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A C^*-Algebra of the Two-dimensional Ising Model

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Abstract. We consider the two-dimensional Ising model and show how correlation functions are determined by a state of a C^*-Clifford algebra. We describe how the phase transition manifests itself in terms of a jump in the index of a Fredholm operator. A connection with the Pfaffian approach is made through the theory of unitary dilations of contraction semigroups.

§ 1. Introduction

The two-dimensional Ising model in zero field has been treated algebraically by many authors, notably Onsager [20], Kaufmann [11], Schultz, Mattis, and Lieb [23], Abraham [1, 2], Abraham and Martin-Löf [3]. They consider an array of spins on a finite lattice, compute correlations using either the Clifford algebra [1, 3, 11] or the Fermi algebra [2, 23] and then pass to the thermodynamic limit. Following Pirogov [22] we consider the Clifford and Fermi algebras associated with the infinite lattice. Other C^*-algebras associated with the Ising model are described by Marinaro and Sewell [16].

We investigate the connection between the Gibbs states of the Ising system and certain states of the Clifford algebra. In this we follow Dobrushin [5] and Landford and Ruelle [12] and regard a Gibbs' state of the infinite system as a family of correlations $\langle\sigma_a\ldots\sigma_{a_n}\rangle$ for finite subsets $\{a_1,\ldots,a_n\}$ of the lattice, σ_a taking on values ± 1. These are obtained as the limit of correlation functions for a sequence of finite sublattices with some prescribed boundary conditions. In particular we denote by $\langle\ldots\rangle^p$, $\langle\ldots\rangle^+$ and $\langle\ldots\rangle^-$ the correlation functions which arise from the periodic, "plus" and "minus" boundary conditions respectively. For a review of boundary conditions and general properties of Ising systems see Gallavotti [7]. The state is translationally invariant if $\langle\sigma_{a_1+a}\ldots\sigma_{a_n+a}\rangle = \langle\sigma_{a_1}\ldots\sigma_{a_n}\rangle$ for all lattice vectors $a\in Z^2$ and all subsets $\{a_1,\ldots,a_n\}$. The set of all translationally invariant equilibrium states is a non-empty convex space. A phase transition is said to occur at inverse temperature β_c if for $\beta>\beta_c$ there is more than one equilibrium state while for $\beta<\beta_c$ a unique state exists. Extending a result of Gallavotti and Miracle-Sole [8], Messager and Miracle-Sole [17] have shown that every translationally invariant equilibrium state $\langle\cdot\rangle$ is such that

$$\langle\cdot\rangle=\alpha\langle\cdot\rangle^+ +(1-\alpha)\langle\cdot\rangle^- \quad \text{for} \quad \text{some} \quad \alpha\in[0,1]\,. \tag{1}$$

Lebowitz [14] has shown that β_c coincides with the Onsager value [21]. The extremal state $\langle\ldots\rangle^+$, $\langle\ldots\rangle^-$ satisfy:

$$\langle\sigma_{a_1}\ldots\sigma_{a_n}\rangle^+=(-1)^n\langle\sigma_{a_1}\ldots\sigma_{a_n}\rangle^- \quad \text{for all} \quad a_1\ldots a_n\in Z^2\,, \tag{2}$$
$$\lim_{|a|\to\infty}\langle\sigma_{a_1}\ldots\sigma_{a_n}\sigma_{a_1+a}\ldots\sigma_{a_n+a}\rangle^+=(\langle\sigma_{a_1}\ldots\sigma_{a_n}\rangle^+)^2\,, \quad \text{for all} \quad a_1,\ldots,a_n, a\in Z^2\,,$$

and so are determined by their common value on products of an even number of spin variables. In § 2 we show how the extremal state $\langle\ldots\rangle^+$ at inverse temperature β corresponds to a state ω_β of the Clifford algebra $\overline{\mathcal{U}(H,s)}$ over a symplectic space $H=E\oplus JE$. Each such ω_β is a Fock state with complex structure A_β on H. For $\beta\neq\beta'$ the operator $|A_\beta-A_{\beta'}|$ is not Hilbert-Schmidt so the corresponding representations are disjoint.

The complex structure A_β has a decomposition

$$A_\beta=PJe^{2J\theta}+QJe^{-2J\theta}\,,$$

where P, Q are the orthogonal projections onto E, JE respectively and θ is self-adjoint. The operator $Je^{2J\theta}$ is Fredholm and its index jumps at the critical temperature:

$$\operatorname{ind}(Je^{2J\theta})=\begin{cases} 0 & \beta<\beta_c \\ -1 & \beta>\beta_c\,. \end{cases}$$

The physical manifestations of the phase transition are shown by the calculated values of the correlations and these depend on the index of $Je^{2J\theta}$.

In this formulation the treatment of translations in the two basic lattice directions appears to be asymmetric, in contrast to the Pfaffian approach [19]. The connection is shown in § 4 by an application of Sz-Nagy's theory of the unitary dilation of contraction semigroups.

§ 2. Algebras and States

We adhere to the notation of Balslev, Manuceau and Verbeure [4]. Let H be an infinite dimensional real Hilbert space, $s(\cdot,\cdot)$ the real inner product on H, and $\overline{\mathcal{U}(H,s)}$ the C^*-Clifford algebra generated by $\{\Gamma(\phi):\phi\in H\}$ where the $\Gamma(\phi)$ satisfy the relations

$$[\Gamma(\phi),\Gamma(\psi)]_+=2s(\phi,\psi)1 \qquad \phi,\psi\in H\,. \tag{3}$$

$\overline{\mathcal{U}(H,s)}=\overline{\mathcal{U}_{ev}(H,s)}\oplus\overline{\mathcal{U}_{od}(H,s)}$, where $\mathcal{U}_{ev}$ is the C^*-subalgebra generated by $\{\Gamma(\phi)\Gamma(\psi),\ \phi,\psi\in H\}$ and $\mathcal{U}_{od}$, the vector subspace spanned by products of an odd number of $\Gamma(\phi)$.

