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On Contraction of Lie Algebra Representations

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Abstract. Given a net $(\mathfrak{g}_\iota)$ of finite-dimensional real Lie algebras contracting into a Lie algebra $\hat{\mathfrak{g}}$, a representation $\hat{\pi}_J$ of $\hat{\mathfrak{g}}$ is constructed explicitly as "limit" of a net (π_ι) of representations, each π_ι being a representation of $\mathfrak{g}_\iota$ on a complex Hilbert space $\mathfrak{H}_\iota$. Conditions are imposed on the net (π_ι) implying that the carrier space of $\hat{\pi}_J$ contain a $\hat{\pi}_J(\hat{\mathfrak{g}})$-stable set of vectors which are analytic for all $\hat{\pi}_J(g)$ $(g \in \mathscr{G})$, where $\mathscr{G}$ is a basis of $\hat{\mathfrak{g}}$. As a corollary, the corresponding result for contractions of representations of simply connected finite-dimensional real Lie groups is derived.

I. Introduction

In this note, we present a theory of contraction of nets of Lie algebra representations. Let J be a directed system (usually a subset of $\mathbf{R}$ with the induced ordering) and, for each $\iota \in J$, let π_ι be a representation of a finite-dimensional real Lie algebra $\mathfrak{g}_\iota$ on a complex Hilbert space $\mathfrak{H}_\iota$. Suppose that, in addition, every $\mathfrak{g}_\iota$ is isomorphic to a reference Lie algebra $\mathfrak{g}$ which is "contracting into $\hat{\mathfrak{g}}$" in a precise sense reviewed in Sect. II.2. We define and investigate a representation $\hat{\pi}_J$ of the contracted Lie algebra $\hat{\mathfrak{g}}$ whose carrier space is constructed in terms of the net $(\mathfrak{H}_\iota)$. The adopted definition permits to give the operators of $\hat{\pi}_J$ directly, without appealing to matrix elements, in a way which seems naturally suited to the problem considered. Since the theory of Lie algebra contraction is rooted in a notion of limit, which is responsible for the fact that $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ are not, in general, isomorphic, the final space cannot be defined in a "canonical" way [for instance, as the Hilbert sum or as a tensor product of the family $(\mathfrak{H}_\iota)$]. Our definition is rather more similar in spirit to Trotter's definition of a Banach space approximated by a sequence of Banach spaces ([1], Sect. 2), with the main difference that Trotter presupposes knowledge of the final space.

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More precisely, the paper is organized as follows. In Sect. II.1, we define the "limit" $\mathfrak{H}_J$ (resp. A_J^F) of a net $(\mathfrak{H}_\iota)$ [resp. (A_ι)] of complex Hilbert spaces [resp. of operators in $(\mathfrak{H}_\iota)$] indexed by a directed system J. We further provide conditions sufficient to assure the nontriviality of the operator A_J^F in $\mathfrak{H}_J$ (Proposition 1). In Proposition 2, conditions are imposed, for each $\iota \in J$, on the action of an operator A_ι on a set $\mathfrak{S}_\iota$ of vectors of $\mathfrak{H}_\iota$ in order that the net (A_ι) should define a nontrivial operator A_J^F in $\mathfrak{H}_J$ and that the linear span of the "limit vectors" of the net $(\mathfrak{S}_\iota)$ should be a set of analytic vectors for A_J^F. In Sect. II.2, we remind the notion of contraction of a net $(\mathfrak{g}_\iota)$ of finite-dimensional real Lie algebras into a Lie algebra $\hat{\mathfrak{g}}$ [2–5]. We also introduce the concept of a contracting net (π_ι) of representations of $(\mathfrak{g}_\iota)$ on $(\mathfrak{H}_\iota)$ and, in Proposition 3, we prove that a contracting net (π_ι) always defines a representation $\hat{\pi}_J$ of $\hat{\mathfrak{g}}$. In Proposition 4, we show the existence of $\hat{\pi}_J$ defined by a net (π_ι) satisfying conditions which are essentially borrowed from the assumptions of Proposition 2. Moreover, we prove that there exists a $\hat{\pi}_J(\hat{\mathfrak{g}})$-stable set of vectors which are analytic for all $\hat{\pi}_J(g)$ with g belonging to an appropriate basis of $\hat{\mathfrak{g}}$. From this it follows that our assertion on the existence of $\hat{\pi}_J$ in Proposition 4 has a counterpart in a result on Lie group representations. This is stated in a corollary at the end of Sect. II.2.

In what follows, J will always denote a directed system and every net considered in the present paper will be indexed by J; we shall always write $(\mathfrak{H}_\iota)$, $(A_\iota), \ldots$ short for $(\mathfrak{H}_\iota)_{\iota \in J}$, $(A_\iota)_{\iota \in J}, \ldots$. The symbol $\mathrm{D}(A)$ will denote the domain of a (bounded or unbounded linear) operator A and, given a subset D of a Hilbert space $\mathfrak{H}$, $\mathrm{sp}(D)$ will stand for the linear span of D in $\mathfrak{H}$.

