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Towards a Constructive Approach of a Gauge Invariant, Massive $P(\phi)_2$ Theory

Robert Schrader

Institut für theoretische Physik, Freie Universität Berlin,
D-1000 Berlin 33, Federal Republic of Germany

Abstract. As part of a possible constructive approach to a gauge invariant $P(\phi)_2$ theory, we consider massive, scalar, polynomially selfcoupled fields ϕ in a fixed external Yang-Mills potential A in two-dimensional euclidean space. For a large class of A's we show that the corresponding euclidean Green's functions for the fields ϕ have a lower mass gap for weak coupling which is uniform in A. The result is obtained by adapting the Glimm-Jaffe-Spencer cluster expansion to the present situation through Kato's inequality, which reflects the diamagnetic effect of the Yang-Mills potential. A discussion of the corresponding gauge covariance is included.

1. Motivation and Outline of Results

There is an increasing belief that Yang-Mills field theories should play an important role in the description of elementary particles. Now the recent attempts to get a rigorous mathematical grip on the problems related to Yang-Mills fields mostly start with a lattice formulation (see e.g. the contributions to the Rome conference on Mathematical Physics and the references quoted there). However, there is *one* aspect which directly allows a continuum discussion and which is the object of the present analysis. For definiteness, we consider a gauge invariant $P(\phi)_2$ theory, but we expect that our arguments may be extended to gauge invariant Y_2 (Yukawa) and ϕ_3^4 theories.

In our case the euclidean Green's functions should formally be given as the moments of a normalized measure μ_f with

$$d\mu_f(\phi^*, \phi, A) = Z_f^{-1} \prod_x \delta(f(A(x))) \exp - \int (\tfrac{1}{4}(F^{\mu\nu}F_{\mu\nu} + P(\phi^*, \phi))(x)dx \cdot \exp - \int (\phi^*(-\Delta_A + m_0^2)\phi)(x)dx \prod_{x,i,\mu} d\phi_i^*(x) d\phi_i(x) dA_\mu(x) \tag{1.1}$$

with

$$\Delta_A = \sum_\mu (\partial_\mu + ieA_\mu)^2 \tag{1.2}$$

(see e.g. Faddeev and Popov [4, 13], Abers and Lee [1]).

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f specifies the gauge and F is the field strength tensor for given A. Each $\phi(x)$ is formally an element of a finite dimensional complex Hilbert space V with components $\phi_i(x)$ (for a fixed orthonormal basis in V) and complex conjugate euclidean fields $\phi_i^*(x)$. Each $A_\mu(x)$ is formally a hermitean operator in V. As a sideremark we note that our discussion may also be carried over to the case where V is a real Hilbert space, the fields $\phi_i(x)$ being real and $iA_\mu(x)$ being skew symmetric.

Furthermore

$$P(\phi^*,\phi)(x)=\sum_{k=0}^{m} a_k(\phi^*,\phi)^k(x) \tag{1.3}$$

with a_k real, $a_m>0$ and $(\phi^*,\phi)(x)=\sum_i \phi_i^*(x)\phi_i(x)$.

We rewrite μ_f as

$$d\mu_f(\phi^*,\phi,A)=Z(A)\cdot Z_f^{-1}d\mu_f(A)d\mu_A(\phi^*,\phi) \tag{1.4}$$

with

$$d\mu_f(A)=\prod_x \delta(f(A(x)))\exp-\int(\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu})(x)dx\prod_{x,\mu} dA_\mu(x) \tag{1.5}$$

and

$$\begin{aligned} d\mu_A(\phi^*,\phi)=Z(A)^{-1}\exp&-\lambda\int P(\phi^*,\phi)(x)dx\\ &-\int\phi^*(-\Delta_A+m_0^2)\phi dx\prod_{x,i} d\phi_i^*(x)d\phi_i(x)\,. \end{aligned} \tag{1.6}$$

$Z(A)$ is chosen so as to make μ_A a normalized measure. In particular,

$$Z_f=\int Z(A)d\mu_f(A)\,. \tag{1.7}$$

The problem consists in giving these expressions a precise meaning, in particular to find nuclear spaces on the duals of which these measures live. In this note we will construct μ_A (or more precisely its moments, since we do not yet know whether μ_A is a measure) and discuss some properties. We leave the problem of the more difficult construction of μ_f aside. μ_A will be given for all small $|\lambda|$ ($\mathrm{Re}\,\lambda\geqq 0$) and all A in a class $\mathscr{P}\subseteqq\mathscr{D}'$ which will be specified in Section 2. We expect this class to be sufficiently large to be close to the support of a would-be measure μ_f. Furthermore we will show that the euclidean Green's functions for a fixed Yang-Mills potential

$$\langle\phi_{i_1}^{(*)}(x_1)\ldots\phi_{i_n}^{(*)}(x_n)\rangle_A=\int\phi_{i_1}^{(*)}(x_1)\ldots\phi_{i_1}^{(*)}(x_1)\ldots\phi_{i_n}^{(*)}(x_n)d\mu_A(\phi^*,\phi) \tag{1.8}$$

define tempered distributions with cluster properties which are uniform in A. In particular, the exponential decay for large separations is (uniformly in A) at least as strong as the one obtained by Glimm et al. [7] (which is the $A=0$ case). This indicates that the full euclidean Green's function

$$\langle\phi_{i_1}^{(*)}(x_1)\ldots\phi_{i_n}^{(*)}(x_n)\rangle_f=\int\langle\phi_{i_1}^{(*)}(x_1)\ldots\phi_{i_n}^{(*)}(x_n)\rangle_A\frac{Z(A)}{Z_f}d\mu_f(A) \tag{1.9}$$

in the gauge f also should have the same cluster properties.

Our results conform with those already obtained on the one-particle level: Combining the discussion of e.g., Combes et al. [2] with the results obtained in Hess et al. [9], the presence of a Yang-Mills potentials has a diamagnetic effect in the sense that it tends to push the spectrum of the Hamiltonian upwards. In multi-particle boson theory the same has also been shown for the case of the ground state energy (Simon [19]).

