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On Local Field Products in Special Wightman Theories*

Klaus Baumann

Bell Laboratories, Murray Hill, New Jersey, USA, and Max-Planck-Institut für Physik und Astrophysik, München, Federal Republic of Germany

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Abstract. We shall try to define local field products under assumptions imposed only on the four-point-function. This idea is based on the work of Schlieder and Seiler [1].

In our framework we shall prove that the two-point-function carries the strongest singularity whenever two arguments in a Wightman function coincide. This will be generalized to the case when more arguments coincide. We shall define "regulated" n-point-functions and study their properties in detail. This will lead us to the definition of arbitrarily high powers of the field-operators as operator-valued distributions over $\mathscr{D}(\mathbb{R}^4)$ in the center coordinate with a dense domain of definition.

1. Introduction and Some Results Stated in [1]

Field products at the same space-time point lead to great difficulties in quantum theories because of the distributional character of the field operators.

Schlieder and Seiler [1] define local products of two field operators under assumptions imposed only on the four-point-function. We want to extend their approach such that it includes local products of three or more field operators. Our investigation is based on axiomatic quantum field theory [2] described in terms of Wightman functions.

Let us first introduce some notations:

$$\underline{z} := (z_0, \ldots, z_n) \in \mathbb{C}^{4(n+1)}$$
$$\underline{\zeta} := (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^{4n} \quad \text{with} \quad \zeta_i = z_i - z_{i-1}$$
$$\tau_n^{\pm} := \{\underline{\zeta} \in \mathbb{C}^{4n} \mid \operatorname{Im} \zeta_i \in V_n^{\pm}\}$$
$$\text{("forward/backward tube")}$$
$$\tau_n' := \{\underline{\zeta} \in \mathbb{C}^{4n} \mid \exists \Lambda \in L_+(\mathbb{C}) : \Lambda \underline{\zeta} \in \tau_n^+\}$$
$$\text{("extended tube")}$$

where $L_+(\mathbb{C})$ denotes the proper complex Lorentz group. For $\pi \in S_{n+1}$ (group of permutations of $\{0, 1, \ldots, n\}$) we define

$$\underline{\zeta}_\pi := (z_{\pi(1)} - z_{\pi(0)}, \ldots, z_{\pi(n)} - z_{\pi(n-1)})$$
$$= \left(\sum_{j=1}^{\pi(1)} \zeta_j - \sum_{j=1}^{\pi(0)} \zeta_j, \ldots, \sum_{j=1}^{\pi(n)} \zeta_j - \sum_{j=1}^{\pi(n-1)} \zeta_j \right)$$
$$\tau_{n,\pi}^{\pm,\prime} := \{\underline{\zeta} \in \mathbb{C}^{4n} \mid \underline{\zeta}_\pi \in \tau_n^{\pm,\prime}\}$$
$$\text{("permuted forward/backward/extended tube")}.$$

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Sometimes we shall decompose $\underline{\zeta}$ in three parts:

$$\underline{\zeta} = (\underset{\rightarrow}{\zeta}, \zeta_j, \vec{\zeta})$$

defined by

$$\underset{\rightarrow}{\zeta} := (\zeta_1, \ldots, \zeta_{j-1}), \qquad \vec{\zeta} := (\zeta_{j+1}, \ldots, \zeta_n).$$

The real scalar field $A(x)$ and the Hilbert space $\mathscr{H}$ are assumed to fulfill Wightman's axioms [2].

$$\Phi(z_0 \ldots, z_n) := \int \exp\left(i \sum_{k=0}^{n} p_k z_k\right) \hat{A}(p_0) \ldots \hat{A}(p_n)\, dp_0 \ldots dp_n \Omega$$

is a vector-valued holomorphic function for

$$z_0, z_1 - z_0, \ldots, z_n - z_{n-1} \in \tau_1^+ \qquad \text{(cf. Jost's book [2])}.$$

We shall call $\Phi(\underline{z})$ a „Jost-state". The span of all Jost-states will be denoted by $\mathscr{J}$ and is a dense subspace of the basic Hilbert space $\mathscr{H}$.

The assumptions made in [1] are:

(A1) *There exists a function* $r(\zeta)$ *with the properties*

a) r *is holomorphic in* τ_1^+.

b) $r(\bar{\zeta}) = \overline{r(-\zeta)}$.

c) r *is invariant under the homogeneous, real Lorentz group* $\mathscr{L}_+^\uparrow$.

d) $r(\xi) \in \mathscr{D}'(\mathscr{U}_r(0))$, *where* $\mathscr{U}_r(0)$ *is a real neighborhood of the point* $0 \in \mathbb{C}^4$ *with respect to the Euclidean norm.*

(A2) $W_4(\zeta_1, \zeta_2, \zeta_3)\, r(\zeta_3)$ *has an analytic continutation to the points* $\tau_2^+ \times \mathscr{U}_r(0)$.

We shall call r a regulating function.

Remarks. (1) A theorem of Hall and Wightman [3] states that r is even invariant under the group $L_+(\mathbb{C})$. Further there is a function $\hat{r}(\sigma)$ holomorphic for $\sigma \in \mathbb{C} \setminus [0, \infty)$, such that

$$r(\zeta) = \hat{r}(\zeta^2).$$

Property (d) implies

$$\hat{r}(\sigma) \in \mathscr{D}'((-\infty, \varrho)) \quad \text{for some} \quad \varrho > 0.$$

(2) Condition (d) is equivalent to (e) formulated in [1] by the virtue of a theorem given in the Appendix 1.

The crucial result on which we shall rely is the following theorem proved by Schlieder and Seiler.

Theorem 0. *Under the assumptions* (A1) *and* (A2) *all the Wightman functions* $W_{n+1}(\underline{\zeta})\, r(\zeta_j)$ *have analytic continuations to the points* $\zeta_j = 0$,

$$(\underset{\rightarrow}{\zeta}, \vec{\zeta}) := (\zeta_1, \ldots, \zeta_{j-1}, \zeta_{j+1}, \ldots, \zeta_n) \in \tau_{n-1}^+.$$

First we want to strengthen this Theorem 0.

