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A Multi-Channel Scattering Theory for Some Time Dependent Hamiltonians, Charge Transfer Problem

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Abstract. Scattering theory for time dependent Hamiltonian $H(t) = -(1/2)\,\Delta + \sum V_j(x - q_j(t))$ is discussed. The existence, asymptotic orthogonality and the asymptotic completeness of the multi-channel wave operators are obtained under the conditions that the potentials are short range: $|V_j(x)| \leqq C_j(1+|x|)^{-2-\varepsilon}$, roughly spoken; and the trajectories $q_j(t)$ are straight lines at remote past and far future, and $|q_j(t) - q_k(t)| \to \infty$ as $t \to \pm\infty (j \neq k)$.

1. Introduction

The purpose of this paper is to study the scattering theory for a class of Schrödinger equations with time dependent potentials

$$i\frac{\partial u}{\partial t}(t, x) = -\frac{1}{2}\,\Delta\, u(t, x) + \sum_{j=1}^{N} V_j(x - q_j(t))u(t, x), \tag{1.1}$$

where $q_j(t) \in \mathbb{R}^n (n \geqq 3)$ are the functions of $t \in \mathbb{R}^1$ which are straight lines at remote past and far future.

Suppose that N-centres of forces are traveling along the given trajectories $q_j(t)$ $(j = 1, 2, \ldots, N)$ each of which acts on a quantum mechanical particle of mass 1 through the potential $V_j(x)$, then the Schrödinger equation for the particle is written as (1.1). If $|q_j(t) - q_k(t)| \to \infty$ as $|t| \to \infty$ sufficiently rapidly in conjunction with the rate of decay of the potentials, one would naturally expect that the behaviour of the particle in far future or remote past are classified into $(N+1)$-ways: (1) The particle behaves like a free particle; (2) the particle travels with one of the centres $q_j(t)$ forming a bound state around the centre $(j = 1, 2, \ldots, N)$. We shall prove in this paper that this is actually what is going on with the equation (1.1) under a suitable condition. In physics literature these centres of forces are usually supposed to be atoms and ions, and the particle to be the electron. In such case the scattering theory

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for (1.1) is nothing but to study how the electron is transfered from some atom or ion to another. This is the reason why the problem is named "charge transfer problem".

Assumption I. (1) For any $j = 1, 2, \ldots, N$, $q_j(t)$ is a continuously differentiable function from $\mathbb{R}^1$ to $\mathbb{R}^n$.
(2) There exist vectors $v_{j,\pm}, a_{j,\pm} \in \mathbb{R}^n$ such that for $\pm t \geqq t_0$,

$$q_j(t) = q_{j,\pm}(t) \equiv tv_{j,\pm} + a_{j,\pm}, \; j = 1, 2, \ldots, N. \tag{1.2}$$

(3) $v_{j,\pm} \neq v_{k,\pm}$ if $j \neq k$.

Assumption II. For any $j = 1, 2, \ldots, N$, $V_j(x)$ is a real valued function on $\mathbb{R}^n$ such that there exist functions $W_{j,1} \in W^{1,s}(\mathbb{R}^n)$ for some $n/2 < s < n$, $W_{j,2} \in W^{1,\infty}(\mathbb{R}^n)$ and a constant $1 < \delta < 3/2$ such that

$$V_j(x) = (1 + |x|^2)^{-\delta}(W_{j,1}(x) + W_{j,2}(x)). \tag{1.3}$$

Here $W^{k,q}(\mathbb{R}^n)$ ($k \geqq 0$ is an integer, $1 \leqq q \leqq \infty$) is the Sobolev space (see Yosida [18], p. 55 for the definition).

Remark 1.1. By Sobolev's embedding theorem (Stein [15], p. 124), $W_{j,1} \in L^p(\mathbb{R}^n)$, $1/p = 1/s - 1/n$. Note $p > n$. Hence by Hölder's inequality $V_j \in L^q(\mathbb{R}^n)$ for any $n/2\delta < q \leqq p$; $A_j(x) = (1 + |x|^2)^{\delta/2} V_j(x) \in L^q(\mathbb{R}^n)$ for any $n/\delta < q \leqq p$. In what follows p is always defined by $1/p = 1/s - 1/n$.

Before stating the last assumption, we state here several preliminary results of Assumption (I) and (II) which can be readily obtained by using the well-known theorems.

(A) Let H_0 be the unique selfadjoint extention of $-\Delta/2|_{C_0^\infty(\mathbb{R}^n)}$ on the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^n)$. By Remark 1.1, the multiplication operator $V_j(x)$ is H_0-compact (Reed–Simon [13], p. 369). Hence for any $t \in \mathbb{R}^1$ and $j = 1, 2, \ldots, N$, $H(t) = H_0 + \sum V_j(x - q_j(t))$, $H_\pm(t) = H_0 + \sum V_j(x - q_{j,\pm}(t))$, $H_j(t) = H_0 + V_j(x - q_j(t))$, $H_{j,\pm}(t) = H_0 + V_j(x - q_{j,\pm}(t))$ and $H_j = H_0 + V_j(x)$ are selfadjoint on $\mathfrak{H}$ with the common domain $W^{2,2}(\mathbb{R}^n) = H^2(\mathbb{R}^n)$.

(B) Since $\partial V_j/\partial x_k \in L^q(\mathbb{R}^n)$ for any $n/2\delta < q \leqq s$,

$$\left\| \frac{d}{dt}[(H_0 + 1)^{-1/2} V_j(x - q_j(t))(H_0 + 1)^{-1/2}] \right\| \leqq C(t) < \infty \tag{1.4}$$

(Reed–Simon [11], Theorem X.19 and Theorem X.20). Therefore by Simon [14], Theorem II.27, $-iH(t)$ generates a unique propagator $U(t, s)$ $(-\infty < t, s < \infty)$:
i) $U(t, s)$ is a unitary operator on $\mathfrak{H}$ and is strongly continuous in (t, s);
ii) $U(t, s)U(s, r) = U(t, r)$, $-\infty < r, s, t < \infty$;
iii) for $f \in D(H_0^{1/2})$, $U(t, s)f \in D(H_0^{1/2})$ and

$$(\partial/\partial t)U(t, s)f = -iH(t)U(t, s)f, \tag{1.5}$$

where the derivative in the L.H.S. of (1.5) is understood as the strong derivative in the space $H^{-1}(\mathbb{R}^n)$ = the dual space of $W^{1,2}(\mathbb{R}^n)$.

The same statements are true for $H_\pm(t)$ and $H_{j,\pm}(t)$ and we write the

corresponding unitary propagators as $U_{\pm}(t,s)$ and $U_{j,\pm}(t,s)$, respectively ($j=1,2,\ldots,N$).
(3) Let us set as $A_j(x)=(1+|x|^2)^{\delta/2}V_j(x)B_j(x)=(1+|x|^2)^{-\delta/2}$ and write as A_j and B_j the corresponding multiplication operators. $\mathbb{R}^1_+=[0,\infty)$. For Banach space $\mathfrak{X}$, $\mathbf{B}(\mathfrak{X})$ is the Banach algebra of all bounded operators on $\mathfrak{X}$. For a closable operator T, $[T]$ is its closure. We write as $r_0(z)=(H_0-z)^{-1}$, $z\notin\mathbb{R}^1_+$.
(i) (Ginibre–Moulin [4], Prop. 3.1.) Let $Q_j(z)=[A_j(H_0-z)^{-1}B_j]$ for $z\in\mathbb{C}^1\backslash\mathbb{R}^1_+$. Then $Q_j(z)$ is a $\mathbf{B}(\mathfrak{H})$-valued analytic function there and can be extended to the closed cut plane (the closure of $\mathbb{C}^1\backslash\mathbb{R}^1_+$ where upper and lower boundaries are distinguished) as a $\mathbf{B}(\mathfrak{H})$-valued Hölder continuous function. $\|Q_j(z)\|\to 0$ as $|z|\to\infty$. We write its boundary values on $\mathbb{R}^1_+$ as $Q_j(\lambda\pm i0)=\lim_{\varepsilon\downarrow 0}Q_j(\lambda\pm i\varepsilon)$.
(ii) (Konno–Kuroda [9].) For any $j=1,2,\ldots,N$, H_j has at most finite number of negative eigenvalues of finite multiplicity. We write the eigenvalues and the corresponding eigenfunctions of H_j as $\mu_{j,1},\ldots,\mu_{j,m_j}$ and $\phi_{j,1},\ldots,\phi_{j,m_j}$ ($\phi_{j,k}$ is normalized). Clearly $H_{j,\pm}(t)$ has the same eigenvalues and the eigenfunctions $\phi_{j,\pm,t,k}=\phi_{j,k}(x-q_{j,\pm}(t))$, $k=1,\ldots,m_j$. We write the projection onto the closed subspace spanned by $\phi_{j,\pm,t,k}$'s as $P_{j,\pm}(t)$.
(iii) (Agmon [1], Lemma 4.2.) For any multi-index $|\alpha|\leqq 2$ and any $\rho\geqq 0$,

$$(1+|x|^2)^{\rho}(\partial/\partial x)^{\alpha}\phi_{j,k}\in L^2(\mathbb{R}^n),\qquad j=1,2,\ldots,N,\quad k=1,2,\ldots,m_j. \tag{1.6}$$

Assumption III. For any $j=1,2,\ldots,N$, $I+Q_j(\lambda\pm i0)$ has its inverse in $\mathbf{B}(\mathfrak{H})$ for $\lambda\geqq 0$.

Remark. By this assumption we assume that H_j(or $H_{j,\pm}(t)$) has no non-negative eigenvalues or resonances.

Now we can state our main theorem in this paper.

Theorem. *Let Assumptions (I), (II) and (III) be satisfied. Then for any $s\in\mathbb{R}^1$, the following statements hold.*
(1) (*Existence of the wave operators.*) *The following limits exist:*

$$\operatorname*{s-lim}_{t\to\pm\infty} U(t,s)^{-1}\exp(-i(t-s)H_0)=W_{0,\pm}(s); \tag{1.7}$$

$$\operatorname*{s-lim}_{t\to\pm\infty} U(t,s)^{-1}U_{j,\pm}(t,s)\exp(ix\cdot v_{j,\pm})P_{j,\pm}(s)=W_{j,\pm}(s). \tag{1.8}$$

(2) (*Asymptotic orthogonality.*) *The ranges $R(W_{j,\pm}(s))$, $j=0,1,\ldots,N$, are orthogonal each other.*
(3) (*Asymptotic completeness.*)

$$\bigoplus_{j=0}^{N} R(W_{j,\pm}(s))=\mathfrak{H}. \tag{1.9}$$

The rest of this paper is devoted to the proof of this theorem. We sketch here the outline of the proof with somewhat crude terminology, displaying the plan of this paper. In Sect. 2, we shall prove the existence and the asymptotic orthogonality of the wave operators by standard methods (Reed–Simon [12], Sect. XI.3, XI.5). We shall prove the asymptotic completeness in Sect. 3 which is devided into five subsections. To prove the completeness we must make a detour. According to Howland [5], we shall introduce bigger Hilbert spaces $\mathfrak{K}_1 = L^2(\mathbb{R}_t^1, L^2(\mathbb{R}_x^n)) \oplus L^2(\mathbb{R}_t^1) \oplus \cdots \oplus L^2(\mathbb{R}_t^1)$ and $\mathfrak{K}_2 = L^2(\mathbb{R}_t^1, L^2(\mathbb{R}_x^n))$ of square integrable functions of time-space variables (t, x): $\mathfrak{K}_1$ is the "channel Hilbert space" and $\mathfrak{K}_2$ is the "basic space". The channel Hamiltonian $K_1 = (-i\partial/\partial t + H_0) \oplus \sum_{j,k} \oplus (-i\partial/\partial t + \mu_{j,k})$ is considered on $\mathfrak{K}_1$ and the Hamiltonain $K_2 = -i\partial/\partial t + H(t)$ is considered on $\mathfrak{K}_2$. Then we shall study the two space scattering between K_1 and K_2 via the identification operator $J: \mathfrak{K}_1 \to \mathfrak{K}_2$, J is defined by (3.2). It will be proved in Subsect. 3.1 that the wave operators $\mathscr{W}_\pm = s\text{-}\lim_{\tau \to \pm\infty} \exp(i\tau K_2) J \exp(-i\tau K_1)$ exist and are the isometries; *the completeness of* $\mathscr{W}_\pm$ *implies that of the original wave operators*. Thus by eliminating the explicit time dependence of the Hamiltonians by this procedure, the problem is reduced to the completeness problem of the wave operator $\mathscr{W}_\pm$ for the time-independent Hamiltonians. Here is an important observation: *If we replace* $-i\partial/\partial t$ *by the kinetic energy* $-\Delta_y/2$ *of certain particle and* $q_j(t)$*'s by* y *in* K_1 *and* K_2, *then* K_1 *and* K_2 *have the same form as the Hamiltonians appearing in three body scattering theory* (see Faddeev [3], Ginibre–Moulin [4], Howland [6], Kato [8] and Yajima [17]). Being suggested by this observation, we shall prove the completeness of $\mathscr{W}_\pm$ by using the methods of three body problem. In Subsect. 3.2, we record the abstract theorem due to Kato [8] by which the completeness will be finally proved. In Subsect. 3.3, an algebraic procedure of the construction of the substitute of the "Faddeev matrix" will be carried out in a way similar to that of Howland [5] in three body case. Various estimates of the operators necessary to apply Kato's theorem will be done in Subsect. 3.4. The proof of the completeness of the original wave operators will be completed in Subsect. 3.5.