Since the true exponential decay for (1.9) is determined by the physical mass, our result indicates that the gap between the ground state and the first excited state for the Bose field (the one-particle state at rest) does not decrease in the presence of a Yang-Mills potential.

In Section 2, as a preparation for the construction of μ_A, we consider the "free" case $\lambda=0$. It is given by a Gaussian measure with covariance $(-\Delta_A+m_0^2)^{-1}$. We discuss the Wick ordering $:\ :_A$ with respect to this measure and show its gauge invariance. This will allow us to perform the only necessary renormalization of $P(\phi^*,\phi)(x)$ by taking $:P(\phi^*,\phi)(x):_A$ instead. Then the euclidean action and its exponential in a finite volume are well defined. The construction of μ_A, the discussion of its gauge covariance, and the proof of the cluster properties follow by adapting the Glimm-Jaffe-Spencer cluster expansion to the present situation and will be given in Section 3. The clue is provided by Kato's inequality for Yang-Mills potentials (Hess et al. [9]) and related inequalities. They are extensions of Kato's inequality for electromagnetic potentials (Kato [11], Simon [16–18]). In particular, the kernel of the covariance $(-\Delta_A+m_0^2)^{-1}$ may be estimated by the kernel of the covariance $(-\Delta+m_0^2)^{-1}$.

Since Section 2 is an adaption of Dimock and Glimm [3] and Section 3 is an adaption of Glimm et al. [7], we will stay as close as possible to the notation used there. Also we will assume the reader familiar with the material and line of arguments presented there.

2. Gaussian Processes with External Yang-Mills Potentials

To fix the notation we start with some definitions. Let V be a finite dimensional complex Hilbert space and let $\langle\,,\rangle$ denote the scalar product in V. $\mathscr{H}_V$ denotes the real linear space of hermitean operators in V. For any measurable subset $B\subseteqq\mathbb{R}^n$, $L^p(B,V)$ is the space of measurable functions f on B with values in V such that

$$\|f\|_p=\left(\int_B \|f(x)\|^p d^n x\right)^{\frac{1}{p}}<\infty\,.$$

Similarly $L^p(B,\mathscr{H}_V)$ is the space of all measurable functions g on B with values in $\mathscr{H}_V$ such that

$$\|g\|_p=\left(\int_B \|g(x)\|_V^p d^n x\right)^{\frac{1}{p}}<\infty\,.$$

$\|\cdot\|_V$ is the operator norm on V. $L^p_{\text{loc}}(\mathbb{R}^n,V)$ and $L^p_{\text{loc}}(\mathbb{R}^n,\mathscr{H}_V)$ are defined similarly. We have $f\in L^p_{(\text{loc})}(\mathbb{R}^n,V)$ if and only if $f_i(x)=\langle e_i,f(x)\rangle\in L^p_{(\text{loc})}(\mathbb{R}^n)$ for all i where $e_i\in V (1\leqq i\leqq \dim V)$ is a fixed orthonormal basis. By $\mathscr{D}(\mathbb{R}^n,V)=C_c^\infty(\mathbb{R}^n,V)$ and $\mathscr{S}(\mathbb{R}^n,V)$ we denote the nuclear spaces consisting of all $f\in\bigcap_p L^p(\mathbb{R}^n,V)$ such that $f_i\in\mathscr{D}(\mathbb{R}^n)$ or $\mathscr{S}(\mathbb{R}^n)$, respectively. We have $\bigcap_p L^p_{\text{loc}}(\mathbb{R}^n,V)\subset\mathscr{D}'(\mathbb{R}^n,V)$. Also

$\mathscr{S}(\mathbb{R}^n, V) \subset \mathscr{S}'(\mathbb{R}^n, V)$ if to each $f' \in \mathscr{S}(\mathbb{R}^n, V)$ we associate the continuous linear form on $\mathscr{S}(\mathbb{R}^n, V)$:

$$f \mapsto (f', f) = \langle f', f \rangle = \int \langle f'(x), f(x) \rangle d^n x .$$

Furthermore, every $g \in L^p_{(\mathrm{loc})}(\mathbb{R}^n, \mathscr{H}_V)$ has a matrix representation $g_{kl}(x) = \langle e_k, g(x) e_l \rangle \in L^p_{(\mathrm{loc})}(\mathbb{R}^n)$. We define $\mathscr{D}(\mathbb{R}^n, \mathscr{H}_V)$ to be the nuclear space consisting of all g with $g_{kl} \in \mathscr{D}(\mathbb{R}^n)$ for all k, l. These definitions obviously do not depend on the particular choice of the basis $\{e_i\}$. We will need the following:

Theorem 2.1. *Let $A = \{A_\mu\}_{1 \leqq \mu \leqq 2}$ be measurable functions on $\mathbb{R}^2$ with values in $\mathscr{H}_V$ such that*

(1) $A_\mu \in L^4_{\mathrm{loc}}(\mathbb{R}^2, \mathscr{H}_V) (\mu = 1, 2)$

(2) $\operatorname{div} A = \sum_{\mu=1}^{2} \frac{\partial}{\partial x_\mu} A_\mu \in L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathscr{H}_V); \qquad x = (x_1, x_2) \in \mathbb{R}^2 .$

Then $\Delta_A = \sum_{\mu=1}^{2} \left(\frac{\partial}{\partial x_\mu} + ie A_\mu(x) \right)^2$ (e real, fixed) is a nonpositive operator on $L^2(\mathbb{R}^2, V)$ which is essentially selfadjoint on $C_c^\infty(\mathbb{R}^2, V)$. For $m_0 > 0$, the kernel of the resolvent $D_A = (-\Delta_A + m_0^2)^{-1}$ satisfies Kato's inequality for resolvents

$$\| D_A(x, y) \|_V \leqq C_\phi(x, y) = (-\Delta + m_0^2)^{-1}(x, y) . \tag{2.1}$$

We only sketch the proof: By assumption $\sum_{\mu=1}^{2} (A_\mu(x))^2 \in L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathscr{H}_V)$ and also the local Stummel norm

$$M_x(A_\mu) = \int_{|x-y|<1} \| A_\mu(x) \|_V |x - y|^{-1} dy$$

is finite. These are Schechter's conditions [14], originally stated for electromagnetic potentials. However, using Kato's inequality for Yang-Mills potentials (Hess et al. [9]) the proof in [14] may easily be taken over to the present situation.