We assume H comes equipped with a fixed complex structure J, satisfying $J^2=-1$, $J^+=-J$ [J^+ the adjoint of J with respect to the inner product $s(\cdot,\cdot)$], such that (H^J, h) is the complexification of (H,s) via

$$(\alpha+i\beta)\phi=\alpha\phi+\beta J\phi \qquad \alpha,\beta\in R \qquad \phi\in H$$
$$h(\phi,\psi)=s(\phi,\psi)+is(J\phi,\psi) \qquad \phi,\psi\in H\,.$$

Let $\{e_n: n \in Z\}$ be an orthonormal basis for (H^J, h), so that $\{e_n, Je_n: n \in Z\}$ is an orthonormal basis for (H, s) and let E be the closed subspace of (H, s) spanned by $\{e_n: n \in Z\}$.

Then $H = E \oplus JE$ and Λ, the conjugation determined by J, defined by

$$\Lambda\phi = \begin{cases} \phi, & \phi \in E, \\ -\phi, & \phi \in JE, \end{cases} \tag{4}$$

satisfies $\Lambda^2 = 1$, $[\Lambda, J]_+ = 0$, and $P = \frac{1+\Lambda}{2}$, $Q = \frac{1-\Lambda}{2}$ are the orthogonal projections onto E, JE respectively.

Let $H_L \subset H$ be the subspace spanned by $\{e_n, Je_n: n = -L, \ldots, L\}$, $s_L(\cdot,\cdot)$ denote the restriction of $s(\cdot,\cdot)$ to H_L, and O_L the restriction to H_L of an operator O on H.

Let $\mathscr{A}_L$ be the Paulion algebra generated by $\{\sigma_j^\alpha: j = -L, \ldots, L, \alpha = x, y, z\}$ which obey the mixed commutation relations

$$[\sigma_j^\alpha, \sigma_k^{\alpha'}]_- = 0, j \neq k, \sigma_j^x \sigma_j^y = i\sigma_j^z \quad \text{et} \quad \text{cyc.}, \qquad (\sigma_j^\alpha)^2 = 1, j = -L, \ldots, L. \tag{5}$$

The Jordan-Wigner transformation [10] is a *-isomorphism $\eta_L: \mathscr{A}_L \to \overline{\mathscr{U}(H_L, S_L)}$ and is defined by

$$\begin{aligned} &\eta_L(\sigma_{-L}^x) = \Gamma(e_{-L}) \\ &\eta_L(\sigma_{-L}^y) = \Gamma(J_L e_{-L}) \\ &\eta_L(\sigma_k^x) = \textstyle\prod_{j=-L}^{k-1} (-i\Gamma(e_j)\Gamma(J_L e_j))\Gamma(e_k) \\ &\qquad k = -L+1, \ldots, L \\ &\eta_L(\sigma_k^y) = \textstyle\prod_{j=-L}^{k-1} (-i\Gamma(e_j)\Gamma(J_L e_j))\Gamma(J_L e_k). \end{aligned} \tag{6}$$

For a finite lattice $\Lambda = \{(i,j) \in Z^2: i = -L, \ldots, L, j = -N, \ldots, N\}$ the algebra of observables is $\mathscr{C}(\{+1, -1\}^\Lambda)$, the space of complex valued continuous functions on the compact set $\{+1, -1\}^\Lambda$, and the expectation value of any observable f is given by the Gibbs formula

$$\langle f \rangle_{NL}^b = (Z_\Lambda^b)^{-1} \textstyle\sum_{x \in \{+1,-1\}^\Lambda} f(x) \exp(-\beta \mathscr{H}_\Lambda^b(x)), \tag{7}$$

where

$$\begin{aligned} &Z_\Lambda^b = \textstyle\sum_{x \in \{+1,-1\}^\Lambda} \exp(-\beta \mathscr{H}_\Lambda^b(x)), \\ &\mathscr{H}_\Lambda^b(x) = -\textstyle\sum_{(i,j) \in \Lambda} (J_1 x_{ij} x_{i+1,j} + J_2 x_{ij} x_{i,j+1}) + \partial \mathscr{H}_\Lambda^b(x). \end{aligned} \tag{8}$$

$J_1, J_2 > 0$ and $\partial \mathscr{H}_\Lambda^b$ is the Hamiltonian interaction between the system Λ and its boundary $\partial\Lambda$.

Correlation functions are expectation values of the functions $\{\sigma_{ij}\}$ where $\sigma_{ij}(X) = x_{ij}$.

Let us introduce a particular representation π_L of $\mathscr{A}_L$ as bounded operators on a Hilbert space $\mathfrak{H}_L = \bigotimes_{-L}^{L} \mathfrak{H}$ where $\mathfrak{H}$ is a 2-dimensional space with orthonormal basis $e_+ = \begin{pmatrix}1\\0\end{pmatrix}$ $e_- = \begin{pmatrix}0\\1\end{pmatrix}$, defined by

$$\pi_L(\sigma_i^\alpha) = 1 \otimes \ldots \otimes \underset{i^{\text{th}} \text{ position}}{\sigma^a} \otimes \ldots 1 \qquad \alpha = x, y, z, \tag{9}$$

where

$$\sigma^x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The array $T_L^b(y^{(m)}, y^{(m+1)})$ defined by

$$T_L^b(y^{(m)}, y^{(m+1)}) = \exp\left\{\frac{K_2}{2}\sum_i^b x_{im}x_{i+1,m} + K_1 \sum_i^b x_{im}x_{im+1} + \frac{K_2}{2}\sum_i^b x_{im+1}x_{i+1,m+1}\right\}, \tag{10}$$

where $y^{(m)} = (x_{-Lm}, \ldots, x_{Lm})$ and $\sum_i^b$ signifies that the limits of the summation are prescribed by the boundary condition, determines an element $V_L^b \in \mathscr{A}_L$ by

$$T_L^b(y^{(m)}, y^{(m+1)}) = \langle \bigotimes_{-L}^L e_{\alpha_i}, \pi_L(V_L^b) \bigotimes_{-L}^L e_{\alpha'_j} \rangle_L, \tag{11}$$

where $\alpha_i = \pm$ when $x_{im} = \pm 1, \alpha'_j = \pm$ when $x_{i,m+1} = \pm 1$ and $\langle \cdot, \cdot \rangle_L$ is the inner-product on $\mathfrak{H}_L$.

$$V_L^b = (2\operatorname{sh} 2K_1)^{n_b} \exp\left(\frac{K_2}{2}\sum_i^b \sigma_i^x \sigma_{i+1}^x\right) \exp(K_1^* \sum_i^b \sigma_i^z) \exp\left(\frac{K_2}{2}\sum_i^b \sigma_i^x \sigma_{i+1}^x\right), \tag{12}$$

where

$$n_b = \sum_i^b (\tfrac{1}{2}) e^{-2K_1^*} = \tanh K_1 . \tag{13}$$

For our purposes we need consider only two boundary conditions: the periodic, and the plus and minus, which give rise to the extremal states. Details omitted here may be found in [3, 24].