II. Contraction of Representations

II.1. Hilbert Spaces of Equivalence Classes of Convergent Nets

For each $\iota \in J$, let $\mathfrak{H}_\iota$ be a complex Hilbert space and let $(\cdot|\cdot)_\iota$ be the inner multiplication on $\mathfrak{H}_\iota$. We say that an element (ϕ_ι) of the product vector space $\prod_\iota \mathfrak{H}_\iota$ of the family $(\mathfrak{H}_\iota)$ is *convergent* if the net $(\|\phi_\iota\|_\iota)$ converges in $\mathbf{R}$. Let $\mathscr{H}_J$ be the vector subspace of all convergent nets $(\phi_\iota) \in \prod_\iota \mathfrak{H}_\iota$ equipped with the positive Hermitian form $(\cdot|\cdot)_J$ defined by

$$((\phi_\iota)|(\phi'_\iota))_J = \lim_\iota (\phi_\iota|\phi'_\iota)_\iota \,. \tag{II.1}$$

We note that (II.1) is meaningful by reason of the polarization identity. If $\mathscr{I}$ is the vector subspace of all nets $(\phi_\iota) \in \mathscr{H}_J$ such that $\|(\phi_\iota)\|_J = 0$, we define a Hilbert space $\mathfrak{H}_J$ as the completion of the quotient vector space $\mathscr{H}_J/\mathscr{I}$ endowed with the extended quotient form which we shall also write $(\cdot|\cdot)_J$.

We shall denote by $[\phi_\iota], [\psi_\iota], \ldots$, *respectively, the equivalence classes modulo* $\mathscr{I}$ of the elements $(\phi_\iota), (\psi_\iota), \ldots$ of $\mathscr{H}_J$. If D is a subset of $\mathscr{H}_J$, the symbol $[D]$ will stand for the subset of all $[\phi_\iota] \in \mathfrak{H}_J$ with $(\phi_\iota) \in D$.

Notice that if $(\phi_\iota) \in \mathscr{H}_J$ and if (γ_ι) is a net of complex numbers converging to γ in $\mathbf{C}$, then $[\gamma_\iota \phi_\iota] = \gamma[\phi_\iota]$.

Remark 1. If there exists a complex Hilbert space $\mathfrak{H}$ such that the net $(\mathfrak{H}_\iota)$ satisfies $\mathfrak{H}_\iota = \mathfrak{H}$ for all $\iota \in J$, then the mapping $\alpha: \phi \mapsto [\phi_\iota]$ with $\phi_\iota = \phi$ for all $\iota \in J$ is a closed injective homomorphism of $\mathfrak{H}$ into $\mathfrak{H}_J$. However, in general, $\alpha(\mathfrak{H})$ is a *proper* subspace of $\mathfrak{H}_J$. For instance, if $J=\mathbf{N}$ and $\mathfrak{H}$ is separable with an orthonormal basis $\{\phi^{(n)}\}_{n\in\mathbf{N}}$, then the element $[\phi_n]$ of $\mathfrak{H}_J$ defined by $\phi_n=\phi^{(n)}$ $(n\in\mathbf{N})$ is orthogonal to $\alpha(\mathfrak{H})$.

A net of operators in$(\mathfrak{H}_\iota)$ is a family (A_ι) of (linear) operators, where every A_ι is an operator in $\mathfrak{H}_\iota$. Let $D_J(A_\iota)$ be the vector subspace of all $(\phi_\iota)\in\mathscr{H}_J$, with $\phi_\iota\in \mathrm{D}(A_\iota)$ for all $\iota\in J$, such that the net $(A_\iota\phi_\iota)$ is convergent and, given an arbitrary vector subspace F of $D_J(A_\iota)$, let $D_u^F(A_\iota)$ be the vector subspace of all $(\phi_\iota)\in F$ such that we have $[A_\iota\phi_\iota]=[A_\iota\phi_\iota']$ whenever (ϕ_ι') is an element of F in the equivalence class of (ϕ_ι) modulo $\mathscr{J}$. Then we define an operator A_J^F in $\mathfrak{H}_J$ by

$$A_J^F[\phi_\iota]=[A_\iota\phi_\iota] \qquad ((\phi_\iota)\in D_u^F(A_\iota)) \tag{II.2}$$

with $\mathrm{D}(A_J^F)=[D_u^F(A_\iota)]$. It can happen that $\mathrm{D}(A_J^F)=\{0\}$, i.e., that A_J^F is trivial. If F' is a vector subspace of F, then $A_J^{F'}\subseteqq A_J^F$, i.e., $\mathrm{D}(A_J^{F'})\subseteqq \mathrm{D}(A_J^F)$ and $A_J^{F'}[\phi_\iota]=A_J^F[\phi_\iota]$ for all $[\phi_\iota]\in \mathrm{D}(A_J^{F'})$ (in words: $A_J^{F'}$ is a restriction of A_J^F).