Theorem 1. *Under the assumptions* (A1) *and* (A2) *all the Wightman functions* $W_{n+1}(\underline{\zeta})\, r(\zeta_j)$ *have analytic continuations to the points*

$$\zeta_j \in \mathscr{U}_r(0) = \{\xi \in \mathbb{R}^4 \mid \xi^2 < r^2\}, \quad (\underset{\rightarrow}{\zeta}, \vec{\zeta}) \in \tau_{n-1}^+ \quad \textit{for some} \quad r > 0.$$

Proof. a) For real, spacelike ζ_j and $(\underrightarrow{\zeta}, \vec{\zeta}) \in \tau_{n-1}^+$ the point $\underline{\zeta} = (\underrightarrow{\zeta}, \zeta_j, \vec{\zeta})$ is in the extended tube τ_n' (cf. Jost [2], p. 84).

b) The case $\zeta_j \in \mathbb{R}^4$, $\zeta_j^2 = 0$ is contained in the Theorem 0 because for every $\delta > 0$ there exists a real Lorentz transformation Λ such that $|\Lambda \zeta_j| < \delta$. ($|\zeta|$ denotes the Euclidean norm!)

c) By Lorentz invariance and because $\hat{r}(\sigma) \in \mathscr{D}'((-\infty, r))$ it is clear that $W_4(\zeta_1, \zeta_2, \zeta_3)\, r(\zeta_3)$ has an analytic continuation to $\zeta_1, \zeta_2 \in \tau_1^+, \zeta_3 \in \mathscr{U}_r(0)$ (see also [10]). This implies that the vector-valued holomorphic function $A(z) A(z+\zeta) r(\zeta) \Omega$ has an analytic continuation to $\zeta \in \mathscr{U}_r(0)$ for every $z \in \tau_1^+$ [1].

d) Now choose $\hat{\zeta}_j \in \mathbb{R}^4$ with $0 < \hat{\zeta}_j^2 < r^2$. Without any restriction we can assume $\hat{\zeta}_j = \begin{pmatrix} c \\ \vec{0} \end{pmatrix}$.

Consider the n-tuple $\underline{\zeta} = (\zeta_1, \ldots, \zeta_n)$ and the permuted one

$$\underline{\zeta}_\pi := \Bigg(\underbrace{\zeta_1, \ldots, \zeta_{j-2}, \sum_{k=j-1}^{n-1} \zeta_k}_{\underrightarrow{\zeta}_\pi}, \underset{\zeta_{\pi,j}}{\underset{\|}{\zeta_n}}, \underbrace{-\sum_{k=j}^{n} \zeta_k, \zeta_j, \ldots, \zeta_{n-2}}_{\vec{\zeta}_\pi} \Bigg).$$

We are looking for a point $\underline{\zeta} = (\zeta_1, \ldots, \zeta_n)$ in the analyticity domain of $W_{n+1}(\underline{\zeta})\, r(\zeta_n)$ and for a complex Lorentz transformation $\Lambda \in L_+(\mathbb{C})$ such that

$$\Lambda \zeta_{\pi,j} = \Lambda \zeta_n = \hat{\zeta}_j \quad \text{and} \quad (\hat{\underrightarrow{\zeta}}, \hat{\vec{\zeta}}) := \Lambda(\underrightarrow{\zeta}_\pi, \vec{\zeta}_\pi) \in \tau_{n-1}^+ .$$

Suppose we have found such a point $\underline{\zeta}$. The analyticity domain is open and therefore there exist complex neighborhoods $\mathscr{U}(\hat{\zeta}_j) \subset \mathbb{C}^4$ and $\mathscr{V}(\hat{\underrightarrow{\zeta}}, \hat{\vec{\zeta}}) \subset \tau_{n-1}^+$ such that $W_{n+1}(\underline{\zeta})\, r(\zeta_j)$ is analytic for $\zeta_j \in \mathscr{U}(\hat{\zeta}_j)$ and $(\underrightarrow{\zeta}, \vec{\zeta}) \in \mathscr{V}(\hat{\underrightarrow{\zeta}}, \hat{\vec{\zeta}})$.

For $g(\xi) \in \mathscr{D}(\mathbb{R}^4 \cap \mathscr{U}(\hat{\zeta}_j))$ let us define

$$\begin{aligned} G(\underrightarrow{\zeta}, \vec{\zeta}) := \int d\xi\, g(\xi) \{ & W_{n+1}(\underrightarrow{\zeta}, \xi + i0, \vec{\zeta})\, r(\xi + i0) \\ & - W_{n+1}(\underrightarrow{\zeta}, \xi - i0, \vec{\zeta})\, r(\xi - i0) \} . \end{aligned}$$

Because of the spectrum condition $G(\underrightarrow{\zeta}, \vec{\zeta})$ is analytic for $(\underrightarrow{\zeta}, \vec{\zeta}) \in \tau_{n-1}^+$. But $G(\underrightarrow{\zeta}, \vec{\zeta}) = 0$ on $\mathscr{V}(\hat{\underrightarrow{\zeta}}, \hat{\vec{\zeta}})$. This implies $G(\underrightarrow{\zeta}, \vec{\zeta}) \equiv 0$ on τ_{n-1}^+. By the edge of the wedge theorem $W_{n+1}(\underline{\zeta})\, r(\zeta_j)$ is analytic for $\zeta_j \in \mathbb{R}^4 \cap \mathscr{U}(\hat{\zeta}_j)$ and fixed $(\underrightarrow{\zeta}, \vec{\zeta}) \in \tau_{n-1}^+$. By the generalized Hartogs theorem [11] we get the analyticity in $\underline{\zeta}$ for $\zeta_j \in \mathbb{R}^4 \cap \mathscr{U}(\hat{\zeta}_j)$ and $(\underrightarrow{\zeta}, \vec{\zeta}) \in \tau_{n-1}^+$.

e) For fixed $z \in \tau_1^+$, $A(z) A(z+\xi) r(\xi) \Omega$ is a vector-values holomorphic function for $\xi \in \mathscr{U}_r(0)$. Therefore there is a complex neighborhood $\mathscr{U} \supset \mathscr{U}_r(0)$ such that $A(z) A(z+\zeta) r(\zeta) \Omega$ is analytic for $\zeta \in \mathscr{U}$. If we write

$$W_{n+1}(\underline{\zeta})\, r(\zeta_n) = (\Phi_{n-1}(\underline{z}), A(z) A(z+\zeta_n) r(\zeta_n) \Omega) ,$$

we see that $W_{n+1}(\underline{\zeta})\, r(\zeta_n)$ is holomorphic if $\zeta_n \in \mathscr{U}$, $\zeta_{n-1} - z \in \tau_1^+$, and $\zeta_1, \ldots, \zeta_{n-2} \in \tau_1^+$. For simplicity we fix $z = i \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}$.

f) Now we choose $0 < \alpha < \dfrac{\pi}{2}$ such that

$$\Lambda(\alpha) \begin{pmatrix} c \\ \vec{0} \end{pmatrix} := \begin{pmatrix} \cos\alpha & 0 & 0 & i \sin\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i \sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} c \\ \vec{0} \end{pmatrix} = \begin{pmatrix} c \cos\alpha \\ \vec{0} \end{pmatrix} + i \begin{pmatrix} \vec{0} \\ c \sin\alpha \end{pmatrix} \in \mathscr{U} .$$

This is always possible because $\begin{pmatrix} c \\ \vec{0} \end{pmatrix} \in \mathscr{U}$ and $\mathscr{U}$ open. Define

$$\begin{aligned}
\zeta_n &:= \begin{pmatrix} c\cos\alpha \\ \vec{0} \end{pmatrix} + i \begin{pmatrix} \vec{0} \\ c\sin\alpha \end{pmatrix} \\
\zeta_{n-1} &:= \begin{pmatrix} \vec{0} \\ (n-j+2)\cot\alpha \end{pmatrix} + i \begin{pmatrix} 2 \\ \vec{0} \end{pmatrix} \\
\zeta_j &= \zeta_{j+1} = \dots = \zeta_{n-2} := i \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \\
\zeta_{j-1} &:= i \begin{pmatrix} 2 \\ \vec{0} \end{pmatrix} \\
\zeta_1 &= \dots = \zeta_{j-2} := i \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \quad \text{and} \\
\Lambda &:= \Lambda(-\alpha)\,.
\end{aligned}$$

This choice of $\underline{\zeta}$ and Λ fulfills the requirements of (d) by construction. This proves our Theorem 1.