The class of all A satisfying the conditions of Theorem 2.1 will be denoted by $\mathscr{P}'$. $\mathscr{P}'$ is euclidean invariant, i.e. with $A \in \mathscr{P}'$ we also have $A_{(a, R)} \in \mathscr{P}'$ where

$$A_{(a,R)}(x) = (R^{-1} A)(Rx + a) (R \in O(2), a \in \mathbb{R}^2).$$

We now discuss gauge transformations: Let $x \mapsto G(x)$ be a measurable map from $\mathbb{R}^2$ into the Lie group $U(V)$ of unitaries on V, i.e. $G \in L^\infty(\mathbb{R}^2, U(V))$. This set has a group structure defined by $(G_1 G_2)(x) = G_1(x) G_2(x)$. Any such G defines a unitary map, also denoted by G, on $L^2(\mathbb{R}^2, V)$ by $(Gf)(x) = G(x) f(x)$. We have, formally,

$$\Delta_A f = G \Delta_{G^{-1}A} G^{-1} f \tag{2.2}$$

with the distributional definition

$$(G^{-1} A)_\mu(x) = G^{-1}(x) A_\mu(x) G(x) + \frac{1}{ie} G^{-1}(x) \frac{\partial}{\partial x_\mu} G(x) . \tag{2.3}$$

More precisely, whenever $A \in \mathscr{P}'$ relation (2.2) may be used to define $\Delta_{G^{-1}A}$ as a selfadjoint operator which is essentially selfadjoint on $G^{-1} C_c^\infty(\mathbb{R}^2, V) \subsetneqq L^2(\mathbb{R}^2, V)$.

Due to relation (2.2) the resolvents are related by

$$G^{-1}D_A G = D_{G^{-1}A} \tag{2.4}$$

such that Kato's inequality for resolvents (2.1) also holds when A is replaced by $G^{-1}A$. If we set

$$\mathscr{P} = \bigcup_{G \in L^\infty(\mathbb{R}^2, U(V))} G^{-1}\mathscr{P}'$$

then $\mathscr{P}$ is also euclidean invariant and trivially

$$G^{-1}\mathscr{P} = \mathscr{P} \quad \text{for} \quad G \in C^\infty(\mathbb{R}^2, U(V)).$$

In what follows A will always be in $\mathscr{P}$. Since $D_A(x,y)$ is measurable, we have in particular $D_A(\cdot,\cdot) \in \bigcap_{1 \leqq p < \infty} L^p_{\mathrm{loc}}(\mathbb{R}^4, \mathscr{H}_V)$ and for fixed y $D_A(\cdot, y) \in \bigcap_{1 \leqq p < \infty} D^p(\mathbb{R}^2, \mathscr{H}_V)$ (see e.g. Dimock and Glimm [3]).

We need a further generalization of the present discussion. As in Glimm et al. [7], let Γ be a subset of $(\mathbb{Z}^2)^* \subseteqq \mathbb{R}^2$, i.e. a set of line segments b in $\mathbb{R}^2$ connecting neighbouring points with integer coordinates. For $A \in G^{-1}\mathscr{P}'$ let $\Delta_{\Gamma, A}$ be the Friedrichs extension of the quadratic form obtained by restricting Δ_A to $G^{-1}C_c^\infty(\mathbb{R}^2 \setminus \Gamma, V)$ (Dirichlet boundary conditions). Then $\Delta_{\phi, A} = \Delta_A$ and $D_{\Gamma, A} = (-\Delta_{\Gamma, A} + m_0^2)^{-1}$ satisfies an estimate similar to (2.3)

$$\|D_{\Gamma, A}(x,y)\|_V \leqq C_\Gamma(x,y) = (-\Delta_\Gamma + m_0^2)^{-1} \tag{2.5}$$

where the notation is as in Glimm et al. [7]. Also relation (2.4) extends to

$$G^{-1}D_{\Gamma, A}G = D_{\Gamma, G^{-1}A}. \tag{2.6}$$

We denote by $\mathscr{D}(A)$ the convex set of all convex combinations D of covariances of the form $D_{\Gamma, A}$. Then the kernel of D satisfies an estimate of the form

$$\|D(x,y)\|_V \leqq C(x,y) \tag{2.6a}$$

where C is formed in the same way from the C_Γ as D is formed with the $D_{\Gamma, A}$.

We now consider euclidean fields $\phi(f)$ to be the complex valued, linear functions on $\mathscr{S}'(\mathbb{R}^2, V)$ given by

$$\phi(f): f' \mapsto (f', f), \quad f' \in \mathscr{S}'(\mathbb{R}^2, V)$$

$\phi(f)$ is additive in f. $\phi^*(f)$ is the complex conjugate field, i.e. the map

$$\phi^*(f): f' \mapsto \overline{(f', f)}.$$

We will write

$$\phi(f) = \int \langle \phi(x), f(x) \rangle d^2x$$

and define $\phi_i(x)$ and $\phi_i^*(x)$ through

$$\phi_i(h) = \phi(e_i h) = \int \phi_i(x) h(x) d^2x$$
$$\phi_i(h)^* = \phi(e_i h)^* = \int \phi_i^*(x) \overline{h(x)} d^2x.$$

To simplify notation, we write

$$\phi_{\underline{i}}=\begin{cases}\phi_i^* & \text{if } \underline{i}=(i,*)\\ \phi_i & \text{if } \underline{i}=(i,\cdot).\end{cases}$$

