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Generalized "Transition Probability"

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Abstract. An operationally meaningful symmetric function defined on pairs of states of an arbitrary physical system is constructed and is shown to coincide with the usual "transition probability" in the special case of systems admitting a quantum-mechanical description. It can be used to define a metric in the set of physical states. Conceivable applications to the analysis of certain aspects of Quantum Mechanics and to its possible modifications are mentioned.

1. Introduction

Let us regard a physical system as specified by its *states* and its *observables*, the former being operationally defined by assigning prescriptions of preparation, the latter by assigning processes of measurement. For the outcome α of any preparation, for each measurement A and for each Borel set E of the real line R let us denote, with Mackey ([1], p. 62), by $p(A, \alpha, E)$ the probability that A performed on α give a result in E. Mackey's axiom I [$p(A, \alpha, \emptyset)=0$, $p(A, \alpha, R)=1$, complete additivity with respect to E] is thus satisfied. The outcomes α and α' of two preparations are not regarded as distinct states if $p(A, \alpha, E)=p(A, \alpha', E)$ for all choices of A and E; similarly two measurements A and A' describe the same observable if $p(A, \alpha, E)=p(A', \alpha, E)$ for all choices of α and E. Thus in $p(A, \alpha, E)$ A and α in fact denote an observable and a state, and Mackey's axiom II is satisfied.

The "probability function" $p(A, \alpha, E)$ can in principle be constructed experimentally with any degree of accuracy: for any given A and any given α one has to repete (many times) the preparation of the state α followed by the measurement of the observable A, and look at the distribution of the results. Therefore any quantity associated with one or more states can be regarded as correctly defined operationally if it is defined in terms of the function p alone. On the other hand, if it is agreed that the possibility of enlarging the set $\mathscr{S}$ of states and the set $\mathcal{O}$ of observables (with a corresponding extension of the function p) would be regarded as giving rise to a *different* physical system, then it can be asserted that p contains *all* the physical information about the given system, and the physical content of any assumption or statement about the latter must in principle be expressible in terms of the probability function p.

Our main purpose here is to remark that for *any* system (satisfying the physically unrestrictive axioms I and II of Mackey's) it is possible to define, in terms of p alone, a function $T(\alpha, \beta)$ which generalizes the quantum-mechanical "transition probability" in the sense that it automatically coincides with the latter on pure states whenever the additional assumption that the system admits a quantum-mechanical description is made.

In the special context of Quantum Mechanics this permits the determination of an important part of the structure (namely, the squared modulus of the scalar product among normalized representatives of the states) on a direct operational basis, without use of any of the specific assumptions or interpretative postulates of the theory (in particular, it is not necessary to make the physically non-obvious assumption that in correspondence with any state α, or even with *some* state α, one can find observables which, when measured on α, give definite results with probability 1: an assumption which is required by the most usual operational definition of the transition probability in Quantum Mechanics. Similarly, it is never necessary to consider the simultaneous measurement of distinct observables).

In a broader context this separation of the "metric" part of the quantum-mechanical structure from its "linear" part (for which no entirely satisfactory *a priori* justification seems to be known) suggests operational criteria to test the validity of the quantum-mechanical scheme as a whole (including linearity), and seems to give some indication towards natural generalizations of the scheme.

2. The Generalized "Transition Probability"

Given any state α and any observable A, we shall denote by α_A the probability measure on the real line R such that $\int_E d\alpha_A = p(A, \alpha, E)$ for any Borel set E. To any pair of states α and β we can associate, in correspondence with A, the finite measure $\sqrt{\alpha_A \beta_A}$ defined, with Mackey ([1], p. 100), by $\int_E d\sqrt{\alpha_A \beta_A} = \int_E \sqrt{\frac{d\alpha_A}{d\sigma}\frac{d\beta_A}{d\sigma}}\, d\sigma$, where σ is any finite measure with respect to which α_A and β_A are absolutely continuous, and $d\alpha_A/d\sigma$, $d\beta_A/d\sigma$ are the Radon-Nikodym derivatives of α_A and β_A with respect to σ.