Let f be a local element of $\mathscr{C}(\{+1, -1\}^{Z^2})$ lying in $\mathscr{C}\{+1, -1\}^{\Lambda})$ say. Using the transfer matrix we have the existence of elements $a_f^p, a_f^\pm \in \mathscr{A}_L$ such that

$$\langle f \rangle_{NL}^p = \frac{\operatorname{tr}_{\mathfrak{H}_L}(\pi_L(a_f^p)\pi_L(V_L^p)^{2N+1})}{\operatorname{tr}_{\mathfrak{H}_L}(\pi_L(V_L^p)^{2N+1})}, \tag{14}$$

$$\langle f \rangle_{NL}^\pm = \frac{\langle \bigotimes_{-L}^L e_\pm \pi_L(V_L^\pm)^{N+1} \pi_L(a_f^\pm) \pi_L(V_L^\pm)^{N+1} \bigotimes_{-L}^L e_\pm \rangle_L}{\langle \bigotimes_{-L}^L e_\pm | \pi_L(V_L^\pm)^{2N+2} | \bigotimes_{-L}^L e_\pm \rangle_L}, \tag{15}$$

where

$$V_L^p = (2\operatorname{sh} 2K_1)^{\frac{2L+1}{2}} \exp\left(\frac{K_2}{2}\sum_{-L}^L \sigma_i^x \sigma_{i+1}^x\right) \exp(K_1^* \sum_{-L}^L \sigma_i^z) \exp\left(\frac{K_2}{2}\sum_{-L}^L \sigma_i^x \sigma_{i+1}^x\right)$$

and σ_{-L}^x is identified with σ_{L+1}^x and

$$V_L^\pm = (2\operatorname{sh} 2K_1)^{\frac{2L-1}{2}} \exp\left(\frac{K_2}{2}\sum_{-L}^{L-1} \sigma_i^x \sigma_{i+1}^x\right) \exp(K_1^* \sum_{-L+1}^{L-1} \sigma_i^z)$$

$$\cdot \exp\left(\frac{K_2}{2}\sum_{-L}^{L-1} \sigma_i^x \sigma_{i+1}^x\right).$$

Define states $\varrho^p_{NL}(\cdot)$, $\varrho^{\pm}_{NL}(\cdot)$ on $\mathscr{A}_L$ by

$$\varrho^b_{NL}(a)=\mathrm{tr}_{\mathfrak{H}_L}(\pi_L(a)\pi_L(V^p_L)^{2N+1})/\mathrm{tr}_{\mathfrak{H}_L}(\pi_L(V^p_L)^{2N+1}\,, \tag{16}$$

$$\varrho^{\pm}_{NL}(a)=\frac{\langle\bigotimes^L_{-L}e_{\pm},\pi_L(V^{\pm}_L)^{N+1}\pi_L(a)\pi_L(V^{\pm}_L)^{N+1}\bigotimes^L_{-L}e_{\pm}\rangle_L}{\langle\bigotimes^L_{-L}e_{\pm},\pi_L(V^{\pm}_L)^{2N+2}\bigotimes^L_{-L}e_{\pm}\rangle_L}\,. \tag{17}$$

Lemma 1. *For any $a\in\mathscr{A}_L$, there exists $f\in\mathscr{C}(\{+1,-1\}^{\Lambda})$ such that $\varrho^b_{NL}(a)=\langle f\rangle^b_{NL}$.*

Proof. $\mathscr{A}_L$ is generated by $\{\sigma^x_k,\sigma^z_k\ k=-L,\dots,L\}$.

Trivially $f=\sigma_{k,0}$ has the property that $\langle\sigma_{k,0}\rangle^b_{NL}=\varrho^b_{NL}(\sigma^x_k)$ $k=-L,\dots,L$. Consider

$$\begin{aligned}f_k=&[\mathrm{ch}2K_1-\mathrm{sh}2K_1\sigma_{k,0}\sigma_{k,1}][\mathrm{ch}K_2-\mathrm{sh}K_2\sigma_{k-1,0}\sigma_{k,0}]\\&\cdot[\mathrm{ch}K_2-\mathrm{sh}K_2\sigma_{k,0}\sigma_{k+1,0}]\,.\end{aligned}$$

Since $\sigma^x_kV^b_L\sigma^x_kV^{b^{-1}}_L=\mathrm{ch}2K^*_1-\mathrm{sh}2K^*_1\sigma^z_k\exp\{K_2(\sigma^x_{k-1}\sigma^x_k+\sigma^x_k\sigma^x_{k+1})\}$ for $b=p$ or $b=\pm$, it is straightforward to show that $\langle f_k\rangle^b_{NL}=\varrho^b_{NL}(\sigma^z_k)$ $k=-L,\dots,L$. We wish to consider the state ω^b_{NL} on $\overline{\mathscr{U}(H_L,s_L)}$ given by

$$\omega^b_{NL}(\gamma)=\varrho^b_{NL}(\eta^{-1}_L(\gamma))\quad\text{for}\quad\text{any}\quad\gamma\in\overline{\mathscr{U}(H_L,s_L)}\,,$$

and in particular to study the limiting state $\omega^b(\cdot)=\lim\limits_{\substack{N\to\infty\\L\to\infty}}\omega^b_{NL}(\cdot)$ on the Clifford algebra $\overline{\mathscr{U}(H,s)}$.

