda^c) + \operatorname{dist}(x', \Lambda^c))), \qquad x, x' \in \Lambda. \tag{2.36}$$

Finally, for $\Omega \subset \Omega_0$ *and*

$$\delta C_{\Lambda}^{(k)}(\Omega, \Omega_0, A) = C_{\Lambda}^{(k)}(\Omega, A) - C_{\Lambda}^{(k)}(\Omega_0, A), \tag{2.37}$$

we have similarly

$$|\delta C_{\Lambda}^{(k)}(\Omega, \Omega_0, A; x, x')|$$
$$\leqq c_0 \exp(-\delta_0(|x-x'| + \operatorname{dist}(x, \Omega^{(k)c}) + \operatorname{dist}(x', \Omega^{(k)c}))), \qquad x, x' \in \Lambda. \tag{2.38}$$

Now we will find recursive relations between the propagators (2.20), (2.30). Using the relations (2.18), (2.19), (2.21), we get after easy calculations

$$Z_{k+1}^{\varepsilon}(\Omega, A) = Z_k^{\varepsilon}(\Omega, A) Z^{(k), L^k \varepsilon}(\Omega, A), \tag{2.39}$$

$$Z_k^{\varepsilon}(\Omega, A) = Z^{(k-1), L^{k-1}\varepsilon}(\Omega, A) \cdot \ldots \cdot Z^{(1), L\varepsilon}(\Omega, A) Z^{(0), \varepsilon}(\Omega, A), \tag{2.40}$$

$$Q_{k+1}(A) G_{k+1}^{\varepsilon}(\Omega, A) = \frac{a a_k}{a_{k+1}} (L^k \varepsilon)^{-2} Q(A) C^{(k), L^k \varepsilon}(\Omega, A) Q_k(A) G_k^{\varepsilon}(\Omega, A), \tag{2.41}$$

$$G_{k+1}^{\varepsilon}(\Omega, A) = a_k^2 (L^k \varepsilon)^{-4} G_k^{\varepsilon}(\Omega, A) Q_k^*(A) C^{(k), L^k \varepsilon}(\Omega, A) Q_k(A) G_k^{\varepsilon}(\Omega, A) + G_k^{\varepsilon}(\Omega, A). \tag{2.42}$$

We can treat these identities as recursive equations of the renormalization group. The last two are the most important. The formula (2.41) allows us to compose the covariances appearing after the successive applications of renormalization transformations. The formula (2.42) is used for similar purposes and it has a fundamental meaning for the analysis of the perturbation expansions, especially for the proof of its renormalizability. More exactly, an equality obtained by solving (2.42), i.e. applying (2.42) k times, has such a meaning. Using the identity $G_1^{\varepsilon}(\Omega, A) = C^{(0), \varepsilon}(\Omega, A)$, we get

$$G_k^{\varepsilon}(\Omega, A) = \sum_{j=1}^{k-1} a_j^2 (L^j \varepsilon)^{-4} G_j^{\varepsilon}(\Omega, A) Q_j^*(A) C^{(j), L^j \varepsilon}(\Omega, A) Q_j(A) G_j^{\varepsilon}(\Omega, A) + C^{(0), \varepsilon}(\Omega, A). \tag{2.43}$$

3. Formal Properties of the Renormalization Procedure

The Lower Bound

In this chapter we begin a proof of the theorem. According to the now well-developed procedure in the papers [13, 2, 3, 14] we will first prove the lower bound in the inequality (1.14). Its proof is essentially simpler than the proof of the upper bound and allows a separation of some aspects of the whole procedure. The proof is based on successive applications of the renormalization transformations. We will describe the operations to be done after each renormalization transformation in a form suitable for the lower bound, but the same operations in a slightly modified form will be done in the next paper in the proof of the upper bound.

A very essential feature of the method is an introduction of the restrictions on the magnitudes of the fields. The restrictions we will use differ from those used in the papers [2, 3, 14]. In the first step we introduce the characteristic functions

$$\chi_0(A)=\prod_{x\in T_\varepsilon}\chi\left(\left\{|A(x)|\leqq\varepsilon^{-\frac{d-2}{2}}p(\varepsilon)\right\}\right)\chi\left(\left\{|(\Delta^\varepsilon A)(x)|\leqq\varepsilon^{-\frac{d+2}{2}}p(\varepsilon)\right\}\right),$$
$$p(\varepsilon)=b_0(1+\log\varepsilon^{-1})^p, \tag{3.1}$$

$$\chi_0(\phi)=\prod_{x\in T_\varepsilon}\chi\left(\left\{|\phi(x)|\leqq\varepsilon^{-\frac{d-2}{2}}p(\varepsilon)\right\}\right)\chi\left(\left\{|(\Delta_A^\varepsilon\phi)(x)|\leqq\varepsilon^{-\frac{d+2}{2}}p(\varepsilon)\right\}\right). \tag{3.2}$$

The restrictions on the fields B, ψ are made by the means of the fields $B^{(1),\varepsilon}$, $\psi^{(1),\varepsilon}$:

$$B^{(1),\varepsilon}=a(L\varepsilon)^{-2}G_1^\varepsilon Q^*B,\qquad \psi^{(1),\varepsilon}=a(L\varepsilon)^{-2}G_1^\varepsilon(B^{(1),\varepsilon})Q^*(B^{(1),\varepsilon})\psi. \tag{3.3}$$

Of course they are defined on the ε-lattice T_ε. We introduce the characteristic functions

$$\chi_1(B)=\prod_{x\in T_\varepsilon}\chi\left(\left\{|B^{(1),\varepsilon}(x)|\leqq(L\varepsilon)^{-\frac{d-2}{2}}p(L\varepsilon)\right\}\right)$$
$$\cdot\chi\left(\left\{|(\Delta^\varepsilon B^{(1),\varepsilon})(x)|\leqq(L\varepsilon)^{-\frac{d+2}{2}}p(L\varepsilon)\right\}\right), \tag{3.4}$$

$$\chi_1(\psi)=\prod_{x\in T_\varepsilon}\chi\left(\left\{|\psi^{(1),\varepsilon}(x)|\leqq(L\varepsilon)^{-\frac{d-2}{2}}p(L\varepsilon)\right\}\right)$$
$$\cdot\chi\left(\left\{|(\Delta^\varepsilon_{B^{(1),\varepsilon}}\psi^{(1),\varepsilon})(x)|\leqq(L\varepsilon)^{-\frac{d+2}{2}}p(L\varepsilon)\right\}\right), \tag{3.5}$$

and the following inequality holds

$$Z^\varepsilon\geqq\int dB\int d\psi\chi_1(B)\chi_1(\psi)T^\varepsilon_{a,L}[T^\varepsilon_{a,L,A}[\chi_0(A)\chi_0(\phi)\exp(-S^\varepsilon)]]. \tag{3.6}$$

We have to calculate $T^\varepsilon_{a,L}[T_{a,L,A}[\chi_0(A)\chi_0(\phi)\exp(-S^\varepsilon)]]$ under the restrictions on the fields B, ψ introduced by the characteristic functions $\chi_1(B)\chi_1(\psi)$.

