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Conservation Laws in the Quantum Version of N-Positional Potts Model

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Abstract. A quantum analogue of the N-positional Potts model is constructed. The system is shown to possess an infinite set of involutory conservation laws in the phase transition point.

1. Introduction

Most of exactly solvable two-dimensional spin models already known are related to the eight-vertex model, analyzed by Baxter [4]. The unique feature of this model is the existence of a complete set of conserved integrals commuting with each other [6]. The transfer matrix of the original model is at the same time the generating function of these integrals [4, 6]. These quantities can be considered as quantum Hamiltonians describing different one-dimensional systems, one of which coincides with the XYZ-model [5]. It is a natural question whether there exist any other systems possessing a hidden symmetry of the same type.

In the present paper it will be shown that this hidden symmetry is indeed present in a wide class of N-positional Potts models. However, conservation laws exist for a certain relation between the original quantum Hamiltonian constants which apparently corresponds to the phase transition point in statistical mechanics. Thus the only way to investigate the critical behaviour of the model is to compute the correlation functions at the phase transition point. To avoid terminological misunderstanding it should be noted that the generating function of conserved integrals in the quantum version of the Potts model cannot be regarded as a transfer matrix of the classical lattice statistical Potts model [7] and thus no exact relation between these models can be indicated.

We employ the representation of the Heisenberg motion equation using the $L-A$ pair which makes it possible to write down directly the generating functional of conservation laws. We illustrate our method considering the Ising chain in Sect. 2 and the anisotropic XYZ-model in Appendix A. In Sect. 3 we formulate the quantum version of the Potts model in a way convenient for further developments. In Sect. 4 the generating functional of motion integrals is written down, some of

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the first integrals are presented explicitly and the involutivity of all the integrals is proved. Appendix B is dedicated to the question of locality of the conservation laws.

2. $L-A$ Pair Equation for the Ising Model

Consider an infinite one-dimensional quantum Ising chain having the Hamiltonian

$$\mathscr{H}=\tfrac{1}{2}\sum_{k=-\infty}^{\infty}(as_k^x+bs_k^zs_{k+1}^z). \tag{2.1}$$

Here s_k^x, s_k^z and $s_k^y=is_k^xs_k^z$ form the Pauli matrix algebra at the k-th site and commute at different sites. The operators

$$s_{k+1/2}=s_k^zs_{k+1}^z \quad \text{and} \quad s_k=s_k^x \tag{2.2}$$

form the following algebra

$$s_k^2=1, \qquad s_ks_{k+1/2}=-s_{k+1/2}s_k, \qquad s_ks_l=s_ls_k(l\neq k\pm 1/2), \tag{2.3}$$

where the indices k and l are integers and half-integers. The operators s_k and $s_{k+1/2}$ are contained in Hamiltonian (2.1) on the equal foot, therefore, the spectrum of the former is a symmetric function of a and b, This corresponds to the well known Kramers-Wannier symmetry [8] in the quantum language. The point $a=b$ corresponds to the phase transition point. The equation of motion takes the form

$$-i\dot{s}_k=[\mathscr{H},s_k]=c_k(s_{k+1/2}s_k-s_ks_{k-1/2}), \tag{2.4}$$

where $c_k=a$ for half-integer and $c_k=b$ for integer k respectively.

We treat the question of conserved integrals representing the motion equations in the $L-A$ pair form:

$$-i\dot{L}_k=A_{k+1/2}L_k-L_kA_k, \tag{2.5}$$

where L_k and A_k are the local functions of the Hamiltonian density operators s_k and of some spectral parameter λ. The locality of L_k and A_k means that L_k depend on s_k at one site and A_k on s_k at several neighbouring sites. The functions L_k and A_k should be chosen in such a way that at any value of spectral parameter the $L-A$ pair Eq. (2.5) are true if, and only if, the equations of motion (2.4) are valid. Due to (2.5) the quantity

$$\mathscr{L}(\lambda)=\prod_{k=+\infty}^{-\infty}L_k(\lambda) \tag{2.6}$$

is conserved. The coefficients of the λ series expansion of (2.6) represent the motion integrals. Assuming that the Hamiltonian is the first coefficient of expansion (2.6) and that L_k depends on s_k and λ only, one can find L_k and A_k using iterations. This

method was proposed by A. M. Polyakov. Suppose

$$L_k = 1 + \sum_{n=1}^{\infty} \lambda^n L_k^{(n)},$$

$$A_k = \sum_{n=1}^{\infty} \lambda^n A_k^{(n)}, \tag{2.7}$$

$$\mathscr{L} = 1 + \sum_{n=1}^{\infty} \lambda^n \mathscr{L}^{(n)}.$$

Then $\mathscr{L}^{(1)} = \sum_k L_k^{(1)}$ and it is natural to assume that $L_k^{(1)} = c_{k+1/2} s_k$, where c_k are just the same as in (2.4).