Remarks. (3) By complex Lorentz transformations we can further enlarge the region of analyticity. Lemma 4 by Schlieder and Seiler [1] ensures the single-valuedness of this continuation.

(4) By a trivial extension of the proof we can write $\underline{\zeta}_\pi$, $\pi \in S_{n+1}$ a permutation, instead of $\underline{\zeta}$ in Theorem 1.

2. Connection between r and the Two-Point-Function W_2

In [4] de Mottoni and Genz argue that the two-point-function has the leading singularity in an expansion of the products of two fields. We shall prove such a behaviour in our framework.

Theorem 2. *Under the assumption* (A) *there is even a function* $r'(\zeta)$ *with the properties*

(1) r' *satisfies the condition* (A).

(2) *Define* $F_2'(\zeta) := W_2(\zeta)\, r'(\zeta)$ *then* $F_2'(0) = 1$.

Remarks. (1) Theorem 1 states the analyticity of F' for $\zeta \in \tau_1^+ \cup \mathscr{U}_r(0)$. Because of $F_2'(0) = 1$

$$r'(\zeta) = \frac{F_2'(\zeta)}{W_2(\zeta)}$$

behaves like $W_2^{-1}(\zeta)$ and

$$r'^{-1}(\zeta) = \frac{W_2(\zeta)}{F_2'(\zeta)}$$

like $W_2(\zeta)$ for sufficiently small $\zeta \in \tau_1^+$.

These two relations imply that $W_2^{-1}(\xi)$ and $r'^{-1}(\xi)$ are elements of $\mathscr{D}'(\mathscr{U}_r(0))$.

(2) The singularities in any n-point-function $W_{n+1}(\underline{\zeta})$ if ζ_j goes to zero cannot be stronger than the singularity of $W_2(\zeta_j)$.

Proof of Theorem 2. (a) $F_2(\zeta) := W_2(\zeta)\, r(\zeta)$ is analytic for $\zeta = 0$. As shown in Appendix 2 there exists a function $\hat{F}_2(\sigma)$ analytic for $\sigma = 0$ with $F_2(\zeta) = \hat{F}_2(\zeta^2)$.

(b) If $F_2(0) = c \neq 0$ then with $r' := \frac{1}{c} \cdot r$ our theorem has been proved. [Because of $F_2(\zeta) = \overline{F_2(-\bar{\zeta})}$ the constant c is real!] Therefore let us assume $F_2(0) = 0$.

(c) **Lemma 1.** *$F_2(0) = 0$ implies $W_4(\zeta)\, r(\zeta_3) = 0$ for $\zeta_3 = i\eta$ with $\eta^2 = 0$ and $i\eta \in \overline{iV^+} \cap \mathscr{U}(0)$.*

Proof. Schwarz's inequality tells us

$$|(A(z)\, A(z_1)\, A(z_2)\Omega, A(z)\Omega)| \leqq \|A(z)\Omega\| \cdot \|A(z)\, A(z_1)\, A(z_2)\Omega\| .$$

Written in terms of Wightman-functions this means

$$|W_4(\zeta_1, \zeta_2, \zeta_3)|^2 \leqq W_2(\zeta_3) \cdot W_6(\zeta_1, \zeta_2, \zeta_3, -\bar{\zeta}_2, -\bar{\zeta}_1)$$

if $\zeta_1, \zeta_2 \in \tau_1^+$ and $\zeta_3 = i\eta \in iV^+$. Because $r(i\eta)$ is real we can multiply both sides with $r^2(\zeta_3) = |r(\zeta_3)|^2$

$$|W_4(\zeta_1, \zeta_2, \zeta_3)\, r(\zeta_3)|^2 \leqq W_2(\zeta_3)\, r(\zeta_3)\, W_6(\zeta_1, \zeta_2, \zeta_3, -\bar{\zeta}_2, -\bar{\zeta}_1)\, r(\zeta_3) .$$

For fixed $\zeta_1, \zeta_2 \in \tau_1^+$ the functions $W_4(\zeta_1, \zeta_2, \zeta_3)\, r(\zeta_3)$, $W_2(\zeta_3)\, r(\zeta_3)$, and $W_6(\zeta_1, \zeta_2, \zeta_3, -\bar{\zeta}_2, -\bar{\zeta}_1)\, r(\zeta_3)$ are holomorphic in ζ_3 within some (complex) neighborhood $\mathscr{U}(0)$ of $0 \in \mathbb{C}^4$.

Therefore the inequality remains true even for $\zeta_3 \in \overline{iV^+} \cap \mathscr{U}(0)$. But $F_2(\zeta_3) = W_2(\zeta_3)\, r(\zeta_3) = 0$ for $\zeta_3^2 = 0$ and $\zeta_3 \in \overline{iV^+} \cap \mathscr{U}(0)$. This proves Lemma 1.

(d) **Lemma 2.** *For fixed $\zeta_1, \zeta_2 \in \tau_1^+$ define*

$$F(\zeta) := W_4(\zeta_1, \zeta_2, \zeta)\, r(\zeta)$$

then $F_2(0) = 0$ implies

1) *$F(\zeta) = 0$ for $\zeta^2 = 0$ and $\zeta \in \mathscr{U}(0)$.*

2) *$\dfrac{F(\zeta)}{\zeta^2}$ is holomorphic in $\mathscr{U}(0)$.*

Proof. (I thank E. Seiler for this proof)

$$f(\zeta^2, \vec{\zeta}) := F(\zeta) = F(\sqrt{\zeta^2 + \vec{\zeta}^2}, \vec{\zeta})$$

is holomorphic in $\zeta^2, \vec{\zeta}$ for $\vec{\zeta} \neq 0$ and $|\zeta^2| < |\vec{\zeta}^2|$. This implies

$$f(\zeta^2, \vec{\zeta}) = \sum_{n=0}^{\infty} b_n(\vec{\zeta}, \pm)\, (\zeta^2)^n$$

in this domain, where $b_n(\vec{\zeta}, \pm)$ are holomorphic functions for $\vec{\zeta} \neq 0$. (The $\pm$ sign in b_n shall indicate that b_n can depend on the choice of the branch of $\sqrt{\vec{\zeta}^2}$.) $F(\zeta) = 0$ for $\zeta^2 = 0$, $\zeta \in \overline{iV^+} \cap \mathscr{U}(0)$ by Lemma 1. This implies $b_0(\vec{\zeta}, +) = 0$ for $\vec{\zeta} \in i\mathbb{R}^3 \cap (\mathscr{U}'(0) \setminus \vec{0})$. But the riemannian manifold of $\sqrt{\vec{\zeta}^2}$ is connected and therefore $b_0(\vec{\zeta}, \pm) \equiv 0$ for $\vec{\zeta} \neq 0$. Lemma 2 follows by the continuity theorem for functions of many complex variables [5], because there can be no singularities on manifolds of real codimension > 2.