Now we will describe briefly the operations to be done in order to calculate this integral and to get an expression which is a starting point for an inductive hypothesis. This integral is rescaled from the ε-lattice to the unit lattice, i.e. we make the transformations (1.23), (1.24) with $\eta=\varepsilon$, $\delta=1$. After the rescaling the integral gets the following form

$$\begin{aligned}\text{const}\int dA\int d\phi\chi_0(A)\chi_0(\phi)\exp\Big[&-\tfrac{1}{2}aL^{d-2}\sum_{y\in T_1'}|B(y)-(QA)(y)|^2\\&-\tfrac{1}{2}\langle A,(-\Delta+\mu_0^2\varepsilon^2)A\rangle-\tfrac{1}{2}aL^{d-2}\sum_{y\in T_1'}|\psi(y)-(Q(A)\phi)(y)|^2\\&-\tfrac{1}{2}\langle\phi,(-\Delta_A+m^2\varepsilon^2)\phi\rangle-\tfrac{1}{2}\delta m^2\varepsilon^2\sum_{x\in T_1}|\phi(x)|^2\\&-\lambda\varepsilon^{4-d}\sum_{x\in T_1}|\phi(x)|^4-E\Big],\end{aligned} \tag{3.7}$$

where the characteristic functions $\chi_0(A)\chi_0(\phi)$ correspond now to the restrictions

$$|A(x)|\leqq p(\varepsilon),\qquad |(\Delta A)(x)|\leqq p(\varepsilon), \tag{3.8}$$

$$|\phi(x)|\leqq p(\varepsilon),\qquad |(\Delta_A\phi)(x)|\leqq p(\varepsilon),\qquad x\in T_1, \tag{3.9}$$

and the scalar product and the operators are defined on the unit lattice. The next step is the translation in the fields A

$$A = A' + aL^{-2}C^{(0)}Q^*B = : A' + B^{(1)} \tag{3.10}$$

separating the quadratic form in the fields A, B in (3.7) into a sum of two forms

$$\begin{aligned}&\tfrac{1}{2}\langle B, \Delta^{(1),L}B\rangle + \tfrac{1}{2}\Big(aL^{d-2}\sum_{y\in T_1'}|(QA')(y)|^2 + \langle A', (-\Delta + \mu_0^2\varepsilon^2)A'\rangle\Big)\\ &= \tfrac{1}{2}\langle B, \Delta^{(1),L}B\rangle + \tfrac{1}{2}\langle A', (C^{(0)})^{-1}A'\rangle .\end{aligned} \tag{3.11}$$

We estimate the integral (3.7) from below by the same integral with additional characteristic functions

$$\chi(A') = \prod_{x\in T_1}\chi(\{|A'(x)| \leqq p_1(\varepsilon)\}), \qquad p_1(\varepsilon) = b_1(1+\log\varepsilon^{-1})^{p_1}. \tag{3.12}$$

Now we will demand that the restrictions on A' introduced by the above characteristic functions and the restrictions on $B^{(1)}$ introduced by the functions $\chi_1(B)$ imply the restrictions (3.8). To meet the demand it is sufficient to require $4dp_1(\varepsilon) + p(L\varepsilon) \leqq p(\varepsilon)$. For example it is easy to see that this condition is satisfied if $p-1 \geqq p_1$, $b_0 p \log L \geqq 4db_1(1+\log L)^{p_1}$. The demand can be written in a form of the equality

$$\chi_1(B)\chi(A')\chi_0(A' + B^{(1)}) = \chi_1(B)\chi(A'), \tag{3.13}$$

so we can remove the functions $\chi_0(A' + B^{(1)})$ from the integral (3.7). We expand the action into a power series in A'. We will write the expansions below for arbitrary fields A, B instead of $A', B^{(1)}$, and for a general case, i.e. for the operators obtained after k transformations. They are considered on the η-lattice, $\eta = L^{-k}$. Thus we have the basic expansion for the function $U(A)$ rescaled to η-lattice

$$\begin{aligned}U(A) &= \exp(\eta q e(L^k\varepsilon)A) = 1 + \sum_{n=1}^{\bar{n}} \frac{(\eta q e(L^k\varepsilon)A)^n}{n!}\\ &\quad + \frac{(\eta q e(L^k\varepsilon)A)^{\bar{n}+1}}{(\bar{n}+1)!} R_{\bar{n}+1}(\eta q e(L^k\varepsilon)A) = : 1 + \eta F_{1,k}(A)\\ &= : 1 + F'_{1,k}(A),\end{aligned} \tag{3.14}$$

where $e(L^k\varepsilon) = e(L^k\varepsilon)^{\frac{4-d}{2}}$ and $R_{\bar{n}+1}(z)$ is an analytic function of z defined by the formula $R_{\bar{n}+1}(z) = (\bar{n}+1)\int_0^1 (1-t)^{\bar{n}} e^{tz}dt$. This implies an expansion of the operator $Q_k(A+B)$

$$\begin{aligned}(Q_k(A+B)\phi)(y) &= L^{-kd}\sum_{x\in B^k(y)} U((A+B)(\Gamma^{(k)}_{y,x}))\phi(x)\\ &= (Q_k(B)\phi)(y) + L^{-kd}\sum_{x\in B^k(y)} F'_{1,k}(A(\Gamma^{(k)}_{y,x}))U(B(\Gamma^{(k)}_{y,x}))\phi(x)\\ &= : (Q_k(B)\phi)(y) + (F_{2,k}(A, B)\phi)(y) .\end{aligned} \tag{3.15}$$

The above two formulas imply an expansion of the covariant Laplace operator with the Neumann boundary conditions on arbitrary Ω

$$\begin{aligned}
-\Delta^{\eta,N}_{A+B,\Omega}+a_kP_k(A+B)= & -\Delta^{\eta,N}_{B,\Omega}-F_{1,k}(-A)^*D^{\eta}_B \\
& -D^{\eta*}_B F_{1,k}(-A)+F_{1,k}(-A)^*F_{1,k}(-A)+a_kP_k(B) \\
& +a_kF_{2,k}(A,B)^*Q_k(B)+a_kQ^*_k(B)F_{2,k}(A,B) \\
& +a_kF_{2,k}(A,B)^*F_{2,k}(A,B)\,. \qquad (3.16)
\end{aligned}$$

We apply these formulas, for $k=1$ and rescaled to the unit lattice, to the expression in the exponential function under the integral (3.7). Further we can estimate the terms containing the matrix $R_{\bar{n}+1}(\cdot)$ using the inequality $|R_{\bar{n}+1}(qA)|\leqq 1$, which holds for arbitrary real A, and using the restrictions on the fields. These terms are estimated by $O(\varepsilon^{\kappa})|T_1|$ for some $\kappa>d$. Thus we get an expression which is a polynomial in the fields ψ,ϕ,A'. The basic quadratic form in the fields ψ,ϕ is equal

$$\tfrac{1}{2}aL^{d-2}\sum_{y\in T'_1}|\psi(y)-(Q(B^{(1)})\phi)(y)|^2+\tfrac{1}{2}\langle\phi,(-\Delta_{B^{(1)}}+m^2\varepsilon^2)\phi\rangle\,, \qquad (3.17)$$

and the remaining terms are interaction terms with coefficients proportional to some power of $\varepsilon^{\frac{4-d}{2}}$.