For given $D\in\mathscr{D}(A)$ let ν_D be the Gaussian measure over $\mathscr{S}'(\mathbb{R}^2,V)$ with mean zero and covariance D. For $D=D_{\phi,A}$ we will also write ν_A. Measurability will be considered with respect to the cylinder sets in $\mathscr{S}'(\mathbb{R}^2,V)$. Thus $\phi(f)$ and $\phi(f)^*$ become random variables with expectations

$$\begin{aligned}&\int\phi(g_1)^*\ldots\phi(g_n)^*\phi(f_1)\ldots\phi(f_{n'})d\nu_D\\&=\begin{cases}\sum\limits_{\pi\in S_n}\prod\limits_{k=1}^{n}\langle g_k, Df_{\pi(k)}\rangle & \text{if } n=n'\\ 0 & \text{otherwise}.\end{cases}\end{aligned}$$

We note that the Markov property holds for ν_A. This follows as in Nelson [12], since Δ_A is a local operator.

We turn to a discussion of Wick ordering $::_A$ w.r.t. the measure ν_A. The generating functional is given by

$$:\exp\phi(f)+\phi(g)^*:_A=\exp(\phi(f)+\phi(g)^*)\left(\int\exp(\phi(f)+\phi(g)^*)d\nu_A\right)^{-1}$$

(see e.g. Schrader [15]) from which by polarization we may obtain terms like $:\prod\limits_{k=1}^{n}\phi_{\underline{i}_k}(f_k):_A(1\leqq i_k\leqq\dim V)$. We are interested in expressions of the form

$$R_A(\{\underline{i}_k^\nu\},W)=\int\prod_{\nu=1}^{r}:\prod_{k=1}^{n_\nu}\phi_{\underline{i}_k^\nu}(y_\nu):_A W(y_1\ldots y_r)dy_1\ldots dy_r$$
$$(1\leqq i_k^\nu\leqq\dim V)\tag{2.7}$$

where $W\in L^2(\mathbb{R}^{2r})$ has compact support.

These expressions may be obtained as in Dimock and Glimm [3] and Glimm et al. [7] by regularizing the fields with a function $\chi_\kappa(y)=\kappa^2\chi(\kappa y)$ $(\kappa>1)$ where χ is nonnegative in $C^\infty(\mathbb{R}^2)$ with $\int\chi=1$ and supported in $|y|\leqq 1$.

Thus we set

$$\begin{aligned}\phi_{\kappa,\underline{i}}(y)&=\int\chi_\kappa(x-y)\phi_{\underline{i}}(x)dx\\&=\phi_{\underline{i}}(\chi_\kappa(\cdot-y))\end{aligned}$$

and

$$R_{A,\kappa}(\{\underline{i}_k^\nu\},W)=\int\prod_{\nu=1}^{r}:\prod_{k=1}^{n_\nu}\phi_{\kappa,\underline{i}_k^\nu}(y_\nu):_A W(y_1\ldots y_r)dy_1\ldots dy_r.$$

The reason that this works is roughly as follows: Although the covariance D in $\mathscr{D}(A)$ do not have regularity properties in general, they may be bounded through Kato's inequality by expressions with the right properties. We only need the following additional lemma which extends and allows to make use of Proposition 7.5 in [7]:

Lemma 2.2. *For any covariance $D \in \mathscr{D}(A)$, the quantity*

$$d(x) = \lim_{y \to x} D_{\phi,A}(x,y) - D(x,y)$$

satisfies the estimate

$$\|d(x)\|_V \leqq c(x) = \lim_{y \to x} (C_\phi(x,y) - C(x,y)) \tag{2.8}$$

where C is constructed in the same way from the C_Γ's as D is from the $D_{\Gamma,A}$'s.

Proof. It is sufficient to consider the case $D = D_{\Gamma,A}$ for some $\Gamma \neq \emptyset$. Assume first the A_μ to be C_c^∞. As in [7] let $dZ_{x,y}^t$ be the conditional Wiener density on paths $Z(\tau)$ in $\mathbb{R}^2$ which start at x at $\tau = 0$ and end at y at $\tau = t$. Let J_b^t be the function

$$J_b^t(Z) = \begin{cases} 0 & \text{if} \quad Z(\tau) \in b, \quad 0 \leqq \tau \leqq t \\ 1 & \text{otherwise} \end{cases}$$

defined on Wiener paths. For the kernel of the semigroup $\exp - t(-\Delta_A + m_0^2)$ we have the relation

$$\begin{aligned} &\exp - t(-\Delta_A + m_0^2)(x,y) \\ &= \int e^{-m_0^2 t} T\left(\exp ie \sum_{\mu=1}^{2} \int_0^t A_\mu(Z(\tau)) dZ_\mu(\tau)\right) \prod_{b \in \Gamma} J_b^t(Z) dZ_{x,y}^t . \end{aligned} \tag{2.9}$$

Here $T(\exp \cdot)$ is the time ordered exponential and

$$G_A(Z) = T\left(\exp + ie \sum_{\mu=1}^{2} \int_0^t A_\mu(Z(\tau)) dZ_\mu(\tau)\right) \in U(V) \tag{2.9a}$$

is a stochastic integral. Relation (2.9) is well known to physicists in the case of electromagnetic potentials A (see e.g. Feynman and Hibbs [5]). Mathematically it follows from a generalization of Ito's integral formula in magnetic fields and is a consequence of Trotters product formula. Thus we have

$$D_{\phi,A}(x,y) - D_{\Gamma,A}(x,y) = \int_0^\infty \int_Z e^{-m_0^2 t} G_A(Z) \left(1 - \prod_{b \in \Gamma} J_b^t\right) dZ_{x,y}^t dt$$

which gives the estimate

$$\begin{aligned} \|D_{\phi,A}(x,y) - D_{\Gamma,A}(x,y)\|_V &\leqq \int_0^\infty \int_Z \left(1 - \prod_{b \in \Gamma} J_b^t\right) dZ_{x,y}^t dt \\ &= C_\phi(x,y) - C_\Gamma(x,y) . \end{aligned}$$