The following notation and conventions are used throughout the paper. For $1 \leqq q \leqq \infty$, $L^q(\mathbb{R}^n)$ is the Banach space of all q-summable functions on $\mathbb{R}^n$ with natural norm. For non-negative integer k, $W^{k,q}(\mathbb{R}^n)$ is the Sobolev space, $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$. For $\gamma \in \mathbb{R}^1$, $L^2_\gamma(\mathbb{R}^n)$ is the weighted L^2-space:

$$L^2_\gamma(\mathbb{R}^n) = \{f \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : \|(1+x^2)^{\gamma/2} f\|_{L^2} = \|f\|_{L^2_\gamma} < \infty\}.$$

The norm of $L^2(\mathbb{R}^n)$ is usually written as $\| \quad \|$ regardless of the dimension of the space $\mathbb{R}^n$; the norms of other spaces are denoted as $\| \quad \|_{L^q}$, $\| \quad \|_{W^{k,q}}$ and etc.

For multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N}$, $(\partial/\partial x)^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$; $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. For multi-induces α and β, $\alpha_{(\leqq)}\beta$ means $\alpha_{j(\leqq)}\beta_j$ for all $j = 1, \ldots, n$; if $\alpha \leqq \beta$, $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \ldots \binom{\alpha_n}{\beta_n}$, where for $a, b \in \mathbb{N}$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. $\mathbb{C}_\pm = \{z \in \mathbb{C} : \operatorname{Im} z \gtrless 0\}$ and for $I \subset \mathbb{R}^1$, $\mathbb{C}_\pm(I) = \{z \in \mathbb{C}_\pm : \operatorname{Re} z \in I\}$. $\mathscr{F}_x$ (or $\mathscr{F}_t$) is the Fourier transform with respect to the variable x (or t). We write $\mathscr{F}f = \hat{f}$, regardless of the variable.

For Hilbert spaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_m$, $\mathfrak{H}_1 \oplus \cdots \oplus \mathfrak{H}_m$ and $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_m$ are their direct product and tensor product. If there exists a linear topological space $\mathfrak{L}$ such that $\mathfrak{H}_j \subset \mathfrak{L}$ for any $j = 1, \ldots, m$, $\mathfrak{H}_1 + \cdots + \mathfrak{H}_m$ is the sum space of $\mathfrak{H}_j$'s. If $A_1, \ldots, A_m$

are closed operators on $\mathfrak{H}_1, \ldots, \mathfrak{H}_m$, respectively, $A_1 \oplus \cdots \oplus A_m$ and $A_1 \otimes \cdots \otimes A_m$ are their direct product and tensor product. $A_1 \otimes I$ and $I \otimes A_2$ are often written as A_1 and A_2 simply. Identity operator is often written as 1. If A is a one-one closed operator from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, the range $R(A)$ of A is considered as a Hilbert space with the norm $\|Au\|_{R(A)} = (\|Au\|^2 + \|u\|^2)^{1/2}$. For a family of Hilbert spaces $\{\mathfrak{H}(t): -\infty < t < \infty\}$, $\int^{\oplus} \mathfrak{H}(t)\,dt$ is the direct integral of $\{\mathfrak{H}(t)\}$. For a closable operator A, $[A]$ is its closure. If $[A]$ is a bounded operator we often use the notation as if the operator A itself is a bounded one. For Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$, $\mathbf{B}(\mathfrak{X}, \mathfrak{Y})$ is the space of all bounded operators from $\mathfrak{X}$ to $\mathfrak{Y}$, $\mathbf{B}_\infty(\mathfrak{X}, \mathfrak{Y})$ the compact operators from $\mathfrak{X}$ to $\mathfrak{Y}$, $\mathbf{B}(\mathfrak{X}) = \mathbf{B}(\mathfrak{X}, \mathfrak{X})$. The symbol $\oplus$ is also used to denote the sum of orthogonal elements in a Hilbert space.

If $m(x)$ is a function, the same symbol $m(x)$ is also used to denote the multiplication operator by the function. The integral without referring to the region of integration is understood to be taken over the whole region of the variable.

2. Existence and Asymptotic Completeness

Here we shall prove the first two statements of the theorem. We start with the following lemma.

Lemma 2.1. *Let Assumptions I and II be satisfied. Then for any $s \in \mathbb{R}^1$ the following statements hold.*

(1) *The following limits exist:*

$$\operatorname*{s-lim}_{t \to \pm\infty} U(t, s)^{-1} U_\pm(t, s) = \Gamma_\pm(s);$$

$$\operatorname*{s-lim}_{t \to \pm\infty} U_\pm(t, s)^{-1} U(t, s) = \Omega_\pm(s).$$

(2) *$\Gamma_\pm(s)$ and $\Omega_\pm(s)$ are unitary operators on $\mathfrak{H}$ and $\Gamma_\pm(s) = \Omega_\pm(s)^{-1}$.*

Proof. Since $H(t) = H_\pm(t)$ for $\pm t \geqq t_0$, $U(t_2, t_1) = U_\pm(t_2, t_1)$ if $\pm t_1 \geqq t_0$ and $\pm t_2 \geqq t_0$. Therefore if $\pm t \geqq t_0$,

$$U(t, s)^{-1} U_\pm(t, s) = U(t_0, s)^{-1} U_\pm(t_0, s);$$

$$U_\pm(t, s)^{-1} U(t, s) = U_\pm(t_0, s)^{-1} U(t_0, s),$$

by Sect. 1, (B), ii). Thus (1) holds trivially. (2) is an immediate consequence of (1). (Q.E.D.)

By Lemma 2.1 and the chain rule for the wave operators (Reed–Simon [12], p. 18), it suffices to prove the theorem under the condition that $q_j(t) = q_{j,+}(t)$ for "+" case ($t \to \infty$), and $q_j(t) = q_{j,-}(t)$ for "−" case ($t \to -\infty$). Since the following argument for "+" case equally applies to "−" case, we shall treat the "+" case only. Thus we assume hereafter that $H(t) = H_+(t)$, $H_j(t) = H_{j,+}(t)$ and we write $v_{j,+}$, etc. as v_j, etc., omitting the suffix "+". Since the cases $a_j \neq 0$ can be treated similarly, we assume $a_j = 0$ $(j = 1, 2, \ldots, N)$ hereafter.

Lemma 2.2. *Let* F, G *be the multiplication operators by* $f, g \in L^q(\mathbb{R}^n)(1 \leqq q \leqq \infty)$ *and* $u \in \mathfrak{H}$. *Then*

(1) $\| F e^{-itH_0} G u \| \leqq (2\pi|t|)^{-n/q} \| f \|_{L^q} \| g \|_{L^q} \| u \|$;

(2) $\lim_{t \to \pm\infty} \| (e^{-itH_0} u)(x) - |t|^{-n/2} e^{i(x^2/2t - n\pi/4)} \hat{u}(x/t) \| = 0.$

Statement (1) is proved by Kato [7], p. 277 and statement (2) is Theorem IX. 31 of Reed–Simon [11].

Lemma 2.3. *For any* $t, s \in \mathbb{R}^1$, $j = 1, 2, \ldots, N$ *and* $k = 1, 2, \ldots, m_j$,

$$(U_j(t, s)(\exp(ix \cdot v_j)\phi_{j,s,k}))(x) \tag{2.1}$$
$$= \exp(ix \cdot v_j - i(t-s)(v_j^2/2 + \mu_{j,k}))\phi_{j,s,k}(x - (t-s)v_j).$$

Proof. If $t = s$, (2.1) obviously holds. By using the equation $H_j(t)\phi_{j,s,k}(x - (t-s)v_j) = \mu_{j,k}\phi_{j,s,k}(x - (t-s)v_j)$, we get by direct calculations that for any $s \in \mathbb{R}^1$

$$(-i\partial/\partial t + H_j(t))(\exp(ix \cdot v_j - i(t-s)(v_j^2/2 + \mu_{j,k}))\phi_{j,s,k}(x - (t-s)v_j)) = 0.$$

Then the uniqueness of the propagator (Sect. 1, (B)) shows that (2.1) holds. (Q.E.D.)

Proof of Statement (1). We prove the case $s = 0$ only. The other cases can be proved similarly. We use the Cook's method (Reed–Simon [12], Theorem XI.4).

(i) We first prove the existence of the limit (1.7). Let $f, g \in C_0^\infty(\mathbb{R}^n)$. Then $(U(t,0)^{-1}\exp(-itH_0)f, g) = (\exp(-itH_0)f, U(t,0)g)$ is continuously differentiable with respect to t, since $\exp(-itH_0)f \in D(H_0)$ and $U(t,0)g$ is continuously differentiable in $H^{-1}(\mathbb{R}^n)$ by Sect. 1, (B), iii). Hence

$$(U(t_2,0)^{-1}\exp(it_2H_0)f, g) - (U(t_1,0)^{-1}\exp(-it_1H_0)f, g)$$
$$= \int_{t_1}^{t_2} \frac{d}{d\sigma}(\exp(-i\sigma H_0)f, U(\sigma,0)g)\,d\sigma$$
$$= \sum_{j=1}^{N} i \int_{t_1}^{t_2} (U(\sigma,0)^{-1} V_j(x - \sigma v_j)\exp(-i\sigma H_0)f, g)\,d\sigma.$$

By Schwartz's inequality we have

$$|(U(t_2,0)^{-1}\exp(-it_2H_0)f - U(t_1,0)^{-1}\exp(-it_1H_0)f, g)|$$
$$\leqq \sum_{j=1}^{N} \int_{t_1}^{t_2} \| V_j(x - \sigma v_j)\exp(-i\sigma H_0)f \|\,d\sigma \| g \|,$$

which obviously implies

$$\| U(t_2,0)^{-1}\exp(-it_2H_0)f - U(t_1,0)^{-1}\exp(-it_1H_0)f \| \tag{2.2}$$
$$\leqq \sum_{j=1}^{N} \int_{t_1}^{t_2} \| V_j(x - \sigma v_j)\exp(-i\sigma H_0)f \|\,d\sigma.$$

Now take q as $n/2\delta \leqq q < n/2$. Then by Hölder's inequality $\| V_j(x - \sigma v_j)$

$\exp(-i\sigma H_0)f\| \leqq \|V_j\|_q \|\exp(-i\sigma H_0)f\|_r$, $1/r+1/q=1/2$. Applying Kato's estimate (Kato [7]), we have $\|\exp(-i\sigma H_0)f\|_r \leqq (2\pi|\sigma|)^{-n/q}\|f\|_{r'}$, $1/r+1/r'=1$. Thus we have $\|V_j(x-\sigma v_j)\exp(-i\sigma H_0)f\| \leqq \|V_j\|_q\|f\|_{r'}(2\pi|\sigma|)^{-n/q}$. Since $n/2\delta \leqq q<n/2$, the integrand in the R.H.S. of (2.2) is integrable on $(1,\infty)$. Therefore $U(t,0)^{-1}\exp(-itH_0)f$ is convergent as $t\to\infty$. Since $U(t,0)^{-1}\exp(-itH_0)$ is unitary for any $0\leqq t<\infty$, this holds for any $f\in\mathfrak{H}$.

(ii) Next we prove the existence of (1.8). Obviously it suffices to prove that $U(t,0)^{-1}U_j(t,0)(\exp(ix\cdot v_j)\phi_{j,0,k})$ is convergent as $t\to\infty$. We omit the suffix 0 in $\phi_{j,0,k}$ hereafter. By a similar argument as to derive (2.2), we have

$$\begin{aligned}\|U(t_2,0)^{-1}U_j(t_2,0)&(\exp(ix\cdot v_j)\phi_{j,k})\\ &-U(t_1,0)^{-1}U_j(t_1,0)(\exp(ix\cdot v_j)\phi_{j,k})\| \\ \leqq \sum_{l\neq j}\int_{t_1}^{t_2}\|V_l(x-&\sigma v_l)U_j(\sigma,0)(\exp(ix\cdot v_j)\phi_{j,k})\|\,d\sigma.\end{aligned} \tag{2.3}$$

By (2.1), $\|V_l(x-\sigma v_l)U_j(\sigma,0)(\exp(ix\cdot v_j)\phi_{j,k})\|=\|V_l(x-\sigma v_l)\phi_{j,k}(x-\sigma v_j)\|=\|V_l(x)\phi_{j,k}(x-\sigma(v_j-v_l))\|$. We write $v_j-v_l=v\neq 0$. Then by Hölder's inequality $\|V_l(x)\phi_{j,k}(x-\sigma v)\|\leqq\|V_l(x)(1+|x|^2)^{\delta/2}\|_{L_n}\|(1+|x|^2)^{-\delta/2}\times(1+|x-\sigma v|^2)^{-\delta/2}\|_{L^\infty}\|(1+|x-\sigma v|^2)^{\delta/2}\phi_{j,k}(x-\sigma v)\|_{L^r}$, $1/r+1/n=1/2$. By Remark 1.1 $\|V_l(x)(1+|x|^2)^{\delta/2}\|_{L^n}<\infty$; by (1.6) and Sobolev's embedding theorem $\|(1+|x|^2)^{\delta/2}\phi_{j,k}(x)\|_{L^r}<\infty$. On the other hand we have by elementary computation that $\|(1+|x|^2)^{-\delta/2}(1+|x-\sigma v|^2)^{-\delta/2}\|_{L^\infty}\leqq 2^{-\delta/2}(1+|\sigma v|^2)^{-\delta/2}$. Since $\delta>1$, the R.H.S. of (2.3) is integrable on $(1,\infty)$. Hence $U(t,0)^{-1}U_j(t,0)(\exp(ix\cdot v_j)\phi_{j,k})$ is convergent as $t\to\infty$. (Q.E.D.)