Now we apply the translation in the fields ϕ

$$\phi=\phi'+aL^{-2}C^{(0)}(B^{(1)})Q^*(B^{(1)})\psi=:\phi'+\psi^{(1)} \qquad (3.18)$$

separating this quadratic form into a sum of two independent forms in the fields ψ,ϕ' respectively:

$$\begin{aligned}
&\tfrac{1}{2}\langle\psi,\Delta^{(1),L}(B^{(1)})\psi\rangle+\tfrac{1}{2}aL^{d-2}\sum_{y\in T'_1}|(Q(B^{(1)})\phi'(y)|^2)+\langle\phi',(-\Delta_{B^{(1)}}+m^2\varepsilon^2)\phi'\rangle \\
&\quad=\tfrac{1}{2}\langle\psi,\Delta^{(1),L}(B^{(1)})\psi\rangle+\tfrac{1}{2}\langle\phi',(C^{(0)}(B^{(1)}))^{-1}\phi'\rangle\,. \qquad (3.19)
\end{aligned}$$

After this we will get some new interaction polynomial. Let us include the constant E_1 to it and let us denote the obtained expression by $V^{(0)}(B^{(1)},\psi,A',\phi')$. Again we introduce restrictions on the fields ϕ' and we estimate the integral from below inserting the characteristic functions

$$\chi(\phi')=\prod_{x\in T_1}\chi(\{|\phi'(x)|\leqq p_1(\varepsilon)\})\,. \qquad (3.20)$$

We will later in a general case prove that these restrictions on ϕ' together with the restrictions introduced by $\chi_1(\psi)$ imply restrictions (3.9) if the inequality $4dp_1(\varepsilon)+p(L\varepsilon)+O(\varepsilon^{\alpha})\leqq p(\varepsilon)$ holds. For example it is easy to see that the above condition is satisfied if $p-1\geqq p_1$, $b_0p\log L\geqq 4db_1(1+\log L)^{p_1}$ and ε is sufficiently small. Again we can write the effect of these conditions in the equality

$$\chi_1(\psi)\chi(\phi')\chi_0(\phi'+\psi^{(1)})=\chi_1(\psi)\chi(\phi')\,. \qquad (3.21)$$

Let us come back to the integral (3.7). It is estimated from below by

$$\begin{aligned}&\mathrm{const}\exp(-\tfrac{1}{2}\langle B, \Delta^{(1),L}B\rangle-\tfrac{1}{2}\langle\psi, \Delta^{(1),L}(B^{(1)})\psi\rangle)\\&\cdot(\int dA\exp(-\tfrac{1}{2}\langle A,(C^{(0)})^{-1}A\rangle))(\int d\phi\exp(-\tfrac{1}{2}\langle\phi,(C^{(0)}(B^{(1)}))^{-1}\phi\rangle))\\&\cdot[\int d\mu_{C^{(0)}}(A')\int d\mu_{C^{(0)}(B^{(1)})}(\phi')\chi(A')\chi(\phi')\\&\cdot\exp(V^{(0)}(B^{(1)},\psi,A',\phi')-E_0+O(\varepsilon^{\kappa})|T_1|)]\,.\end{aligned}\tag{3.22}$$

To calculate the integral in the square bracket [...] we use the cumulant expansion formula of the form

$$\langle\exp(V)\rangle=\exp\left[\langle V\rangle+\frac{1}{2!}\langle V^2\rangle^T+\frac{1}{3!}\langle V^3\rangle^T+\dots\right],\tag{3.23}$$

where $\langle\cdot\rangle$ denotes the expectation value with respect to the measure $d\mu_{C^{(0)}}(A')d\mu_{C^{(0)}(B^{(1)})}(\phi')$, $V=V^{(0)}$ and $\langle V^n\rangle^T$ denotes the truncated expectation of a product of n polynomials V, thus

$$\langle V^2\rangle^T=\langle V^2\rangle-\langle V\rangle^2,\langle V^3\rangle^T=\langle V^3\rangle-3\langle V^2\rangle\langle V\rangle+\langle V\rangle^3,$$

and so on. On the right side of (3.23) there is a formal series, so obviously we can use only some truncated form of this expansion. The coefficients of the polynomial V are proportional to some positive powers of ε. The smallest such power is $\varepsilon^{1/2}$ and $\langle V^n\rangle^T$ is the expression corresponding to the sum of connected graphs with exponentially decaying propagators. Hence $\langle V^n\rangle^T=O(\varepsilon^{\kappa})|T_1|$ with $\kappa>d$ for n sufficiently large, e.g. $n>6$. The terms of the formal series in (3.23) with n large are thus uninteresting for us and we would like to have a cumulant expansion formula in the form taking into account the existence of the characteristic functions also

$$\langle\chi\exp(V)\rangle=\exp\left[\langle V\rangle+\frac{1}{2!}\langle V^2\rangle^T+\dots+\frac{1}{\bar{n}!}\langle V^{\bar{n}}\rangle^T+O(\varepsilon^{\kappa})|T_1|\right],\quad\kappa>d.\tag{3.24}$$

There is one obvious way of proving this formula, namely by a cluster expansion, but it is a long and tedious way. Instead we will rely on the results of Benfatto et al. [2]. The lemma formulated on p. 152 of this paper can be applied in our situation because all the assumptions are satisfied.

The method used by Gawędzki and Kupiainen in [14] can be adapted here also.

If we denote the expression under the exponential on the right side of this formula by $\mathscr{P}'^{(1),L}(B^{(1)},\psi)$, then we can write

(the integral in the square bracket [...] in (3.22))

$$\geqq\exp(\mathscr{P}'^{(1),L}(B^{(1)},\psi)-E_0+O(\varepsilon^{\kappa})|T_1|)\,.\tag{3.25}$$

The expression $\mathscr{P}'^{(1),L}$ is a polynomial in ψ and its terms can be represented by suitable connected graphs. This graphical representation will be introduced and used in a separate paper treating the properties of the perturbation expansions. Here we can remark only that using the connectedness of the graphs, the exponential decay of propagators and the restrictions on the field ψ we can estimate all the terms in $\mathscr{P}'^{(1),L}$ of the order in coupling constants higher than $\bar{n}$ by

$O(\varepsilon^{\kappa})|T_1|$ with $\kappa > d$. This will be proved in the general case in that paper also. Let us denote by $\mathscr{P}^{(1),L}$ the sum of the remaining terms of the order $\leqq \bar{n}$.

The last operation is a rescaling of $\mathscr{P}^{(1),L}$ from the L-lattice to the $L\varepsilon$-lattice. The results we have obtained and the expressions we have arrived at serve as a basis of an induction hypothesis. We assume that after k renormalization transformations we get an action $S^{(k),L^k\varepsilon}(A,\phi)$ for the fields A,ϕ on $L^k\varepsilon$-lattice $T^{(k)}_{L^k\varepsilon}$. For this action the following fundamental inequality holds

$$Z^{\varepsilon} \geqq \int dA \int d\phi \chi_k(A)\chi_k(\phi)\exp(-S^{(k),L^k\varepsilon}(A,\phi) + \sum_{j=0}^{k-1} O(1)(L^j\varepsilon)^{\kappa_0}|T_\varepsilon|), \tag{3.26}$$

where

$$\chi_k(A) = \prod_{x\in T_\varepsilon} \chi\left(\left\{|A^{(k),\varepsilon}(x)| \leqq (L^k\varepsilon)^{-\frac{d-2}{2}} p(L^k\varepsilon)\right\}\right)$$
$$\cdot \chi\left(\left\{|(\Delta^{\varepsilon}A^{(k),\varepsilon})(x)| \leqq (L^k\varepsilon)^{-\frac{d+2}{2}} p(L^k\varepsilon)\right\}\right), \tag{3.27}$$

$$\chi_k(\phi) = \prod_{x\in T_\varepsilon} \chi\left(\left\{|\phi^{(k),\varepsilon}(x)| \leqq (L^k\varepsilon)^{-\frac{d-2}{2}} p(L^k\varepsilon)\right\}\right)$$
$$\cdot \chi\left(\left\{|(\Delta^{\varepsilon}_{A^{(k),\varepsilon}}\phi^{(k),\varepsilon})(x)| \leqq (L^k\varepsilon)^{-\frac{d+2}{2}} p(L^k\varepsilon)\right\}\right), \tag{3.28}$$

$$A^{(k),\varepsilon} = a_k(L^k\varepsilon)^{-2} G^{\varepsilon}_k Q^*_k A, \quad \phi^{(k),\varepsilon} = a_k(L^k\varepsilon)^{-2} G^{\varepsilon}_k(A^{(k),\varepsilon}) Q^*_k(A^{(k),\varepsilon})\phi. \tag{3.29}$$