(e) Lemma 1 and 2 together imply that if $F_2(0)=0$ then $\frac{1}{\zeta^2} r(\zeta)$ satisfies conditions (A). If

$$\hat{F}_2(\zeta^2)=\sum_{n=k}^{\infty} a_n(\zeta^2)^n, \qquad a_k \neq 0$$

then with

$$r'(\zeta)=\frac{1}{a_k(\zeta^2)^k}\, r(\zeta).$$

Theorem 2 is fulfilled.

Remark 3. Because of Theorem 2 we can always normalize our regulating function r in such a way that $F_2(0)=W_2(0)\, r(0)=1$. If r_1 and r_2 are two normalized regulating functions then there is a meromorphic function h with $h(0)=1$ and

$$r_1(\zeta)=r_2(\zeta)\, h(\zeta).$$

This defines an equivalence relation.

3. Representation of the *n*-Point-Functions

The first aim of [1] was to define the product of two field operators under assumptions only on the four-point-function. But to our great surprise the assumptions (A) are sufficient to make definite statements about the singularities occurring in n-point-functions if an arbitrary number of arguments comes very close together.

Let us define

$$R_{n+1}(\underline{\zeta})=R_{n+1}(\zeta_1,\ldots,\zeta_n):=\prod_{0\leqq i<j\leqq n} r(z_j-z_i)$$
$$=\prod_{1\leqq i\leqq j\leqq n} r(\zeta_i+\cdots+\zeta_j) \quad \text{with} \quad \zeta_i=z_i-z_{i-1}, \qquad i=1,\ldots,n$$

and

$$F_{n+1}(\underline{\zeta})=\mathscr{W}_{n+1}(z_0,\ldots,z_n)\prod_{0\leqq i<j\leqq n} r(z_j-z_i)$$
$$=W_{n+1}(\underline{\zeta})\, R_{n+1}(\underline{\zeta}).$$

Theorem 3. *$F_{n+1}(\underline{\zeta})$ has an analytic continuation to the point $\underline{\zeta}=0$.*

Proof. a) Properties of $F_{n+1}(\underline{\zeta})$:

Combining the well established properties of W_{n+1} and R_{n+1} we get

(1) $F_{n+1}(\underline{\zeta})$ is holomorphic for $\underline{\zeta}\in \bigcup_{\pi\in S_{n+1}} \tau'_{n,\pi}$ and invariant under the group $L_+(\mathbb{C})$.

(2) $F_{n+1}(\underline{\zeta})=F_{n+1}(\underline{\zeta}_\pi)$ for every permutation $\pi\in S_{n+1}$.

(3) $\lim\limits_{V_n^+\ni \underline{\eta}_\pi\to 0} F_{n+1}(\underline{\xi}+i\underline{\eta})\in \mathscr{D}'\left(\left[\frac{1}{n}\,\mathscr{U}_r(0)\right]^n\right)$.

b) Consequences of Theorem 1:

For every permutation $\pi\in S_{n+1}$ the function $W_{n+1}(\underline{\zeta}_\pi)\, r(\zeta_{\pi,j})$ is holomorphic for $\zeta_{\pi,j}\in\mathscr{U}_r(0)$ and $(\underline{\zeta}_\pi,\vec{\zeta}_\pi)\in\tau^+_{n-1}$. Because of the analytic structure of $R_{n+1}(\underline{\zeta})$ we get

(4) $F_{n+1}(\underline{\zeta})$ holomorphic for $\zeta_{\pi,j}\in\mathscr{U}_r(0)$, $(\underline{\zeta}_\pi,\vec{\zeta}_\pi)\in\tau^+_{n-1}$.

c) Idea of proof:

$F_{n+1}(\underline{\zeta})$ is holomorphic in τ_n^+ and $F_{n+1}(\underline{\zeta}_\pi):=F_{n+1}(-\zeta_n,\ldots,-\zeta_1)$ is holomorphic in τ_n^-. If the boundary values $F_{n+1}(\underline{\xi}+i\underline{0})$ and $F_{n+1}(\underline{\xi}-i\underline{0})$ are equal (in the sense of distributions) then by the edge of the wedge theorem, Theorem 3 has been proved. For this purpose we shall put in $\frac{n}{2}(n-1)-1$ other boundary values between $F_{n+1}(\underline{\xi}+i\underline{0})$ and $F_{n+1}(\underline{\xi}-i\underline{0})$ such that by (b) every boundary value equals its neighbor.

d) Definition of the boundary values:

Without any restriction we can choose $\mathscr{U}_r^{(n)}(0) \subset \frac{1}{n}\mathscr{U}_r(0)$. Let us define for $g \in \mathscr{D}([\mathscr{U}_r^{(n)}(0)]^n)$ and $\pi \in S_{n+1}$

(5) $G(\underline{\eta}_\pi) := \int d\underline{\xi}\, g(\underline{\xi})\, F_{n+1}(\underline{\xi}+i\underline{\eta}), \underline{\eta}_\pi \in V_n^+$.

The boundary value

(6) $G(\underline{0}_\pi) := \lim\limits_{V_n^+ \ni \underline{\eta}_\pi \to \underline{0}} G(\underline{\eta}_\pi)$

exists by property (3).

f) **Lemma 3.** *Let $\pi \in S_{n+1}$ be a permutation and $\tau=(j-1,j)$ be the transposition which permutes $j-1$ with j then*

(7) $G(\underline{0}_\pi) = G(\underline{0}_{\pi\circ\tau})$.

Proof. 1) By definition

$$\zeta_{\pi\circ\tau,j} := z_{\pi\circ\tau(j)} - z_{\pi\circ\tau(j-1)} = z_{\pi(j-1)} - z_{\pi(j)} = -\zeta_{\pi,j}$$

and therefore $\eta_{\pi\circ\tau,j} = -\eta_{\pi,j}$. This means

(8) $\lim\limits_{V^- \ni \eta_{\pi,j}\to 0} G(\underline{\eta}_\pi) = \lim\limits_{V^+ \ni \eta_{\pi\circ\tau,j}\to 0} G(\underline{\eta}_{\pi\circ\tau})$.