Setting

$$T_A(\alpha, \beta) = \int_R d\sqrt{\alpha_A \beta_A}\,, \tag{1}$$

we shall define the *generalized transition probability* from the state α to the state β as the infimum of $T_A(\alpha, \beta)$ as A runs through the set $\mathcal{O}$ of all observables:

$$T(\alpha, \beta) = \inf_{A \in \mathcal{O}} T_A(\alpha, \beta)\,. \tag{2}$$

It is immediately obvious from the definition that $T(\alpha, \beta)$ is intrinsically symmetric ($T(\alpha, \beta) = T(\beta, \alpha)$), non-negative, and equal to 1 whenever $\alpha = \beta$. It is also easy to show that $T(\alpha, \beta) \leqq 1$, with a strict inequality whenever $\alpha \neq \beta$. In fact, if for given A, α and β the measure σ is chosen as above, then $d\alpha_A/d\sigma$ and $d\beta_A/d\sigma$ determine two elements of unit norm, with scalar product $T_A(\alpha, \beta)$, in the real Hilbert space $L^2(R, \sigma)$ of the real functions on R which are square-integrable with respect to σ; thus, from the Schwarz inequality,

$$T_A(\alpha, \beta) \leqq 1\,, \tag{3}$$

which implies $T(\alpha, \beta) \leqq 1$. On the other hand, if α and β are distinct states, there exists at least one observable A and a Borel set E such that $\int_E d\alpha_A \neq \int_E d\beta_A$, so that $d\alpha_A/d\sigma$ and $d\beta_A/d\sigma$ determine distinct elements of $L^2(R, \sigma)$ and (3) holds with the inequality: consequently $T(\alpha, \beta)$ is strictly smaller than 1.

Let us show that, under the additional specific assumption that the system admits a quantum-mechanical description, $T(\alpha, \beta)$ agrees, on pure states, with the ordinary quantum-mechanical transition probability $T_q(\alpha, \beta) = |\langle \underset{\sim}{\alpha}, \underset{\sim}{\beta} \rangle|^2$ (where $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ are normalized representatives of the states α and β in the Hilbert space H of the theory, and $\langle , \rangle$ denotes the scalar product in H). Denoting by $\underset{\sim}{A}$ the self-adjoint operator in H associated with the observable A, let us express $\underset{\sim}{A}$ as the sum of two operators $\underset{\sim}{A}'$ and $\underset{\sim}{A}''$, where $\underset{\sim}{A}'$ has a pure point spectrum and $\underset{\sim}{A}''$ a purely continuous spectrum. We shall denote by $\lambda_1, \lambda_2, \ldots$ the eigenvalues of $\underset{\sim}{A}'$, by $\underset{\sim}{x}_{i1}, \underset{\sim}{x}_{i2}, \ldots$ an orthonormal basis in the subspace of H whose elements are eigenvectors of $\underset{\sim}{A}'$ with eigenvalue λ_i, by $\mathscr{E}_\lambda$ the spectral function of $\underset{\sim}{A}''$ and by $\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots$ a maximal set of mutually orthogonal vectors of H such that, for each k, $\underset{\sim}{A}''$ has a simple spectrum on the closed subspace generated by the elements $\mathscr{E}_\lambda \underset{\sim}{y}_k (-\infty < \lambda < \infty)$. If the choice of the vectors $\underset{\sim}{y}_k$ gives rise to a normal subdivision of the spectral function, the scalar product of $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ can be expressed in the form (see for example [2], Chapter 5)

$$\langle \underset{\sim}{\alpha}, \underset{\sim}{\beta} \rangle = \sum_i \sum_s \langle \underset{\sim}{\alpha}, \underset{\sim}{x}_{is} \rangle \overline{\langle \underset{\sim}{\beta}, \underset{\sim}{x}_{is} \rangle} + \int_R \sum_R \alpha_k(\lambda) \underline{\beta}_k(\lambda) \varrho_k'(\lambda) d\varrho_1(\lambda) ,$$

where $\varrho_k(\lambda) = \langle \mathscr{E}_\lambda \underset{\sim}{y}_k, \underset{\sim}{y}_k \rangle$, $\varrho_k' = d\varrho_k / d\varrho_1$ and $\int_{-\infty}^{\lambda} \alpha_k(\mu) d\varrho_k(\mu) = \langle \underset{\sim}{\alpha}, \mathscr{E}_\lambda \underset{\sim}{y}_k \rangle$. Hence