Remark 2. Even if the elements (ϕ_ι) and (ϕ_ι') of $\mathscr{H}_J$ are in the same equivalence class modulo $\mathscr{J}$, it can occur that $(A_\iota\phi_\iota)$ is convergent but $(A_\iota\phi_\iota')$ is not.

Remark 3. The above definitions are related to those introduced, in another context, by Trotter [1] and Kurtz [6]. For each $\iota\in J$, we can define a linear mapping P_ι of $\mathscr{H}_J/\mathscr{J}$ into $\mathfrak{H}_\iota$ such that

$$\lim_\iota \|P_\iota[\phi_{\iota'}]\|_\iota=\|[\phi_{\iota'}]\|_J$$

for all $(\phi_{\iota'})\in\mathscr{H}_J$. But, contrarily to the corresponding Trotter's mappings, the P_ι are, in general, unbounded [see however Proposition 4(v)].

Of particular importance in the sequel will be nets (A_ι) of operators in $(\mathfrak{H}_\iota)$ satisfying the following condition

(K) *For each $\iota\in J$, each $\phi\in\mathrm{D}(A_\iota)$, and each real (resp. imaginary) scalar λ, we have*

$$\|\lambda\phi-A_\iota\phi\|_\iota\geqq|\lambda|\,\|\phi\|_\iota\,. \tag{II.3}$$

Notice that nets of skew-symmetric (resp. symmetric) operators satisfy Condition (K), as can be checked by developing the square of the left-hand side of (II.3).

Proposition 1. *Let (A_ι) be a net of operators in $(\mathfrak{H}_\iota)$ satisfying Condition* (K). *If F is a vector subspace of $D_J(A_\iota)\cap D_J(A_\iota^2)$, then $D_u^F(A_\iota)=F$.*

Proof. Our proof is modeled on that of Kurtz's Lemma (1.1) [6]. Let (ϕ_ι), (ϕ_ι') be two elements of F in the same equivalence class modulo $\mathscr{J}$ and put $(\psi_\iota)=(\phi_\iota)-(\phi_\iota')$. Then, if λ is any real (resp. imaginary) scalar, we have by virtue of (II.3)

$$\lim_\iota\|\{\lambda\mathrm{Id}_{\mathfrak{H}_\iota}-A_\iota\}\{A_\iota\psi_\iota+\lambda\psi_\iota\}\|_\iota=\|[A_\iota^2\psi_\iota]\|_J\geqq|\lambda|\,\|[A_\iota\psi_\iota]\|_J\,,$$

whence $[A_\iota\psi_\iota]=0$. $\square$

Let (A_ι), (B_ι) be two nets of operators both in $(\mathfrak{H}_\iota)$ and let F be a vector subspace of $D_J(A_\iota B_\iota)$ such that $(A_\iota\phi_\iota)\in F$, $(B_\iota\phi_\iota)\in F$ whenever $(\phi_\iota)\in F$. Then we define an operator $(AB)_J^F$ in $\mathfrak{H}_J$ by

$$(AB)_J^F[\phi_\iota]=[A_\iota B_\iota\phi_\iota] \qquad ((\phi_\iota)\in D_u^F(A_\iota B_\iota))$$

with $\mathrm{D}((AB)_J^F)=[D_u^F(A_\iota B_\iota)]$.

Remark 4. If F is as in the definition of $(AB)_J^F$ and if, in addition, $D_u^F(A_\iota) = D_u^F(B_\iota)=F$, we have $D_u^F(A_\iota B_\iota)=F$ and $(AB)_J^F=A_J^F B_J^F$.

Proposition 2. *Let (A_ι) be a net of operators in $(\mathfrak{H}_\iota)$ satisfying Condition* (K), *let S be a subset of* $\mathbf{R}$, *and let $\mathfrak{S}_J=\{[\phi_\iota^{(s)}]\}_{s\in S}$ be a norm-bounded subset of $\mathfrak{H}_J$ with $\phi_\iota^{(s)}\in \mathrm{D}(A_\iota)$ for all $\iota\in J$ and all $s\in S$. Suppose that, for each $\iota\in J$ and each $s\in S$, we have*

$$A_\iota\phi_\iota^{(s)}=\sum_{m=-k}^{k} c_{\iota,s,m}\phi_\iota^{(s+m)}\,, \tag{II.4}$$

where k is a positive integer, the $c_{\iota,s,m}$ are complex numbers with $c_{\iota,s,m}=0$ whenever $s+m\notin S$, and, for any fixed pair s,m, the net $(c_{\iota,s,m})$ converges in $\mathbf{C}$ *to $c_{s,m}$. If there exist two real numbers v and t such that $|c_{s,m}|\leqq v(|s|+|t|)$ for $-k\leqq m\leqq k$ and all $s\in S$, then*

(i) $\mathrm{sp}(\{(\phi_\iota^{(s)})\}_{s\in S})$ *is a vector subspace of* $W=\bigcap_{n\in\mathbf{N}^*} D_J(A_\iota^n)$.