The action has the form:

$$\begin{aligned} S^{(k),L^k\varepsilon}(A,\phi) = & -\log Z_k - \log Z_k(A^{(k),\varepsilon}) \\ & + \tfrac{1}{2}\langle A, \Delta^{(k),L^k\varepsilon}A\rangle + \tfrac{1}{2}\langle \phi, \Delta^{(k),L^k\varepsilon}(A^{(k),\varepsilon})\phi\rangle \\ & - \mathscr{P}^{(k),L^k\varepsilon}(A^{(k),\varepsilon},\phi) + E_0, \end{aligned} \tag{3.30}$$

and for the factors $Z_k, Z_k(A^{(k),\varepsilon})$ we have the formulas:

$$Z_k = \left(\frac{a_k\,(L^k\varepsilon)^{d-2}}{2\pi}\right)^{\frac{d}{2}|T_1^{(k)}|} \int dA \exp(-\tfrac{1}{2}\langle A, (G^{\varepsilon}_k)^{-1}A\rangle), \tag{3.31}$$

$$Z_k(A^{(k),\varepsilon}) = \left(\frac{a_k(L^k\varepsilon)^{d-2}}{2\pi}\right)^{\frac{N}{2}|T_1^{(k)}|} \int d\phi \exp(-\tfrac{1}{2}\langle \phi, (G^{\varepsilon}_k(A^{(k),\varepsilon}))^{-1}\phi\rangle). \tag{3.32}$$

The most difficult task is to describe $\mathscr{P}^{(k),L^k\varepsilon}(A^{(k),\varepsilon},\phi)$. We will do it giving another formula for the whole action $S^{(k),L^k\varepsilon}$. At first let us notice that after k successive renormalization transformations together with the corresponding translations, the field A with which we have started in the first step is represented as

$$A = A'^{(0),\varepsilon} + A'^{(1),\varepsilon} + \ldots + A'^{(k-1),\varepsilon} + A^{(k),\varepsilon}, \tag{3.33}$$

where $A'^{(j),\varepsilon}$ are given by the formula (3.29) with A'_j instead of A.

The fields A'_j defining the components of (3.33) are independent Gaussian random variables with the covariances $C^{(j),L^j\varepsilon}$. Also they are independent of the field A on the $L^k\varepsilon$-lattice, defining the configuration $A^{(k),\varepsilon}$. We introduce an additional convention. It concerns the expressions $U(A(\Gamma_{y,x}^{(k)}))$ in the operators $Q_k(A)$. We will change the definition of $A(\Gamma_{y,x}^{(k)})$ if we take the right side of (3.33) instead of A.

Making use of the definition (2.2) we assume

$$A(\Gamma_{y,x}^{(k)}) = A^{(k),\varepsilon}(\Gamma_{y,x}^{(k)}) + \sum_{j=1}^{k-1} A'^{(j),\varepsilon}(\Gamma_{x_{j+1},x}^{(j+1)}), \tag{3.34}$$

where we have denoted $x_k = y$, $x_0 = x$.

Now the perturbative formula for the action $S^{(k),L^k\varepsilon}$ can be written. This formula for $k=1$ follows from the close inspection of the procedures used and the formulas obtained in the first step. Let us introduce at first the function

$$\begin{aligned} E_k(e',\lambda',eA^{(k),\varepsilon},\phi) = -\log\Bigg[\bigg(\frac{a(L^k\varepsilon)^{d-2}}{2\pi}\bigg)^{\frac{d}{2}|T_1^{(k)}|} \\ \cdot \int dA'_{k-1}\exp(-\tfrac{1}{2}\langle A'_{k-1},(C^{(k-1),L^{k-1}\varepsilon})^{-1}A'_{k-1}\rangle)\cdot\ldots \\ \cdot\bigg(\frac{a(L\varepsilon)^{d-2}}{2\pi}\bigg)^{\frac{d}{2}|T_1^{(1)}|}\int dA'_0\exp(-\tfrac{1}{2}\langle A'_0,(C^{(0),\varepsilon})^{-1}A'_0\rangle) \\ \cdot T^{\varepsilon}_{a_k,L^k,eA^{(k),\varepsilon}+e'\sum_{j=0}^{k-1}A'^{(j),\varepsilon}}\bigg[\exp\bigg(-\tfrac{1}{2}\Big\langle\phi',\Big(-\Delta^{\varepsilon}_{eA^{(k),\varepsilon}+e'\sum_{j=0}^{k-1}A'^{(j),\varepsilon}}\Big)\phi'\Big\rangle \\ -\sum_{x\in T_\varepsilon}\varepsilon^d\mathscr{P}(e',\lambda',\phi(x))-E\bigg)\bigg]\Bigg], \\ \mathscr{P}(e',\lambda',\phi) = \lambda'|\phi|^4+\tfrac{1}{2}\delta m^2(e',\lambda')|\phi|^2. \end{aligned} \tag{3.35}$$

The action $S^{(k),L^k\varepsilon}$ can be written in the form

$$\begin{aligned} S^{(k),L^k\varepsilon}(A,\phi) = \tfrac{1}{2}\langle A,\Delta^{(k),L^k\varepsilon}A\rangle \\ + \sum_{0\leqq\alpha+\beta\leqq\bar{n}}\frac{1}{\alpha!\beta!}e^\alpha\lambda^\beta\bigg(\frac{\partial^{\alpha+\beta}}{\partial e'^\alpha\partial\lambda'^\beta}E_k(e',\lambda',eA^{(k),\varepsilon},\phi)\bigg)\bigg|_{e'=\lambda'=0}. \end{aligned} \tag{3.36}$$

From these formulas we can read out a graphical form of the expression above, i.e. its vertices and propagators. We will not need them here.

Now we will consider a general case, i.e. all the operations to be done after $k+1^{\text{st}}$ application of the renormalization transformation. From (3.26) and (2.9) we have

$$\begin{aligned} Z^\varepsilon \geqq \int dB\int d\psi\chi_{k+1}(B)\chi_{k+1}(\psi)T^{L^k\varepsilon}_{a,L}[T^{L^k\varepsilon}_{a,L,A^{(k),\varepsilon}}[\chi_k(A)\chi_k(\phi) \\ \cdot\exp(-S^{(k),L^k\varepsilon}(A,\phi))]]\exp\bigg(\sum_{j=0}^{k-1}O(1)(L^j\varepsilon)^{\kappa_0}|T_\varepsilon|\bigg) \end{aligned} \tag{3.37}$$

and we have to calculate the internal integral above.