This proves the lemma for A_μ being C_c^∞. For general $A \in \mathscr{P}'$ let $A_{\varepsilon,\mu}(x) = \int \delta_\varepsilon(x-y) \zeta_\varepsilon(y) A(y) dy$ be regularizations of A, where the $\delta_\varepsilon \in C_c^\infty(\mathbb{R}^2)$ are approximative δ-functions and the $\zeta_\varepsilon \in C_c^\infty(\mathbb{R}^2)$ are approximation to 1 such that $\delta_\varepsilon \to \delta$ and $\zeta_\varepsilon \to 1$ in $\mathscr{D}'(\mathbb{R}^2)$ as $\varepsilon \to 0$. Then the $\Delta_{\Gamma,A_\varepsilon}$ tend to $\Delta_{\Gamma,A}$ on the common core $C_c^\infty(\mathbb{R}^2 \setminus \Gamma, V)$. Therefore, we have strong convergence of D_{Γ,A_ε} to $D_{\Gamma,A}$ (see e.g. Kato [10], Chap. VIII). Thus estimate (2.8) also holds for $A \in \mathscr{P}'$. The general case $A \in \mathscr{P}$ then follows from relation (2.6), concluding the proof of Lemma 2.2.

Proposition 2.3. *For any $D \in \mathscr{D}(A)$, the $R_{A,\kappa}(\{i_k^\nu\}, W)$ are elements in $L^p(\mathscr{S}'(\mathbb{R}^2, V), dv_D)(1 \leqq p < \infty)$ and converge there as $\kappa \to \infty$ to an element denoted by $R_A(\{i_k^\nu\}, W)$ and written as in expression* (2.7).

The proof follows along the lines of [7] making use of the estimates given above. We omit the details.

Next we introduce the euclidean action, which will be well defined due to Proposition 2.3. For any measurable function h on $\mathbb{R}^2$ with compact support and $0 \leqq h \leqq 1$ we set

$$P_A(\phi^*, \phi)(h) = \sum_{k=0}^{m} a_k \int : \left(\sum_{i=1}^{\dim V} \phi_i^*(y) \phi_i(y) \right)^k :_A h(y) dy .$$

The next proposition is the analogue of Theorem 9.5 in [7].

Proposition 2.4. *Let* $\mathrm{Re}\,\lambda \geqq 0$. *Then* $\exp - \lambda P_A(\phi^*, \phi)(h)$ *is in* $L^p(\mathscr{S}'(\mathbb{R}^2, V), dv_D)(1 \leqq p < \infty, D \in \mathscr{D}(A))$ *and there is a constant K independent of h, A, and* $D \varepsilon \mathscr{D}(A)$ *such that*

$$|Z_{h,D}(A)| = |\int e^{-\lambda P_A(\phi^*, \phi)(h)} dv_D| < e^{K\, measure(\mathrm{supp} h)}$$

Corollary 2.5. *For any* λ *with* $\mathrm{Re}\,\lambda \geqq 0$, $R_A e^{-\lambda P_A(\phi^*, \phi)(h)}$ *is in* $L^p(\mathscr{S}'(\mathbb{R}^2, V), dv_D)(1 \leqq p < \infty, D \in \mathscr{D}(A))$.

Remark 2.6. Proposition 2.3 and Corollary 2.5 may be sharpened to analogues of Theorem 9.4 and Corollary 9.6 in Glimm et al. [7] with the same estimates uniformly in A and $D \in \mathscr{D}(A)$.

We turn to a discussion of the gauge invariance of the euclidean action. Let $G \in \mathscr{O}_M(\mathbb{R}^2, U(V))$, such that G maps $\mathscr{S}(\mathbb{R}^2, V)$ homeomorphically onto itself. Thus the field $G\phi$ with $(G\phi)(f) = \phi(G^{-1}f)$ is well defined such that $G_1(G_2\phi) = (G_1G_2)\phi$ and formally

$$\begin{aligned} (G\phi)_j(x) &= \sum_i G_{ij}^{-1}(x) \phi_i(x) \\ (G\phi)_j^*(x) &= \sum_i \bar{G}_{ij}^{-1}(x) \phi_i^*(x) . \end{aligned} \tag{2.10}$$

We first discuss the gauge covariance of the measures v_D. Note that if $D \in \mathscr{D}(A)$, then $GDG^{-1} \in \mathscr{D}(GA)$. Hence, by definition of v_D we have

$$\begin{aligned} &\int (G\phi)^{(*)}(f_1) \dots (G\phi)^{(*)}(f_n) dv_D \\ &= \int \phi^{(*)}(G^{-1}f_1) \dots \phi^{(*)}(G^{-1}f_n) dv_D \\ &= \int \phi^{(*)}(f_1) \dots \phi^{(*)}(f_n) dv_{GDG^{-1}} . \end{aligned}$$

This may easily be extended to smooth, polynomially bounded cylinder functions $F = \hat{F}(\phi^{(*)}(f_1), \dots \phi^{(*)}(f_n))$ on $\mathscr{S}'(\mathbb{R}^2, V)$ in the sense that

$$\begin{aligned} &\int \hat{F}((G\phi)^{(*)}(f_1), \dots (G\phi)^{(*)}(f_n)) dv_D \\ &= \int \hat{F}(\phi^{(*)}(f_1), \dots \phi^{(*)}(f_n)) dv_{GDG^{-1}} . \end{aligned}$$

Finally it may be extended in $L^p(\mathcal{S}'(\mathbb{R}^2, V), d\nu_D)$ to limits of such functions. We will write this as

$$d\nu_{GDG^{-1}}(\phi^*, \phi) = d\nu_D((G\phi)^*, G\phi) = : d\nu_D^{G^{-1}}(\phi^*, \phi)$$

such that

$$\nu^G_{GDG^{-1}} = \nu_D \tag{2.11}$$

which expresses the gauge covariance of the Gaussian measures under consideration. In particular we have

$$\nu^G_{GA} = \nu_A . \tag{2.12}$$

The Wick ordering is compatible with this structure in the following sense: Consider

$$R^G_{GA,\kappa}(\{\underline{i}^\nu_k\}, W) = \int \prod_{\nu=1}^{r} \prod_{k=1}^{n_\nu} (G^{-1}\phi)_{\kappa, \underline{i}^\nu_k}(y_\nu) :_{GA} W(y_1, \dots y_r) dy_1 \dots dy_r$$

for $D \in \mathcal{D}(A)$, where we note that $(G\phi)_{\kappa,i}(y) = \sum_j \phi_j(G^{-1}_{ji}(\cdot)\chi_\kappa(\cdot - y))$.