(ii) $\mathrm{sp}(\mathfrak{S}_J)$ *is a set of analytic vectors for A_J^F, where F is any vector subspace of W containing* $\mathrm{sp}(\{(\phi_\iota^{(s)})\}_{s\in S})$.

Proof. Assertion (i) follows from (II.4). To prove (ii), we remark that we have $D_u^F(A_\iota^n)=D_u^F(A_\iota)=F$ for all $n\in\mathbf{N}^*$ by virtue of Proposition 1. Hence $\mathrm{sp}(\mathfrak{S}_J)$ is a set of C^∞-vectors for A_J^F because $(A^n)_J^F=(A_J^F)^n$ for all $n\in\mathbf{N}^*$. Now let r be a real number $\geqq 1$ such that $\|[\phi_\iota^{(s)}]\|_J\leqq r$ for all $s\in S$. For each $n\in\mathbf{N}^*$ and each $s\in S$, we have (with $-k\leqq m_i\leqq k$ and $1\leqq i\leqq n$)

$$\begin{aligned}
&\|(A_J^F)^n[\phi_\iota^{(s)}]\|_J\\
&=\lim_\iota\Big\|\sum_{m_\iota} c_{\iota,s,m_1}c_{\iota,s+m_1,m_2}\cdots c_{\iota,s+m_1+\ldots+m_{n-1},m_n}\phi_\iota^{(s+m_1+m_2+\ldots+m_n)}\Big\|_\iota\\
&\leqq\sum_{m_\iota}|c_{s,m_1}|\ldots|c_{s+m_1+\ldots+m_{n-1},m_n}|r\\
&\leqq(2k+1)^n v^n(|s|+|t|)(|s|+|t|+k)\ldots(|s|+|t|+(n-1)k)r\leqq d^n n!\,,
\end{aligned}$$

where d is some positive real number depending on k,r,v, and $(|s|+|t|)$, but not on n. Therefore $[\phi_\iota^{(s)}]$ is an analytic vector for A_J^F and $\mathrm{sp}(\mathfrak{S}_J)$ is a set of analytic vectors for A_J^F. □

II.2. Contraction of Nets of Lie Algebra Representations

To begin with, let us recollect some facts about contraction of Lie algebras. In this Section, V will always stand for a *finite-dimensional real vector space.*

Let $\mathfrak{M}_V$ be the algebraic set of all Lie multiplications on V, i.e., the set of all bilinear alternating mappings of $V\times V$ into V satisfying the Jacobi identity. Notice that, by choosing a basis of V, we can identify $\mathfrak{M}_V$ with the set of all families of structure constants of Lie algebras with underlying vector space V. We give $\mathfrak{M}_V$ the topology induced by the canonical Hausdorff topology of the vector space of all bilinear alternating mappings of $V \times V$ into V. We shall denote by $\operatorname{alg}(V, \mu)$ the Lie algebra with underlying vector space V and Lie multiplication μ.

A net $(\operatorname{alg}(V, \mu_\iota))$ of Lie algebras is said to be *contracting* with respect to a Lie algebra $\operatorname{alg}(V, \mu)$ if all its elements are isomorphic to $\operatorname{alg}(V, \mu)$ and if the net (μ_ι) converges in $\mathfrak{M}_V$ to some Lie multiplication $\hat{\mu}$. In other words, the net $(\operatorname{alg}(V, \mu_\iota))$ is contracting into $\operatorname{alg}(V, \hat{\mu})$ if there exists a net (Γ_ι) of automorphisms of V such that, for each $\iota \in J$, the mapping Γ_ι is an isomorphism of $\operatorname{alg}(V, \mu_\iota)$ onto $\operatorname{alg}(V, \mu)$ and

$$\lim_\iota \Gamma_\iota^{-1}(\mu(\Gamma_\iota(g), \Gamma_\iota(g'))) = \hat{\mu}(g, g')$$

for all g, g' in V. Alternatively, if we are given the net (Γ_ι), then the net (μ_ι) is defined by

$$\mu_\iota(g, g') = \Gamma_\iota^{-1}(\mu(\Gamma_\iota(g), \Gamma_\iota(g'))) .$$

The Lie algebra $\operatorname{alg}(V, \hat{\mu})$ is said to be the *contracted Lie algebra* of the net $(\operatorname{alg}(V, \mu_\iota))$ and the operation performed is called a *contraction* (of a net of Lie algebras). By abuse of language, we shall also say that $\operatorname{alg}(V, \hat{\mu})$ is a contraction of $\operatorname{alg}(V, \mu)$. We shall call $\operatorname{alg}(V, \mu)$ the reference Lie algebra and (Γ_ι) the reference net of automorphisms of V. Notice that we do not exclude the trivial case of a contracted Lie algebra isomorphic to the reference Lie algebra.