Expanding (2.5) into the λ series one can obtain

$$-i\dot{L}_k^{(1)} = A_{k+1/2}^{(1)} - A_k^{(1)} = ab(s_{k+1/2} s_k - s_k s_{k-1/2}) \tag{2.8}$$

which yields $A_k^{(1)} = abs_k s_{k-1/2}$. The second iteration results in the following expression

$$-i\dot{L}_k^{(2)} = A_{k+1/2}^{(1)} L_k^{(1)} - L_k^{(1)} A_k^{(1)} + A_{k+1/2}^{(2)} - A_k^{(2)}. \tag{2.9}$$

Substitution of $A_k^{(1)}$ and $L_k^{(1)}$ gives

$$-i\dot{L}_k^{(2)} = abc_{k+1/2}(s_{k+1/2} - s_{k-1/2}) + A_{k+1/2}^{(2)} - A_k^{(2)}.$$

Hence, $L_k^{(2)} = 0$ and

$$A_k^{(2)} = -ab(c_k s_k + c_{k-1/2} s_{k-1/2}).$$

Further iterations do not change the structure of L_k and A_k, i.e., L_k depends on s_k and A_k depends on s_k and $s_{k-1/2}$. Most general functions of the above variables are

$$\begin{aligned} L_k &= 1 + x_k s_k \\ A_k &= u_k s_k + v_k s_{k-1/2} + w_k s_k s_{k-1/2}, \end{aligned} \tag{2.10}$$

where each of x, u, v, w takes only two values at integer and half-integer k, respectively. Inserting (2.10) into (2.5) one obtains the following equations for the coefficients:

$$\begin{aligned} &w_k = w_{k+1/2} = \frac{c_{k+1/2} x_k}{1 - x_k^2} = \frac{c_k x_{k+1/2}}{1 - x_{k+1/2}^2}, \\ &u_k = v_{k+1/2} = -x_{k+1/2} w_k, \qquad u_{k+1/2} = v_k = -x_k w_{k+1/2}. \end{aligned} \tag{2.11}$$

The easiest way to parametrize (2.11) is to introduce a parameter λ:

$$\begin{aligned} w_k(\lambda) &= w_{k+1/2}(\lambda) = ab\lambda, \\ x_k(\lambda) &= \frac{(1 + 4c_k^2 \lambda^2)^{1/2} - 1}{2c_k \lambda}, \\ u_k = v_{k+1/2} &= \frac{c_k}{2}[1 - (1 + 4c_{k+1/2}^2 \lambda^2)^{1/2}], \\ u_{k+1/2} = v_k &= \frac{c_{k+1/2}}{2}[1 - (1 + 4c_k^2 \lambda^2)^{1/2}]. \end{aligned} \tag{2.12}$$

The conservation laws obtained from (2.7) are non-local. In order to obtain local conservation laws one should take a logarithm of the generating functional $\mathscr{L}(\lambda)$ (2.6) and then expand it into a series. The theorem proved in appendix of paper [6] states that the coefficient at λ^n in the $\log \mathscr{L}(\lambda)$ expansion is the local integral $\mathscr{C}_n$ of the n-th order:

$$\log \mathscr{L}(\lambda) = \sum_{n=1}^{\infty} \mathscr{C}_n \lambda^n, \tag{2.13}$$

where

$$\mathscr{C}_n = \sum_k C_n(k).$$

Note that $C_n(k)$ is a function of operators s_k in no more than n points positioned successively and $C_n(k+1)$ is connected with $C_n(k)$ by the unit translation of all variables. The above mentioned theorem is formulated in Appendix B, where also recurrent formulas for computing $\mathscr{C}_n$ and the scheme of proof are given.