To take the limit $N\to\infty$ it is necessary to have the spectrum of the transfer matrix. Under the Jordan-Wigner transformation we can write (see [1], [11])

$$\begin{aligned}\eta_L(V^p_L)=&(2\mathrm{sh}2K_1)^{\frac{2L+1}{2}}\\&\cdot\{\eta_L(V^p_{2L})^-\eta_L(V^p_{1L})\eta_L(V^p_{2L})^-P_L+\eta_L(V^p_{2L})^+\eta_L(V^p_{1L})\eta_L(V^p_{2L})^+Q_L\}\,,\end{aligned}$$

where

$$P_L,Q_L=\tfrac{1}{2}(1\pm U_L),\qquad U_L=\eta_L(\textstyle\prod^L_{-L}(-\sigma^z_k))$$

and

$$\begin{aligned}\eta_L(V^p_{2L})&=\exp\left(-i\frac{K_2}{2}\textstyle\sum^L_{-L}\Gamma(J_Le_i)\Gamma(W^{\pm}_Le_i)\right)\\\eta_L(V^p_{1L})&=\exp\left(-iK^*_1\textstyle\sum^L_{-L}\Gamma(e_i)\Gamma(J_Le_i)\right)\end{aligned}$$

$W^{\pm}_L:H_L\to H_L$ defined by

$$\begin{aligned}&W^{\pm}_Le_n=e_{n+1}\qquad n=-L,\dots,L-1\\&W^{\pm}_Le_L=\pm e_{-L}\\&[W^{\pm}_L,J_L]=0\,.\end{aligned} \tag{19}$$

Similarly

$$\eta_L(V^{\pm}_L)=(2\mathrm{sh}2K_1)^{\frac{2L-1}{2}}\eta_L(V^{\pm}_{2L})\eta_L(V^{\pm}_{1L})\eta_L(V^{\pm}_{2L})\,,$$

where

$$\eta_L(V_{2L}^{\pm}) = \exp\left(-i\frac{K_2}{2}\sum_{-L}^{L}\Gamma(J_L e_i)\Gamma(W_L e_i)\right)$$

$$\eta_L(V_{1L}^{\pm}) = \exp\left(-iK_1^*\sum_{-L}^{L}\Gamma(e_i)\Gamma(J_{L-1}e_i)\right)$$

and $W:H\to H$ is the bilateral shift

$$\begin{aligned} We_n &= e_{n+1} \qquad n\in Z\,, \\ [W,J]_- &= 0\,. \end{aligned} \tag{20}$$

Let us define operators on H_L by

$$\cosh\gamma_L^{\pm} = \mathrm{ch}2K_1^*\mathrm{ch}2K_2 1 - \mathrm{sh}2K_1^*\mathrm{sh}2K_2\left(\frac{W_L^{\pm}+(W_L^{\pm})^{-1}}{2}\right), \tag{21a}$$

$$\mathrm{sh}\,\gamma_L^{\pm}\cos\delta_L^{*\pm} = \mathrm{ch}2K_1^*\mathrm{sh}2K_2 1 - \mathrm{sh}2K_1^*\mathrm{ch}2K_2\left(\frac{W_L^{\pm}+(W_L^{\pm})^{-1}}{2}\right), \tag{21b}$$

$$\mathrm{sh}\,\gamma_L^{\pm}\sin\delta_L^{*\pm} = -\mathrm{sh}2K_1^* J_L\left(\frac{W_L^{\pm}-(W_L^{\pm})^{-1}}{2}\right), \tag{21c}$$

$$\begin{aligned} A_L^{\pm} &= -J_L\exp(J_L\Lambda_L\delta_L^{*\pm})\cdot((W_L^{\pm})^{-1}P_L + W_L^{\pm}Q_L) \\ &= J_L\exp(2J_L\Lambda_L\theta_L^{\pm}) = S_L^{\pm}J_L(S_L^{\pm})^+ \end{aligned} \tag{21d}$$

$$S_L^{\pm} = \exp(-J_L\Lambda_L\theta_L^{\pm})\,. \tag{21e}$$

Let ω_{J_L} be the Fock state on (H_L, s_L) corresponding to complex structure J_L, the representation defined in terms of creation operators

$a^*(x) = \frac{1}{2}(\Gamma(x) - i\Gamma(J_L x))$ and vacuum vector $|\Omega_L>$.

It is straightforward to verify that the Bogoliubov automorphisms $\alpha_{S_L}^{\pm}:\Gamma(x)\to\Gamma(S_L^{\pm}x)$ of $\overline{\mathscr{U}(H_L, s_L)}$ induced from the orthogonal operators $S_L^{\pm}$ are implemented in the above representation by

$$\mathscr{S}_L^{\pm} = \exp\left[\frac{i}{2(2L+1)}\sum_{k=-L}^{L}\theta(\omega_k^{\pm})\{a^*(-\omega_k^{\pm})a^*(\omega_k^{\pm}) + a(\omega_k^{\pm})a(-\omega_k^{\pm})\}\right], \tag{22}$$

where $a^*(\omega) = \sum_{-L}^{L} e^{-in\omega}a^*(e_n)$

$$\omega_k^+ = 2\pi ik/2L+1, \qquad \omega_k^- = \pi i(2k+1)/2L+1 \qquad k=-L,\ldots,L\,, \tag{23}$$

$$\cosh\gamma(\omega) = \mathrm{ch}2K_1^*\,\mathrm{ch}2K_2 - \mathrm{sh}2K_1^*\,\mathrm{sh}2K_2\cos\omega\,, \tag{24a}$$

$$\mathrm{sh}\,\gamma(\omega)\cos\delta(\omega) = \mathrm{ch}2K_1^*\,\mathrm{sh}2K_2 - \mathrm{sh}2K_1^*\,\mathrm{ch}2K_2\cos\omega\,, \tag{24b}$$

$$\mathrm{sh}\,\gamma(\omega)\sin\delta^*(\omega) = \mathrm{sh}2K_1^*\sin\omega\,, \tag{24c}$$

$$2\theta(\omega) = \delta^*(\omega) + \omega - \pi\,. \tag{24d}$$

Theorem 1. *For $\beta<\beta_c$ the state $\omega_L^p(\cdot)$ is a quasi-free Fock state over $\overline{\mathscr{U}(H_L, s_L)}$, described by complex structure $A_L^- = S_L^- J_L(S_L^-)^+$.*

Proof. For $\beta<\beta_c$ the principal eigenvalue of $\pi_L(V_L^p)$ is non-degenerate and its eigenvector is $|\Phi_L^-\rangle=\pi_L\eta_L^{-1}\mathcal{S}_L^-|\Omega_L\rangle$. Taking the limit $N\to\infty$ in (16) in the usual way we have

$$\begin{aligned}\omega_L^p(\cdot)&=\langle\Phi_L^-|\pi_L\eta_L^{-1}(\cdot)|\Phi_L^-\rangle\\&=(\omega_{J_L}\circ\alpha_{(S_L^-)^+})(\cdot)\end{aligned}$$

which is a Fock state as it is related to the Fock state ω_{J_L} by a Bogoliubov transformation.