We remind that a *representation* of a Lie algebra $\mathfrak{g} = \operatorname{alg}(V, \mu)$ on a complex Hilbert space $\mathfrak{H}$ is an ordered pair $(\pi, \mathrm{D}(\pi))$, where π is a mapping of $\mathfrak{g}$ into the set of all operators in $\mathfrak{H}$ and $\mathrm{D}(\pi)$ is a $\pi(\mathfrak{g})$-stable vector subspace of $\bigcap_{g \in \mathfrak{g}} \mathrm{D}(\pi(g))$ dense in $\mathfrak{H}$, such that

$$\pi(\gamma g + \gamma' g')\phi = \gamma \pi(g)\phi + \gamma' \pi(g')\phi ,$$
$$(\pi(g)\pi(g') - \pi(g')\pi(g))\phi = \pi(\mu(g, g'))\phi$$

for all γ, γ' in $\mathbf{R}$, all g, g' in $\mathfrak{g}$, and all $\phi \in \mathrm{D}(\pi)$. In what follows, we shall simply say "the representation π", tacitly understanding that $\mathrm{D}(\pi)$, which is called the *domain of* π, is also given. By the *restriction of* π to a $\pi(\mathfrak{g})$-stable vector subspace D' of $\mathrm{D}(\pi)$, we shall mean the representation π' of $\mathfrak{g}$ on the closure of D' in $\mathfrak{H}$ defined by $\pi'(g)|D' = \pi(g)|D'$ and with $\mathrm{D}(\pi') = D'$. We shall denote by $\operatorname{Env}(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$; thus $\pi^{(\mathrm{Env})}$ will stand for the canonical extension of π to a representation of $\operatorname{Env}(\mathfrak{g})$ on $\mathfrak{H}$ [with domain $\mathrm{D}(\pi)$]. The representation π is said to be symmetric (resp. skew-symmetric) if $\pi(g)$ is symmetric (resp. skew-symmetric) for all $g \in \mathfrak{g}$. The meaning of "irreducibility" and "(unitary) equivalence" of Lie algebra representations should be clear.

Given a net $(\mathfrak{g}_\iota) = (\operatorname{alg}(V, \mu_\iota))$ of Lie algebras contracting into $\hat{\mathfrak{g}} = \operatorname{alg}(V, \hat{\mu})$ with respect to $\mathfrak{g} = \operatorname{alg}(V, \mu)$ by means of a reference net (Γ_ι) of automorphisms of V and, for each $\iota \in J$, a representation π_ι of $\mathfrak{g}_\iota$ on a complex Hilbert space $\mathfrak{H}_\iota$, let $\tilde{\pi}_\iota$ be the representation $\pi_\iota \circ \Gamma_\iota^{-1}$ of $\mathfrak{g}$ on $\mathfrak{H}_\iota$. We put

$$D_J(\pi_\iota) = \bigcap_{x \in \operatorname{Env}(\mathfrak{g})} D_J(\tilde{\pi}_\iota^{(\mathrm{Env})}(x)) ,$$

we denote by $D_u(\pi_\iota)$ the vector subspace of all $(\phi_\iota) \in D_J(\pi_\iota)$ such that we have

$$[\tilde{\pi}_\iota^{(\mathrm{Env})}(x)\phi_\iota] = [\tilde{\pi}_\iota^{(\mathrm{Env})}(x)\phi'_\iota]$$

for all $x \in \mathrm{Env}(\mathfrak{g})$ whenever (ϕ'_ι) is an element of $D_J(\pi_\iota)$ in the equivalence class of (ϕ_ι) modulo $\mathscr{J}$, and we write $D_u^q(\pi_\iota)$ instead of $[D_u(\pi_\iota)]$. Notice that

$$D_u(\pi_\iota) = \bigcap_{x \in \mathrm{Env}(\mathfrak{g})} D_u^{D_J(\pi_\iota)}(\tilde{\pi}_\iota^{(\mathrm{Env})}(x)) .$$

A net (π_ι) of representations of $(\mathfrak{g}_\iota)$ on $(\mathfrak{H}_\iota)$ is said to be *contracting* if $D_u(\pi_\iota) = D_J(\pi_\iota) \neq \{0\}$. In other words, (π_ι) is contracting if $D_J(\pi_\iota) \neq \{0\}$ and

$$D_u^{D_J(\pi_\iota)}(\tilde{\pi}_\iota^{(\mathrm{Env})}(x)) = D_J(\pi_\iota)$$

for all $x \in \mathrm{Env}(\mathfrak{g})$. For each $g \in V$, we shall denote by $\pi_J(g)$ the operator defined by (II.2) with $A_\iota = \pi_\iota(g)$ and $F = D_J(\pi_\iota)$. Moreover, $\hat{\pi}_J$ will stand for the mapping of V into the set of all operators in $\mathrm{cl}(D_u^q(\pi_\iota))$ [the closure of $D_u^q(\pi_\iota)$ in $\mathfrak{H}_J$] obtained by restricting to $D_u^q(\pi_\iota)$ all the operators $\pi_J(g)$.