The system of integrals $\mathscr{C}_n$ is equivalent to a standard set of integrals [12] that can be obtained by the Jordan-Wigner transformation. In our notations it can be written in the following way

$$s_k = i\psi_k \chi_{k-1/2}, \qquad s_{k+1/2} = i\chi_{k+1/2}\psi_k. \tag{2.14}$$

Here ψ_k and $\chi_{k+1/2}$ (k takes integer values only) are Clifford variables obeying the following commutation relations

$$\{\psi_k, \psi_{k'}\}_+ = 2\delta_{k,k'}, \qquad \{\chi_{k+1/2}, \chi_{k'+1/2}\}_+ = 2\delta_{k,k'}, \qquad \{\chi_{k+1/2}, \psi_{k'}\}_+ = 0.$$

The equations of motion in terms of fermion variables are of the form

$$\begin{cases} \dot{\psi}_k = a\chi_{k-1/2} - b\chi_{k+1/2} \\ \dot{\chi}_{k+1/2} = b\psi_k - a\psi_{k+1}. \end{cases} \tag{2.15}$$

The following quantities are found to be the integrals of these equations

$$\begin{aligned} \mathscr{C}_n^{(+)} &= \sum_k (\psi_k \psi_{k+n} + \chi_{k+1/2}\chi_{k+n+1/2}), \\ \mathscr{C}_n^{(-)} &= \sum_k a(\chi_{k-1/2}\psi_{k+n} - \psi_k \chi_{k+n-1/2}) \\ &\quad + \sum_k b(\psi_k \chi_{k+n+1/2} - \chi_{k-1/2}\psi_{k+n}). \end{aligned} \tag{2.16}$$

These integrals can be expressed in terms of initial variables using (2.14). It should be noted that in the case of a finite chain with cyclic boundary conditions formulas (2.7) and (2.13) do not yield integrals of motion. The difficulties arise from the fact that the product of any finite number of $L_k(\lambda)$ is not of a cyclic character, unlike (2.7). Let the ordered product of $L_k(\lambda)$ be defined as a product beginning in an arbitrary point and ending in the neighbouring point. The coefficients at the λ powers in the expansion of the logarithm of the above expression can be used to obtain local integrals. For this purpose they must be corrected by introducing additional terms to provide their cyclic character. In the present paper we restrict ourselves by this observation as far as boundary conditions are concerned.

3. Models with Z_N Symmetry and the Quantum Potts Model

Consider a one-dimensional lattice model of a magnet in which there are N possible and equivalent positions of a spin at each site. The direction of spin at the k-th site can be described by the complex variable σ_k, taking the values $\exp\left(\frac{2\pi i m}{N}\right)$, $m=0, 1, \ldots, N-1$. Assuming the interaction to be local and dependent on relative spin orientations only, the potential energy may be written in the form:

$$\mathbf{U}=\sum_k V(\sigma_k \sigma_{k+1}^*), \tag{3.1}$$

where V is the real function of its argument. The wave function of the system is a function of the variables σ_k at all sites. It seems convenient to introduce diagonal unitary operators $\tilde{\sigma_k}$ acting upon $|\{\sigma_k\}\rangle$ as follows:

$$\tilde{\sigma_k}|\{\sigma_k\}\rangle=\sigma_k|\{\sigma_k\}\rangle. \tag{3.2}$$

Obviously $\tilde{\sigma_k}$ commute at different sites and

$$\tilde{\sigma_k}^N=1. \tag{3.3}$$

The operator of potential energy has the same form as (3.1) with the substitutions $\sigma_k \to \tilde{\sigma_k}$ and $\sigma_{k+1}^* \to \tilde{\sigma_{k+1}}^+$. The kinetic energy is responsible for transition processes between the states with different configurations $\{\sigma_k\}$. Since the matrix elements of the Hamiltonian between the states $\{\sigma_k\}$ and $\{\sigma_k'\}$ depend only on the relative orientations of σ_k and σ_k' at every site it seems natural to introduce local unitary shift operators $\tilde{s_k}$:

$$\tilde{s_k}|\ldots, \sigma_{k-1}, \sigma_k, \sigma_{k+1}, \ldots\rangle=|\ldots \sigma_{k-1}, \omega\sigma_k, \sigma_{k+1}, \ldots\rangle, \tag{3.4}$$

where $\omega=\exp\left(\frac{2\pi i}{N}\right)$. It can be seen from the definition (3.4) that $\tilde{s_k}$ commute with each other and with $\tilde{\sigma_l}$ at different sites and

$$\tilde{s_k}^N=1. \tag{3.5}$$

It follows from (3.2) and (3.4) that

$$\tilde{\sigma_k}\tilde{s_k}=\omega\tilde{s_k}\tilde{\sigma_k} \quad \text{and} \quad \tilde{\sigma_k}\tilde{s_k}^+=\omega^*\tilde{s_k}^+\tilde{\sigma_k}. \tag{3.6}$$

The requirement for the kinetic energy to be local leads to the following expression:

$$\mathbf{T}=\sum_k T(\tilde{s_k}), \tag{3.7}$$

where $T(x)$ is a real function of its argument. The operators $\tilde{\sigma_k}$ and $\tilde{s_k}$ have been introduced earlier in [9].