By property (4) $F_{n+1}(\underline{\zeta})$ is holomorphic if $\zeta_{\pi,j} \in \mathscr{U}_r^{(n)}(0)$ and $(\underset{\to}{\zeta_\pi}, \vec{\zeta}_\pi) \in \tau_{n+1}^+$. Hence

(9) $\lim\limits_{V^+ \ni \eta_{\pi,j}\to 0} G(\underline{\eta}_\pi) = \lim\limits_{V^- \ni \eta_{\pi,j}\to 0} G(\underline{\eta}_\pi) = \lim\limits_{V^+ \ni \eta_{\pi\circ\tau,j}\to 0} G(\underline{\eta}_{\pi\circ\tau})$ if $(\underset{\to}{\eta_\pi}, \vec{\eta}_\pi) \in V_{n-1}^+$.

2) $G(\underline{0}_\pi) = \lim\limits_{V_{n-1}^+ \ni (\underset{\to}{\eta}, \vec{\eta})_\pi \to 0} \left[\lim\limits_{V^+ \ni \eta_{\pi,j}\to 0} G(\underline{\eta}_\pi) \right]$

$= \lim\limits_{V_{n-1}^+ \ni (\underset{\to}{\eta}, \vec{\eta})_\pi \to 0} \left[\lim\limits_{V^+ \ni \eta_{\pi\circ\tau,j}\to 0} G(\underline{\eta}_{\pi\circ\tau}) \right] = G(\underline{0}_{\pi\circ\tau})$.

This proves Lemma 3.

g) Let π be the permutation

$$\begin{pmatrix} 0 & 1 & 2 & & n \\ & & & \cdots & \\ n & n-1 & n-2 & & 0 \end{pmatrix},$$

then $\underline{\zeta}_\pi = (-\zeta_n, \ldots, -\zeta_1)$. The theorem has been proved if we can show that $G(\underline{0}) = G(\underline{0}_\pi)$. But we can write π as the product

$$\pi = [(01)(12)\ldots(n-1\; n)]\,[(01)\ldots(n-2\; n-1)]\ldots[(01)(12)]\,[(01)]\,.$$

Now by applying Lemma 3 we have proved the theorem because

$$G(\underline{0}) = G(\underline{0}_{(12)}) = G(\underline{0}_{(01)(12)}) = \cdots = G(\underline{0}_\pi)\,.$$

Corollaries. 1) By following the same line as in the proof of Theorem 3 we can prove similar statements, e.g.:

a) $W_{2n}(\underline{\zeta})\, r(\zeta_1)\, r(\zeta_3) \ldots r(\zeta_{2n-1})$ has an analytic continuation to $\zeta_1, \zeta_3, \ldots, \zeta_{2n-1} \in \mathscr{U}_r(0)$, $\zeta_2, \zeta_4, \ldots, \zeta_{2n-2} \in \tau_1^+$.

b) $W_{n+1}(\underline{\zeta})\, r(\zeta_j)\, r(\zeta_j+\zeta_{j+1})\, r(\zeta_{j+1})$ has an analytic continuation to $\zeta_j, \zeta_{j+1} \in \mathscr{U}_r(0)$, $\zeta_1, \ldots, \zeta_{j-1}, \zeta_{j+2}, \ldots, \zeta_n \in \tau_1^+$. For more complicated expressions, analogous results hold.

2) Theorem 3 gives the following representation of the n-point-functions:

$$W_{n+1}(\underline{\zeta}) = F_{n+1}(\underline{\zeta}) \prod_{1 \leqq i \leqq j \leqq n} r^{-1}(\zeta_i + \cdots + \zeta_j)\,.$$

By Remark 1 of Part 2 we can replace r^{-1} by $\dfrac{W_2}{F_2}$ in a small neighborhood $\mathscr{V}(0)$:

$$W_{n+1}(\underline{\zeta}) = F_{n+1}(\underline{\zeta}) \prod_{1 \leqq i \leqq j \leqq n} \frac{W_2}{F_2}(\zeta_i + \cdots + \zeta_j)\,.$$

But $F_2^{-1}(\zeta)$ is analytic for $\zeta = 0$ and therefore the singularities of $W_{n+1}(\underline{\zeta})$ are controlled by those of $\prod_{1 \leqq i \leqq j \leqq n} W_2(\zeta_i + \cdots + \zeta_n)$.

This result is surprising, because assumptions (A) are conditions only imposed on the four-point-function. Nevertheless, we get statements about all possible singularities of n-point-functions.

Examples. Perhaps we should illustrate our approach by two examples

1) Free field with mass zero [6]

$$A(x) = \varphi(x) \quad \text{with} \quad \Box\, \varphi(x) = 0\,.$$

The two-point-function is of the form

$$\mathscr{W}_2(z_0, z_1) = W_2(\zeta) = \frac{-1}{(2\pi)^2}\, \frac{1}{\zeta^2}\,.$$

All the higher n-point-functions are built up from W_2.

$$\mathscr{W}_n(\underline{z}) = 0 \quad n \text{ odd}$$

$$\mathscr{W}_{2n}(z_1, \ldots, z_{2n}) = \sum_{\substack{\text{all} \\ \text{partitions } i}} \mathscr{W}_2(z_{i_1}, z_{i_2}) \ldots \mathscr{W}_2(z_{i_{2n-1}}, z_{i_{2n}})\,.$$

This means for the four-point-function

$$W_4(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{(2\pi)^4} \left\{ \frac{1}{\zeta_1^2 \zeta_3^2} + \frac{1}{(\zeta_1+\zeta_2)^2 (\zeta_2+\zeta_3)^2} + \frac{1}{(\zeta_1+\zeta_2+\zeta_3)^2\, \zeta_2^2} \right\}.$$

As a normalized regulating function we can use

$$r(\zeta) = -(2\pi)^2 \zeta^2 = \frac{1}{W_2(\zeta)}\,.$$

The assumptions (A) are trivially fulfilled. The Wick product is defined by

$$\varphi(z_1)\, \varphi(z_2) = W_2(z_2 - z_1) + {:}\varphi(z_1)\, \varphi(z_2){:}\,.$$

Hence we get

$$\varphi(z)\, \varphi(z+\zeta)\, r(\zeta) = 1 - (2\pi)^2 \zeta^2\, {:}\varphi(z)\, \varphi(z+\zeta){:}\,.$$

The properties of the Wick product [6] imply that for all $\zeta \in \mathbb{R}^4$ the "regularized" product $\varphi(z)\,\varphi(z+\zeta)\,r(\zeta)$ is defined on $\mathscr{J}_z$ (see Part 4 for definition) and analytic in ζ if applied to a vector $\Phi \in \mathscr{J}_z$.

Starting from the above example we can construct a whole class of examples-namely all Wick polynomials in the free field $\varphi(x)$ with mass zero.