Proof of Asymptotic Orthogonality. Again we prove the case $s=0$ only. Other cases can be proved similarly. For $j,k=0,1,2,\ldots,N$,

$$(W_j(0)f,W_k(0)g)=\lim_{t\to\infty}(U_j(t,0)e^{ix\cdot v_j}P_j(0)f,\ U_k(t,0)e^{ix\cdot v_k}P_k(0)g),$$

where $v_0=0$, $P_0(0)=I$ and $U_0(t,s)=\exp(-i(t-s)H_0)$. Therefore it suffices to prove

$$\lim_{t\to\infty}(U_0(t,0)f,\ U_j(t,0)(\exp(ix\cdot v_j)\phi_{j,k}))=0; \tag{2.4}$$

$$\lim_{t\to\infty}(U_j(t,0)(\exp(ix\cdot v_j)\phi_{j,k}),\ U_l(t,0)(\exp(ix\cdot v_l)\phi_{l,m}))=0, \tag{2.5}$$

for $j,l=1,2,\ldots,N$, $k=1,2,\ldots,m_j$, $m=1,2,\ldots,m_m$, $j\neq l$ and $f\in C_0^\infty(\mathbb{R}^n)$. Since

$$\begin{aligned}&|(U_j(t,0)(\exp(ix\cdot v_j)\phi_{j,k}),\ U_l(t,0)(\exp(ix\cdot v_l)\phi_{l,m}))|\\ &\quad\leqq\int_{\mathbb{R}^n}|\phi_{j,k}(x-v_jt)\phi_{l,m}(x-v_lt)|\,dx,\end{aligned}$$

(2.5) obviously holds. By Lemma 2.2, (2) and (1.6),

$$\begin{aligned}&\limsup_{t\to\infty}|(\exp(-itH_0)f,\ U_j(t,0)(\exp(ix\cdot v_j)\phi_{j,k}))|\\ &\quad\leqq\limsup_{t\to\infty}\int_{\mathbb{R}^n}t^{-n/2}|\hat f(x/t)\phi_{j,k}(x-v_jt)|\,dx\end{aligned}$$

$$\leq \limsup_{t\to\infty} t^{-n/2} \|\hat{f}\|_{L^\infty} \|(1+|x|^2)^{(n+1)/2}\phi_{j,k}\| \|(1+|x|^2)^{-(n+1)/2}\|$$
$$= 0.$$

This proves (2.4) (Q.E.D.)

3. Asymptotic Completeness

This section is devoted to the proof of the asymptotic completeness of the wave operators, the third statement of the theorem.

3.1. Reduction to the Stationary Problem

According to Howland [5], we shall prove the asymptotic completeness, reducing the problem to the stationary one. Let us first introduce two accessory Hilbert spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$ as follows.

$$\begin{cases} \mathfrak{K}_1 = (L^2(\mathbb{R}^1)\otimes L^2(\mathbb{R}^n)) \oplus L^2(\mathbb{R}^1) \oplus L^2(\mathbb{R}^1) \oplus \ldots \oplus L^2(\mathbb{R}^1), \sum m_j\text{-copies} \\ \mathfrak{K}_2 = L^2(\mathbb{R}^1)\otimes L^2(\mathbb{R}^n), \end{cases} \tag{3.1}$$

We write the generic element of $\mathfrak{K}_1$ as $\tilde{u} = {}^\dagger(u, \sigma_{1,1}, \ldots, \sigma_{N,m_N})$, where † stands for the transpose. We define the identification operator J from $\mathfrak{K}_1$ to $\mathfrak{K}_2$ as

$$J\tilde{u} = u + \sum_{j=1}^{N} T_{v_j}\left(\sum_{k=1}^{m_j} \sigma_{j,k}\otimes\phi_{j,k}\right), \tag{3.2}$$

where for any vector $v\in\mathbb{R}^n$ the operator T_v is defined as

$$(T_v f)(t,x) = \exp(i(x\cdot v - v^2 t/2)) f(t, x - vt)$$

for $f\in L^2(\mathbb{R}^1)\otimes L^2(\mathbb{R}^n) = L^2(\mathbb{R}^{n+1})$. Obviously $J\in\mathbf{B}(\mathfrak{K}_1, \mathfrak{K}_2)$. Now we define one parameter families of operators $\mathscr{U}_1(\tau)$ and $\mathscr{U}_2(\tau)$ $(-\infty<\tau<\infty)$ on the spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively as follows:

$$\begin{aligned}(\mathscr{U}_1(\tau)\tilde{u})(t) = (&\exp(-i\tau H_0)u(t-\tau), \exp(-i\tau\mu_{1,1})\sigma_{1,1}(t-\tau), \\ &\ldots, \exp(-i\tau\mu_{N,m_N})\sigma_{N,m_N}(t-\tau));\end{aligned} \tag{3.3}$$

$$(\mathscr{U}_2(\tau)u)(t) = U(t, t-\tau)u(t-\tau). \tag{3.4}$$

$\mathscr{U}_1(\tau)$ is obviously a strongly continuous unitary group on $\mathfrak{K}_1$ and so is $\mathscr{U}_2(\tau)$ on $\mathfrak{K}_2$ since $U(t,s)$ is strongly continuous in both variables t and s and is unitary. Hence by Stone's theorem there exist selfadjoint operators K_1 on $\mathfrak{K}_1$ and K_2 on $\mathfrak{K}_2$ such that

$$\mathscr{U}_1(\tau) = \exp(-i\tau K_1), \quad \mathscr{U}_2(\tau) = \exp(-i\tau K_2). \tag{3.5}$$

By (3.3) we readily see that

$$K_1 = L_0 \oplus \sum_{j=1}^{N} \oplus \left(\sum_{k=1}^{m_j} \oplus L_{j,k}\right),$$
$$L_0 = -i\partial/\partial t + H_0; \quad L_{j,k} = -i\partial/\partial t + \mu_{j,k}. \tag{3.6}$$

The following lemma plays an import role.

Lemma 3.1. *Let Assumption (I) and (II) be satisfied. Then the following statements hold.*

(1) *For any* $\tilde{u} \in \mathfrak{K}_1$,

$$\lim_{\tau \to \infty} \| J \exp(-i\tau K_1)\tilde{u} \| = \| \tilde{u} \|.$$

(2) *The following limit exists*:

$$s\text{-}\lim_{t \to \infty} \exp(i\tau K_2)\, J \exp(i\tau K_1) = \mathscr{W}.$$

(3) *The operator* $\mathscr{W}$ *is isometry from* $\mathfrak{K}_1$ *to* $\mathfrak{K}_2$.

(4) *If the range* $R(\mathscr{W}) = \mathfrak{K}_2$, *the statement (3) of the theorem holds.*

Proof. By definition

$$\begin{aligned} &J \exp(-i\tau K_1)\tilde{u}(t) \\ &= e^{-i\tau H_0} u(t-\tau) + \sum_{j=1}^{N} T_{v_j}\left(\sum_{k=1}^{m_j} e^{-i\tau\mu_{j,k}} \sigma_{j,k}(\cdot - \tau) \otimes \phi_{j,k} \right)(t). \end{aligned} \tag{3.7}$$

Each term under the $\sum$-sign in the R.H.S. of (3.7) can be written by (2.1) as

$$\begin{aligned} &e^{i(x\cdot v_j - tv_j^2/2)} \phi_{j,k}(x - v_j t) e^{-i\tau\mu_{j,k}} \sigma_{j,k}(t-\tau) \\ &= e^{-i(\tau - t)\mu_{j,k}} \sigma_{j,k}(t-\tau) \otimes (U_j(t,0) e^{ix\cdot v_j} \phi_{j,k}). \end{aligned} \tag{3.8}$$

We prove (1) first. Since T_v is unitary, it suffices to prove that for $j, l = 1, 2, \ldots, N$, $k = 1, 2, \ldots, m_j$, $m = 1, 2, \ldots, m_l$ such that $(j, k) \neq (l, m)$,

$$\lim_{\tau \to \infty} (e^{-itH_0} u(t-\tau), \phi_{j,k}(x - v_j t) e^{-i(\tau\mu_{j,k} - x.v_j + tv_j^2/2)} \sigma_{j,k}(t-\tau))_{\mathfrak{K}_2} = 0; \tag{3.9}$$

$$\begin{aligned} &\lim_{\tau \to \infty} (\phi_{j,k}(x - v_j t) e^{-i(\tau\mu_{j,k} - x\cdot v_j + tv_j^2/2)} \sigma_{j,k}(t-\tau), \\ &\quad \phi_{l,m}(x - v_l t) e^{-i(\tau\mu_{l,m} - x\cdot v_l + tv_l^2/2)} \sigma_{l,m}(t-\tau))_{\mathfrak{K}_2} = 0. \end{aligned} \tag{3.10}$$

Let us prove (3.9) first. The inner product is majorized by

$$\begin{aligned} &\int dt \int dx |(e^{-i\tau H_0} u(t-\tau, \cdot))(x) \phi_{j,k}(x - v_j t) \sigma_{j,k}(t-\tau)| \\ &= \int |\sigma_{j,k}(t)| dt \int dx |(e^{-i\tau H_0} u(t, \cdot))(x) \phi_{j,k}(x - v_j(t+\tau))|. \end{aligned} \tag{3.11}$$

By Schwartz's inequality the integral by x in the R.H.S. of (3.11) is majorized by $\| u(t,.) \|$ which is square integrable with respect to t. On the other hand if we fix t, this integral coverges to zero as $\tau \to \infty$. This can be proved exactly in the same way as to prove (2.4). Hence (3.9) is an immediate consequence of Lebesgue's dominated convergence theorem. Next we prove (3.10). If $j = l$ and $k \neq m$, (3.10) is obvious, since $\phi_{j,k}$ and $\phi_{l,m}$ are orthogonal each other then. Suppose $j \neq l$. The inner product is majorized by

$$\int dt |\sigma_{j,k}(t) \sigma_{l,m}(t)| \int dx |\phi_{j,k}(x - v_j(t+\tau)) \phi_{l,m}(x - v_l(t+\tau))|. \tag{3.12}$$

The integral by x in (3.12) is uniformly bounded by 1; for each fixed $t\in\mathbb{R}^1$ this converges to zero as $\tau\to\infty$, since $\phi_{j,k}$ and $\phi_{l,m}$ are normalized and $v_j\neq v_l$. Thus (3.10) is an immediate consequence of Lebesgue's dominated convergence theorem.

Next we prove (2). By (3.4), (3.7) and (3.8) we have

$$\begin{aligned}
&(\exp(i\tau K_2)\,J\exp(-i\tau K_1)\tilde{u})(t)\\
&\quad= U(t,t+\tau)(J\exp(-i\tau K_1)\tilde{u})(t+\tau)\\
&\quad= U(t,0)[U(0,t+\tau)e^{-i(t+\tau)H_0}e^{itH_0}u(t)\\
&\qquad+\sum_{j,k} e^{it\mu_{j,k}}\sigma_{j,k}(t)\otimes U(0,t+\tau)U_j(t+\tau,0)e^{ix\cdot v_j}\phi_{j,k}]
\end{aligned}\tag{3.13}$$

By (3.13) and statement (1) of the theorem, for any fixed $t\in\mathbb{R}^1$, we have

$$\begin{aligned}
&s\text{-}\lim_{\tau\to\infty}(\exp(i\tau K_2)J\exp(-i\tau K_1)\tilde{u})(t)\\
&\quad= U(t,0)[W_0(0)e^{itH_0}u(t)+\sum_{j,k}e^{it\mu_{j,k}}\sigma_{j,k}(t)\otimes W_j(0)\,\phi_{j,k}],
\end{aligned}\tag{3.14}$$

where the limit is understood in the sense of the strong convergence in $\mathfrak{H}$. Here in the R.H.S. of (3.14) all summands are orthogonal each other by statement (2) of the theorem and the isometry property of $W_j(0)$'s. On the other hand $\|(\exp(i\tau K_2)\,J\,\exp(-i\tau K_1)\tilde{u})(t)\|\leqq\|u(t)\|+\sum|\sigma_{j,k}(t)|$, and the R.H.S. of (3.14) is also majorized by $\|u(t)\|+\sum|\sigma_{j,k}(t)|$. Since $(\|u(t)\|+\sum|\sigma_{j,k}(t)|)^2$ is integrable with respect to $t\in\mathbb{R}^1$, Lebesgue's dominated convergence theorem implies that

$$s\text{-}\lim_{\tau\to\infty}\exp(i\tau K_2)\,J\exp(-i\tau K_1)=\mathscr{W}$$

exists on $\mathfrak{K}_2$.

Statement (3) is a direct consequence of statements (1) and (2). Finally we prove statement (4). Let the operators T on $\mathfrak{K}_2$, O on $\mathfrak{K}_1$ and $\overline{\mathscr{W}}$ from $\mathfrak{K}_1$ to $\mathfrak{K}_2$ be defined as

$$\begin{aligned}
&(Tu)(t)=U(t,0)u(t),\quad u\in\mathfrak{K}_2;\\
&(O\tilde{u})(t)={}^{\dagger}(e^{itH_0}u(t),e^{it\mu_{1\,1}}\sigma_{1,1}(t)\ldots,e^{it\mu_{N,m_N}}\sigma_{N,m_N}(t));\\
&(\overline{\mathscr{W}}\tilde{u})(t)=W_0(0)u(t)\oplus\Big(\sum_{j,k}\sigma_{j,k}(t)\otimes W_j(0)\phi_{j,k}\Big).
\end{aligned}$$

T is unitary on $\mathfrak{K}_2$ and O is unitary on $\mathfrak{K}_1$. By (3.14) we have $\mathscr{W}=T\overline{\mathscr{W}}\,O$, hence $R(\mathscr{W})=TR(\overline{\mathscr{W}})$ and if $R(\mathscr{W})=\mathfrak{K}_2$, $R(\overline{\mathscr{W}})=\mathfrak{K}_2$. By asymptotic orthogonality,

$$\begin{aligned}
R(\overline{\mathscr{W}})&=(L^2(\mathbb{R}^1)\otimes R(W_0(0)))\oplus\Big(\sum_{j,k}\oplus(L^2(\mathbb{R}^1)\otimes\langle W_j(0)\phi_{j,k}\rangle)\Big)\\
&=L^2(\mathbb{R}^1)\otimes(R(W_0(0))\oplus\sum_{j,k}\oplus\langle W_j(0)\phi_{j,k}\rangle),
\end{aligned}$$

where $\langle W_j(0)\phi_{j,k}\rangle$ stands for the one dimentional subspace spanned by

$W_j(0)\phi_{j,k}$. Hence if $R(\overline{\mathscr{W}}) = \mathfrak{K}_2 = L^2(\mathbb{R}^1) \otimes L^2(\mathbb{R}^n)$,

$$\mathfrak{H} = R(W_0(0)) \oplus \sum_{j,k} \oplus \langle W_j(0)\phi_{j,k} \rangle ,$$

which obviously implies statement (4). (Q.E.D.)