The first step in the calculation is a rescaling of all the fields and the propagators from $L^k\varepsilon$-lattice $T_{L^k\varepsilon}^{(k)}$ to 1-lattice $T_1^{(k)}$. After the rescaling the integral transforms into the integral

$$\text{const}\,\chi_{k+1}(B)\chi_{k+1}(\psi)\int dA\int d\phi\chi_k(A)\chi_k(\phi)\exp\Big[-\tfrac{1}{2}aL^{d-2}\sum_{y\in T_L^{(k+1)}}|B(y)-(QA)(y)|^2$$
$$-\tfrac{1}{2}aL^{d-2}\sum_{y\in T_L^{(k+1)}}|\psi(y)-(Q(A^{(k)})\phi)(y)|^2-\tfrac{1}{2}\langle A,\Delta^{(k)}A\rangle$$
$$-\tfrac{1}{2}\langle\phi,\Delta^{(k)}(A^{(k)})\phi\rangle+\log Z_k+\log Z_k(A^{(k)})+\mathscr{P}^{(k)}(A^{(k)},\phi)-E_0\Big], \tag{3.38}$$

where now

$$Z_k=\left(\frac{a_k(L^k\varepsilon)^{d-2}}{2\pi}\right)^{\frac{d}{2}|T_1^{(k)}|}(L^k\varepsilon)^{-\frac{d(d-2)}{2}|T_1|}\int dA\exp(-\tfrac{1}{2}\langle A,G_k^{-1}A\rangle), \tag{3.39}$$

$$Z_k(A^{(k)})=\left(\frac{a_k(L^k\varepsilon)^{d-2}}{2\pi}\right)^{\frac{N}{2}|T_1^{(k)}|}(L^k\varepsilon)^{-\frac{N(d-2)}{2}|T_1|}\int d\phi\exp(-\tfrac{1}{2}\langle\phi,G_k(A^{(k)})^{-1}\phi\rangle)$$
$$=\left(\frac{a_k(L^k\varepsilon)^{d-2}}{2\pi}\right)^{\frac{N}{2}|T_1^{(k)}|}(L^k\varepsilon)^{-\frac{N(d-2)}{2}|T_1|}(2\pi\eta^d)^{-\frac{1}{2}N|T_1|}(\det G_k(A^{(k)})^{-1})^{-1/2}. \tag{3.40}$$

The second formula for $Z_k(A^{(k)})$ will be used in the sequel. We can easily deduce a form of the expression $\mathscr{P}^{(k)}$. Of course a graphical description is the same as before, only the coefficients at the vertices are changed. Further we will describe more exactly some properties of these expressions.

The next operation is a translation in the fields A and an expansion of the action with respect to a small field produced by the translation. This translation has the form

$$A=A'+aL^{-2}C^{(k)}Q^*B \tag{3.41}$$

and it separates the quadratic form in the fields A, B in the exponential function under the integral (3.38) into a sum of the forms $\frac{1}{2}\langle B,\Delta^{(k+1),L}B\rangle+\frac{1}{2}\langle A',(C^{(k)})^{-1}A'\rangle$. In the remaining part of the action the field A occurs only through the function $A^{(k)}$, so let us write the effect of the translation on the field $A^{(k)}$,

$$A^{(k)}=a_kG_kQ_k^*A'+a_kaL^{-2}G_kQ_k^*C^{(k)}Q^*B=a_kG_kQ_k^*A'$$
$$+a_{k+1}L^{-2}G_{k+1}^{\eta}Q_{k+1}^*B=A'^{(k)}+B^{(k+1)}. \tag{3.42}$$

We estimate the integral (3.38) from below introducing the characteristic functions

$$\chi(A')=\prod_{x\in T_1^{(k)}}\chi(\{|A'(x)|\leqq p_1(L^k\varepsilon)\}), \tag{3.43}$$

and we expand the action with respect to $A'^{(k)}$. This expansion was described in the formulas (3.19), (3.15), and (3.16). Let us write only the resulting expansion of the propagator $G_k(\Omega,A+B)$:

$$G_k(\Omega,A+B)=G_k(\Omega,B)+G_k(\Omega,B)[F_{1,k}(-A)^*D_B^{\eta}$$
$$+D_B^{\eta *}F_{1,k}(-A)-F_{1,k}(-A)^*F_{1,k}(-A)-a_kF_{2,k}(A,B)^*Q_k(B)$$
$$-a_kQ_k^*(B)F_{2,k}(A,B)-a_kF_{2,k}(A,B)^*F_{2,k}(A,B)]\,G_k(\Omega,A+B). \tag{3.44}$$

Iterating this equation we get an expansion of the propagator $G_k(\Omega, A+B)$ with respect to A. Of course only the first few terms of the expansion are important, the remaining ones give an expression proportional to $(L^k\varepsilon)^\kappa$, $\kappa > d$. Let us denote the operator in the square bracket in (3.44) by V_k. Applying the formula (3.44) $\bar{n}$ times we get the expansion

$$G_k(\Omega, A+B) = \sum_{n=0}^{\bar{n}} G_k(\Omega, B)[V_k G_k(\Omega, B)]^n + G_k(\Omega, B)[V_k G_k(\Omega, B)]^{\bar{n}} V_k G_k(\Omega, A+B). \tag{3.45}$$

This expansion is inserted in each place in the action $S^{(k)}$ where the propagator $G_k(\Omega, A+B)$ occurs. The formulas (3.14)–(3.16), (3.44), and (3.45) are sufficient to expand all the expressions in the action $S^{(k)}$, except the factor $Z_k(A^{(k)})$. Taking into account the form (3.40) of this factor it is sufficient to find an expansion of $(\det G_k^{-1}(A'^{(k)} + B^{(k+1)}))^{-1/2}$. Using the formula (3.16) we get

$$\begin{aligned}&(\det G_k(A'^{(k)} + B^{(k+1)})^{-1})^{-1/2} \\ &\quad = (\det G_k(B^{(k+1)})^{-1})^{-1/2} [\det(I - G_k(B^{(k+1)})^{1/2} V_k G_k(B^{(k+1)})^{1/2})]^{-1/2}. \end{aligned} \tag{3.46}$$

Obviously the operators standing in the above determinants are positive and symmetric. Thus the inequality $\log(1+\lambda) \leqq \lambda - \frac{\lambda^2}{2} + \frac{\lambda^3}{3} + \ldots + \frac{(-1)^{n-1}\lambda^n}{n}$ holding for n odd and λ real, $\lambda > -1$, implies the following inequality for the second determinant on the right side of (3.46):

$$[\det(I - G_k(B^{(k+1)})^{1/2} V_k G_k(B^{(k+1)})^{1/2}]^{-1/2} \geqq \exp \sum_{j=1}^{n} \frac{1}{2j} \operatorname{Tr}(G_k(B^{(k+1)}) V_k)^j. \tag{3.47}$$

We take n odd and $n \geqq \bar{n}$. The expression in the exponent above is obviously represented by the graphs with exactly one loop of the scalar field propagators. The expansion of the integrand in (3.38) with respect to $A'^{(k)}$ field is accomplished.

We got some complicated expression and at first we would like to get rid of undesirable terms, especially the terms which are not polynomials with respect to $A'^{(k)}$. These are the terms having the additional factors of the form $R_{\bar{n}+1}(\eta q e(L^k\varepsilon) A'^{(k)}(\Gamma_{y,x}^{(k)}))$ or $R_{\bar{n}+1}(\eta q e(L^k\varepsilon) A_b'^{(k)})$ besides the "normal", polynomial factors. The existence of such factors implies that there is the factor $e(L^k\varepsilon)^{\bar{n}+1}$ also and it gives the factor $(L^k\varepsilon)^\kappa$ for some $\kappa > d$. Such terms will be called R-terms. We have the following theorem concerning the expression we arrived at after the expansion.