2) The exponential function of a free field with arbitrary mass in two dimensions [7]

$$A(x) = :e^{\varphi}:(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\, :\varphi^k:(x) \quad \text{with} \quad (\Box + m^2)\,\varphi(x) = 0\,.$$

The restriction on two dimensions is necessary for the Wightman distributions to be temperate. We shall denote the two-point-function of the underlying free field $\varphi(x)$ by $W_2(\zeta)$ and the Wightman functions of $A(x)$ by $\mathscr{V}_n(\underline{z})$ or $V_n(\underline{\zeta})$.

$$\mathscr{V}_2(z_0, z_1) = V_2(\zeta) = \exp\{W_2(\zeta)\}$$

$$\mathscr{V}_{n+1}(\underline{z}) = \prod_{0 \leqq i < j \leqq n}^{\cdot} \mathscr{V}_2(z_i, z_j)$$

e.g.: $$V_4(\underline{\zeta}) = V_2(\zeta_1)\, V_2(\zeta_1+\zeta_2)\, V_2(\zeta_1+\zeta_2+\zeta_3)\, V_2(\zeta_2)\, V_2(\zeta_2+\zeta_3)\, V_2(\zeta_3)\,.$$

Because of this factorization property, it is clear that $r(\zeta) := \dfrac{1}{V_2(\zeta)}$ defines a normalized regulating function. By Wick ordering we get [7]

$$A(z_1)\,A(z_2) = V_2(z_2 - z_1)\, :A(z_1)\,A(z_2):$$

or

$$A(z)\,A(z+\zeta)\,r(\zeta) = :A(z)\,A(z+\zeta):\,.$$

Therefore $A(z)\,A(z+\zeta)\,r(\zeta)$ has the domain $\mathscr{J}_z$ and is analytic in ζ for $\zeta \in \mathbb{R}^4$ if applied to $\Phi \in \mathscr{J}_z$. For the functions $F_{n+1}(\underline{\zeta})$ defined above we get $F_{n+1}(\underline{\zeta}) = 1$. Because of this we think of this example as a very typical one.

4. Operator Products

Up to here all statements and conclusions have been formulated in terms of Wightman functions. Now we want to define operator products, specify their domains, and characterize their analytic behaviour.

For this purpose we need the subspace $\mathscr{J}_z$ of the space $\mathscr{J}$ of all Jost-states defined by

$$\mathscr{J}_z := e^{ip\cdot z}\, \mathscr{J}$$

where $e^{ip\cdot z}$ is the translation operator. This means

$$\Phi_n(z_1, \ldots, z_n) \in \mathscr{J}_z \quad \text{iff} \quad \Phi_n(z_1 - z, \ldots, z_n - z) \in \mathscr{J}\,.$$

Theorem 4.

$$A(x_1) \ldots A(x_l) \prod_{1 \leqq i < j \leqq l} r(x_j - x_i)$$

exists as a sesquilinear form with domain $\mathscr{J} \times \mathscr{J}$ *and is analytic in* $x_1, \ldots, x_l$ *if* $x_1, \ldots, x_l \in \mathscr{U}_r(x)$ *for some neighborhood* $\mathscr{U}_r(x) = \{x\} + \mathscr{U}_r(0)$.

Proof. Because of the structure of $\mathscr{J}$ it is sufficient to prove the theorem for the special states $\Phi_m(\underline{z}')\otimes\Phi_n(\underline{z}'')$

$$\begin{aligned}
&A(x_1)\ldots A(x_l)\prod_{i<j} r(x_j-x_i)\,(\Phi_m(z_1',\ldots,z_m')\otimes\Phi_n(z_1'',\ldots,z_m''))\\
&:=(\Omega, A(\bar z_m')\ldots A(\bar z_1')\,A(x_1)\ldots A(x_l)\,A(z_1'')\ldots A(z_n'')\Omega)\prod_{i<j} r(x_j-x_i)\\
&=\mathscr{W}_{m+l+n}(\bar z_m',\ldots,\bar z_1',x_1,\ldots,x_l,z_1'',\ldots,z_n'')\prod_{i<j} r(x_j-x_i)\,.
\end{aligned}$$

But this is analytic in $x_1,\ldots,x_l$ because of Theorem 3.

Theorem 5. *If* $z\in\tau_1^+$ *then* $A(z_1)\ldots A(z_l)\prod_{i<j} r(z_j-z_i)$ *defines an operator with domain* $\mathscr{J}_z$ *and is analytic in* $z_1\ldots z_l$ *if* $z_1\ldots z_l\in\mathscr{U}_r(z):=\{z\}+\mathscr{U}_r(0)$ *for some neighborhood* $\mathscr{U}_r(0)$.

Proof. 1) Let Φ be the state $A(z_1')\ldots A(z_n')\Omega\in\mathscr{J}_z$. We can write the vector

$$A(z_1)\ldots A(z_l)\prod_{i<j} r(z_j-z_i)\,A(z_1')\ldots A(z_n')\Omega$$

in two ways with different regions of analyticity:

$$\begin{aligned}
\Phi_1(\underline{\zeta}):=\;&A(z+\zeta_1)\,A(z+\zeta_1+\zeta_2)\ldots A(z+\zeta_1+\cdots+\zeta_l)\\
&\cdot\prod_{2\leq i\leq j\leq l} r(\zeta_i+\cdots+\zeta_i)\,A(z_1')\ldots A(z_n')\Omega
\end{aligned}$$

holomorphic in $\zeta_1,\ldots,\zeta_l$ if $\zeta_1\ldots\zeta_l\in\tau_1^+$ and $z_1'-z-\zeta_1-\cdots-\zeta_l\in\tau_1^+$

$$\begin{aligned}
\Phi_2(\underline{\zeta}):=\;&A(z+\zeta_1+\cdots+\zeta_l)\ldots A(z+\zeta_1)\prod_{2\leq i\leq j\leq l} r(-\zeta_i-\cdots-\zeta_j)\\
&\cdot A(z_1')\ldots A(z_n')\Omega
\end{aligned}$$

holomorphic in $\zeta_1,\ldots,\zeta_l$ if $\zeta_1\ldots\zeta_l\in\tau_1^-$ and $z+\zeta_1+\cdots+\zeta_l\in\tau_1^+$.