3.2. Abstract Stationary Theory

By Lemma 3.1, the proof of the completeness of the wave operators is reduced to prove $R(\mathscr{W}) = \mathfrak{K}_2$. For proving this we shall apply the following abstract theorem due to Kato [8].

Theorem 3.2. (*Kato* [8]). *For $j = 1, 2$, K_j be a selfadjoint operator on the Hilbert space $\mathfrak{K}_j$ with the resolvent $R_j(z) = (K_j - z)^{-1}$ and the spectral measure $E_j(d\lambda)$. Let $J \in \mathbf{B}(\mathfrak{K}_1, \mathfrak{K}_2)$ be the identification operator and $I \subset \mathbb{R}^1$ be a Borel measurable subset of $\mathbb{R}^1$. Suppose the following conditions be satisfied.*

(*1*) $\lim_{\tau \to \pm\infty} \| J \exp(-i\tau K_1)u \| = \| u \|, \quad u \in K_1.$

(*2*) *There exists a linear manifold $\mathfrak{X}_j$ of $\mathfrak{K}_j$ such that there is no proper subspace of $\mathfrak{K}_j$ invariant under K_j and containing $\mathfrak{X}_j$ and such that $\mathfrak{X}_j$ is a normed space with its own norm (we write the completion of $\mathfrak{X}_j$ as $\bar{\mathfrak{X}}_j$) satisfying the following conditions:*
(*2.i*) *For $x, y \in \mathfrak{X}_1$, $f_0(z, x, y) = \pi^{-1} |\mathrm{Im}\, z| (R_1(z)x, R_1(z)y)$ defined for $z \in \mathbb{C}_\pm(I)$ has a continuous boundary value for $z = \lambda \in I$.*
(*2ii*) *There is a strong continuous family of operators $Y(z) \in \mathbf{B}(\bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_1)$ defined for $z \in \mathbb{C}_\pm(I) \cup I$ such that if $z \in \mathbb{C}_\pm(I)$, $Y(z)$ maps $\mathfrak{X}_2$ into $\mathfrak{X}_1$ with*

$$R_2(z)y = JR_1(z)Y(z)y, \quad \text{for} \quad y \in \mathfrak{X}_2. \tag{3.15}$$

Then K_1 and K_2 are spectrally absolutely continuous on I and there exists $Z_\pm \in \mathbf{B}(\mathfrak{K}_2, \mathfrak{K}_1)$ which is partially isometric with initial set $E_2(I)$ and final set $E_1(I)$ and such that

$$s\text{-Abel} \lim_{\tau \to \pm\infty} \exp(i\tau K_2) J \exp(-i\tau K_1) Z_\pm = E_2(I). \tag{3.16}$$

In particular, if $\mathscr{W}_\pm = s\text{-}\lim_{\tau \to \pm\infty} \exp(i\tau K_2) J \exp(-i\tau K_1)$ exists on $E_1(I)\mathfrak{K}_1$, then

$$R(\mathscr{W}_\pm) \supset E_2(I)\mathfrak{K}_2. \tag{3.17}$$

3.3. The Faddeev's Matrix

In the following subsections we shall prove that the conditions of Theorem 3.2 are satisfied for our operators K_1 and K_2, taking the spaces $\mathfrak{X}_1$, $\mathfrak{X}_2$ and the set I appropriately. In this subsection, we shall derive the decomposition formula (3.15) for the resolvent $R_2(z) = (K_2 - z)^{-1}$, postponing the proof of various estimates necessary for its justification until next subsection, although we shall prove some preliminary lemmas here. To avoid unnecessary complexity, we assume here and hereafter that $m_j = 1$ for $j = 1, 2, \dots, N$ and write $\mu_{j,k}$, $\phi_{j,k}$ and $L_{j,k}$ as μ_j, ϕ_j and L_j simply. The general case can be treated by a simple modification of the formulas which will appear in what follows.

We set $\mathfrak{K}_3 = \bigoplus_{j=1}^{N} (L^2(\mathbb{R}^1) \otimes L^2(\mathbb{R}^n))$ and define the operators $\mathscr{A}$ and $\mathscr{B}$ from $L^2(\mathbb{R}^1) \otimes L^2(\mathbb{R}^n)$ to $\mathfrak{K}_3$ as

$$\begin{cases} \mathscr{A}u = {}^\dagger(\mathscr{A}_1 u(t, x), \dots, \mathscr{A}_N u(t, x)), \\ \mathscr{B}u = {}^\dagger(\mathscr{B}_1 u(t, x), \dots, \mathscr{B}_N u(t, x)), \end{cases} \tag{3.18}$$

where

$$\begin{aligned} &\mathscr{A}_j u(t, x) = A_j(x - v_j t) u(t, x), \\ &\mathscr{B}_j u(t, x) = B_j(x - v_j t) u(t, x). \end{aligned} \tag{3.19}$$

We write $G_j(z) = (L_j - z)^{-1}, j = 0, 1, \dots, N$.

Lemma 3.3. *Let $\mathscr{M}(z)$ be any one of $\mathscr{A}G_0(z)\mathscr{B}^*$, $\mathscr{B}G_0(z)\mathscr{A}^*$, $\mathscr{A}G_0(z)\mathscr{A}^*$, $\mathscr{B}G_0(z)\mathscr{B}^*$. Then*
(1) *$\mathscr{M}(z)$ is a $\mathbf{B}(\mathfrak{K}_2)$-valued analytic function of $z \in \mathbb{C}_\pm$ and is uniformly bounded there.*
(2) *$\mathscr{M}(z)$ can be extended to the closed cut plane as a $\mathbf{B}(\mathfrak{K}_3)$-valued Hölder continuous function of z.*
(3) $\|\mathscr{M}(z)\| \to 0$ as $|\mathrm{Im}\, z| \to \infty$.
In particular $\mathscr{A}$ and $\mathscr{B}$ are L_0-smooth in the sense of Kato [7]. *Let us write as $\mathscr{M}(\lambda \pm i0)$ ($\lambda \in \mathbb{R}^1$) the boundary values on the reals.*

Proof. Let us write the generic element of $\mathfrak{K}_3$ as $u = (u_1, \dots, u_N)$. By the definition, for $\mathrm{Im}\, z > 0$ (we prove only this case, the other case can be proved similarly),

$$\begin{aligned} &(\mathscr{A}G_0(z)\mathscr{B}^* u)_j(t) \\ &\quad = \sum_{k=1}^{N} i \int_0^\infty e^{isz} A_j(x - v_j t) e^{-isH_0} B_k(x - v_k(t-s)) u_k(t-s)\, ds. \end{aligned} \tag{3.20}$$

Therefore taking $q \geqq 1$ as $n/\delta < q < n$, we have by Lemma 2.2, (1),

$$\begin{aligned} &\|(\mathscr{A}G_0(z)\mathscr{B}^* u)_j(t)\| \\ &\quad \leqq C \sum_{k=1}^{N} \int_0^\infty e^{-s\mathrm{Im}\, z} \min_{p,q}(\|A_j\|_p \|B_k\|_p |s|^{-n/p}, \|A_j\|_q \|B_k\|_q |s|^{-n/q}) \|u_k(t-s)\|\, ds. \end{aligned} \tag{3.21}$$

Here in this section $\|f\|_p$'s are L^p-norm of f. For $\mathrm{Im}\, z \geqq 0$,

$$C_{j,k}(z) = \int_0^\infty e^{-s\mathrm{Im}\, z} \min_{p,q}(\|A_j\|_p \|B_k\|_p s^{-n/p}, \|A_j\|_q \|B_k\|_q s^{-n/q})\, ds \tag{3.22}$$

is uniformly bounded and

$$C_{j,k}(z) \to 0 \quad \text{as} \quad \mathrm{Im}\, z \to \infty. \tag{3.23}$$

Young's inequality shows that

$$\|(\mathscr{A}G_0(z)\mathscr{B}^*u)_j\|_{\mathfrak{K}_2} \leqq \sum_{k=1}^{N} C_{j,k}(z)\|u_k\|_{\mathfrak{K}_2}. \tag{3.24}$$

Hence by Schwartz's inequality we get

$$\|\mathscr{A}G_0(z)\mathscr{B}^*u\|_{\mathfrak{K}_3} \leqq \left(\sum_{j,k} C_{j,k}(z)^2\right)^{1/2} \|u\|_{\mathfrak{K}_3}. \tag{3.25}$$

By (3.22) and (3.25), we see that $\mathscr{A}G_0(z)\mathscr{B}^*$ has the bounded closed extension $[\mathscr{A}G_0(z)\mathscr{B}^*]$ and this is obviously a $\mathbf{B}(\mathfrak{K}_3)$-valued analytic function of $z \in \mathbb{C}_+$. By (3.23) and (3.25), we also see that

$$\lim_{\operatorname{Im} z \to \infty} \|\mathscr{A}G_0(z)\mathscr{B}^*\| = 0. \tag{3.26}$$

For proving that $\mathscr{A}G_0(z)\mathscr{B}^*$ can be extended to $\mathbb{C}_+ \cup \mathbb{R}^1$ as a $\mathbf{B}(\mathfrak{K}_3)$-valued Hölder continuous function, it suffices to prove that $\mathscr{A}G_0(z)\mathscr{B}^*$ is uniformly Hölder continuous on $\mathbb{C}_+$. We first note that if $\operatorname{Im} z$, $\operatorname{Im} z' > 0$ and $s > 0$,

$$|e^{isz} - e^{isz'}| \leqq s^\alpha |z - z'|^\alpha$$

for any $0 \leqq \alpha \leqq 1$. Hence by (3.20) and Lemma 2.2, (1),

$$\begin{aligned} &\|(\mathscr{A}G_0(z)\mathscr{B}^*u)_j(t) - (\mathscr{A}G_0(z')\mathscr{B}^*u)_j(t)\| \\ &\leqq \sum_{k=1}^{N} \int_0^\infty s^\alpha |z - z'|^\alpha \min_{p,q}(\|A_j\|_p \|B_k\|_p s^{-n/p}, \|A_j\|_q \|B_k\|_q s^{-n/q}) \|u_k(t-s)\| \, ds. \end{aligned}$$

By taking $0 < \alpha < (n/q) - 1$, we see that

$$C_{j,k}^\alpha \equiv \int_0^\infty s^\alpha \min(\|A_j\|_p \|B_k\|_p s^{-n/p}, \|A_j\|_q \|B_k\|_q s^{-n/q}) ds < \infty.$$

Hence by Young's and Schwartz's inequalities we get

$$\begin{aligned} &\|\mathscr{A}G_0(z)\mathscr{B}^*u - \mathscr{A}G_0(z')\mathscr{B}^*u\|_{\mathfrak{K}_3} \\ &\leqq |z - z'|^\alpha \left(\sum_{j,k} (C_{j,k}^\alpha)^2\right)^{1/2} \|u\|_{\mathfrak{K}_3}. \end{aligned}$$

Thus we see that $\mathscr{A}G_0(z)\mathscr{B}^*$ is uniformly Hölder continuous on $\mathbb{C}_+$ with exponent α, α is an arbitrary number satisfying $0 < \alpha < (n/q) - 1$, $\alpha \leqq 1$. This completes the proof of the statements (1)–(3) for $\mathscr{A}G_0(z)\mathscr{B}^*$. The forgoing arguments obviously apply for any other operator being considered here. The L_0-smoothness of $\mathscr{A}$ (or $\mathscr{B}$) is obvious by the statement (1) for $\mathscr{A}G_0(z)\mathscr{A}^*$ (or $\mathscr{B}G_0(z)\mathscr{B}^*$) and the definition of the L_0-smoothness (see Reed–Simon [13], Theorem XIII. 25).

(Q.E.D.)

Remark. By (3.21) we have

$$C_{j,k}(z) \leqq C_{p,q} \max_{p,q} (\|A_j\|_p \|B_k\|_{p'} \|A_j\|_q \|B_k\|_q)$$

where $C_{p,q}$ is a constant depending only on p and q. Hence

$$\| \mathscr{A} G_0(z)\mathscr{B}^* \| \leqq C_{p,q} \max_{p,q} (\| A_j \|_p \| B_k \|_{p'} \| A_j \|_q \| B_q \|_q : j, k = 1, \dots, N). \tag{3.27}$$

Therefore if all A_j's (or B_k's) converge to zero in $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ $(1 \leqq q < n/2 < p \leqq \infty)$, then the operator $\mathscr{A} G_0(z)\mathscr{B}^*$ converges to zero in operator norm. Similar result obviously holds for other operators being considered in Lemma 3.3. This result will be used in the proof of Lemma 3.8.

We need the following auxiliary operators.