Integrating over $\nu_{GDG^{-1}}$ we obtain a sum of products of elements of the form

$$(\chi_\kappa * (\mathrm{D})_{i_1, i_2} * \chi_\kappa)(y_\nu, y_{\nu'})(\nu \neq \nu', i_1 \in \{i^\nu_k\}_{1 \leqq k \leqq n_\nu}, \quad i_2 \in \{i^{\nu'}_k\}_{1 \leqq k \leqq n_{\nu'}})$$

or

$$(\chi_\kappa * (D)_{i_1, i_2} * \chi_\kappa)(y_\nu, y_\nu)$$
$$- (\chi_\kappa * G^{-1} D_{GA} G * \chi_\kappa)(y_\nu, y_\nu) i_1, i_2 \in \{i^\nu_k\}_{1 \leqq k \leqq n_\nu}$$

multiplied by $W(y_1, \dots y_r)$ and integrated over. Thus we have

$$\int R^G_{GA,\kappa}(\{\underline{i}^\nu_k\}, W) d\nu_{GDG^{-1}} = \int R_{A,\kappa}(\{\underline{i}^\nu_k\}, W) d\nu_D \tag{2.13}$$

and Proposition 2.3 therefore gives:

Lemma 2.7. *The $R^G_{GA,\kappa}(\{\underline{i}^\nu_k\}, W)$ converge in $L^p(\mathcal{S}'(\mathbb{R}^2, V), d\nu_{GDG^{-1}})$ $(1 \leqq p < \infty$, $D \in \mathcal{D}(A))$ as $\kappa \to \infty$ to an element written as*

$$R^G_{GA}(\{\underline{i}^\nu_k\}, W) = \int \prod_{\nu=1}^{r} : \prod_{k=1}^{n_\nu} (G^{-1}\phi)_{\underline{i}^\nu_k}(y_\nu) :_{GA} W(y_1 \dots y_r) dy_1 \dots dy_r \tag{2.14}$$

such that

$$\int R^G_{GA}(\{\underline{i}^\nu_k\}; W) d\nu_{GDG^{-1}} = \int R_A(\{\underline{i}^\nu_k\}, W) d\nu_D \tag{2.15}$$

for $D \in \mathcal{D}(A)$.

We will call R^G_{GA} the gauge transform of R_A. Thus, from the class of (measurable) functions on $\mathcal{S}'(\mathbb{R}^2, V)$ considered, we conclude a gauge transformation G maps $L^p(\mathcal{S}'(\mathbb{R}^2, V), d\nu_D)$ onto $L^p(\mathcal{S}'(\mathbb{R}^2, V) d\nu_{GDG^{-1}})$. In particular it is isometric if p is even and hence for all $p \geqq 2$ by the Riesz-Thorin interpolation theorem.

Now the formal relations (2.10) and the multilinearity of the Wick ordering (over $\mathbb{C}$) suggest the following:

Lemma 2.8. *In $L^p(\mathscr{S}'(\mathbb{R}^2, V), d\nu_{GDG^{-1}})$ $(1 \leqq p < \infty, D \in \mathscr{D}(A))$ the following relations holds*

$$R^G_{GA}(\{\underline{i}^\nu_k\}, W) = \sum_{\underline{j}^\nu_k} \int \prod_{\nu=1}^{r} : \prod_{k=1}^{n_\nu} \phi_{\underline{j}^\nu_k}(y_\nu) :_{GA} \prod_{\nu=1}^{r} \prod_{k=1}^{n_\nu} G_{\underline{j}^\nu_k \underline{i}^\nu_k}(y_\nu) \cdot W(y_1, \ldots y_r) dy_1 \ldots dy_r \tag{2.16}$$

with the notational convention

$$\underline{j}^\nu_k = \begin{cases} (j^\nu_k, *) & \text{if } \underline{i}^\nu_k = (i^\nu_k, *) \\ (j^\nu_k, \cdot) & \text{if } \underline{i}^\nu_k = (i^\nu_k, \cdot) \end{cases}$$

and

$$G_{\underline{i}\underline{i}'} = \begin{cases} \overline{G_{ii'}} & \text{if } \underline{i} = (i, *), \underline{i}' = (i', *) \\ G_{ii'} & \text{if } \underline{i} = (i, \cdot), \underline{i}' = (i', \cdot). \end{cases}$$