2) Theorem 3 tells us that the matrix element

$$\begin{aligned}
\Big(&\Omega, A(\bar z_n')\ldots A(\bar z_1')\,A(\bar z_l)\ldots A(\bar z_1)\prod_{i<j} r(\bar z_i-\bar z_j)\\
&\cdot A(z_1)\ldots A(z_l)\prod_{i<j} r(z_j-z_i)\,A(z_1')\ldots A(z_n')\Omega\Big)
\end{aligned}$$

is analytic in $\bar z_1\ldots\bar z_l$ if $\bar z_1\ldots\bar z_l\in\mathscr{U}_r(\bar z)$ and in $z_1\ldots z_l$ if $z_1\ldots z_l\in\mathscr{U}_r(z)$. ($\bar z_i, z_i$ are thought of as independent variables!) This implies

$$\begin{aligned}
&\lim_{V_n^+\ni\underline{\eta},\underline{\eta}'\to 0}\|\Phi_1(\underline{\xi}+i\underline{\eta})-\Phi_2(\underline{\xi}-i\underline{\eta}')\|^2\\
&=\lim_{V_n^+\ni\underline{\eta},\underline{\eta}'\to 0}\{(\Phi_1(\underline{\xi}+i\underline{\eta}),\Phi_1(\underline{\xi}+i\underline{\eta}))+(\Phi_2(\underline{\xi}-i\underline{\eta}'),\Phi_2(\underline{\xi}-i\underline{\eta}'))\\
&\quad-(\Phi_1(\underline{\xi}+i\underline{\eta}),\Phi_2(\underline{\xi}-i\underline{\eta}'))-(\Phi_2(\underline{\xi}-i\underline{\eta}'),\Phi_1(\underline{\xi}+i\underline{\eta}))\}=0
\end{aligned}$$

because each term tends to the same limit. By the edge of the wedge theorem we get that for all $\Psi\in\mathscr{H}$

$$\Big(\Psi, A(z_1)\ldots A(z_l)\prod_{i<j} r(z_j-z_i)\,A(z_1')\ldots A(z_n')\Omega\Big)$$

is analytic in $z_1, \ldots, z_l$ if $z_1, \ldots, z_l \in \mathscr{U}_r(z)$. But weak analyticity implies strong analyticity and the theorem has been proved.

As a more complicated problem we now attack the existence of the boundary value

$$\lim_{V^+ \ni y \to 0} A(x+iy+\xi_1) \ldots A(x+iy+\xi_l) \prod_{i<j} r(\xi_j - \xi_i).$$

Of course it will be no longer an operator without smearing in the center coordinate x!

Theorem 6. *For $\xi_1, \ldots, \xi_l \in \mathscr{U}_r(0)$, $\mathscr{U}_r(0)$ some real neighborhood, the product*

$$A(x+\xi_1) \ldots A(x+\xi_l) \prod_{i<j} r(\xi_j - \xi_i)$$

defines an operator-valued distribution (in x) over $\mathscr{D}(\mathbb{R}^4)$ with domain $\mathscr{J}$ and is infinitely differentiable in the relative coordinates $\xi_1, \ldots, \xi_l$.

Proof. 1) We choose for Ψ the vector $\Psi = A(z_1) \ldots A(z_n)\Omega \in \mathscr{J}$. Then there exists a point $y \in V^+$ such that $z_1 - iy \in \tau_1^+$. Now by Theorem 5

$$A(x+\xi_1+iy) \ldots A(x+\xi_l+iy) \prod_{i<j} r(\xi_j - \xi_i) A(z_1) \ldots A(z_n)\Omega$$

is an analytic vector-valued function in the relative coordinates $\xi_1, \ldots, \xi_l$.

2) Let us consider the norm:

$$\left\| A(x+\xi_1+iy) \ldots A(x+\xi_l+iy) \prod_{i<j} r(\xi_j - \xi_i) A(z_1) \ldots A(z_n)\Omega \right\|^2$$
$$= (\Omega, A(\bar{z}_n) \ldots A(\bar{z}_1) A(x+\xi_l-iy) \ldots A(x+\xi_1-iy) A(x+\xi_1+iy) \ldots$$
$$\ldots A(x+\xi_l+iy) A(z_1) \ldots A(z_n)\Omega) \prod_{i<j} |r(\xi_j - \xi_i)|^2 .$$

Now we multiply with $\prod_{1 \leqq i,j \leqq l} r(\xi_j - \xi_i + 2iy)$ and divide afterwards by the same expression

$$= \Big\{(\Omega, A(\bar{z}_n) \ldots A(x+\xi_1-iy) A(x+\xi_1+iy) \ldots A(z_n)\Omega)$$
$$\cdot \prod_{i<j} r(\xi_j - \xi_i) r(\xi_i - \xi_j) \prod_{i,j} r(\xi_j - \xi_i + 2iy)\Big\} \cdot \prod_{i,j} r^{-1}(\xi_j - \xi_i + 2iy).$$

The term between curly brackets { } has a finite value as $y \in V^+$ goes to 0. Therefore $\prod_{i,j} r^{-1}(\xi_j - \xi_i + 2iy)$ controls the growth of

$$\lim_{V^+ \ni y \to 0} \|A(x+\xi_1+iy) \ldots A(x+\xi_l+iy) \prod_{i<j} r(\xi_j - \xi_i)\Psi\|^2 .$$

But as stated in Remark 1 of Part 2 $r^{-1}(x+i0)$ is a distribution over $\mathscr{D}(V_r(0))$ and therefore there is an inequality like

$$|r^{-1}(x+iy)| < \tilde{C}(y^2)^{-q} \quad \text{if} \quad x \in \mathscr{V}_r(0).$$

By this we have proved

$$\left\| A(x+\xi_1+iy) \ldots A(x+\xi_l+iy) \prod_{i<j} r(\xi_j - \xi_i)\Psi \right\| < C(y^2)^{-\frac{1}{2}ql^2}$$

if $\xi_1, \ldots, \xi_l \in \mathscr{U}_r(0)$. This implies that

$$A(x+\xi_1) \ldots A(x+\xi_l) \prod_{i<j} r(\xi_j - \xi_i) \Psi$$

is a vector-valued distribution (in x) over $\mathscr{D}(\mathbb{R}^4)$ if $\xi_1, \ldots, \xi_l \in \mathscr{U}_r(0)$.

3) Now let D^α be a differential operator on the components of $\xi_1, \ldots, \xi_l$. By similar reasoning as in (2) we get

$$\left\| D^\alpha A(x+\xi_1+iy) \ldots A(x+\xi_l+iy) \prod_{i<j} r(\xi_j - \xi_i) \Psi \right\| < C(y^2)^{-Q},$$

where Q depends on the order of D^α. Therefore

$$D^\alpha A(x+\xi_1) \ldots A(x+\xi_l) \prod_{i<j} r(\xi_j - \xi_i) \Psi$$

defines a vector-valued distribution over $\mathscr{D}(\mathbb{R}^4)$ too.

4) So far we have proven that for $g \in \mathscr{D}(\mathbb{R}^4)$ and $\Psi \in \mathscr{J}$ the limit

$$\lim_{V^+ \ni y \to 0} \int g(x) D^\alpha A(x+\xi_1+iy) \ldots A(x+\xi_l+iy) \prod_{i<j} r(\xi_j - \xi_i) \Psi$$

exists. If we can show that this limit is equal to

$$D^\alpha \lim_{V^+ \ni y \to 0} \int g(x) A(x+\xi_1+iy) \ldots A(x+\xi_l+iy) \prod_{i<j} r(\xi_j - \xi_i) \Psi$$

our theorem has been proved. But this problem is solved by the Appendix too.