Remark 5. For each $g \in V$, each $(\phi_\iota) \in D_u(\pi_\iota)$, and each $(\phi'_\iota) \in \mathscr{H}_J$, we have

$$(\hat{\pi}_J(g)[\phi_\iota] | [\phi'_\iota])_J = \lim_\iota (\pi_\iota(g)\phi_\iota | \phi'_\iota)_\iota .$$

We can now realize as follows the proposal of Inönü and Wigner ([3], Sect. IIb) for the study of representations of contracted Lie algebras.

Proposition 3. *Let* $(\mathfrak{g}_\iota) = (\mathrm{alg}(V, \mu_\iota))$ *be a net of Lie algebras contracting into* $\hat{\mathfrak{g}} = \mathrm{alg}(V, \hat{\mu})$ *with respect to* $\mathfrak{g} = \mathrm{alg}(V, \mu)$ *by means of a reference net* (Γ_ι) *of automorphisms of* V, *and let* (π_ι) *be a contracting net of representations of* $(\mathfrak{g}_\iota)$ *on a net* $(\mathfrak{H}_\iota)$ *of complex Hilbert spaces. Then* $\hat{\pi}_J$ *is a representation of* $\hat{\mathfrak{g}}$ *on* $\mathrm{cl}(D_u^q(\pi_\iota))$ *with* $\mathrm{D}(\hat{\pi}_J) = D_u^q(\pi_\iota)$.

Proof. The mapping $\hat{\pi}_J$ is obviously linear. Let n be the dimension of V, let $\mathscr{G} = \{g_j\}_{1 \leqq j \leqq n}$ be a basis of V, and, for $1 \leqq j, k, l \leqq n$, let $\hat{c}_{jk}^l$ (resp. $c_{(\iota)jk}^l$) be the structure constants of $\hat{\mathfrak{g}}$ [resp. $\mathfrak{g}_\iota$ $(\iota \in J)$] with respect to $\mathscr{G}$. By Remark 4, we have

$$\begin{aligned}
&(\hat{\pi}_J(g_j)\hat{\pi}_J(g_k) - \hat{\pi}_J(g_k)\hat{\pi}_J(g_j))[\phi_\iota] \\
&= [\pi_\iota(\mu_\iota(g_j, g_k))\phi_\iota] = \sum_{l=1}^n [c_{(\iota)jk}^l \pi_\iota(g_l)\phi_\iota] = \sum_{l=1}^n \hat{c}_{jk}^l [\pi_\iota(g_l)\phi_\iota] \\
&= \hat{\pi}_J(\hat{\mu}(g_j, g_k))[\phi_\iota]
\end{aligned}$$

for all pairs g_j, g_k of elements of $\mathscr{G}$ and all $[\phi_\iota] \in D_u^q(\pi_\iota)$. □

The operation performed to obtain $\hat{\pi}_J$ is called a *contraction* (of a net of Lie algebra representations).

Let $\mathfrak{H}$ be a Hilbert space, let $(\mathfrak{H}_\iota)$ be a net of Hilbert spaces, and, for each $\iota \in J$, *let* P_ι *be a continuous linear mapping of* $\mathfrak{H}$ *into* $\mathfrak{H}_\iota$. *The net* $(\mathfrak{H}_\iota)$ *is said to approximate* $\mathfrak{H}$ *with respect to the net* (P_ι) ([1], Sect. 2) if $\|P_\iota\| \leqq 1$ for all $\iota \in J$ and $\lim_\iota \|P_\iota \phi\|_\iota = \|\phi\|_{\mathfrak{H}}$ for all $\phi \in \mathfrak{H}$. Then a net $(\phi_\iota) \in \prod_\iota \mathfrak{H}_\iota$ is (P_ι)-*convergent to* $\phi \in \mathfrak{H}$, and we shall write $\phi = (P_\iota) - \lim \phi_\iota$, if $\lim_\iota \|\phi_\iota - P_\iota \phi\|_\iota = 0$; a net (A_ι) of operators in $(\mathfrak{H}_\iota)$ is (P_ι)-*convergent to the operator* A *in* $\mathfrak{H}$, and we shall write $A = (P_\iota) - \lim A_\iota$, if $A\phi = (P_\iota) - \lim A_\iota P_\iota \phi$ for all $\phi \in \mathrm{D}(A)$.