Lemma 2. *For $\beta>\beta_c$ the states $\omega_L^\pm(\cdot)$ have the property $\omega_L^\pm(\gamma)=0$ for any $\gamma\in\overline{\mathcal{U}_{od}(H_{L-1},S_{L-1})}$*

Proof. For $\beta>\beta_c$ the largest eigenvalue of $\pi_L(V_L^\pm)$ has an exact degeneracy between $|\Phi_L^\pm\rangle$ and $|D_L^\pm\rangle=\frac{1}{2}\pi_L\eta_L^{-1}(\Gamma(e_{-L})-i\,\Gamma(J_Le_L))|\Phi_L^\pm\rangle$.

Letting $N\to\infty$ in (17) and using the parity of $|\Phi_L^\pm\rangle$ and $|D_L^\pm\rangle$ we obtain

$$\omega_L^\pm(\gamma)=\pm\tfrac{1}{2}\mathrm{Re}\langle\Phi_L^\pm|\pi_L\eta_L^{-1}(\gamma)|D_L^\pm\rangle\quad\gamma\in\overline{\mathcal{U}_{od}(H_L,S_L)}\,.$$

If $\gamma\in\overline{\mathcal{U}_{od}(H_{L-1},S_{L-1})}\subset\overline{\mathcal{U}_{od}(H_L,S_L)}$ then

$$\omega_L^\pm(\gamma)=0\quad\text{since}\quad\pi_L\eta_L^{-1}(\Gamma(e_{-L})+i\Gamma(J_Le_L))|\Phi_L^\pm\rangle=0\,.$$

Lemma 3. *For any $\gamma_L\in\mathcal{U}_{ev}(H_L,s_L)$ and any $M>L$ we have*

$$\omega_M^P(\gamma_L)-\tfrac{1}{2}(\omega_{J_M}\circ\alpha_{(S_M^-)^+}+\omega_{J_M}\circ\alpha_{(S_M^+)^+})(\gamma_L)=0\left(\frac{2L+1}{\sqrt{2M+1}}\right)\|\gamma_L\|\,.$$

Proof. When $\beta>\beta_c$ there is an asymptotic degeneracy of $\pi_M(V_M^p)$ between

$$|\Phi_M^-\rangle\quad\text{and}\quad\pi_M\eta_M^{-1}B_+^*(g_0^+/\|g_0^+\|)|\Phi_M^+\rangle$$

where

$$g_0^+=\textstyle\sum_{-M}^M e_j,\ B_+^*(x)=\frac{1}{2}(\Gamma(S_M^+x)-i\Gamma(S_M^+J_Mx))$$

and the respective eigenvalues $\lambda_{\max}^-$, λ_0^+ have the property

$$\lambda_0^+/\lambda_{\max}^-=1-O(e^{-\tau M})\,,$$

where τ is the surface tension [2]. Consequently for any $\gamma_L\in\overline{\mathcal{U}_{ev}(H_L,s_L)}$.

$$\begin{aligned}\omega_M^P(\gamma_L)=&\tfrac{1}{2}\langle\Phi_M^-|\pi_M\eta_M^{-1}(\gamma_L)|\Phi_M^-\rangle\\&+\langle\Phi_M^+|\pi_M\eta_M^{-1}\{B_+(g_0^+/\|g_0^+\|)\gamma_LB_+^*(g_0^+/\|g_0^+\|)\}|\Phi_M^+\rangle+O(e^{-\tau M})\|\gamma_L\|\,.\end{aligned}$$

But for any $x\in H_L$ $|h(x,g_0^+/\|g_0^+\|)|\leq\dfrac{2L+1}{\sqrt{2M+1}}\|x\|$ and $B_+(g_0^+)B_+^*(g_0^+)|\Phi_M^+\rangle=\|g_0^+\|^2|\Phi_M^+\rangle$, therefore after successive application of the anticommutation relations,

$$\omega_M^P(\gamma_L)=\tfrac{1}{2}(\omega_{J_M}\circ\alpha_{(S_M^-)^+}+\omega_{J_M}\circ\alpha_{(S_M^+)^+})(\gamma_L)+O\left(\frac{2L+1}{\sqrt{2M+1}}\right)\|\gamma_L\|\,.$$

Let ω_A be the Fock state on $\overline{\mathscr{U}(H,s)}$ determined by complex structure A on H where

$$\operatorname{ch}\gamma = \operatorname{ch}2K_1^* \operatorname{ch}2K_2 1 - \operatorname{sh}2K_1^* \operatorname{sh}2K_2 \left(\frac{W+W^{-1}}{2}\right), \tag{25a}$$

$$\operatorname{sh}\gamma \cos\delta^* = \operatorname{ch}2K_1^* \operatorname{sh}2K_2 1 - \operatorname{sh}2K_1^* \operatorname{ch}2K_2 \left(\frac{W+W^{-1}}{2}\right), \tag{25b}$$

$$\operatorname{sh}\gamma \sin\delta^* = -J \operatorname{sh}2K_1^* \left(\frac{W-W^{-1}}{2}\right), \tag{25c}$$

$$\begin{aligned} A &= -J\exp(J\Lambda\delta^*)(W^{-1}P + WQ) \\ &= J\exp(2J\Lambda\theta), \end{aligned} \tag{25d}$$

$$= SJS^+ \qquad S = \exp(-J\Lambda\theta). \tag{25e}$$

The following extends Theorem 3 of [22].