For $j = 1, 2, \dots, N$, define one parameter unitary group $\mathscr{V}_j(\tau)$ on $\mathfrak{K}_2$ as

$$(\mathscr{V}_j(\tau) f)(t) = U_j(t, t-\tau) f(t-\tau) \tag{3.28}$$

and write the generator of this group as $\mathscr{L}_j$:

$$\mathscr{V}_j(\tau) = e^{-i\tau\mathscr{L}_j}, \quad -\infty < \tau < \infty. \tag{3.29}$$

We write as

$$\mathscr{G}_j(z) = (\mathscr{L}_j - z)^{-1}, \quad \operatorname{Im} z \neq 0. \tag{3.30}$$

Lemma 3.4. *For* $\operatorname{Im} z \neq 0$, $j = 1, \dots, N$,

$$R_2(z) = G_0(z) - [\mathscr{B} G_0(\bar{z})]^* (1 + \mathscr{Q}_0(z))^{-1} \mathscr{A} G_0(z), \tag{3.31}$$

$$\mathscr{G}_j(z) = G_0(z) - [\mathscr{B}_j G_0(\bar{z})]^* (1 + \mathscr{Q}_j(z))^{-1} \mathscr{A}_j G_0(z), \tag{3.32}$$

where $\mathscr{Q}_0(z) = \mathscr{A} G_0(z)\mathscr{B}^*$ *and* $\mathscr{Q}_j = \mathscr{A}_j G_0(z)\mathscr{B}_j^*$.

This lemma can be proved similarly as Lemma 3.3 of Yajima [16], using Lemma 3.3 above and (B) of Sect. 1. Hence the proof is omitted here.

Now we proceed to the derivation of (3.15). Let

$D(z) = (N \times N)$-diagonal matrix with (j, j)-element $\mathscr{A}_j G_0(z)\mathscr{B}_j^*$.

$F_0(z) = Q(z) - D(z)$.

By (3.32) we have

$$(1 + D(z))^{-1} = (N \times N)\text{-diagonal matrix with } (j, j)\text{-element}$$
$$1 - \mathscr{A}_j \mathscr{G}_j(z) \mathscr{B}_j^*; \tag{3.33}$$

$$(1 + D(z))^{-1} \mathscr{A} G_0(z) = {}^{\dagger}(\mathscr{A}_1 \mathscr{G}_1(z), \dots, \mathscr{A}_N \mathscr{G}_N(z)); \tag{3.34}$$

$$(1 + D(z))^{-1} F_0(z) = (N \times N)\text{-matrix with} \tag{3.35}$$
$$(j, k)\text{-element } \hat{\delta}_{j,k} \mathscr{A}_j \mathscr{G}_j(z) \mathscr{B}_k,$$

where

$$\hat{\delta}_{j,k} = \begin{cases} 1 & \text{if } j \neq k \\ 0 & \text{if } j = k. \end{cases} \tag{3.36}$$

We write the R.H.S. of (3.34) and (3.35) as

$$\hat{A}\hat{G}(z) = {}^{\dagger}(\mathscr{A}_1 \mathscr{G}_1(z), \dots, \mathscr{A}_N \mathscr{G}_N(z)), \tag{3.37}$$

$$F(z) = (\hat{\delta}_{j,k} \mathscr{A}_j \mathscr{G}_j(z) \mathscr{B}_k)_{j,k}. \tag{3.38}$$

Thus combining (3.33)–(3.38) with (3.31), we have

$$\begin{aligned} &R_2(z)\\ &\quad = G_0(z) - [\mathscr{B}G_0(\bar{z})]^*(1 + (1 + D(z))^{-1}F_0(z))^{-1}(1 + D(z))^{-1}\mathscr{A}G_0(z)\\ &\quad = G_0(z) - [\mathscr{B}G_0(\bar{z})]^*(1 + F(z))^{-1}\hat{A}\hat{G}(z). \end{aligned} \tag{3.39}$$

To go further we need the following lemma.

Lemma 3.5.

$$T_{v_j}(-i\partial/\partial t \otimes I + I \otimes H_j - z)^{-1}T_{v_j}^* = \mathscr{G}_j(z). \tag{3.40}$$

Proof. Note that

$$(T_{v_j}^* f)(t, x) = e^{-ix\cdot v_j - iv_j^2 t/2} f(t, x + v_j t). \tag{3.41}$$

Then simple calculations show

$$T_{v_j}G_0(z)T_{v_j}^* = G_0(z), \tag{3.42}$$

$$T_{v_j}^*\mathscr{A}_j T_{v_j} = I \otimes A_j, \quad T_{v_j}^*\mathscr{B}_j^* T_{v_j} = I \otimes B_j^*. \tag{3.43}$$

We apply to the both sides of (3.32), T_{v_j} from the right and $T_{v_j}^*$ from the left. Then we get by (3.42) and (3.43) that

$$\begin{aligned} T_{v_j}^*\mathscr{G}_j(z)T_{v_j} &= T_{v_j}^*G_0(z)T_{v_j} - [T_{v_j}^*\mathscr{B}_j T_{v_j}(T_{v_j}^*G_0(\bar{z})T_{v_j})]^*\\ &\quad \times (1 + T_{v_j}^*\mathscr{A}_j T_{v_j}(T_{v_j}^*G_0(z)T_{v_j})(T_{v_j}^*\mathscr{B}_j T_{v_j}))^{-1}\\ &\quad \times T_{v_j}^*\mathscr{A}_j T_{v_j}(T_{v_j}^*G_0(z)T_{v_j})\\ &= G_0(z) - [(I \otimes B_j)G_0(\bar{z})]^*(1 + (I \otimes A_j)G_0(z)(I \otimes B_j))^{-1}\\ &\quad \times (I \otimes A_j)G_0(z). \end{aligned} \tag{3.44}$$

We regard as $L^2(\mathbb{R}^1) \otimes L^2(\mathbb{R}^n) = \int^{\oplus} L^2(\mathbb{R}^n)dt$. Then the last member of (3.44) is written as $\mathscr{F}_t^{-1}\int^{\oplus}(r_0(z-t) - [r_0(z-\tau)B_j](1 + Q_j(z-\tau))^{-1}[Ar_0(z-\tau)])d\tau\mathscr{F}_t = \mathscr{F}_t^{-1} \times \int^{\oplus}(H_j - z + \tau)^{-1}d\tau\mathscr{F}_t = (-i\partial/\partial t \otimes I + I \otimes H_j - z)^{-1}$. This proves (3.40). (Q.E.D.)

By spectral decomposition we have

$$H_j = H_j(1 - P_j) + \mu_j P_j \equiv H_j^c + \mu_j P_j.$$

We write as $1 - P_j = P_j^c$. Then

$$\begin{aligned} &(-i\partial/\partial t \otimes I + I \otimes H_j - z)^{-1}\\ &\quad = (-i\partial/\partial t \otimes I + I \otimes H_j^c - z)^{-1}(I \otimes P_j^c) + G_j(z) \otimes P_j. \end{aligned} \tag{3.45}$$

We write

$$T_{v_j}(-i\partial/\partial t \otimes I + I \otimes H_j^c - z)^{-1}(I \otimes P_j^c)T_{v_j}^* = \mathscr{G}_j^c(z) \tag{3.46}$$

and set $\Gamma_j \in \mathbf{B}(L^2(\mathbb{R}^1) \otimes L^2(\mathbb{R}^n), L^2(\mathbb{R}^1))$ as

$$(\Gamma_j u)(t) = \int \phi_j(x)u(t, x)dx.$$

Then equations (3.40), (3.45) and (3.46) imply

$$\mathscr{G}_j(z) = \mathscr{G}_j^c(z) + T_{v_j}\Gamma_j^* G_j(z)\Gamma_j T_{v_j}^*. \tag{3.47}$$

Now we proceed to further decomposition of (3.39). We define $(1 \times 2N)$-matrix $L(z)$, $(N \times 2N)$-matrix $X(z)$ and $(2N \times 2N)$-matrix $A(z)$ as follows:

$$L(z) = {}^\dagger(\mathscr{A}_1\mathscr{G}_1^c(z), \Gamma_1 T_{v_1}^*, \ldots, \mathscr{A}_N\mathscr{G}_N^c(z), \Gamma_N T_{v_N}^*); \tag{3.48}$$

$$X(z) = (X_{jk}(z))_{j=1,\ldots,N:\,k=1,\ldots,2N}, \tag{3.49}$$

$$X_{jk}(z) = \begin{cases} I, & k = 2j-1, \quad j = 1, \ldots, N, \\ \mathscr{A}_j T_{v_j}\Gamma_j^* G_j(z), & k = 2j, \quad j = 1, \ldots, N, \\ 0, & \text{otherwise}; \end{cases}$$

$$A(z) = (A_{jk}(z))_{j,k=i,\ldots,2N}, \tag{3.50}$$

$$A_{jk}(z) = \begin{cases} \mathscr{A}_m\mathscr{G}_m^c(z)\mathscr{B}_l^*, & j = 2m-1, \quad k = 2l-1, \quad m \neq l, \\ \mathscr{A}_m\mathscr{G}_m^c(z)\mathscr{B}_l^*\mathscr{A}_l T_{v_l}\Gamma_l^* G_l(z), & j = 2m-1, \quad k = 2l, \quad m \neq l. \\ \Gamma_m T_{v_m}^*\mathscr{B}_l^* & j = 2m, \quad k = 2l-1, \quad m \neq l. \\ \Gamma_m T_{v_m}^*\mathscr{B}_l^*\mathscr{A}_l T_{v_l}\Gamma_l^* G_l(z), & j = 2m, \quad k = 2l, \quad m \neq l. \\ 0, & \text{otherwise}. \end{cases}$$

$A(z)$ is a substitute of so-called Faddeev matrix in three body problem. By (3.32) and (3.47), we can easily see that

$$\hat{A}\hat{G}(z) = X(z)L(z); \tag{3.51}$$

$$F(z)X(z) = X(z)A(z); \tag{3.52}$$

and hence

$$(1 + F(z))^{-1}X(z) = X(z)(1 + A(z))^{-1}. \tag{3.53}$$

Here the existence of the inverse in both side of (3.53) can be proved by the standard way (Faddeev [3], p. 50, Lemma 7.5). Combining (3.39) with (3.51) and (3.53), we get

$$R_2(z) = G_0(z) - [\mathscr{B}G_0(\bar{z})]^* X(z)(1 + A(z))^{-1}L(z). \tag{3.54}$$

By definition,

$$\begin{aligned} &[\mathscr{B}G_0(\bar{z})]^* X(z) \\ &\quad = {}^\dagger(G_0(z)\mathscr{B}_1, G_0(z)\hat{V}_1 T_{v_1}\Gamma_1^* G_1(z), \ldots, G_0(z)\mathscr{B}_N, G_0(z)\hat{V}_N T_{v_N}\Gamma_N^* G_N(z)). \end{aligned} \tag{3.55}$$

On the other hand by (3.32), (3.40) and (3.44),

$$\begin{aligned} G_0(z)\hat{V}_j T_{v_j}\Gamma_j^* G_j(z) &= G_0(z)\hat{V}_j\mathscr{G}_j(z)T_{v_j}\Gamma_j^* \\ &= (G_0(z) - G_j(z))T_{v_j}\Gamma_j^* \\ &= G_0(z)T_{v_j}\Gamma_j^* - T_{v_j}\Gamma_j^* G_j(z). \end{aligned} \tag{3.56}$$

Here in (3.55) and (3.56) we wrote as $\hat{V}_j = \mathscr{A}_j\mathscr{B}_j^*$ simply. We write as

$$(1 + A(z))^{-1}L(z) = {}^\dagger(\mathscr{H}_1(z), h_1(z_1), \ldots, \mathscr{H}_N(z), h_N(z)); \tag{3.57}$$

$$Y_0(z) = 1 - \sum_{j=1}^{N} (\mathscr{B}_j \mathscr{H}_j(z) + T_{v_j} \Gamma_j^* \mathscr{k}_j(z)); \tag{3.58}$$

$$Y_j(z) = -\mathscr{k}_j(z) \quad (j = 1, 2, \ldots, N). \tag{3.59}$$

Then combining (3.54)–(3.59), we finally get

$$\begin{aligned} R_2(z) &= G_0(z)\left(1 - \sum_{j=1}^{N} (\mathscr{B}_j \mathscr{H}_j(z) + T_{v_j} \Gamma_j^* \mathscr{k}_j(z))\right) - \sum_{j=1}^{N} T_{v_j} \Gamma_j^* G_j(z) \mathscr{k}_j(z) \\ &= G_0(z) Y_0(z) + \sum_{j=1}^{N} T_{v_j}(G_j(z) Y_j(z) \otimes \phi_j) \\ &= J R_1(z) Y(z), \end{aligned} \tag{3.60}$$

where, of course,

$$Y(z) = {}^{\dagger}(Y_0(z), Y_1(z), \ldots, Y_N(z)). \tag{3.61}$$

3.4. Estimates of Operators

In this subsection we give several estimates of the operators which are necersary for the application of Theorem 3.2. To start with, we prove the following lemma.

Lemma 3.6. *Let $\gamma > 1/2$ and $j = 1, 2, \ldots, N$. Then $G_j(z)$ satisfies the following properties.*
(1) $G_j(z)$ is a $\mathbf{B}(L^2_\gamma(\mathbb{R}^1), L^2_{-\gamma}(\mathbb{R}^1))$-valued uniformly bounded analytic function of $z \in \mathbb{C}_\pm$ and can be extended to $\mathbb{C}_\pm \cup \mathbb{R}^1$ as a Hölder continuous function.

(2) $G_j(z) \in \mathbf{B}_\infty(L^2_\gamma(\mathbb{R}^1), L^2_{-\gamma}(\mathbb{R}^1))$.
(3) $\|G_j(z)\|_{\mathbf{B}(L^2_\gamma, L^2_{-\gamma})} \to 0$ as $|\mathrm{Im}\, z| \to \infty$.