Proposition 3.1. *The sum of R-terms of the previously described expansion of the action $S^{(k)}$ is of the order $O(1)(L^k\varepsilon)^\kappa |T_1^{(k)}|$ for some $\kappa > d$. Also of the same order is the sum of terms, for which the sum of the powers of the factors $L^k\varepsilon$ occurring at the vertices is greater than d. In these estimates* $0(1)$ *is independent of k, it depends only on the coupling constants and the number $\bar{n}$.*

This theorem will be obtained as a corollary of an analysis of the perturbation expansions and its renormalization. Using the above theorem we can estimate the sum of all the terms of an order higher than $\bar{n}$ by $O(1)(L^k\varepsilon)^\kappa |T_1^{(k)}|$ and we are left with an action which is a polynomial in the fields $A'^{(k)}$, ϕ, ψ.

Now we will do a translation in the fields ϕ. The translation has the form

$$\phi = \phi' + aL^{-2}C^{(k)}(B^{(k+1)})Q^*(B^{(k+1)})\psi, \tag{3.48}$$

and it separates again the basic quadratic form for scalar fields into a sum of the corresponding forms in the fields ψ and ϕ'. The rest of the action changes in an obvious manner. Let us notice that if (3.48) is done in

$$\phi^{(k)} = a_k G_k(B^{(k+1)})Q_k^*(B^{(k+1)})\phi,$$

then we use (2.66), (3.70) and we get

$$\phi^{(k)} = \phi'^{(k)} + \psi^{(k+1)}. \tag{3.49}$$

Finally we estimate from below the integral introducing the characteristic functions

$$\chi(\phi') = \prod_{x \in T_1^{(k)}} \chi(\{|\phi'(x)| \leqq p_1(L^k\varepsilon)\}). \tag{3.50}$$

Let us denote by $V^{(k)}$ a polynomial in the fields $A^{(k)}, \phi', \psi$ appearing when the translation (3.48) is done. After these operations and estimates we have the inequality

$$\begin{aligned}(3.38) \geqq {} & \operatorname{const} Z_k Z_k(B^{(k+1)}) \exp(-\tfrac{1}{2}\langle B, \Delta^{(k+1),L}B\rangle - \tfrac{1}{2}\langle \psi, \Delta^{(k+1),L}(B^{(k+1)})\psi\rangle) \\ & \cdot \chi_{k+1}(B)\chi_{k+1}(\psi) \cdot \int dA' \int d\phi' \chi_k(A' + aL^{-2}C^{(k)}Q^*B)\chi_k(\phi' + aL^{-2}C^{(k)}(B^{(k+1)}) \\ & \cdot Q^*(B^{(k+1)})\psi) \cdot \chi(A')\chi(\phi') \exp[-\tfrac{1}{2}\langle A', (C^{(k)})^{-1}A'\rangle \\ & -\tfrac{1}{2}\langle \phi', (C^{(k)}(B^{(k+1)}))^{-1}\phi'\rangle + V^{(k)}(B^{(k+1)}, \psi, A'^{(k)}, \phi') \\ & - E_0 + O(1)(L^k\varepsilon)^{\kappa}|T_1^{(k)}|].\end{aligned} \tag{3.51}$$

Now we will analyze in detail the restrictions on the fields implied by the characteristic functions in the above integral. We will do it for scalar fields only, the considerations for vector fields are even simpler. At first let us notice that the restrictions introduced by $\chi_{k+1}(\psi)$ or $\chi_k(\phi)$ imply some restrictions on the fields ψ, ϕ, for example we have:

$$\begin{aligned}|\psi(y)| &= |(Q_{k+1}^*(B^{(k+1)})\psi)(x)| \\ &= a_{k+1}^{-1}L^2|((-\Delta^{\eta}_{B^{(k+1)}} + a_{k+1}L^{-2}P_{k+1}(B^{(k+1)}) + m^2(L^k\varepsilon)^2)\psi^{(k+1)})(x)| \\ &\leqq a_{k+1}^{-1}L^{-\frac{d-2}{2}}p(L^{k+1}\varepsilon) + L^{-\frac{d-2}{2}}p(L^{k+1}\varepsilon) = cp(L^{k+1}\varepsilon) \leqq cp(L^k\varepsilon)\end{aligned} \tag{3.52}$$

for some c independent of k, ε. The function $\chi_k(\phi' + aL^{-2}C^{(k)}(B^{(k+1)})\,Q^*(B^{(k+1)})\psi)$ gives the restrictions on the field

$$\begin{aligned}& a_k G_k(A'^{(k)} + B^{(k+1)})Q_k^*(A'^{(k)} + B^{(k+1)})\phi' \\ & \quad + a_k a L^{-2} G_k(A'^{(k)} + B^{(k+1)})Q_k^*(A'^{(k)} + B^{(k+1)})C^{(k)}(B^{(k+1)})Q^*(B^{(k+1)})\psi.\end{aligned} \tag{3.53}$$

From (3.50) it follows that the first term above and its covariant Laplacian can be estimated by $c_1 p_1(L^k\varepsilon)$ with c_1 independent of k, ε. Now if we expand the second term in (3.53), and the Laplacian of this term, with respect to the field $A'^{(k)}$ using (3.15), (3.16), and (3.44), then the first term of this expansion is equal to $\psi^{(k+1)}$ or $\Delta^{\eta}_{B^{(k+1)}}\psi^{(k+1)}$ and the sum of the remaining terms can be estimated by $c_2(L^k\varepsilon)^{\alpha}$ for some $\alpha>0$ with a constant c_2 independent of k, ε. Hence the expression (3.53) can be estimated by $c_1p_1(L^k\varepsilon)+L^{-\frac{d-2}{2}}p(L^{k+1}\varepsilon)+c_2(L^k\varepsilon)^\alpha$ and the Laplacian $\Delta^\eta_{A'^{(k)}+B^{(k+1)}}$ of this expression can be estimated by $c_1p_1(L^k\varepsilon)+L^{-\frac{d+2}{2}}p(L^{k+1}\varepsilon)+c_2(L^k\varepsilon)^\alpha$. Then we have

$$\chi_{k+1}(\psi)\chi_k(\phi'+aL^{-2}C^{(k)}(B^{(k+1)})Q^*(B^{(k+1)})\psi)\chi_k(\phi')=\chi_{k+1}(\psi)\chi_k(\phi')$$

if $c_1p_1(L^k\varepsilon)+p(L^{k+1}\varepsilon)+c_2(L^k\varepsilon)^\alpha\leqq p(L^k\varepsilon)$, and this inequality is satisfied if e.g.