$$|\langle \underset{\sim}{\alpha}, \underset{\sim}{\beta} \rangle| \leqq \sum_i \left(\sum_s |\langle \underset{\sim}{\alpha}, \underset{\sim}{x}_{is} \rangle|^2\right)^{1/2} \left(\sum_r |\langle \underset{\sim}{\beta}, \underset{\sim}{x}_{ir} \rangle|^2\right)^{1/2} + \int_R \left(\sum_k |\alpha_k|^2 \varrho_k'\right)^{1/2} \left(\sum_l |\beta_l|^2 \varrho_l'\right)^{1/2} d\varrho_1 .$$

But according to the postulates of Quantum Mechanics one has

$$p(A, \alpha, E) = \sum_{i | \lambda_i \in E} \sum_s \langle \underset{\sim}{\alpha}, \underset{\sim}{x}_{is} \rangle|^2 + \int_E \sum_k |\underset{\sim}{\alpha}_k|^2 \varrho_k' d\varrho_1 ,$$

so that the square of the right-hand side of the last inequality is just $T_A(\alpha, \beta)$ as defined by (1), while the square of the left-hand side is equal to $T_q(\alpha, \beta)$. Thus for every observable A we have $T_q(\alpha, \beta) \leqq T_A(\alpha, \beta)$, and therefore

$$T_q(\alpha, \beta) \leqq T(\alpha, \beta) . \tag{4}$$

If we now choose A to be the observable associated with the projection operator on the one-dimensional subspace generated by $\underset{\sim}{\alpha}$, then the probability measure α_A is concentrated at the pofnt $\lambda = 1$, while $p(A, \beta, 1) = |\langle \underset{\sim}{\alpha}, \underset{\sim}{\beta} \rangle|^2$, i.e. $T_A(\alpha, \beta) = T_q(\alpha, \beta)$, which implies $T(\alpha, \beta) \leqq T_q(\alpha, \beta)$; and by comparison with (4) we see that the equality sign must hold.

3. Metric Structure of the Space of Physical States

Coming back to the general case, $T(\alpha, \beta)$ can be used to define in the space $\mathscr{S}$ of physical states a metric $d(\alpha, \beta)$ which agrees, in the case of Quantum Mechanics, with the natural "distance function" in the projective Hilbert space whose points correspond one-to-one to the pure states of the physical system (see for example [3]). Such a metric can be defined by setting

$$d(\alpha, \beta) = [2(1 - T^{1/2}(\alpha, \beta)]^{1/2} . \tag{5}$$

From the properties of $T(\alpha, \beta)$ one gets immediately $d(\alpha, \alpha) = 0$, $d(\alpha, \beta) = d(\beta, \alpha)$ and $d(\alpha, \beta) > 0$ whenever $\alpha \neq \beta$. The triangle inequality can be proved by setting

$$d_A(\alpha, \beta) = [2(1 - T_A^{1/2}(\alpha, \beta)]^{1/2} \tag{6}$$

in correspondence with any observable A, and by remarking that the triangle inequality in the Hilbert space $L^2(R, \sigma)$ considered in the previous section is expressed by $d_A(\alpha, \beta) \leqq d_A(\alpha, \gamma) + d_A(\gamma, \beta)$. Since $d(\alpha, \beta) = \sup_{A \in \mathcal{O}} d_A(\alpha, \beta)$ [on account of (2), (5) and (6)], the last inequality yields $d(\alpha, \beta) \leqq d(\alpha, \gamma) + d(\gamma, \beta)$.

We conclude by remarking that any characterization of the conditions which the metric $d(\alpha, \beta)$ should satisfy to make $\mathcal{S}$ isometric to a projective Hilbert space would amount to requirements [expressible in terms of $p(A, \alpha, E)$, and therefore operationally meaningful] on the physical system in order that it admit a quantum-mechanical description. Should a physical system not satisfy such requirements, the present approach might suggest a setting for a modified theory retaining the *metric* aspect of Quantum Theory but possibly giving up some of the features related with the *linearity* of the Hilbert structure.

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