Proof. Let $m(t) = (1 + t^2)^{-\gamma/2}$ and M be the multiplication operator by $m(t)$. Then M is a unitary operator from $L^2(\mathbb{R}^1)$ to $L^2_\gamma(\mathbb{R}^1)$ as well as from $L^2_{-\gamma}(\mathbb{R}^1)$ to $L^2(\mathbb{R}^1)$. Hence for proving (1)–(3), it suffices to prove that
(1)′ $G_j'(z) = M G_j(z) M$ is $\mathbf{B}(L^2(\mathbb{R}^1))$-valued uniformly bounded analytic function of $z \in \mathbb{C}_\pm$ and can be extended to $\mathbb{C}_\pm \cup \mathbb{R}^1$ as a Hölder continuous function;
(2)′ $G_j'(z) \in \mathbf{B}_\infty(L^2(\mathbb{R}^1))$;
(3)′ $\|G_j'(z)\|_{\mathbf{B}(L^2(\mathbb{R}^1))} \to 0$ as $|\mathrm{Im}\, z| \to \infty$.

We prove (1)′–(3)′ for $z \in \mathbb{C}_+$. The case $z \in \mathbb{C}_-$ can be proved similarly. Since $G_j(z)$ is a convolution operator with the function $i \exp(i(z - \mu_j)t)\theta(t)$, $\theta(t)$ is the Heaviside function, $G_j'(z)$ is the integral operator with the kernel

$$K_{j,\gamma}(t, s; z) = i(1 + t^2)^{-\gamma/2} e^{i(z - \mu_j)(t - s)} \theta(t - s)(1 + s^2)^{-\gamma/2}.$$

$$\begin{aligned} &\int |K_{j,\gamma}(t, s; z)|^2 dt\, ds \\ &\quad \leqq \int (1 + t^2)^{-\gamma}(1 + s^2)^{-\gamma} e^{-\mathrm{Im}\, z |t - s|} dt\, ds. \end{aligned} \tag{3.62}$$

The R.H.S. of (3.62) is uniformly bounded for $z \in \mathbb{C}_+$ and converges to zero as $\mathrm{Im}\, z \to \infty$, since $\gamma > 1/2$. Hence $G_j'(z)$ is a Hilbert–Schimdt operator on $L^2(\mathbb{R}^1)$ and the Hilbert–Schmidt norm converges to zero as $\operatorname{Im} z \to \infty$. This proves (2)′ and (3)′ and a part of (1)′. Thus for completing the proof it suffices to prove that $G_j'(z)$ is a uniformly Hölder continuous $\mathbf{B}(L^2(\mathbb{R}^1))$-valued function. This is simple, since for any $0 < \alpha < 2\gamma - 1$ and $\alpha \leqq 1$,

$$\begin{aligned}&\int |K_{j,\gamma}(t, s; z) - K_{j,\gamma}(t, s; z')|dt\, ds\\ &\quad \leqq |z - z'|^\alpha \int |t - s|^\alpha (1 + t^2)^{-\gamma}(1 + s^2)^{-\gamma} dt\, ds\end{aligned}$$

and the integral in the R.H.S. is finite. (Q.E.D.)

We define the spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$ as follows:

$$\begin{cases} \mathfrak{X}_1 = R(\mathscr{B}^*) \oplus \sum_{j=1}^{N} L_\delta^2(\mathbb{R}^1); \\ \mathfrak{X}_2 = L_\delta^2(\mathbb{R}^{n+1}). \end{cases} \tag{3.63}$$

$\mathfrak{X}_1$ and $\mathfrak{X}_2$ are equipped with the natural Hilbert space structure. Obviously $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are dense linear submanifolds of $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively.

Lemma 3.7. *$\mathfrak{X}_1$ and K_1 satisfy the condition* (2.*i*) *of Theorem* 3.2.

Proof. Since $R_1(z)\tilde{u} = (K_1 - z)^{-1}\tilde{u} = {}^\dagger(G_0(z)u, G_1(z)\sigma_1, \ldots, G_N(z)\sigma_N)$ and $f_0(z, x, y) = |\operatorname{Im} z|(R_1(z)x, R_1(z)y) = (1/2\pi i)((R_1(z) - R_1(\bar{z}))x, y)$, it suffices to prove that $\mathscr{B}^* G_0(z)\mathscr{B}$ (or $G_j(z)$) can be extended to the closed cut plane $\mathbb{C}_\pm \cup \mathbb{R}^1$ as a $\mathbf{B}(L^2(\mathbb{R}^{n+1}))$ (or $\mathbf{B}(L_\delta^2(\mathbb{R}^1), L_{-\delta}^2(\mathbb{R}^1))$)-valued continuous function of z. For $G_j(z)$ $(j = 1, 2, \ldots, N)$ this is proved in Lemma 3.6, and for $\mathscr{B}^* G_0(z)\mathscr{B}$ in Lemma 3.3. (Q.E.D.)

The following is the key lemma in this section.

Lemma 3.8. *If $j \neq k$, $\mathscr{A}_j G_0(z)\mathscr{B}_k^*$ is a compact operator on $L^2(\mathbb{R}^{n+1})$ for any z in the closed cut plane $\mathbb{C}_\pm \cup \mathbb{R}^1$.*

For proving the lemma we need the following preliminary lemmas. In Lemmas 3.9, 3.10 and Corollary 3.11, n is any positive integer.

Lemma 3.9. *Let $\sigma, \rho > 0$. If $K \in \mathbf{B}(L^2(\mathbb{R}^n), L_\rho^2(\mathbb{R}^n))$ and $K \in \mathbf{B}(L^2(\mathbb{R}^n), H^\sigma(\mathbb{R}^n))$, then K is a compact operator on $L^2(\mathbb{R}^n)$.*

Proof. Let $R > 0$ and χ_R be the characteristic function of the ball $\{|x| \leqq R\}$. Then $\chi_R K$ is a compact operator on $L^2(\mathbb{R}^n)$ by Rellich's compactness theorem. On the other hand,

$$\begin{aligned}\|Ku - \chi_R Ku\| &\leqq \left(\int_{|x| \geqq R} |Ku(x)|^2 dx \right)^{1/2}\\ &\leqq (1 + R^2)^{-\rho/2} \|Ku\|_{L_\rho^2} \leqq (1 + R^2)^{-\rho/2} \|K\|_{B(L^2, L_\rho^2)} \|u\|.\end{aligned}$$

Therefore as $R \to \infty$, $\|K - \chi_R K\| \to 0$. This proves Lemma 3.9. (Q.E.D.)

Lemma 3.10. *Let* $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and *ξ, η be the conjugate variables of x, y (m may be* *zero). Let $g(\xi, \eta)$ be a function such that for any multi-index α,* $\sup_{\xi,\eta}|(\partial/\partial\xi)^\alpha g(\xi,\eta)| \leqq C_\alpha < \infty$. *Let $G(D_x, D_y)$ be the operator defined as*

$$G(D_x, D_y) = \mathscr{F}^{-1}_{(x,y)} g(\xi, \eta) \mathscr{F}_{(x,y)}.$$

Then for any $\rho \in \mathbb{R}^1$,

$$\|(1+|x|^2)^\rho G(D_x, D_y)(1+|x|^2)^{-\rho}\| < \infty. \tag{3.64}$$

Proof. Let $\rho \geqq 0$ be an integer. By Parseval's relation and Leibniz's rule we have

$$\begin{aligned}
&\|(1+|x|^2)^\rho G(D_x, D_y)(1+|x|^2)^{-\rho} u(x,y)\| \\
&\quad = \|(1-\Delta_\xi)^\rho g(\xi,\eta)(1-\Delta_\xi)^{-\rho}\hat{u}(\xi,\eta)\| \\
&\quad = \Bigg\| \sum_{0\leq\sigma\leq\rho} \binom{\rho}{\sigma}(-1)^\sigma \sum_{0\leqq\beta\leqq 2\bar\sigma} \binom{2\bar\sigma}{\beta} (\partial/\partial\xi)^{2\bar\sigma-\beta} g(\xi,\eta)(\partial/\partial\xi)^\beta \\
&\qquad \cdot (1-\Delta_\xi)^{-\rho}\hat{u}(\xi,\eta) \Bigg\| \\
&\quad \leqq \sum_{0\leq\sigma\leq\rho} \sum_{0\leq\beta\leq 2\bar\sigma} \binom{\rho}{\sigma}\binom{2\bar\sigma}{\beta} C_{2\bar\sigma-\beta} \|(\partial/\partial\xi)^\beta (1-\Delta_\xi)^{-\rho}\hat{u}(\xi,\eta)\|,
\end{aligned} \tag{3.65}$$

where $2\bar\sigma = (2\sigma, 2\sigma, \ldots, 2\sigma)$. Since $0 \leqq \sigma \leqq \rho$, $0 \leqq \beta \leqq 2\bar\sigma$,

$$\|(\partial/\partial\xi)^\beta (1+\Delta_\xi)^{-\rho}\hat{u}(\xi,\eta)\| = \|x^\beta(1+x^2)^{-\rho}u(x,y)\| \leqq \|u(x,y)\|.$$

Combining this with (3.65), we get (3.64) for ρ non-negative integers. For general $\rho \geqq 0$, (3.64) is the consequence of the above case and the interpolation theorem (Lions–Magenes [19], p. 27). We now prove the case $\rho < 0$. Let $u, h \in C_0^\infty(\mathbb{R}^n)$. Then the result for the case $\rho > 0$ implies

$$\begin{aligned}
&|((1+|x|^2)^\rho G(D_x, D_y)(1+|x|^2)^{-\rho} u, h)| \\
&\leqq |(u, (1+|x|^2)^{-\rho} \bar{G}(D_x, D_y)(1+|x|^2)^\rho h)| \leqq C\|u\|\,\|h\|.
\end{aligned}$$

This obviously implies (3.64). (Q.E.D.)

The Fourier transform of Lemma 3.10 implies the following

Corollary 3.11. *Let $f \in C^\infty(\mathbb{R}^n)$ and its derivatives are all bounded functions. Then the multiplication operator by f is a bounded operator on $H^s(\mathbb{R}^n)$ for any $s \in \mathbb{R}^1$*

Proof of Lemma 3.9. (i) We first show that it suffices to prove that $A(x-vt)G_0(\pm i)B(x)$ is a compact operator on $\mathfrak{K}_2$ when $A, B \in C_0^\infty(\mathbb{R}^n)$ and $v \neq 0$. Since $\mathscr{A}_j G_0(z)\mathscr{B}_k$ is a $\mathbf{B}(\mathfrak{K}_2)$-valued continuous function of $z \in \mathbb{C}_\pm \cup \mathbb{R}^1$ by Lemma 3.3, it suffices to prove that $\mathscr{A}_j G_0(\pm i)\mathscr{B}_k \in \mathbf{B}_\infty(\mathfrak{K}_2)$. Choose $\omega(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\omega(x) \geqq 0$, $\omega(x) \equiv 1$ near $x=0$ and $\int \omega(x)dx = 1$. For $\varepsilon > 0$, $\omega_\varepsilon(x) = \varepsilon^{-n}\omega(x/\varepsilon)$. Set $A_{j,\varepsilon}(x) = \omega(\varepsilon x)(A_j * \omega_\varepsilon)(x)$ and $B_{k,\varepsilon}(x) = \omega(\varepsilon x)(B_k * \omega_\varepsilon)(x)$. Then $A_{j,\varepsilon}$, $B_{k,\varepsilon} \in C_0^\infty(\mathbb{R}^n)$ and for $n/\delta < q < n$ and p, $1/p = 1/s - 1/n$, $\|A_{j,\varepsilon} - A_j\|_{L^p}$, $\|B_{k,\varepsilon} - B_k\|_{L^p}$, $\|A_{j,\varepsilon} - A_j\|_{L^q}$ and $\|B_{k,\varepsilon} - B_k\|_{L^q}$ converge to zero as $\varepsilon \to 0$. Therefore by the remark following Lemma 3.3, $\|\mathscr{A}_{j,\varepsilon} G_0(z)\mathscr{B}_{k,\varepsilon} - \mathscr{A}_j G_0(z)\mathscr{B}_k\| \to 0$ as $\varepsilon \to 0$. Hence we may assume A_j, $B_k \in C_0^\infty(\mathbb{R}^n)$. By (3.42) and (3.43), $T^*_{v_k}\mathscr{A}_j G_0(z)\mathscr{B}_k T_{v_k}$

$= T^*_{v_k}\mathscr{A}_j T_{v_k} G_0(z)(I \otimes B_j)$. Since $I \otimes B_j$ is the multiplication by $B_j(x)$ and $T^*_{v_k}\mathscr{A}_j T_{v_k}$ is the multiplication by $A_j(x - v_j - v_k)t)$, $v_j - v_k \neq 0$, we get the desired result.
(ii) Let $f(\tau) \in C^\infty(\mathbb{R}^1)$ be such that $f(\tau) = 1$ for $|\tau| \geqq 2$ $f(\tau) = 0$ for $|\tau| \leqq 1$ and $|f(\tau)| \leqq 1$ for $1 \leqq |\tau| \leqq 2$. Let F_R be the operator defined as $F_R = \mathscr{F}_t^{-1} f(\tau/R)\mathscr{F}_t$.

$$\begin{aligned} A(x-vt)G_0(\pm i)B(x) &= A(x-vt)G_0(\pm i)F_R B(x) \\ &\quad + A(x-vt)G_0(\pm i)(1-F_R)B(x). \end{aligned} \tag{3.66}$$