Proposition 4. *Let $(\mathfrak{g}_\iota)$, $\hat{\mathfrak{g}}$, $\mathfrak{g}$, (Γ_ι) be as in Proposition 3 and let $\mathcal{G}$ be a basis of V. For each $\iota \in J$, let $\mathfrak{H}_\iota$ be a complex Hilbert space of dimension at most* Card$(\mathbf{R})$, *let π_ι be a skew-symmetric representation of $\mathfrak{g}_\iota$ on $\mathfrak{H}_\iota$, let S_ι be a subset of $\mathbf{R}$ such that $S_{\iota'} \subseteq S_\iota$ whenever $\iota' < \iota$, and let $\{\phi_\iota^{(s)}\}_{s\in S_\iota}$ be an orthonormal basis of $\mathfrak{H}_\iota$ contained in* $\mathrm{D}(\pi_\iota)$. *Suppose that for $-k \leqq m \leqq k$, where k is a fixed positive integer, for each $s \in S = \bigcup_\iota S_\iota$, and each $g \in \mathcal{G}$, we have a net $(c_{\iota,s,m}(g))$ of complex numbers, which are* 0 *whenever $s \notin S_\iota$ or $s+m \notin S_\iota$, converging in $\mathbf{C}$ to $c_{s,m}(g)$. If*

$$\pi_\iota(g)\phi_\iota^{(s)} = \sum_{m=-k}^{k} c_{\iota,s,m}(g)\phi_\iota^{(s+m)} \tag{II.5}$$

for all $\iota \in J$, all $s \in S_\iota$, all $g \in \mathcal{G}$, and if there exist two real numbers $v(g)$, $t(g)$ such that

$$|c_{s,m}(g)| \leqq v(g)\,(|s| + |t(g)|) \tag{II.6}$$

for $-k \leqq m \leqq k$, all $s \in S$, and all $g \in \mathcal{G}$, then

(i) *For each $s \in S$, the net $(\psi_\iota^{(s)}) \in \prod_\iota \mathfrak{H}_\iota$, where $\psi_\iota^{(s)} = \phi_\iota^{(s)}$ whenever $s \in S_\iota$ and $\psi_\iota^{(s)} = 0$ otherwise, is convergent and $\mathfrak{S}_J = \{[\psi_\iota^{(s)}]\}_{s\in S}$ is an orthonormal system in $\mathfrak{H}_J$.*

(ii) *$\hat{\pi}_J$ is a skew-symmetric representation of $\hat{\mathfrak{g}}$ on* $\mathrm{cl}\,(D_u^q(\pi_\iota))$ *with* $\mathrm{D}(\hat{\pi}_J) = D_u^q(\pi_\iota)$.

(iii) *For each $g \in \mathcal{G}$,* $\mathrm{sp}\,(\mathfrak{S}_J)$ *is a $\hat{\pi}_J(\hat{\mathfrak{g}})$-stable set of analytic vectors for $\hat{\pi}_J(g)$.*

(iv) *The restriction $\hat{\pi}$ of $\hat{\pi}_J$ to* $\mathrm{sp}\,(\mathfrak{S}_J)$ *is a skew-symmetric representation of $\hat{\mathfrak{g}}$ on the closure $\mathfrak{H}$ of* $\mathrm{sp}\,(\mathfrak{S}_J)$ *in $\mathfrak{H}_J$; for each $g \in \mathcal{G}$ and each $s \in S$, we have*

$$\hat{\pi}(g)\,[\psi_\iota^{(s)}] = \sum_{m=-k}^{k} c_{s,m}(g)\,[\psi_\iota^{(s+m)}]\,.$$

(v) *The net $(\mathfrak{H}_\iota)$ approximates $\mathfrak{H}$ with respect to the net (P_ι) defined in $\mathfrak{S}_J$ by $P_\iota[\psi_{\iota'}^{(s)}] = \psi_\iota^{(s)}$ $(s \in S)$ and extended to $\mathfrak{H}$ by linearity and continuity. For each $g \in V$, we have $\hat{\pi}(g) = (P_\iota) - \lim \pi_\iota(g)$.*

Proof. Assertion (i) follows at once from the remark that, by reason of the assumption on the sets S_ι $(\iota \in J)$ we have $(\psi_\iota^{(s)}) \in \mathscr{H}_J$ and $[\psi_\iota^{(s)}] \neq 0$ for all $s \in S$. Now for each $g \in V$ and each $n \in \mathbf{N}^*$, we have

$$D_u^{D_J(\pi_\iota)}(\pi_\iota(g)^n) = D_J(\pi_\iota)$$

by virtue of Proposition 1; it follows, on account of Remark 4 and by using the Poincaré-Birkhoff-Witt theorem [7], that $D_u(\pi_\iota) = D_J(\pi_\iota)$. On the other hand, $\mathrm{sp}\,(\{(\psi_\iota^{(s)})\}_{s\in S}) \subseteq D_J(\pi_\iota)$ by (II.5), so that the net (π_ι) is contracting. Then Proposition 3 implies (ii), the skew-symmetry of $\hat{\pi}_J$ being a consequence of that of the π_ι $(\iota \in J)$.