As in the case of the Ising model, it is convenient to introduce equivalent operators $\tilde{s}_{k+1/2}$ at the half-integer sites:

$$\tilde{s}_{k+1/2}=\tilde{\sigma_k}\tilde{\sigma}_{k+1}^+. \tag{3.8}$$

It follows from the definition (3.8) and the algebra (3.3), (3.6) that $s^{\sim}_{k+1/2}$ are unitary and commute with each other and with every $s^{\sim}_{l}$ except for those at the neighbouring sites. For the last case we have:

$$\begin{aligned} &s^{\sim}_{k+1/2}s^{\sim}_{k} = \omega s^{\sim}_{k}s^{\sim}_{k+1/2}, \quad s^{\sim}_{k+1/2}s^{\sim\,+}_{k} = \omega^* s^{\sim\,+}_{k}s^{\sim}_{k+1/2}, \\ &s^{\sim}_{k+1}s^{\sim}_{k+1/2} = \omega s^{\sim}_{k+1/2}s^{\sim}_{k+1}, \quad s^{\sim\,+}_{k+1}s^{\sim}_{k+1/2} = \omega^* s^{\sim}_{k+1/2}s^{\sim\,+}_{k+1}, \end{aligned} \tag{3.9}$$

and

$$s^{\sim\,N}_{k+1/2} = 1. \tag{3.10}$$

Operators $s^{\sim}_{k}$ and $s^{\sim}_{l+1/2}$ participate in algebraic relations in the same way. Having introduced the index l acquiring both integer and half-integer values one can construct a unified description of algebra (3.5), (3.9), (3.10):

$$s^{\sim}_{l}s^{\sim\,+}_{l} = s^{\sim\,+}_{l}s^{\sim}_{l} = 1, \quad s^{\sim\,N}_{l} = 1. \tag{3.11}$$

$$\begin{aligned} &s^{\sim}_{l+1/2}s^{\sim}_{l} = \omega s^{\sim}_{l}s^{\sim}_{l+1/2}, \quad s^{\sim}_{l+1/2}s^{\sim\,+}_{l} = \omega^* s^{\sim\,+}_{l}s^{\sim}_{l+1/2}, \\ &s^{\sim}_{l}s^{\sim}_{k} = s^{\sim}_{k}s^{\sim}_{l}, \quad s^{\sim}_{l}s^{\sim\,+}_{k} = s^{\sim\,+}_{k}s^{\sim}_{l} \; (|k-l| > 1/2). \end{aligned} \tag{3.12}$$

The only reason for integer and half-integer points to be non-equivalent is the Hamiltonian:

$$\mathscr{H} = \sum_{k} T(s^{\sim}_{k}) + V(s^{\sim}_{k+1/2}). \tag{3.13}$$

The functions T and V are defined by N parameters in conformity with (3.11). Either the Fourier coefficients $t_m, v_m (m = 0, 1, \ldots, N-1)$

$$T(s^{\sim}) = \sum_{m=0}^{N-1} t_m s^{\sim\,m}$$

$$V(s^{\sim}) = \sum_{m=0}^{N-1} v_m s^{\sim\,m}$$

or their own eigenvalues may be used to define these functions. The eigenvalues of $s^{\sim}$ are equal to $\omega^m (m = 0, 1, \ldots, N-1)$. Accordingly, there are N eigenvalues of $T(s^{\sim})$ and $V(s^{\sim})$, namely $T(\omega^m)$ and $V(\omega^m) (m = 0, 1, \ldots, N-1)$, which are related to the Fourier coefficients in an obvious way.

Already, for $N = 3$ the iterations analogous to those presented in Sect. 2 show that the local $L-A$ pair exists for the Hamiltonian with $T(s^{\sim}) = V(s^{\sim})$ and when the last function is chosen in a special way.