We first prove $A(x-vt)G_0(\pm i)(1-F_R)B(x) \in \mathbf{B}_\infty(K_2)$ for any $R > 0$. Let $E_\pm(\tau, \xi) = (1 - f(\tau/R))(\tau + |\xi|^2/2 \pm i)^{-1}$. Obviously

$$G_0(\pm i)(1-F_R) = F^{-1}_{(t,x)} F_\pm(\tau, \xi) F_{(t,x)}; \tag{3.67}$$

$$|(\partial/\partial\tau)^\alpha(\partial/\partial\xi)^\beta E_\pm(\tau, \xi)| \leqq C^\gamma_{R,\alpha,\beta}(1+|\tau|)^{-\gamma}(1+|\xi|)^{-2}, \tag{3.68}$$

for any multi-index α and β, and any $\gamma \geqq 0$. Hence $G_0(\pm i)(1-F_R) \in \mathbf{B}(L^2(\mathbb{R}^{n+1}), H^2(\mathbb{R}^{n+1}))$ and by Corollary 3.11

$$A(x-vt)G_0(\pm i)(1-F_R)B(x) \in \mathbf{B}(L^2(\mathbb{R}^{n+1}), H^2(\mathbb{R}^{n+1})). \tag{3.69}$$

On the other hand by Lemma 3.10, (3.67), (3.68) and the obvious inequality

$$(1+|x-vt|)^{-\rho}(1+|x|)^{-\rho} \leqq C_\rho(1+|x|+|t|)^{-\rho}, \tag{3.70}$$

we get

$$A(x-vt)G_0(\pm i)(1-F_R)B(x) \in \mathbf{B}(L^2(\mathbb{R}^{n+1}), L^2_\rho(\mathbb{R}^{n+1})), \tag{3.71}$$

for any $\rho > 0$. Hence by Lemma 3.9, $A(x-vt)G_0(\pm i)(1-F_R)B(x)$ is a compact operator on $L^2(\mathbb{R}^{n+1}) = \mathfrak{K}_2$.
(iii) Finally we prove

$$\lim_{R\to\infty} \|A(x-vt)G_0(\pm i)F_R B(x)\| = 0, \tag{3.72}$$

which completes the proof of Lemma 3.8. Since $A, B \in C_0^\infty(\mathbb{R}^n)$, $A(x-vt)\mathscr{F}^{-1}_{(t,x)} \in \mathbf{B}(L^2(\mathbb{R}^{n+1}))$ and $\mathscr{F}_{(t,x)}B(x) \in \mathbf{B}(L^2(\mathbb{R}^{n+1}), L^2(\mathbb{R}^1) \otimes H^k(\mathbb{R}^n))$ for any $k \geqq 0$. Therefore for proving (3.72), it suffices to prove that the multiplication operator by the function $f(\tau/R)(\tau + \xi^2/2 \pm i)^{-1}$ regarded as an operator from $L^2(\mathbb{R}^1) \otimes H^k(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{n+1})$ converges to zero in operator norm as $R \to \infty$, for some $k > 0$. We take $k = 1$. Then by Sobolev's embedding theorem (see Kuroda [10], p. 4.13, Theorem 1 and p. 4.26, Theorem 1′), we have for $u \in H^1(\mathbb{R}^n)$,

$$\sup_{\rho \geqq 0} \rho^{(n-1)/2} \|u(\rho\cdot)\|_{L^2(S^{n-1})} \leqq C\|u\|_{H^1(\mathbb{R}^n)}, \tag{3.73}$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Let $N < R$. Then

$$\begin{aligned} &\int \frac{|f(\tau/R)u(\tau,\xi)|^2}{|\tau + \xi^2/2 \pm i|^2} d\tau\, d\xi \\ &\leqq \int_{\substack{|\tau+\xi^2/2| \leqq N \\ |\tau| \geqq R}} + \int_{\substack{|\tau+\xi^2/2| \geqq N \\ |\tau| \leqq R}} \frac{|u(\tau,\xi)|^2}{(\tau+\xi^2/2)^2 + 1} d\tau\, d\xi \end{aligned} \tag{3.74}$$

$$\leqq \int_{\substack{\sqrt{2\tau - 2N} \leqq |\xi| \leqq \sqrt{2\tau + 2N} \\ \tau \geqq R}} |u(-\tau, \xi)|^2 d\tau d\xi + (N^2+1)^{-1} \|u\|^2$$

$$= \int_R^\infty d\tau \left\{ \int_{\sqrt{2\tau - 2N}}^{\sqrt{2\tau + 2N}} d\rho \left(\rho^{n-1} \int_{S^{n-1}} |u(-\tau, \rho\omega)|^2 d\omega \right) \right\} + (N^2+1)^{-1} \|u\|^2$$

$$\leqq C \int_R^\infty d\tau \int_{\sqrt{2\tau - 2N}}^{\sqrt{2\tau + 2N}} \|u(-\tau, \cdot)\|^2_{H^1(\mathbb{R}^n)} d\rho + (N^2+1)^{-1} \|u\|^2$$

$$\leqq C \int_R^\infty (\sqrt{2\tau + 2N} - \sqrt{2\tau - 2N}) \|u(-\tau, \cdot)\|^2_{H^1(\mathbb{R}^n)} d\rho + (N^2+1)^{-1} \|u\|^2$$

$$\leqq C\sqrt{2N}(R-N)^{-1/2} \int_{-\infty}^{\infty} \|u(\tau, \cdot)\|^2_{H^1(\mathbb{R}^n)} d\tau + (N^2+1)^{-1} \|u\|^2$$

$$\leqq (C\sqrt{2N}(R-N)^{-1/2} + (N^2+1)^{-1}) \|u\|^2_{L^2(\mathbb{R}^1) \otimes H^1(\mathbb{R}^n)}.$$

Here we used (3.73) in the fourth step. Deviding both sides of (3.74) by $\|u\|^2_{L^2(\mathbb{R}^1) \otimes H^1(\mathbb{R}^n)}$ and letting $R \to \infty$ first and then $N \to \infty$, we get the desired result easily. This completes the proof of Lemma 3.8. (Q.E.D.)

Lemma 3.12 *For any* $j = 1, 2, \ldots, N$ *and* $z \in \mathbb{C}_\pm \cup \mathbb{R}^1$, $1 + \mathscr{A}_j G_0(z) \mathscr{B}_j^*$ *has an inverse in* $\mathbf{B}(\mathfrak{K}_2)$. *Moreover* $(1 + \mathscr{A}_j G_0(z) \mathscr{B}_j^*)^{-1}$ *is uniformly bounded and Hölder continuous* in $z \in \mathbb{C}_\pm \cup \mathbb{R}^1$.

Proof. By (3.42) and (3.43),

$$\begin{aligned} 1 + \mathscr{A}_j G_0(z) \mathscr{B}_j^* &= T_{v_j}(1 + (1 \otimes A_j)(-i\partial/\partial t \otimes I + I \otimes H_0 - z)^{-1}(I \otimes B_j)) T_{v_j}^* \\ &= T_{v_j} \mathscr{F}_t^{-1}(1 + (I \otimes A_j)(\tau + I \otimes H_0 - z)^{-1}(I \otimes B_j)) \mathscr{F}_t T_{v_j}^*, \end{aligned} \tag{3.75}$$

where τ is the conjugate variable of t. Now we regard as $L^2(\mathbb{R}^1) \otimes L^2(\mathbb{R}^n) = \int^{\oplus} L^2(\mathbb{R}^n) d\tau$; we write as $Q_j(z) = A_j(H_0 - z)^{-1} B_j$. Then by (3.75),

$$1 + \mathscr{A}_j G_0(z) \mathscr{B}_j^* = T_{v_j} \mathscr{F}_t^{-1} \left(\int^{\oplus} (1 + Q_j(z - \tau)) d\tau \right) \mathscr{F}_t T_{v_j}^*. \tag{3.76}$$

Since $(1 + Q_j(z))^{-1}$ exists for $z \in \mathbb{C}_\pm \cup \mathbb{R}^1$ and is uniformly bounded and Hölder continuous there by Sect. 1, (C). (i) and Assumption (III), (3.76) implies that

$$(1 + \mathscr{A}_j G_0(z) \mathscr{B}_j^*)^{-1} = T_{v_j} \mathscr{F}_t^{-1} \left(\int^{\oplus} (1 + Q_j(z - \tau))^{-1} d\tau \right) \mathscr{F}_t T_{v_j}^* \tag{3.77}$$

exists and is uniformly bounded and Hölder continuous in $z \in \mathbb{C}_\pm \cup \mathbb{R}^1$.

(Q.E.D.)

Lemma 3.13. *Let* $j \neq k$, $j, k = 1, 2, \ldots, N$. *Then*

(1) $\Gamma_j T_{v_j}^* \mathscr{B}_k \in \mathbf{B}(L^2(\mathbb{R}^{n+1}), L^2_\delta(\mathbb{R}^1))$;

(2) $\mathscr{A}_j T_{v_k} \Gamma_k^*, \mathscr{B}_j T_{v_k} \Gamma_k^* \in \mathbf{B}(L^2_\rho(\mathbb{R}^1), L^2_{\rho+\delta}(\mathbb{R}^{n+1}))$, $\rho \leqq 0$;

(3) $\Gamma_j T_{v_j}^* \mathscr{B}_k^* \mathscr{A}_k T_{v_k} \Gamma_k^*$ *is a multiplication operator by a function* $f_{j,k}(t)$ *satisfying* $|f_{j,k}(t)| \leqq C(1+t^2)^{-\delta}$.

Proof. We frequently use the following two results:
(i) If $v \neq v'$ and $\rho \leqq 0$,

$$(1+|x-vt|^2)^\rho(1+|x-v't|^2)^\rho \leqq C(1+|x|^2+|t|^2)^\rho, \tag{3.78}$$

where C is a constant depending only on ρ and $v-v'$;
(ii) By (1.6) and Sobolev's embedding theorem, for any $0 \leqq \rho$, $(1+|x|^2)^\rho \phi_j(x) \in L^q(\mathbb{R}^n)$, where $q = 2n/(n-4)$ if $n \geqq 5$; $1 \leqq q < \infty$ is arbitrary if $n = 4$; and $1 \leqq q \leqq \infty$ is arbitrary if $n = 3$. Therefore by Hölder's inequality,

$$(1+|x|^2)^\delta \phi_j(x) \in L^{2n/(n-2)}(\mathbb{R}^n), \quad n \geqq 3. \tag{3.79}$$

(1) $|(\Gamma_j T_{v_j}^* \mathscr{B}_k u)(t)| \leqq \int |\phi_j(x)(1+|x-(v_k-v_j)t|^2)^{-\delta/2} u(x-v_j t, t)|\, dx$.
By Schwartz's inequality and (3.78), the R.H.S. is majorized by the square root of

$$C(1+t^2)^{-\delta}\left(\int (1+x^2)^\delta |\phi_j(x)|^2 dx\right)\left(\int |u(x,t)|^2 dx\right).$$

Hence $\int (1+t^2)^{-\delta} |(\Gamma_j T_{v_j}^* \mathscr{B}_k u)(t)|^2 dt \leqq C \|(1+x^2)^{\delta/2}\phi_j\|^2 \|u\|^2$.
(2) $|(\mathscr{A}_j T_{v_k} \Gamma_k^* \sigma)(t,x)| = |A_j(x-v_j t)\phi_k(x-v_k t)\sigma(t)|$.
Hence by Remark 1.1, (1.6), (3.78), (3.79) and Hölder's inequality, we have

$$\begin{aligned}
&\int (1+x^2+t^2)^{\rho+\delta} |(\mathscr{A}_j T_{v_k} \Gamma_k^* \sigma)(t,x)|^2 dx\, dt \\
&\quad \leqq C \int (1+x^2+t^2)^\rho |W_{j,1}(x-v_j t) + W_{j,2}(x-v_j t)|^2 \\
&\qquad \cdot (1+|x-v_k t|^2)^\delta |\phi_k(x-v_k t)|^2 |\sigma(t)|^2 dx\, dt \\
&\quad \leqq C(\|W_{j,1}\|_{L^n}^2 \|(1+x^2)^\delta \phi_k\|_{L^{2n/(n-2)}}^2 + \|W_{j,2}\|_{L^\infty}^2 \|(1+x^2)^\delta \phi_k\|_{L^2}^2) \|\sigma\|_{L^2_\beta}^2.
\end{aligned}$$

The other is easier to prove and the proof is omitted.
(3) $\Gamma_j T_{v_j}^* \mathscr{A}_k \mathscr{B}_k^* T_{v_k} \Gamma_k^*$ is obviously a multiplication operator by a function $f_{j,k}(t)$ which is majorized by

$$\int |\phi_j(x) V_k(x-(v_k-v_j)t)\phi_k(x-v_k t)|\, dx. \tag{3.80}$$

By Hölder's inequality, (3.78) and (3.79), (3.80) is majorized by constant times $\|(1+x^2)^\delta \phi_j\|_{L^{2n/(n-2)}} \|(1+x^2)^\delta \phi_k\|_{L^{2n/(n-2)}} \|V_k\|_{L^{n/2}} (1+t^2)^{-\delta}$.

(Q.E.D.)