Proof. First, we note that the right-hand side of relation (2.16) is well defined by Proposition 2.3. More precisely, denote by $R^{G'}_{GA,\kappa}$ the regularization of the right-hand side of (2.16) obtained from replacing $\phi_{\underline{j}^\nu_k}$ by $\phi_{\underline{j}^\nu_k,\kappa}$. Then $R^{G'}_{GA,\kappa}$ tends to the right-hand side of (2.16) in the desired way as $\kappa \to \infty$. By arguments similar to those leading to relation (2.13), it is easily checked that

$$\int R^{G'}_{GA,\kappa} d\nu_{GDG^{-1}} = \int R'_{A,\kappa} d\nu_D \tag{2.17}$$

where $R'_{A,\kappa}$ is obtained from expression (2.7) by replacing $\phi(x)$ with its regularization

$$\phi'_\kappa(y) = \phi(\chi^G_\kappa(\cdot, y)) \tag{2.18a}$$

where

$$\chi^G_\kappa(x, y) = G^{-1}(x)\chi_\kappa(x-y)G(y). \tag{2.18b}$$

χ^G_κ is also an admissible regularization: For given G there is $C < \infty$ such that

$$\|G^{-1}(x)G(y) - \mathbb{1}\|_V < C \cdot |x-y|$$

for all $x \in \mathbb{R}^2$ and $y \in \operatorname{suppt} W$. Therefore,

$$\|\chi^G_\kappa(\cdot, y) - \chi_\kappa(\cdot - y)\|_{L^1(\mathbb{R}^2, \operatorname{End} V)} < C'\kappa^{-1}$$

uniformly for $y \in \operatorname{suppt} W$ for some $C' > 0$.

Hence, by the arguments leading to the proof of Proposition 2.3 in Dimock and Glimm [3] and the estimates given above, $R'_{A,\kappa}$ tends to R_A in $L^p(\mathscr{S}'(\mathbb{R}^2, V), d\nu_D)(1 \leqq p < \infty, D \in \mathscr{D}(A))$. Hence, by relations (2.15) and (2.17) $R^{G'}_{GA,\kappa}$ tends to R^G_{GA} in $L^p(\mathscr{S}'(\mathbb{R}^2, V), d\nu_{GDG^{-1}})$ concluding the proof of the lemma.

Theorem 2.9. *The euclidean action is gauge invariant in the sense that*

$$P_A((G\phi)^*, (G\phi)(h)) = P_A(\phi^*, \phi)(h) \tag{2.19}$$

as an equality in $L^p(\mathscr{S}'(\mathbb{R}^2, V), d\nu_D)(1 \leqq p < \infty, D \in \mathscr{D}(A))$.

The proof is immediate from the definition of the euclidean action, Lemma 2.8, and the unitarity of the $G(y)$.

3. Convergence of the Glimm-Jaffe-Spencer Cluster Expansion in External Yang-Mills Potentials

In what follows h_Λ will denote the characteristic function of a union Λ of lattice squares. For $\operatorname{Re}\lambda \geqq 0$ we set

$$Z_\Lambda(A) = \int \exp - \lambda P_A(\phi^*, \phi)(h_\Lambda) dv_A. \tag{3.1}$$

Below we will show that $Z_\Lambda(A) \neq 0$. Therefore, we may define the complex measures $\mu_{\Lambda, A}$ by

$$d\mu_{\Lambda, A} = Z_\Lambda^{-1}(A) \exp - \lambda P_A(\phi^*, \phi)(h_\Lambda) dv_A. \tag{3.2}$$

We let

$$\mathfrak{S}^n_{\Lambda, A}(x_1, \underline{i}_1, \dots x_n, \underline{i}_n) = \int \phi_{\underline{i}_1}(x_1) \dots \phi_{\underline{i}_n}(x_n) d\mu_{\Lambda, A} \in \mathscr{S}'(\mathbb{R}^{2n}) \tag{3.3}$$

be the approximative euclidean Green's functions in the external Yang-Mills potential A. We have

$$\int R^G_{GA} d\mu_{\Lambda, GA} = \int R_A d\mu_{\Lambda, A}. \tag{3.4}$$

Indeed,

$$\begin{aligned}\int R^G_{GA} \exp - \lambda P_{GA}(\phi^*, \phi) dv_{GA} &= \int R^G_{GA} \exp - \lambda P_{GA}((G\phi)^*, G\phi) dv_{GA} \\ &= \int R_A \exp - \lambda P_A(\phi^*, \phi) dv_A\end{aligned}$$

where the first identity follows from Theorem 2.9 and the second from relation (2.15). Thus $Z_\Lambda(GA) = Z_\Lambda(A)$ and relation (3.4) follows.

In particular, the euclidean Green's functions $\mathfrak{S}_{\Lambda, A}$ satisfy the following gauge covariance properly:

$$\begin{aligned}&\sum_{\substack{j_k=1 \\ 1 \leqq k \leqq n}}^{\dim V} \prod_{k=1}^{n} G_{\underline{i}_k, \underline{j}_k}(x_k) \mathfrak{S}^n_{\Lambda, GA}(x_1, \underline{j}_1; \dots; x_n, \underline{j}_n) \\ &\quad = \mathfrak{S}^n_{\Lambda, A}(x_1, \underline{i}_1; \dots x_n, \underline{i}_n).\end{aligned} \tag{3.5}$$

Consider now λ in the half circle $0 < |\lambda| < \varepsilon$, $-\frac{\pi}{2} < \arg\lambda < \frac{\pi}{2}$. As in Glimm et al. [7] we have the following:

Theorem 3.1. *For λ belonging to the half circle above and for $\frac{\varepsilon}{m_0^2}$ sufficiently small $\int R_A d\mu_{\Lambda, A}$ converges (uniquely) as $h_\Lambda \to 1$ for all R_A uniformly in A. In particular, the euclidean Green's functions $\mathfrak{S}^n_{\Lambda, A}$ converge in $\bigotimes_n \mathscr{S}'(\mathbb{R}^2, V)$ to euclidean Green's functions $\mathfrak{S}^n_A$ which are analytic in λ. They satisfy the following estimate*

$$\left|\int \mathfrak{S}^n_A(x_1, \underline{i}_1; \dots x_n, \underline{i}_n) f(x_1, \dots x_n) dx_1 \dots dx_n\right| < C^n (n!)^{3/2} \|f\|_{3n} \tag{3.6}$$

uniformly in n and $f \in \mathscr{S}(\mathbb{R}^{2n})$ *with* $c = c(\varepsilon, m_0^2)$. *Furthermore, there is a uniform cluster property in the following sense: For some* $m > 0$ *depending only on* ε *and* m_0^2

$$|\mathfrak{S}_A^{n_1+n_2}(f_1, i_1; \ldots; f_{n_1}, i_{n_1}; g_1^a, j_1 \ldots g_{n_2}^a, j_{n_2}) - \mathfrak{S}_A^{n_1}(f_1, i_1; \ldots; f_{n_1}, i_{n_1})$$
$$\mathfrak{S}_A^{n_2}(g_1^a, j_1; \ldots g_{n_2}^a, j_{n_2})| \leqq C(\{f\}, \{g\}) e^{-m|a|}$$

$(f_i, g_j \in C_c^\infty(\mathbb{R}^2), g^a(x) = g(x-a))$.
The $\mathfrak{S}_A^n$ *satisfy the same gauge covariance properties as the* $\mathfrak{S}_{\Lambda,A}^n$.