It turns out that such a Hamiltonian possesses a local $L-A$ pair in an arbitrary model with Z_N symmetry. Namely,

$$\begin{aligned} &\mathscr{H} = \sum_{l} F(\omega^{m_l} s^{\sim}_{l}) \\ &F(\omega^{m_l} s^{\sim}_{l}) \equiv H_l = \frac{1}{N} \sum_{n=0}^{N-1} (\omega^{m_l} s^{\sim}_{l})^n. \end{aligned} \tag{3.14}$$

Since $\sum_{n=0}^{N-1} \omega^{pn} = N\delta_{p,0}$, $(p = 0, 1, \ldots, N-1)$ and by virtue of the algebra (3.11), (3.12)

the density of the Hamiltonian chosen in such a way obeys the following relations:

$$H_k^2 = H_k, \tag{3.15}$$

$$H_k H_{k\pm 1/2} H_k = \frac{1}{N} H_k, \tag{3.16}$$

$$H_k H_l = H_l H_k (|k-l| > 1/2). \tag{3.17}$$

The above properties (3.15)–(3.17) are sufficient to obtain $L-A$ pair and a set of motion integrals. Owing to (3.15) H_k is a projective operator and, as is seen from (3.14) all its eigenvalues are equal to zero except for one equal to unity, which corresponds to that eigenvector for which the $s_l^{\sim}$ eigenvalue is ω^{*m_l}.

Note that, since m_l can be chosen arbitrarily at different points, the Hamiltonian (3.14) may be regarded as inhomogeneous.

Since H_k is a projective operator with the only eigenvalue equal to unity, one may naturally regard the model under consideration as a quantum version of the N-positional lattice Potts model [7]. By analogy with the case of the Ising model the present Hamiltonian can be considered as a particular version of a more general Hamiltonian of the Potts model, corresponding to the phase transition point of the latter.

4. $L-A$ Pair of the N-Positional Potts Model Phase Transition Point: Conservation Laws

Hamiltonian (3.14) involves integer and halfinteger points in an equivalent way. With this in mind and to make further notations more convenient we extend the scale by a factor of two and consequently in what follows k will acquire integer values only

$$\mathscr{H} = \sum_k H_k, \tag{4.1}$$

$$H_k^2 = H_k, \tag{4.2}$$

$$H_k H_{k\pm 1} H_k = \frac{1}{N} H_k, \tag{4.3}$$

$$H_k H_l = H_l H_k (|k-l| > 1). \tag{4.4}$$

The motion equations for H_k acquire the following form

$$-i\dot{H}_k = [\mathscr{H}, H_k] = [H_{k+1}, H_k] - [H_k, H_{k-1}]. \tag{4.5}$$

The corresponding $L-A$ pair is

$$L_k = 1 + \lambda H_k, \tag{4.6}$$

$$A_k = \frac{\lambda}{1+\lambda} H_k H_{k-1} - \lambda H_{k-1} H_k, \tag{4.7}$$

$$-i\dot{L}_k = A_{k+1} L_k - L_k A_k. \tag{4.8}$$

Its form can be found by using iterations entirely analogous to those presented in Sect. 2.

Owing to (4.8) the generating functional of conservation laws is given by the formula

$$\mathscr{L}(\lambda)=\prod_{k=+\infty}^{-\infty} L_k(\lambda). \tag{4.9}$$

The local conservation laws are obtained by the λ series expansion

$$\log \mathscr{L}(\lambda)=\sum_{n=1}^{\infty} \mathscr{C}_n \lambda^n. \tag{4.10}$$

It is rather convenient to employ the recurrent formulas for the density $\mathscr{C}_n$ (see Appendix B)

$$\mathscr{C}_1=\sum_k H_k=\mathscr{H}, \tag{4.11}$$

$$\mathscr{C}_2=\tfrac{1}{2}\sum_k [H_{k+1}, H_k]-\tfrac{1}{2}\mathscr{C}_1, \tag{4.12}$$

$$\mathscr{C}_3=\tfrac{1}{3}\sum_k ([H_{k+2}, [H_{k+1}, H_k]]+H_{k+1}H_k)-\tfrac{4}{3}\mathscr{C}_2-\tfrac{1}{3}\left(2+\frac{1}{N}\right)\mathscr{C}_1. \tag{4.13}$$

Now we formulate a *theorem of involutivity* of the motion integrals $\mathscr{C}_n$. It is sufficient to show the commutativity of the two generating functionals $\mathscr{L}(\lambda)$ and $\mathscr{L}(\mu)$ at arbitrary λ and μ.

$$[\mathscr{L}(\lambda), \mathscr{L}(\mu)]=0, \tag{4.14}$$

$$[\log \mathscr{L}(\lambda), \log \mathscr{L}(\mu)]=0 \tag{4.15}$$

then owing to (4.10) $[\mathscr{C}_n, \mathscr{C}_k]=0$ at arbitrary n and k. To prove that (4.14) is valid, we write a two-parameter $L-A$ pair equation for the nonlocal Hamiltonian $\mathscr{L}(\mu)$

$$[\mathscr{L}(\mu), L_k(\lambda)]=A_{k+1}(\lambda, \mu)L_k(\lambda)-L_k(\lambda)A_k(\lambda, \mu). \tag{4.16}$$