Theorem 2. *For all β and for each $\gamma \in \mathscr{U}(H,s)$, lim $\omega^P(\gamma)$ and* $\lim \omega_L^{\pm}(\gamma)$ both *exist and*

$$\lim_{L\to\infty} \omega_L^p(\gamma) = \lim_{L\to\infty} \omega_L^{\pm}(\gamma) = \omega_A(\gamma).$$

Proof. In the case of periodic boundary conditions the result follows immediately from Theorem 1 and Lemma 3, together with the fact that

$$\operatorname*{s-lim}_{L\to\infty} W_L^- = \operatorname*{s-lim}_{L\to\infty} W_L^+ = W.$$

If $\gamma \in \mathscr{U}_{od}(H,s)$, then $\lim_{L\to\infty} \omega_L^{\pm}(\gamma) = 0$ for $\beta > \beta_c$ follows immediately from Lemma 2. A careful consideration of the degeneracies of $\pi_L(V_L^{\pm})$ when $\beta < \beta_c$ from [3] shows that $\lim_{L\to\infty} \omega_L^{\pm}(\gamma) = 0$ for $\beta < \beta_c$ also when $\gamma \in \mathscr{U}_{od}(H,s)$. To show that $\omega^{\pm}$ agrees with ω_A on $\mathscr{U}_{ev}(H,s)$ we require the following Lemma.

Let $\mathscr{E}v(\{+1,-1\}^{Z^2}) = \{f : f(-x) = f(x) \quad x \in \{+1,-1\}^{Z^2}\}$.

Lemma 4. *For any boundary condition b and inverse temperature β, given $f \in \mathscr{E}v(\{+1,-1\}^{Z^2})$ there exists $\gamma_f \in \mathscr{U}_{ev}(H,s)$ such that*

$$\langle f\rangle^b = \omega_A(\gamma_f)$$

and conversely given $\gamma \in \mathscr{U}_{ev}(H,s)$ there exists $f_\gamma \in \mathscr{E}v(\{+1,-1\}^{Z^2})$ such that $\langle f_\gamma\rangle^b = \omega_A(\gamma)$ for any boundary condition b.

Proof. The first part follows from Eq. (1) and the fact that it is true for $b=p$. The converse follows essentially from Lemma 1. It is sufficient to show f_{γ_k}, f_{μ_k} exist for $\{\gamma_k = \Gamma(e_k)\Gamma(Je_k) : k \in Z\}$, $\{\mu_k = \Gamma(Je_k)\Gamma(e_{k+1}) : k \in Z\}$. Direct verification shows that $f_{\mu_k} = i\sigma_{k-1,0}\sigma_{k,0}$ and $f_{\gamma_k} = if_k$, f_k as in (18).

Since ω_A is W-invariant we take the Fourier transform

$$\mathscr{F} : H^J \to L^2(S) = \left\{ f : S \to C : \|f\|^2 = \int_{-\pi}^{\pi} |f(e^{ip})|^2 \frac{dp}{2\pi} < \infty \right\}$$

determined by $e_n \to e^{inp}$, so that for each $\phi \in H^J$ we have

$$(\mathscr{F}\phi)(e^{ip}) = \hat{\phi}(e^{ip}) = \sum_Z h(e_n, \phi)e^{inp}.$$

For every operator $T:H \to H$ such that $[T, J]_- = [T, W]_- = 0$ there exists $t(\cdot) \in L^\infty(S)$ such that

$$(\hat{T\phi})(e^{ip}) = t(e^{ip})\hat{\phi}(e^{ip})$$

and $\|T\| = \|t\|_\infty$.

Theorem 3. *The finite temperature ω_A are obtained from the infinite temperature state ω_J by a Bogoliubov transformation induced from the orthogonal operator S^+ on H i.e. $\omega_A = \omega_J \circ \alpha_{S^+}$.*

The automorphism $\Gamma(\phi) \to \Gamma(S^+\phi)$ is not unitarily implemented in the Fock representation determined by ω_J.

Proof. The non-implementability follows from Theorem 2 of [15]. The operator $|A - J|$ is not Hilbert-Schmidt since it has continuous spectrum.

Let the Fock representations determined by the states ω_J and ω_A have creation operators $a^*(\phi) = \frac{1}{2}(\Gamma(\phi) - i\Gamma(J\phi))$ and $b_A^*(\phi) = \frac{1}{2}(\Gamma(\phi) - i\Gamma(A\phi))$ and vacuum Ω_0, Ω_β respectively.

Let $b^*(\phi) = b_A^*(S\phi) = \frac{1}{2}(\Gamma(S\phi) - i\Gamma(SJ\phi))$. The Bogoliubov transformation has the form

$$b^*(\phi) = a^*(\cos\theta\phi) - ia(J\Lambda\sin\theta\phi). \tag{27}$$

Introducing the operator-valued distributions $a^*(p)$, $b^*(p)$ by

$$a^*(\phi) = \int_{-\pi}^{\pi} \phi(p)a^*(p)\frac{dp}{2\pi}$$

$$b^*(\phi) = \int_{-\pi}^{\pi} \phi(p)b^*(p)\frac{dp}{2\pi},$$

it takes the form

$$b^*(p) = \cos\theta(p)a^*(p) + i\sin\theta(p)a(-p). \tag{28}$$

Let $V:H \to H$ be the operator such that $[V, J]_- = [V, W]_- = 0$ and $(\hat{V\phi})(e^{ip}) = e^{-\gamma(p)}\hat{\phi}(e^{ip})$, and let V_F be the operator on the Fock space $\mathscr{F}(L^2(S'))$ determined by $\hat{V}$ on $L^2(S')$. The following extends Theorem 4 of [22] and is immediate.

Theorem 4. *For any boundary conditions, the transfer matrix normalised by dividing out the maximum eigenvalue tends strongly to the operator V_F on $\mathscr{F}(L^2(S'))$. Consequently $V_F\Omega_\beta = \Omega_\beta$ and $V_F^n b^*(\phi)V_F^{-n} = b^*(V^n\phi)$.*

The operator V_F is unitarily equivalent to the operator P_∞ in [18] when the magnetic field equals zero.

§ 2. Index

We have shown how to compute expectation values of observables $f \in \mathscr{C}(\{+1, -1\}^{Z^2})$ using the Fock state ω_A, at all temperatures. Odd correlations are in principle determined by the clustering properties [Eq. (2)] and convexity [Eq. (1)].

In principle therefore the correlation functions are all determined by the complex structure A.