$$\begin{aligned}p-1\geqq p_1, \quad & b_0p\log L\geqq c_1b_1(1+\log L)^{p_1},\\ & \tfrac{1}{2}b_0p(p-1)\log L\geqq c_2(L^k\varepsilon)^\alpha,\end{aligned} \tag{3.54}$$

where we have assumed $L^{k+1}\varepsilon\leqq 1$. In the sequel we will assume that the conditions (3.54) are satisfied. Let us notice that the first two can be satisfied by a proper choice of b_0, b_1, p, p_1, but the third demands additionally $L^k\varepsilon\leqq\varepsilon_0$ for sufficiently small ε_0. Hence we have

$$\begin{aligned}(\text{the right side of}\,(3.51))\geqq\ & \text{const}\, Z_kZ_k(B^{(k+1)})\\ & \cdot\exp(-\tfrac{1}{2}\langle B,\Delta^{(k+1),L}B\rangle-\tfrac{1}{2}\langle\psi,\Delta^{(k+1),L}(B^{(k+1)})\psi\rangle)\\ & \cdot(\textstyle\int dA'\exp(-\tfrac{1}{2}\langle A',(C^{(k)})^{-1}A'\rangle))\\ & (\textstyle\int d\phi'\exp(-\tfrac{1}{2}\langle\phi',(C^{(k)}(B^{(k+1)}))^{-1}\phi'\rangle))\\ & \cdot\chi_{k+1}(B)\chi_{k+1}(\psi)\textstyle\int d\mu_{C^{(k)}}(A')\int d\mu_{C^{(k)}(B^{(k+1)})}(\phi')\chi(A')\chi(\phi')\\ & \cdot\exp(V^{(k)}-E_0+O(1)(L^k\varepsilon)^\kappa|T_1^{(k)}|)\,.\end{aligned} \tag{3.55}$$

The next step is a calculation of the integral

$$\int d\mu_{C^{(k)}}(A')\int d\mu_{C^{(k)}(B^{(k+1)})}(\phi')\chi(A')\chi(\phi')\exp(V^{(k)})\,. \tag{3.56}$$

Again we will use the lemma from [2]. The following theorem clarifies that the assumptions of the lemma are satisfied in our case.

Proposition 3.2. *The function* $V^{(k)}(B^{(k+1)},\psi,A'^{(k)},\phi')$ *has the form*

$$\begin{aligned}V^{(k)}(B^{(k+1)},\psi,A'^{(k)},\phi')=\ & \sum_{n,m=0}^{n(\bar n)}\ \sum_{x_1,\dots,x_n,y_1,\dots,y_m\in T_1^{(k)}}\\ & \cdot\sum_{j_1,\dots,j_n=1}^{N}\ \sum_{\mu_1,\dots,\mu_m=1}^{d} v^{(k)}_{j_1,\dots,j_n;\mu_1,\dots,\mu_m}\\ & \cdot(B^{(k+1)},\psi;x_1,\dots,x_n,y_1,\dots,y_m)\phi'_{j_1}(x_1)\dots\phi'_{j_n}(x_n)\\ & \cdot A'_{\mu_1}(y_1)\dots A'_{\mu_m}(y_m)\,,\end{aligned} \tag{3.57}$$

and the coefficients in the above representation satisfy the inequalities

$$\begin{aligned}&|v^{(k)}_{j_1,\dots,j_n;\mu_1,\dots,\mu_m}(B^{(k+1)},\psi;x_1,\dots,x_n,y_1,\dots,y_m)|\\ &\quad\leqq O(1)(L^k\varepsilon)^{\kappa_0}\exp(-\delta_0 d(x_1,\dots,x_n,y_1,\dots,y_m))\end{aligned} \tag{3.58}$$

for some independent of k *positive constants* κ_0, δ_0, *and* $O(1)$, *where* $d(x_1,\dots,x_n,y_1,\dots,y_m)$ *denotes a length of the shortest graph connecting the points* $x_1,\dots,x_n,y_1,\dots,y_m$.

In this theorem the representation (3.57) is obvious and the essential content of it is in the inequalities (3.58). A proof of this theorem will be given together with the proofs of the other properties of the perturbation expansions. Of course the restrictions (3.52) are used. The properties (3.57) and (3.58) and the properties formulated in the Propositions 2.2 and 2.3 and concerning the operators defining the basic quadratic forms of the action are sufficient for the assumptions made in the lemma [2]. Let us notice that the assumptions of Propositions 2.2 and 2.3 concerning the regularity of a vector field configuration are satisfied, the configuration $B^{(k+1)}$ is regular on the basis of the restrictions introduced by $\chi_{k+1}(B)$. Also let us notice that this lemma was proved in [2] for a Gaussian measure defined by the operator $-\Delta+c$, but the proof applies with some obvious modifications to the more general measures defined by the operators satisfying the properties formulated in Proposition 2.2 and 2.3. Using the lemma we get

$$\begin{aligned}(3.56) &= \langle \chi(A')\chi(\phi')\exp(V^{(k)})\rangle \\ &= \exp\left[\sum_{n=1}^{\bar{n}} \frac{1}{n!}\langle (V^{(k)})^n\rangle^T + O(1)(L^k\varepsilon)^\kappa |T_1^{(k)}|\right], \qquad \kappa > d. \end{aligned} \tag{3.59}$$

It is worth mentioning that κ_0 in (3.58) can be taken as $\kappa_0=\frac{1}{2}-\beta$, where β is an arbitrarily small positive number, so for the expansion (3.59) to hold it is sufficient to take $\bar{n}\geqq 6$. The terms of this expansion are represented by the graphs which we get connecting the legs of the graphs representing $V^{(k)}$ by the lines corresponding to the suitable propagators. Further we can estimate all the terms of the expansion of order higher than $\bar{n}$ by $O(1)(L^k\varepsilon)^\kappa |T_1^{(k)}|$. Let us denote the expression we get by $\mathscr{P}^{(k+1),L}(B^{(k+1)},\psi)$, so we have

$$(3.56) \geqq \exp(\mathscr{P}^{(k+1),L}(B^{(k+1)},\psi) + O(1)(L^k\varepsilon)^\kappa |T_1^{(k)}|)\,. \tag{3.60}$$

The last operation is a rescaling of the expression we got from the L-lattice for the fields B,ψ to the $L^{k+1}\varepsilon$-lattice $T_{L^{k+1}\varepsilon}^{(k+1)}$. Taking into account the expressions and the inequalities (3.38), (3.51), (3.55), and (3.60), we get the inequalities and the expressions of the induction hypothesis (3.26)–(3.32) but with $k+1$ instead of k. Finally we have to verify that the action $S^{(k+1),L^{k+1}\varepsilon}$ is given by a formula corresponding to (3.36). At first let us write the action as a perturbative expansion of some integral expression defined by the action $S^{(k),L^k\varepsilon}$. Let us introduce the function

$$\begin{aligned} E(e',\lambda',B,\psi) &= -\log[T_{a,L}^{L^k\varepsilon}[T_{a,L,A^{(k)\varepsilon}}^{L^k\varepsilon}[\exp(-S^{(k),L^k\varepsilon}(A,\phi))]]] \\ &= \tfrac{1}{2}\langle B, \Delta^{(k+1),L^{k+1}\varepsilon}B\rangle \\ &\quad - \log\Bigg[\left(\frac{a(L^{k+1}\varepsilon)^{d-2}}{2\pi}\right)^{\frac{d}{2}|T_1^{(k)}|} \int dA_k' \exp(-\tfrac{1}{2}\langle A_k', (C^{(k),L^k\varepsilon})^{-1}A_k'\rangle) \\ &\quad \cdot T_{a,L,eB^{(k+1)\varepsilon}+e'A'^{(k)\varepsilon}}^{L^k\varepsilon}\Bigg[\exp\Bigg[-\sum_{0\leqq\alpha+\beta\leqq n}\frac{1}{\alpha!\beta!}e'^\alpha\lambda'^\beta \\ &\quad \cdot\left(\frac{\partial^{\alpha+\beta}}{\partial e''^\alpha\partial\lambda''^\beta}E_k(e'',\lambda'',eB^{(k+1)\varepsilon}+e'A'^{(k)\varepsilon},\phi)\right)\Bigg|_{e''=\lambda''=0}\Bigg]\Bigg]\Bigg]. \end{aligned} \tag{3.60}$$