Due to (4.4) the expression for commutator in the left-hand side of (4.16) may be written in the following way

$$\begin{aligned}[\mathscr{L}(\mu), L_k(\lambda)]=&\left(\prod_{n=+\infty}^{k+2} L_n(\mu)\right)\{[L_{k+1}(\mu), L_k(\lambda)]L_k(\mu)L_{k-1}(\mu)\\ &+L_{k+1}(\mu)L_k(\mu)[L_{k-1}(\mu), L_k(\lambda)]\}\left(\prod_{n=k-2}^{-\infty} L_n(\mu)\right).\end{aligned} \tag{4.17}$$

Now formula (4.17) can help to guess the form of $A_k(\lambda, \mu)$:

$$A_k(\lambda, \mu)=\left(\prod_{n=+\infty}^{k+1} L_n(\mu)\right) B_k(\lambda, \mu)\left(\prod_{n=k-2}^{-\infty} L_n(\mu)\right), \tag{4.18}$$

where $B_k(\lambda, \mu)$ depends on H_k and H_{k-1}. Then, owing to (4.16)–(4.18), the following relation takes place

$$\begin{aligned}&[L_{k+1}(\mu), L_k(\lambda)]L_k(\mu)L_{k-1}(\mu)\\&\quad + L_{k+1}(\mu)L_k(\mu)[L_{k-1}(\mu), L_k(\lambda)]\\&= B_{k+1}(\lambda, \mu)L_{k-1}(\mu)L_k(\lambda) - L_k(\lambda)L_{k+1}(\mu)B_k(\lambda, \mu).\end{aligned} \tag{4.19}$$

The form of $B_k(\lambda, \mu)$ may be obtained using two-parameter iterations. The result is

$$B_k(\lambda, \mu) = \mu\lambda\left[\frac{1+\mu(1+\mu/N)}{1+\lambda(1+\mu/N)}H_kH_{k-1} - H_{k-1}H_k\right]. \tag{4.20}$$

Thus, the existence of an infinite set of local involutive motion integrals for the system under consideration has been proved. This fact allows one to hope that the spectrum, eigenvectors and correlation functions of the system can be computed exactly.

We are grateful to A. M. Polyakov under whose quidance the work has been done.

Appendix A

Here the computation of an $L-A$ pair for the anisotropic XYZ model is considered applying the method used in Sect. 2 for computing an $L-A$ pair of the Ising model. The Heisenberg chain Hamiltonian has the form

$$\mathscr{H} = \tfrac{1}{2}\sum_k \sum_{i=1}^{3} J_i \sigma_k^i \sigma_{k+1}^i, \tag{A.1}$$

where the operators $\sigma_k^i (i = 1, 2, 3)$ form algebra of Pauli matrices on the k-th site, and they commute at different sites. The equations of motion can be given as follows:

$$\dot{\sigma}_k^i = \sum_{l,m=1}^{3} \varepsilon_{ilm} J_l (\sigma_{k+1}^l + \sigma_{k-1}^l)\sigma_k^m. \tag{A.2}$$

The iterative method just similar to that presented in Sect. 2 allows one to conclude that the $L-A$ pair has the following form

$$L_k = 1 + \sum_{i=1}^{3} \alpha_i \sigma_k^i \sigma_{k+1}^i, \tag{A.3}$$

$$\begin{aligned}A_k = &\sum_{i=1}^{3} [\beta_i(\sigma_{k+1}^i + \sigma_{k-1}^i)\sigma_k^i + \gamma_i \sigma_{k+1}^i \sigma_{k-1}^i]\\&+ i \sum_{i,l,m=1}^{3} \varepsilon_{ilm} \varrho_l \sigma_{k+1}^i \sigma_k^l \sigma_{k-1}^m.\end{aligned} \tag{A.3}$$