Now $A = A_1 + \Lambda A_2 = P(A_1 + A_2) + Q(A_1 - A_2)$
$= PJe^{2J\theta} + QJe^{-2J\theta}$

and elementary manipulation of Eq. (26)

$$(Je^{2J\theta}\phi)\hat{}(e^{ip}) = ie^{2i\theta(p)}\hat{\phi}(e^{ip}) = a(e^{ip})\hat{\phi}(e^{ip})$$

where

$$e^{2i\theta(p)} = -e^{ip}\sqrt{\frac{B}{A}}\left\{\frac{(e^{ip}-A)(e^{ip}-B^{-1})}{(e^{ip}-A^{-1})(e^{ip}-B)}\right\}^{\frac{1}{2}} \tag{29}$$

and

$$A = \coth K_2 \coth K_1^* \qquad B = \coth K_1^* \tanh K_2 \tag{30}$$

so that

$$\begin{aligned} &A^{-1} < B^{-1} < 1 < B < A \quad \text{for} \quad K_1^* < K_2 \\ &A^{-1} < B < 1 < B^{-1} < A \quad \text{for} \quad K_1^* > K_2 \,. \end{aligned} \tag{31}$$

If $\phi : S \to C$ is a continuous function, the index of ϕ, $I(\phi)$, is given by

$$2\pi I(\phi) = \arg(\phi(e^{i\pi})) - \arg(\phi(e^{i\pi})) \,. \tag{32}$$

Lemma 5. $I(a) = 1 \quad \beta > \beta_c$
$\qquad = 0 \quad \beta < \beta_c \,.$

Proof. Form (31) $B \gtrless 1$ if and only if $\beta \gtrless \beta_c$ and the lemma follows from direct computation.

When $\beta = \beta_c$ the function $a(\cdot)$ is not continuous. It is not even locally sectorial in the sense of [6], so we cannot assign an index to it in the same way.

§ 3. Spontaneous Magnetisation

We compute m^* by one the standard methods

$$m^{*2} = \lim_{n \to \infty} \langle \sigma_{00} \sigma_{n0} \rangle \,.$$

Using the state ω_A we have

$$\begin{aligned} \langle \sigma_{00} \sigma_{n0} \rangle &= \omega_A([-i\Gamma(Je_0)\Gamma(e_1)][-i\Gamma(Je_1)\Gamma(e_2)]\dots[-i\Gamma(Je_{n-1})\Gamma(e_n)]) \\ &= \det D^{(n)} \,, \end{aligned}$$

$D^{(n)}$ an $n \times n$ matrix with entries

$$D^{(n)}_{j,k} = s(AJW^{-1}e_j, e_k) = \int_{-\pi}^{\pi} e^{i(j-k)p} D(p)\, dp/2\pi \,,$$

where $D(p) = \exp(i\delta^*(p))$.

From (24d) and Lemma 5

$$I(D)=0 \qquad \beta>\beta_c$$
$$=-1 \qquad \beta<\beta_c .$$

Let $H^2(S)$ denote the Hardy space $=\left\{\phi\in L^2(S):\check{\phi}(n)=\int_{-\pi}^{\pi}\phi(p)e^{-inp}\frac{dp}{2\pi}=0, n<0\right\}$, and P^+ the orthogonal projection $L^2(S)\to H^2(S)$.

For each $\phi\in L^\infty(S)$, let T_ϕ denote the Toeplitz operator on $H^2(S)$ determined by ϕ by $T_\phi f=P^+(\phi\cdot f)$ $f\in H^2(S)$.

Lemma 6. $\|T_\phi\|=\|\phi\|_\infty=\sup_{p\in[-\pi,\pi]}|\phi(e^{ip})|$.

Theorem 5. (Douglas and Widom [6]). *If ϕ is continuous and bounded away from zero then T_ϕ is a Fredholm operator and*

$$\operatorname{ind} T_\phi=\dim(\ker T_\phi)-\dim(\operatorname{coker} T_\phi)=-I(\phi). \tag{33}$$

Moreover T_ϕ is invertible if and only if $I(\phi)=0$.

Theorem 6. *Let T_ϕ be a Fredholm operator on $H^2(S)$. For $n=0,1,2,\ldots$ let P_n^+ be the projection of $H^2(S)$ onto the span of $\{e_j: j=0,1,2,\ldots n\}$, and let $T_\phi^{(n)}=P_n^+T_\phi P_n^+$. Then if $I(\phi)\neq 0$ and $\|T_\phi\|\leqq 1$, $\det(T_\phi^{(n)})\to 0$ as $n\to\infty$.*

Proof. If $\ker T_\phi\neq\{0\}$, then there exists a unit vector $f\in\ker T_\phi$ and an integer n_0 s.t. for $n>n_0$ the component P_n^+f is non-zero.

Then for all $n>0$, there exist operators U_n such that $U_nP_n^+=P_n^+U_n$ and $U_n\hat{e}_0=P_n^+f/\|P_n^+f\|$ and so $\|f-U_n\hat{e}_0\|\to 0$.

By Hadamard's inequality

$$\begin{aligned}|\det T_\phi^{(n)}| &\leqq \|T_\phi U_n\hat{e}_0\|\,\|T_\phi U_n\hat{e}_L\|\ldots\|TU_n\hat{e}_n\|\\ &\leqq \|T_\phi\|^{n-1}(\|T_\phi f\|+\|T_\phi\|\,\|f-U_n\hat{e}_0\|)\\ &\leqq \|f-U_n\hat{e}_0\|\to 0 .\end{aligned}$$

If $\ker T_\phi=\{0\}$, then $\operatorname{coker} T_\phi\neq\{0\}$, which on a Hilbert space means $\ker T_\phi^*\neq\{0\}$. Since $|\det P_n^+T_\phi^*P_n^+|=|\det P_n^+T_\phi P_n^+|$ the same conclusion holds.

Corollary. $m^*=0$ *for* $\beta<\beta_c$.

Let $\mathcal{N}=\{\phi\in L^2(S):\mathcal{N}(\phi)^2=\sum_{-\infty}^{\infty}|n|\,|\check{\phi}(n)|^2<\infty\}$.