Lemma 3.14. *For any* $j, k = 1, 2, \ldots, N$,

$$\mathscr{B}_j^{-1} T_{v_j} \Gamma_j^* \in \mathbf{B}(L^2(\mathbb{R}^1), \quad L^2(\mathbb{R}^{n+1})); \tag{3.81}$$

$$\mathscr{A}_j T_{v_j} \Gamma_j^* \Gamma_j T_{v_j}^* \mathscr{B}_j^{-1}, \quad \mathscr{B}_k^{-1} T_{v_j} \Gamma_j^* \Gamma_j T_{v_j}^* \mathscr{B}_k \in \mathbf{B}(L^2(\mathbb{R}^{n+1})). \tag{3.82}$$

Proof. (3.81) is obvious by (1.6). We prove (3.82). By Schwartz's inequality, $f_j(t,x) = (\mathscr{A}_j T_{v_j} \Gamma_j^* \Gamma_j T_{v_j}^* \mathscr{B}_j^{-1} u)(t,x)$ satisfies $|f_j(t,x)| = |A_j(x-v_j t)\phi_j(x-v_j t)| \cdot \|\phi_j(x)(1+x^2)^{\delta/2}\| \|u(t,\cdot)\|$. Hence by Hölder's inequality, $\|f_j\| \leqq \|(1+x^2)^{\delta/2}\phi_j\| \cdot \|\phi_j\|_{L^{2n/(n-2)}} \|A_j\|_{L^n} \|u\|$. We prove the second. The case $j = k$ is obvious by (3.18). We assume $j \neq k$. By Lemma 3.13, (1), it suffices to prove

$\mathscr{B}_k^{-1}T_{v_j}\Gamma_j^* \in \mathbf{B}(L_\delta^2(\mathbb{R}^1),\ L^2(\mathbb{R}^{n+1}))$. By triangle inequality $(1+t^2)^{-\delta/2}(1+|x-vt|^2)^{\delta/2} \leqq C(1+x)^{\delta/2}$. Hence

$$\int |(\mathscr{B}_k^{-1}T_{v_j}\Gamma_j^*\sigma)(t,x)|^2 dt\,dx \leqq C\|(1+x^2)^{\delta/2}\phi_j\|^2 \|(1+t^2)^{\delta/2}\sigma\|^2, \tag{3.83}$$

from which the desired result follows. (Q.E.D.)

Lemma 3.15. *For any $j, k = 1, 2, \ldots, N$, $\mathscr{M}_{j,k}(z) = \mathscr{A}_j\mathscr{G}_j^c(z)\mathscr{B}_k$ satisfies the following properties.*

(1) *$\mathscr{M}_{j,k}(z)$ is a $\mathbf{B}(\mathfrak{K}_2)$-valued analytic function of $z \in \mathbb{C}_\pm$ and is uniformly bounded there.*

(2) *$\mathscr{M}_{j,k}(z)$ can be extended to the closed cut plane $\mathbb{C}_\pm \cup \mathbb{R}^1$ as a $\mathbf{B}(\mathfrak{K}_2)$-valued Hölder continuous function of z.*

(3) *If $j \neq k$, $\mathscr{M}_{j,k}(z)$ is a compact operator on $\mathfrak{K}_2$ for any $z \in \mathbb{C}_\pm \cup \mathbb{R}^1$.*

(4) $\|\mathscr{M}_{j,k}(z)\| \to 0$ as $|\mathrm{Im}\, z| \to \infty$.

Proof. By the resolvent equation

$$\begin{aligned}(-i\partial/\partial t \otimes I + I \otimes H_j - z)^{-1} \\ = G_0(z) - G_0(z)(I \otimes V_j)(-i\partial/\partial t \otimes I + I \otimes H_j - z)^{-1}.\end{aligned} \tag{3.84}$$

Multiply to the both sides of (3.84), $T_{v_j}A_j$ from the left and $(I \otimes P_j^c)T_{v_j}^*\mathscr{B}_k$ from the right. Then by (3.42), (3.43) and (3.46), we get

$$\mathscr{M}_{j,k}(z) = \mathscr{A}_jG_0(z)T_{v_j}(I \otimes P_j^c)T_{v_j}^*\mathscr{B}_k - \mathscr{A}_jG_0(z)\mathscr{B}_j\mathscr{M}_{j,k}(z). \tag{3.85}$$

Therefore by Lemma 3.12,

$$\mathscr{M}_{j,k}(z) = (1 + \mathscr{A}_jG_0(z)\mathscr{B}_j)^{-1}\mathscr{A}_jG_0(z)T_{v_j}(I \otimes P_j^c)T_{v_j}^*\mathscr{B}_k. \tag{3.86}$$

Here $\mathscr{A}_jG_0(z)T_{v_j}(I \otimes P_j^c)T_{v_j}^*\mathscr{B}_k = \mathscr{A}_jG_0(z)\mathscr{B}_k - \mathscr{A}_jG_0(z)\mathscr{B}_k\mathscr{B}_k^{-1}T_{v_j}\Gamma_j^*\Gamma_jT_{v_j}^*\mathscr{B}_k$. Since $\mathscr{A}_jG_0(z)\mathscr{B}_k$ satisfies Lemma 3.3 and is compact if $j \neq k$ by Lemma 3.8; $\mathscr{B}_k^{-}T_{v_j}\Gamma_j^*\Gamma_jT_{v_j}^*\mathscr{B}_k \in \mathbf{B}(\mathfrak{K}_2)$ by (3.82) and $(1 + \mathscr{A}_jG_0(z)\mathscr{B}_j)^{-1}$ satisfies Lemma 3.12, $\mathscr{M}_{j,k}(z)$ satisfies all the properties of Lemma 3.15.

(Q.E.D.)

We set as

$$\mathfrak{Y} = \bigoplus_{j=1}^{N} (L^2(\mathbb{R}^{n+1}) \oplus L_\delta^2(\mathbb{R}^1)) \tag{3.87}$$

with natural Hilbert space structure.

Lemma 3.16. (1) *$L(z)$ is a $\mathbf{B}(\mathfrak{X}_2, \mathfrak{Y})$-valued analytic function of $z \in \mathbb{C}_\pm$ and is uniformly bounded there.*

(2) *$L(z)$ can be extended to $\mathbb{C}_\pm \cup \mathbb{R}^1$ as a $\mathbf{B}(\mathfrak{X}_2, \mathfrak{Y})$-valued Hölder continuous function.*

Proof. Lemma 3.13, (1) implies that $\Gamma_jT_{v_j}^* \in \mathbf{B}(\mathfrak{X}_2, L_\delta^2(\mathbb{R}^1))$. Hence applying Lemma 3.15 to $\mathscr{A}_j\mathscr{G}_j^c(z)$, we get easily the statements of the lemma.

(Q.E.D.)

Finally we prove the following lemma.

Lemma 3.17. *The operator valued function $A(z)$ satisfies the following properties:*

(1) *$A(z)$ is a $\mathbf{B}(\mathfrak{Y})$-valued analytic function of $z\in\mathbb{C}_\pm$ and is uniformly bounded there.*
(2) *$A(z)$ can be extended to $\mathbb{C}_\pm\cup\mathbb{R}^1$ as a $\mathbf{B}(\mathfrak{Y})$-valued Hölder continuous function of z.*
(3) *For any $z\in\mathbb{C}_\pm\cup\mathbb{R}^1$, $A^2(z)$ is a compact operator on $\mathfrak{Y}$.*

(4) $\lim_{|\mathrm{Im}\, z|\to\infty} \|A^2(z)\|=0.$

Corollary 3.18. *There exists a closed null set $e\subset\mathbb{R}^1$ such that $(1+A(z))^{-1}$ can be extended to $\mathbb{C}_\pm\cup(\mathbb{R}^1\backslash e)$ as a $\mathbf{B}(\mathfrak{Y})$-valued locally Hölder continuous function of z.*

Proof. 1) $\mathscr{A}_j\mathscr{G}_j^c(z)\mathscr{B}_k$ satisfies the statments of Lemma 3.15.
2) $\Gamma_j T_{v_j}^*\mathscr{B}_k$ satisfies Lemma 3.13, (1).
3) Put $C_{j,k}(z)=\Gamma_j T_{v_j}^*\mathscr{B}_k^*\mathscr{A}_k T_{v_k}\Gamma_k^* G_k(z)$. Since $\Gamma_j T_{v_j}^*\mathscr{B}_k^*\mathscr{A}_k^* T_{v_k}\Gamma_k^*$ is a multiplication operator by $f_{j,k}(t)$ satisfying the estimate $|f_{j,k}(t)|\leqq C(1+t^2)^{-\delta}$ by Lemma 3.13, (3), Lemma 3.6 implies that $C_{j,k}(z)$ is a $\mathbf{B}_\infty(L_\delta^2(\mathbb{R}^1))$-valued bounded analytic function of $z\in\mathbb{C}_\pm$ and can be extended to $\mathbb{C}_\pm\cup\mathbb{R}^1$ as a Hölder continuous function; $\|C_{j,k}(z)\|\to 0$ as $|\mathrm{Im}\, z|\to\infty$.
4) Put $\mathscr{N}_{j,k}(z)=\mathscr{A}_j\mathscr{G}_j^c(z)\mathscr{B}_k^*\mathscr{A}_k T_{v_k}\Gamma_k^* G_k(z)$. We show that $\mathscr{N}_{j,k}(z)$ is a $\mathbf{B}_\infty(L_\delta^2(\mathbb{R}^1), L^2(\mathbb{R}^{n+1}))$-valued bounded analytic function of $z\in\mathbb{C}_\pm$ and can be extended to $\mathbb{C}_\pm\subset\mathbb{R}^1$ as a Hölder continuous function; $\|\mathscr{N}_{j,k}(z)\|\to 0$ as $|\mathrm{Im}\, z|\to\infty$ and is uniformly bounded. Using (3.85) for $\mathscr{A}_j G_j^c(z)\mathscr{B}_k$we get

$$\begin{aligned}\mathscr{N}_{j,k}(z)=&\mathscr{A}_j G_0(z) T_{v_j}(I\otimes P_j^c)T_{v_j}^*\mathscr{B}_k^*\mathscr{A}_k T_{v_k}\Gamma_k^* G_k(z)\\&-\mathscr{A}_j G_0(z)\mathscr{B}_j\mathscr{N}_{j,k}(z).\end{aligned}\tag{3.88}$$

Therefore by Lemma 3.12, we have

$$\begin{aligned}\mathscr{N}_{j,k}(z)&=(1+\mathscr{A}_j G_0(z)\mathscr{B}_j)^{-1}\mathscr{A}_j G_0(z) T_{v_j}(I\otimes P_j^c)T_{v_j}^*\mathscr{B}_k^*\mathscr{A}_k T_{v_k}\Gamma_k^* G_k(z).\\&=(1+\mathscr{A}_j G_0(z)\mathscr{B}_j)^{-1}\mathscr{A}_j G_0(z)\mathscr{B}_k^*(1-\mathscr{B}_k^{-1}T_{v_j}\Gamma_j^*\Gamma_j T_{v_j}^*\mathscr{B}_k^*)\mathscr{A}_k T_{v_k}\Gamma_k^* G_k(z).\end{aligned}\tag{3.89}$$

Now we apply Lemma 3.12 to the first factor; Lemma 3.15 to $\mathscr{A}_j G_0(z)\mathscr{B}_k^*$; Lemma 3.14, (8.82) to $(1-\mathscr{B}_k^{-1}T_{v_j}\Gamma_j^*\Gamma_j T_{v_j}^*\mathscr{B}_k^*)$; Lemma 3.6 and Lemma 3.13, (2) to $\mathscr{A}_k T_{v_k}\Gamma_k^* G_k(z)$. Then we get the desired result.
Combining these results (1)–(4), we get easily the statements of Lemma 3.17. Corollary 3.18 is a well-known result of Lemma 3.17.

(Q.E.D.)

3.5. Completion of the Proof of the Theorem

What is left to be proved is $R(\mathscr{W})=\mathfrak{H}_2$. We take $e\subset\mathbb{R}^1$ as in Corollary 3.18. We set $I=\mathbb{R}^1\backslash e$; $\mathfrak{H}_1$ and $\mathfrak{H}_2$ as (3.1); K_1 and K_2 as (3.3), (3.4) and (3.5); J as (3.2); $Y(z)$ as (3.60); $\mathfrak{X}_1$ and $\mathfrak{X}_2$ as (3.63). We first check that all the assumptions of Theorem 3.2 are satisfied. Condition (1) is satisfied by Lemma 3.1. In condition (2), for $j=1,2$, $\mathfrak{X}_j$ is a dense linear submanifold of $\mathfrak{H}_j$ and is a Hilbert space. (2.i) is satisfied by Lemma 3.7. The equation (3.15) is satisfied by (3.59). $Y(z)$ is a $\mathbf{B}(\mathfrak{X}_2,\mathfrak{X}_1)$-valued strongly continuous function of $z\in\mathbb{C}_\pm(I)\cup I$, since $(1+A(z))^{-1}L(z)$ is a $\mathbf{B}(\mathfrak{X}_2,\mathfrak{Y})$-valued strongly continuous function of $z\in\mathbb{C}_\pm(I)\cup I$ by Lemma 3.16 and Corollary 3.18; the injection operator from $\mathfrak{X}_2$ into $R(\mathscr{B})^*$ is bounded and $T_{v_j}\Gamma_j^*$ satisfies (3.81). Hence the all assumptions are satisfied. Therefore by Theorem 3.2. we see that K_1 and K_2 are absolutely continuous on I; $R(\mathscr{W}) \supset E_2(I)\mathfrak{K}_2$. Let us admit the following lemma for a moment.

Lemma 3.19. *K_2 is spectrally absolutely continuous on $\mathfrak{K}_2$.*
By Lemma 3.19, $E_2(I)\mathfrak{K}_2 = \mathfrak{K}_2$ since $\mathbb{R}^1 \backslash I$ has Lebesgue measure zero. Thus $R(\mathscr{W}) = \mathfrak{K}_2$. This completes the proof of the theorem.

Proof of Lemma 3.19. Let us define a one parameter unitary group $I(\tau)$, $-\infty < \tau < \infty$, and a unitary operator T on $\mathfrak{K}_2$ as

$$(I(\tau)u)(t, x) = u(t - \tau, x), \tag{3.90}$$

$$(Tu)(t, \cdot) = U(t, 0)u(t, \cdot), \quad u \in \mathfrak{K}_2. \tag{3.91}$$

By Sect. 1, (B), (ii), (3.4) and (3.5), we can easily see that

$$\exp(-i\tau K_2) = TI(\tau)T^*. \tag{3.92}$$

Since $I(\tau)$ has the absolutely continuous generator $-i(\partial/\partial t)$, (3.92) obviously implies that K_2 is absolutely continuous.

(Q.E.D.)

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