The equation

$$-i\dot{L}_k = [\mathscr{H}, L_k] = A_{k+1}L_k - L_kA_k \tag{A.4}$$

yields a system of equations for the coefficients α_i, β_i, γ_i, ϱ_i in (A.3). The following equations are the consequence of the above system

$$\alpha_i + \alpha_k \alpha_l = \lambda J_i \tag{A.5}$$

which holds for any permutation of three noncoincident indices $i, k, l = 1, 2, 3$ and where λ is a spectral parameter. The dependence of $\alpha_i(\lambda)$ is defined by Eq. (A.5). The coefficients β_i, γ_i, ϱ_i can be expressed by Eq. (A.5) in terms of $\alpha_i(\lambda)$ and λ

$$\begin{aligned}
\beta_i &= \frac{\lambda^2 J_i}{\Delta}\left[J_k^2 + J_l^2 - \lambda^2 \frac{(J_k^2 - J_l^2)^2}{(1-\alpha_i^2)^2}\right], \\
\varrho_i &= \frac{\lambda J_k J_l}{\Delta}(\alpha_i^2 - \alpha_k^2 - \alpha_l^2 + 1), \\
\gamma_1 &= \gamma_2 = \gamma_3 = -\frac{2\lambda^2 J_1 J_2 J_3}{\Delta},
\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}
\Delta = &(\alpha_1 + \alpha_2 + \alpha_3 - 1)(\alpha_1 + \alpha_2 - \alpha_3 + 1) \\
&\cdot (\alpha_1 - \alpha_2 + \alpha_3 + 1)(-\alpha_1 + \alpha_2 + \alpha_3 + 1)
\end{aligned}$$

and (i, k, l) is an arbitrary interchange of three noncoincident indices $(1, 2, 3)$.

As is known, there exists a matrix $L - A$ pair for the XYZ model as well [5, 10]. Equation (A.4) are equivalent to (A.2), if L_k and A_k are defined in the following way

$$\begin{aligned}
L_k &= 1 + \sum_{i=1}^{3} a_i \sigma_k^i \tau^i, \\
A_k &= \sum_{i=1}^{3} [b_i(\sigma_k^i + \sigma_{k-1}^i)\tau^i + c_i \sigma_k^i \sigma_{k-1}^i] \\
&\quad + i \sum_{i,l,m=1}^{3} \varepsilon_{ilm} d_i \sigma_k^l \sigma_{k-1}^m \tau^i.
\end{aligned} \tag{A.7}$$

Here τ^i are auxiliary Pauli matrices and σ_k^i are spin variables at the k-th site as in the previous case. The coefficients a_i are related by the following conditions

$$a_i^2 - a_k^2 - a_l^2 + 1 = \mu J_i, \tag{A.8}$$

where μ is the spectral parameter. Formula (A.8) permits expressing the coefficients a_i explicitly

$$a_i^2 = 1 - \frac{\mu}{2}(J_k + J_l). \tag{A.9}$$

(i, k, l) in formulas (A.8) and (A.9) is an arbitrary interchange of three noncoincident indices $(1, 2, 3)$. The coefficients b_i, c_i and d_i with regard to (A.8) are

expressed by the formulas

$$c_i = \frac{J_i}{\Delta}[(a_k^2 - a_l^2)^2 - (a_k + a_i a_l)^2 - (a_l + a_i a_k)^2],$$

$$b_i = 2\frac{J_i}{\Delta}(a_k + a_i a_l)(a_l + a_i a_k), \tag{A.10}$$

$$d_i = -\mu\frac{J_k J_l}{\Delta}(a_i + a_k a_l).$$

$$\Delta = (a_1 + a_2 + a_3 - 1)(a_1 + a_2 - a_3 + 1)(a_1 - a_2 + a_3 + 1)(-a_1 + a_2 + a_3 + 1)$$

in formulas (A.10).

The coefficients a_i and α_i of the matrix (A.7) and scalar (A.3) $L-A$ pairs, respectively, are connected by simple transformation. It is convenient to introduce projective coordinates in order to formulate the transformation:

$$a_i = \frac{a_i'}{a_4'}, \qquad \alpha_i = \frac{\alpha_i'}{\alpha_4'}. \tag{A.11}$$

If a_i' and α_i' are related in the following way

$$\begin{pmatrix} \alpha_1' \\ \alpha_2' \\ \alpha_3' \\ \alpha_4' \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} a_1' \\ a_2' \\ a_3' \\ a_4' \end{pmatrix},$$

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \\ a_4' \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha_1' \\ \alpha_2' \\ \alpha_3' \\ \alpha_4' \end{pmatrix} \tag{A.12}$$

then the conditions (A.5) and (A.9) are equivalent. The first transformation of (A.12) (from a_i to α_i) was performed in papers [5, 6] in terms of generating functionals. The problem of conservation laws for the XYZ model was elaborated in [6]. Their explicit form of motion integrals for the XXX model was obtained in [13].