More exactly the formula is not well defined because the expression in the exponent is a high order polynomial in the fields ϕ, A_k, which is not necessarily positive. We can give a sense to this formula introducing the corresponding characteristic functions or adding some positive polynomial in these fields of a sufficiently high order and with a coefficient which will be put equal to 0 after the expansions. It is not interesting enough to make these remarks more precise. Analyzing carefully the successive steps we get the following perturbative formula

$$S^{(k+1),L^{k+1}\varepsilon}(B,\psi)=\sum_{0\leqq\alpha+\beta\leqq\bar{n}}\frac{1}{\alpha!\beta!}e^{\alpha}\lambda^{\beta}\left(\frac{\partial^{\alpha+\beta}}{\partial e'^{\alpha}\partial\lambda'^{\beta}}E(e',\lambda',B,\psi)\right)\Bigg|_{e'=\lambda'=0}. \quad (3.62)$$

Because we take here an expansion until the order $\bar{n}$ only, so in (3.61) we can take a whole function E_k instead of its expansion until the order $\bar{n}$. Hence we have

$$S^{(k+1),L^{k+1}\varepsilon}(B,\psi)=\sum_{0\leqq\alpha+\beta\leqq\bar{n}}\frac{1}{\alpha!\beta!}e^{\alpha}\lambda^{\beta}\left(\frac{\partial^{\alpha+\beta}}{\partial e'^{\alpha}\partial\lambda'^{\beta}}E'(e',\lambda',B,\psi)\right)\Bigg|_{e'=\lambda'=0}, \quad (3.63)$$

where E' is an expression given by the formula (3.61) only instead of the Taylor expansion of the function E_k in the exponent we have the function $E_k(e',\lambda',eB^{(k+1)\varepsilon}+e'A'^{(k)\varepsilon},\phi)$ itself. Using the formula (3.35) for it we get

$$\begin{aligned}E'(e',\lambda',B,\psi)&=\tfrac{1}{2}\langle B,\Delta^{(k+1),L^{k+1}\varepsilon}B\rangle\\&-\log\Bigg[\prod_{j=0}^{k}\left(\frac{a(L^{j+1}\varepsilon)^{d-2}}{2\pi}\right)^{\frac{d}{2}|T_1^{(j+1)}|}\int dA'_j\exp(-\tfrac{1}{2}\langle A'_j,(C^{(j),L^j\varepsilon})^{-1}A'_j\rangle)\\&\cdot T^{L^k\varepsilon}_{a,L,eB^{(k+1)\varepsilon}+e'A'^{(k)\varepsilon}}\Bigg[T^{\varepsilon}_{a_k,L^k,eB^{(k+1)\varepsilon}+e'\sum_{j=0}^{k}A'^{(j)\varepsilon}}\\&\cdot\Bigg[\exp\Bigg[-\tfrac{1}{2}\Big\langle\phi',\Big(-\Delta^{\varepsilon}_{eB^{(k+1)\varepsilon}+e'\sum_{j=0}^{k}A'^{(j)\varepsilon}}\Big)\phi'\Big\rangle\\&-\sum_{x\in T_\varepsilon}\varepsilon^d\mathscr{P}(e',\lambda',\phi'(x))-E\Bigg]\Bigg]\Bigg]\Bigg].\end{aligned} \quad (3.64)$$

Taking into account the law of composition of the renormalization transformations (2.14) and the convention (3.34) we obtain the following equality

$$E'(e',\lambda',B,\psi)=\tfrac{1}{2}\langle B,\Delta^{(k+1),L^{k+1}\varepsilon}B\rangle+E_{k+1}(e',\lambda',eB^{(k+1)\varepsilon},\psi). \quad (3.65)$$

If we substitute in the formula (3.63) the above expression for E' and if we change B,ψ into A,ϕ, then we get precisely the formula (3.35) for $S^{(k+1),L^{k+1}\varepsilon}$. Thus we have finished the inductive proof of the formula (3.63) for the k^{th} action.

Now we can finish the proof of the lower bound. We take K such that $L^K\varepsilon\leqq\varepsilon_0$, but $L^{K+1}\varepsilon>\varepsilon_0$, and then we have (3.26)–(3.32) with $k=K$.

We use the fact that the field $A^{(K)\varepsilon}$ is small and we expand the whole action with respect to this field. We use Proposition 3.1 for this expansion and we get

$$\begin{aligned}Z^{\varepsilon}\geqq\int dA\int d\phi\,\chi_K(A)\chi_K(\phi)Z_KZ_K(0)\exp[&-\tfrac{1}{2}\langle A,\Delta^{(K),L^K\varepsilon}A\rangle\\&-\tfrac{1}{2}\langle\phi,\Delta^{(K),L^K\varepsilon}(0)\phi\rangle+V^{(K)\varepsilon}(0,0;A,\phi)-E_0+0(1)|T_\varepsilon|].\end{aligned} \quad (3.66)$$

Using representation (3.57) in Proposition 3.2 and the restrictions on the fields A, ϕ we can estimate the absolute value of the interaction $V^{(K)\varepsilon}$ by $O(1)(L^K\varepsilon)^{\chi_0}|T_1^{(K)}|$. Further we can estimate

$$Z^\varepsilon \geqq \int dA \int d\phi \prod_{x\in T^{(K)}_{L^{(K)}\varepsilon}} \chi\left(\left\{|A(x)| \leqq c_1^{-1}(L^K\varepsilon)^{-\frac{d-2}{2}} p(L^K\varepsilon)\right\}\right)$$
$$\cdot \chi\left(\left\{|\phi(x)| \leqq c_1^{-1}(L^K\varepsilon)^{-\frac{d-2}{2}} p(L^K\varepsilon)\right\}\right) Z_K Z_K(0)$$
$$\cdot \exp\left[-\tfrac{1}{2}\langle A, \Delta^{(K), L^K\varepsilon} A\rangle - \tfrac{1}{2}\langle \phi, \Delta^{(K), L^K\varepsilon}(0)\phi\rangle - E_0 + O(1)|T_\varepsilon|\right]. \quad (3.67)$$

The quatratic forms in the exponential above are bounded, so the integral can be estimated from below by $\exp(-O(1)|T_\varepsilon|)$ with a constant $O(1)$ which in general depends on $L^K\varepsilon$, thus on ε_0. We get

$$Z^\varepsilon \geqq Z_K Z_K(0)\exp(-E_0 + O(1)|T_\varepsilon|). \quad (3.68)$$

It is easily seen that $Z_K Z_K(0)$ is almost equal $\exp(E_0)$, more exactly we have

$$\begin{aligned} Z_K Z_K(0)\exp(-E_0) &= [\det(I + a_K(L^K\varepsilon)^{d-2} P_K G^\varepsilon)]^{-1/2} \\ &\quad \cdot [\det(I + a_K(L^K\varepsilon)^{d-2} P_K G^\varepsilon(0))]^{-1/2} \exp(O(1)|T_\varepsilon|) \\ &\geqq \exp[-\tfrac{1}{2} a_K(L^K\varepsilon)^{d-2}(\mathrm{Tr}\, P_K G^\varepsilon + \mathrm{Tr}\, P_K G^\varepsilon(0)) + O(1)|T_\varepsilon|] \\ &= \exp(O(1)|T_\varepsilon|), \end{aligned} \quad (3.69)$$

and we obtain finally the required lower bound.

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