Due to the uniqueness of the limit, we have the following euclidean covariance property:

Corollary 3.2. *Under euclidean transformations* (a, R) *the* $\mathfrak{S}_A$ *transform like*

$$\mathfrak{S}_{A(a,R)}^n(R^{-1}(x_1 - a), i_1; \ldots R^{-1}(x_n - a), i_n) = \mathfrak{S}_A^n(x_1, i_1; \ldots x_n, i_n).$$

The proof of Theorem 3.1 follows by adapting the cluster expansion to the present situation. This means effectively that the covariances C of [7] are replaced by the corresponding covariances $D \in \mathscr{D}(A)$. Therefore it is only necessary to repeat all estimates involving covariances, most of which have already been done by estimating the D's by the C's through Kato's and related inequalities. Then the quantities involved are estimated by the corresponding quantities for the $A=0$ case to which the usual cluster expansion estimates of Glimm et al. (with a dim V-component Bose field) applies. In particular, m is exactly the lower bound for the physical mass thus obtained. Only ε could become smaller than the one obtained from [7] for the multicomponent Bose theory due to the special arguments needed for the proof of Lemma 3.4 below (which fixes ε). We now present the remaining estimates. With otherwise the same notational convention as in [7] we have:

Lemma 3.3. *For any* $D \in \mathscr{D}(A)$

$$\|D(x, y)\|_V \leqq C(x, y) \tag{3.7}$$

and

$$\|\partial^\gamma D(s)(x, y)\|_V \leqq \partial^\gamma C(s). \tag{3.8}$$

Here D and $D(s)$ are formed in the same way from the $D_{\Gamma, A}$ as C and $C(s)$ are from the C_Γ.

Proof. Estimate (3.7) has already been stated in estimate (2.6a). Estimate (3.8) follows along the same lines as the proof of Lemma 2.2, i.e. by using the stochastic representation of $\partial^\gamma D(s)$ and $\partial^\gamma C(s)$.

The next lemma fixes ε in Theorem 3.1:

Lemma 3.4. *Let* ε *be sufficiently small. Then for all* $m_0^2 > 1$ *and all* $\lambda (|\lambda| < \varepsilon, \operatorname{Re} \lambda \geqq 0)$

$$Z_{h_\Delta, \partial\Delta}(A) = \int \exp - \lambda P_A(\phi^*, \phi)(h_\Delta) d\nu_{D'}$$
$$D' = D_{\partial\Delta, A}$$

satisfies

$$\tfrac{1}{2} \leqq |Z_{h_\Delta, \partial\Delta}(A)| \leqq 2 \tag{3.9}$$

uniformly for all lattice squares Δ *and all* $A \in \mathscr{P}$.

Proof. First choose $0<\varepsilon_1<1$ so small that

$$|\int e^{-\lambda P_A(\phi^*,\phi)(h_\Delta)} dv_{D'}| \leqq 2 \tag{3.10}$$

for all λ with $|\lambda|<2\varepsilon_1$, $\mathrm{Re}\,\lambda \geqq 0$ uniformly in A. This is possible due to the arguments leading to Proposition 2.5. Next

$$\begin{aligned} &|\int e^{-\lambda P_A(\phi^*,\phi)(h_\Delta)} dv_{D'} - 1| \\ &= \left|\lambda \int_0^s \int e^{-\lambda s P_A(\phi^*,\phi)(h_\Delta)} P_A(\phi^*,\phi)(h_\Delta) dv_{D'} ds\right| \\ &\leqslant |\lambda| |\int\int e^{-2\mathrm{Re}\,\lambda s P_A(\phi^*,\phi)(h\Delta)} dv_{D'} ds|^{1/2} (\int |P_A(\phi^*,\phi)(h_\Delta)|^2 dv_{D'})^{1/2} \end{aligned} \tag{3.11}$$

By estimate (3.10) and by now familar arguments the right-hand side of (3.11) may be made smaller than $\frac{1}{2}$ for any λ with $|\lambda|<\varepsilon<\varepsilon_1$, $\mathrm{Re}\,\lambda \geqq 0$ uniformly in A. This concludes the proof of Lemma 3.4.

Corollary 3.5. *For all A, $\Lambda \neq \emptyset$, all $m_0^2>1$ and all λ with $|\lambda|<\varepsilon$, $\mathrm{Re}\,\lambda \geqq 0$ (ε as in Lemma 3.4)*

$$Z_\Lambda(A) \neq 0.$$

The proof follows from Lemma 3.4 using the arguments of Glimm et al. [7].

This concludes the proof of estimates involving covariances. They lead to the cluster expansion of Glimm et al. which again give Theorem 3.1, except for estimate (3.6). The additional arguments leading to (3.6) may be taken over from e.g. Schrader [15].

Remark 3.6. In the case $A=0$, estimate (3.6) may be improved to conclude that the euclidean Green's functions are moments of a measure $\mu_{A=0}$ (Fröhlich [6], Glimm and Jaffe [8]). Unfortunately, neither of the methods employed there is directly applicable to the general situation $A \in \mathscr{P}$. It would be interesting to see whether these and other methods employed in analyzing the structure of $P(\phi)_2$ theories may be adapted to the present situation.

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