Appendix B

Here presented are both the scheme of the proof of locality of integrals $\mathscr{C}_n$ and the recurrent formulas facilitating the computation of $\log \mathscr{L}(\lambda)$. The main theorems are proved in papers [6, 11].

Suppose there is a set of totally symmetric functions $G_n^I(k_1, \ldots, k_n)$ which arguments are integer valued and label the sites of a one-dimensional lattice. The generating functional for the above functions using an arbitrary field $J(k)$ is

defined by the following equation

$$F\{J\} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k_1,\ldots,k_n} G_n^I(k_1,\ldots,k_n) J(k_1)\ldots J(k_n). \tag{B.1}$$

Theorem 1. *The generating functional*

$$Z\{J\} = \exp(F\{J\}) \tag{B.2}$$

is defined by the analogous formula

$$Z\{J\} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1,\ldots,k_n} G_n(k_1,\ldots,k_n) J(k_1)\ldots J(k_n), \tag{B.3}$$

where $G_0 = 1$ *and other functions* $G_n(k_1,\ldots,k_n)$ *are expressed through* $G_n^I(k_1,\ldots,k_n)$ *as follows*

$$G_n(k_1,\ldots,k_n) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{\substack{\text{over all possible} \\ \text{separations of } (k_1,\ldots,k_n) \\ \text{into groups } A_1,\ldots,A_m \\ \text{of the length} \\ n_1,\ldots,n_m, \text{ respectively} \\ n_1+\ldots+n_m=n}} G_{n_1}^I(A_1)\ldots G_{n_m}^I(A_m). \tag{B.4}$$

The above theorem connects the coefficients in the λ expansion of $\mathscr{L}(\lambda)$ and $\log\mathscr{L}(\lambda)$. Namely, it is sufficient to put

$$J(k) \equiv \lambda, \qquad F\{\lambda\} = \log\mathscr{L}(\lambda), \qquad Z\{\lambda\} = \mathscr{L}(\lambda). \tag{B.5}$$

In conformity with (B.1)–(B.3)

$$\mathscr{L}(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{k_1,\ldots,k_n} G_n(k_1,\ldots,k_n), \tag{B.6}$$

$$\log\mathscr{L}(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \sum_{k_1,\ldots,k_n} G_n^I(k_1,\ldots,k_n). \tag{B.7}$$

According to (4.10) and (B.7)

$$\mathscr{C}_n = \frac{1}{n!} \sum_{k_1,\ldots,k_n} G_n^I(k_1,\ldots,k_n). \tag{B.8}$$

$G_n^I(k_1,\ldots,k_n)$ are expressed through $G_n(k_1,\ldots,k_n)$ in accordance with (B.4). In the case of the N-positional Potts model in the phase transition point the calculation of $G_n(k_1,\ldots,k_n)$ is trivial, namely

$$\frac{1}{n!} \sum_{k_1,\ldots,k_n} G_n(k_1,\ldots,k_n) J(k_1)\ldots J(k_n) = \sum_{k_1>k_2>\ldots>k_n} H_{k_1}\ldots H_{k_n} J(k_1)\ldots J(k_n).$$

If

$$\mathscr{L}(\lambda) = \prod_{k=+\infty}^{-\infty} L_k(\lambda) \quad \text{and} \quad L_k(0) = 1$$

and only the neighbouring $L_k(\lambda)$ do not commute with each other which is true for all models considered in the paper, then $G_n(k_1, \ldots, k_n)$ possess the property of breaking into parts. Namely, let $(k_1, \ldots, k_n)$ break down into clusters $A_1, \ldots, A_m$ $(m \leqq n)$ which are separated by intervals strictly exceeding unity, then

$$G_n(k_1, \ldots, k_n) = G_{n_1}(A_1) \ldots G_{n_m}(A_m) \tag{B.9}$$

and $[G_{n_i}(A_i), G_{n_k}(A_k)] = 0$.

The following theorem can be formulated on the basis of the property (B.9).

Theorem 2. *Let* $G_n(k_1, \ldots, k_n)$ *and* $G_n^I(k_1, \ldots, k_n)$ *be connected by formula* (B.4) *and* (B.9) *be valid. Then the same conditions are sufficient for irreducibility of* $G_n^I(k_1, \ldots, k_n)$*:*

$$G_n^I(k_1, \ldots, k_n) = 0, \quad [G_{n_i}^I(A_i), G_{n_k}^I(A_k)] = 0. \tag{B.10}$$

The theorem can be proved by induction. It shows the locality of the integrals $\mathscr{C}_n$ and together with formulas (B.4) and (B.8) gives the prescription for their computation.

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