Assertion (iii) follows from Proposition 2, whereas (iv) is obvious. To prove (v) it is sufficient to note that, for each $g \in \mathcal{G}$ and each $s \in S$, we have

$$P_\iota \hat{\pi}(g)\,[\psi_{\iota'}^{(s)}] = \sum_{m=-k}^{k} c_{s,m}(g)\psi_\iota^{(s+m)}\,,$$

whence

$$\lim_\iota \|\pi_\iota(g)P_\iota[\psi_{\iota'}^{(s)}] - P_\iota\hat{\pi}(g)\,[\psi_{\iota'}^{(s)}]\|_\iota$$

$$\leqq \sum_{m=-k}^{k} \lim_\iota |c_{\iota,s,m}(g) - c_{s,m}(g)| = 0\,. \quad \square$$

Remark 6. By virtue of Proposition 1, Proposition 4 is still true if "skew-symmetric" is everywhere replaced by "symmetric".

Remark 7. Since the topological space $\mathfrak{M}_V$ is metrizable, it is always possible to use countable index sets in the study of Lie algebra contractions. But this is, a priori, no longer the case when contractions of representations are considered.

Let G be a finite-dimensional real Lie group, let $\mathrm{Lie}(G)$ be its Lie algebra, and let $\mathfrak{H}$ be a complex Hilbert space. We remind that a strongly continuous unitary representation U of G on $\mathfrak{H}$ defines a skew-symmetric representation dU of $\mathrm{Lie}(G)$ on $\mathfrak{H}$ by

$$dU(g)\phi = \lim_{t\to 0} \frac{U(\exp(tg))\phi - \phi}{t} \qquad (\phi \in \mathfrak{H}^{\infty}(G)),$$

where $\mathfrak{H}^{\infty}(G) = \mathrm{D}(dU)$ is the vector subspace of all $\phi \in \mathfrak{H}$ such that the mapping $x \mapsto U(x)\phi$ of G into $\mathfrak{H}$ is of class C^{∞}. The representation dU is called the *differential* of U. A skew-symmetric representation π on $\mathfrak{H}$ of a finite-dimensional real Lie algebra $\mathfrak{g}$ is said to be *integrable* if, for every simply connected Lie group G whose Lie algebra is isomorphic to $\mathfrak{g}$, there exist a (necessarily unique) strongly continuous unitary representation U of G on $\mathfrak{H}$ and an isomorphism θ of $\mathfrak{g}$ onto $\mathrm{Lie}(G)$ such that $\pi(g) \subseteq dU(\theta(g))$ for all $g \in \mathfrak{g}$.

By ([8], Theorem 1), we now have the

Corollary. *Let G be a simply connected finite-dimensional real Lie group and let $\mathfrak{g} = \mathrm{alg}(V, \mu)$ be its Lie algebra. For each $\iota \in J$, let U_ι be a strongly continuous unitary representation of G on a complex Hilbert space $\mathfrak{H}_\iota$ of dimension at most* Card (**R**), *let (Γ_ι) be a reference net of automorphisms of V for a contraction $\hat{\mathfrak{g}}$ of $\mathfrak{g}$, and put $\pi_\iota = dU_\iota \circ \Gamma_\iota$. If $\mathscr{G}$, S_ι, $\{\phi_\iota^{(s)}\}_{s\in S_\iota}$, and $(c_{\iota,s,m}(g))$ are as in Proposition 4 and if* (II.5), (II.6) *are satisfied, then the skew-symmetric representation $\hat{\pi}$ of $\hat{\mathfrak{g}}$ on $\mathfrak{H}$ defined in Proposition 4*(iv) *is integrable.*

References

1. Trotter, H.F.: Approximation of semi-groups of operators. Pacific J. Math. **8**, 887–919 (1958)
2. Segal, I.E.: A class of operator algebras which are determined by groups. Duke Math. J. **18**, 221–265 (1951)
3. Inönü, E., Wigner, E.P.: On the contraction of groups and their representations. Proc. Natl. Acad. Sci. U.S.A. **39**, 510–524 (1953)
4. Saletan, E.J.: Contraction of Lie groups. J. Math. Phys. **2**, 1–21 (1961)
5. Lévy-Nahas, M.: Deformation and contraction of Lie algebras. J. Math. Phys. **8**, 1211–1222 (1967)
6. Kurtz, T.G.: Extensions of Trotter's operator semigroup approximation theorems. J. Funct. Anal. **3**, 354–375 (1969)
7. Jacobson, N.: Lie algebras. New York: Interscience 1962
8. Flato, M., Simon, J., Snellman, H., Sternheimer, D.: Simple facts about analytic vectors and integrability. Ann. Sci. Éc. Norm. Sup. (4ème série) **5**, 423–434 (1972)

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