Theorem 7. (Devinatz [9]). *Let $\phi\in\mathcal{N}$ be such that*

(i) *ϕ is continuous.*
(ii) *$\phi(e^{ip})\neq 0$ for $p\in[-\pi,\pi]$.*
(iii) *$P^+\log\phi$ and $(1-P^+)\log\phi$ are continuous.*
(iv) *$I(\phi)=0$.*

Then $\lim_{n\to\infty}(\det T_\phi^{(n)})FG^{-n-1}=1$, *where if*

$$k_n=\int_{-\pi}^{\pi}\log\phi(e^{ip})e^{-inp}\frac{dp}{2\pi}$$
$$F=\exp(-\textstyle\sum_0^\infty mk_mk_{-m})$$
$$G=e^{k_0}.$$

It is straightforward to verify that for $\beta > \beta_c$, the function $D(\cdot)$ satisfies the conditions of the above theorem, and by the usual computation (see [19]) we obtain:

Corollary. $m^* = \{1 - (\operatorname{sh} 2K_1 \operatorname{sh} 2K_2)^{-2}\}^{1/8}$ *for* $\beta > \beta_c$.

§ 4. Dilations of a Semigroup

In an algebraic treatment which incorporates the transfer matrix translations along the two basic lattice directions are seemingly represented in an asymmetric way. Perpendicular to the transfer direction translation is described by the automorphism $\alpha_w : \Gamma(\phi) \to \Gamma(W\phi)$ of $\mathfrak{A}(H, s)$ whereas along the transfer direction it is described by the automorphism $\alpha_v : \Gamma(\phi) \to V_F \Gamma(\phi) V_F^{-1}$. Translation invariance of the state ω_A is a consequence of $V_F \Omega_\beta = \Omega_\beta$ and $[A, W]_- = 0$.

The Pfaffian approach [19], however, does not distinguish one lattice direction from the other. Even correlation functions in this approach can be calculated from knowledge of $\{F_{n_1, n_2}\}$ given in the appendix to [19]

$$F_{n_1, n_2} = \frac{1}{(2\pi)^2} \iint_{-\pi}^{\pi} \frac{\exp i(\phi_1 n_1 + \phi_2 n_2) d\phi_1 d\phi_2}{a - \gamma_1 \cos\phi_1 - \gamma_2 \cos\phi_2}, \tag{34}$$

where

$$a = (1 + Z_1^2)(1 + Z_2^2)$$
$$\gamma_1 = 2Z_1(1 - Z_2^2) \qquad Z_i = \tanh K_i$$
$$\gamma_2 = 2Z_2(1 - Z_1^2).$$

From Theorem 4 we have

$$V_F^n b^*(\phi) V_F^{-n} = b^*(V^n \phi),$$

where

$$(\hat{V}\phi)(e^{ip}) = e^{-\gamma(p)} \hat{\phi}(e^{ip})$$

and $\|V\| = e^{-\gamma(0)} < 1$ for $\beta \neq \beta_c$.

Let $G = \{\hat{V}^n : n > 0\}$ denote the contraction semigroup on $L^2(S)$.

Theorem 8. (Sz-Nagy [25]). *Let T be a contraction on a Hilbert space $\mathfrak{H}$, then $\mathfrak{H}$ can be imbedded in a larger Hilbert space $\mathfrak{K}$ on which there is a unitary operator U in such a way that $T^n = \pi U^n$ $n > 0$ on $\mathfrak{H}$, where π is the projection of $\mathfrak{K}$ onto $\mathfrak{H}$.*

We will use the Lax-Phillips [13] construction of the unitary dilation of the semigroup G.

Let $\hat{\mathfrak{K}} = l^2(-\infty, \infty; \mathcal{N})$ $\mathcal{N}$ some auxiliary Hilbert space, and let $U : \hat{\mathfrak{K}} \to \hat{\mathfrak{K}}$ be the shift operator

$$(U\{\underline{x}\})_n = x_{n-1} \qquad \{\underline{x}\} = (\ldots, x_{-1}, x_0, x_1, \ldots) \qquad x_j \in \mathcal{N}.$$

Let $\mathcal{F} : \hat{\mathfrak{K}} \to \mathfrak{K}$ be the Fourier transform onto $\mathfrak{K} = L^2(S; \mathcal{N})$ given by

$$\mathcal{F} : \{\underline{x}\} \to \sum_{-\infty}^{\infty} x_j e^{ij\theta}$$

so that $(\mathscr{F} U \mathscr{F}^{-1} g)(e^{i\theta}) = e^{i\theta} g(e^{i\theta}) g \in L^2(S, \mathscr{N})$. Choose $\mathscr{N} = L^2(S; \mu)$ for some measure μ on S, so that

$$\mathfrak{K} = L^2(S; \mathscr{N}) = L^2(S \times S; \mu \times \mu_0),\ \mu_0 \text{ the Lebesgue measure on } S.$$

The Lax-Phillips construction is unique up to unitary equivalence of $\mathscr{N}$.

The map $L^2(S) \to \mathfrak{K}$ given by

$$f(p) \to (1 - e^{-\gamma(p)} e^{-i\theta})^{-1} f(p)$$

is an isometric imbedding if and only if

$$\mu(dp) = (1 - e^{-2\gamma(p)}) \frac{dp}{2\pi} \text{ a.e.}$$

The map $L^2(S) \to \mathfrak{K}$ given by

$$f(p) \to \frac{\operatorname{sh} \gamma(p)}{\cosh \gamma(p) - \cos\theta} f(p)$$

is an isometric imbedding if and only if $\mu(dp) = \operatorname{th} \gamma(p) \dfrac{dp}{2\pi}$ a.e. The realisation of Theorem 8 with this second imbedding becomes

$$\frac{e^{-n\gamma(p)}}{\operatorname{sh} \gamma(p)} = \int_{-\pi}^{\pi} \frac{e^{in\theta}}{\cosh \gamma(p) - \cos\theta} \frac{d\theta}{2\pi} \quad \text{on} \quad L^2(S). \tag{35}$$

From (24a) we have

$$\cosh \gamma(\phi_1) = \gamma_2 (a - \gamma_1 \cos \phi_1)$$

i.e.

$$F_{n_1, n_2} = \frac{1}{\gamma_2} \int e^{in_1\phi_1} \int \frac{e^{in_2\phi_2}}{\cosh \gamma(\phi_1) - \cos\phi_2} d\phi_2 d\phi_1 .$$

The relation between (34) and (35) is evident.

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