These results can be extended in various directions:

a) One can enlarge the domain of definition in Theorem 6 such that it contains states like

$$A(z) A(z+\xi) r(\xi) \Omega, \qquad \xi \in \mathscr{V}_r(0)$$

or

$$A(z_1) A(z_1+\xi_1) r(\xi_1) A(z_2) A(z_2+\xi_2) r(\xi_2) \Omega, \qquad \xi_1, \xi_2 \in \mathscr{V}_r(0).$$

b) We can prove the locality of the operator products in the sense of sesquilinear forms on $\mathscr{J} \times \mathscr{J}$.

c) It is easy to derive from Theorem 6 a Wilson-Zimmermann expansion [8] for arbitrarily high products. Let us consider the product of three field operators as an example:

$$A(z-\xi) A(z) A(z+\eta) r(\xi) r(\xi+\eta) r(\eta)$$

acting on a state $\Phi \in \mathscr{J}_z$, $z \in \tau_1^+$ is analytic for $\xi, \eta \in \mathscr{U}_r(0)$ (cf. Theorem 5). The boundary value

$$A(x-\xi) A(x) A(x+\eta) r(\xi) r(\xi+\eta) r(\eta)$$

is an operator-valued distribution in x, infinitely differentiable in ξ, η for $\xi, \eta \in \mathscr{U}_r(0)$, and has domain $\mathscr{J}$ (cf. Theorem 6). Therefore we get the following asymptotic expansion:

$$A(x-\xi) A(x) A(x+\eta) \sim \sum_{m,n=0}^{\infty} \frac{1}{m!} \frac{1}{n!} \frac{\xi^{\mu_1} \ldots \xi^{\mu_m} \eta^{\nu_1} \ldots \eta^{\nu_n}}{r(\xi) r(\xi+\eta) r(\eta)} C_{\mu_1 \ldots \mu_m, \nu_1 \ldots \nu_n}(x).$$

The "composite operators" $C_{\mu_1 \ldots \mu_m, \nu_1 \ldots \nu_n}(x)$ are local [in the sense of (b)], operator-valued distributions with common domain $\mathscr{J}$.

Finally we should mention that the assumption (A) is too restrictive to be considered seriously for more realistic Wightman theories. One should try to start with a finite decomposition of the four-point-function

$$W_4(\zeta_1, \zeta_2, \zeta_3) = \sum_{k=1}^{l} u_k(\zeta_3)\, G_k(\zeta_1, \zeta_2, \zeta_3).$$

The functions $u_k(\zeta_3)$ are analytic in τ_1^+, Lorentz invariant, and characterize the possible singularities in ζ_3. The functions $G_k(\zeta_1, \zeta_2, \zeta_3)$ are assumed to be analytic for $\underline{\zeta} \in \tau_2^+ \times \{\tau_1^+ \cup \mathscr{U}_r(0)\}$. Such an ansatz is suggested by the free fields and their Wick polynomials.

This problem is still under investigation and has not yet been solved in a convincing manner.

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Appendix 1. Boundary Values of Functions Holomorphic in the Forward Tube

Let be

$$\mathscr{U} \subset \mathbb{R}^{4n} \text{ a connected open set}$$

$$\mathscr{C} \subset V_n^+ \text{ an open convex cone}$$

$$\mathscr{C}_\delta := \mathscr{C} \cap \{\underline{y} \in \mathbb{R}^{4n} \mid |\underline{y}| < \delta\} \quad \text{with} \quad |\underline{y}|^2 = \sum_{k=1}^{n} \sum_{\mu=0}^{3} |y_k^\mu|^2$$

$\overline{\mathscr{C}}'$ a compact subcone.

Proposition. *Let $f(\underline{z}, \underline{\xi})$ be continuous for $(\underline{z}, \underline{\xi}) \in \tau_n^+ \times G$. $G \subset \mathbb{C}^{4m}$ compact, and*

$$\bigwedge_{\xi \in G} (f(\underline{z}, \underline{\xi}) \text{ holomorphic in } \underline{z}).$$

For $g \in \mathscr{D}(\mathscr{U})$ let us define

$$F_{\underline{y}}(g, \underline{\xi}) := \int d\underline{x}\, g(\underline{x})\, f(\underline{x} + i\underline{y}, \underline{\xi}).$$

Then the following statements are equivalent:

(A) *For all $g \in \mathscr{D}(\mathscr{U})$ $\lim_{\underline{y} \to 0} F_{\underline{y}}(g, \underline{\xi})$ exists and $F_{\underline{y}}(g, \underline{\xi})$ is continuous for $(\underline{y}, \underline{\xi}) \in \overline{\mathscr{C}}_\delta' \times G$ for every compact subcone $\overline{\mathscr{C}}'$ and some $\delta > 0$.*

(B) *For every compact set $K \subset \mathscr{U}$ and compact subcone $\overline{\mathscr{C}}' \subset V_n^+$ there are $\delta > 0$, $M > 0$, and $\mathbb{N} \ni m \geqq 0$ such that*

$$|f(\underline{x} + i\underline{y}, \underline{\xi})| \leqq \frac{M}{|\underline{y}|^m} \quad \text{for all} \quad (\underline{x} + i\underline{y}, \underline{\xi}) \in [K + i\overline{\mathscr{C}}_\delta' \setminus \{0\}] \times G.$$

For the proof of the theorem, we refer to [9].

Corollary. *Let $f(\underline{z}, \underline{\xi})$ be an infinitely differentiable function in $\underline{\xi} \in G \subset \mathbb{R}^m$ and let f together with all derivatives $D_{\underline{\xi}} f$ fulfill condition (A) of the proposition. Then for all $g \in \mathscr{D}(\mathscr{U})$*

$$\lim_{\underline{y} \to 0} (D_{\underline{\xi}} F)_{\underline{y}}(g, \underline{\xi}) = D_{\underline{\xi}} \lim_{\underline{y} \to 0} F_{\underline{y}}(g, \underline{\xi}).$$

Proof. It is sufficient to prove the corollary for f and $\frac{\partial}{\partial \xi_i} f$ only. Without any restriction we can assume that f depends only on $\underline{z}$ and $\xi = \xi_i$. Let $g \in \mathscr{D}(\mathscr{U})$. By assumption

$$\left(\frac{\partial^2}{\partial \xi^2} F\right)_{\underline{y}} (g, \xi) := \int d\underline{x} g(\underline{x}) \frac{\xi^2}{\partial \xi^2} f(\underline{x} + i\underline{y}, \xi)$$

is continuous for $(\underline{y}, \xi) \in \overline{\mathscr{C}}'_\delta \times G$. The mean value theorem tells us

$$\left| F_{\underline{y}}(g, \xi + \Delta) - F_{\underline{y}}(g, \xi) - \Delta \left(\frac{\partial F}{\partial \xi}\right)_{\underline{y}} (g, \xi)\right| = \frac{\Delta^2}{2} \left|\left(\frac{\partial^2 F}{\partial \xi^2}\right)_{\underline{y}} (g, \xi + \Theta \Delta)\right|$$

with $0 < \Theta < 1$. But for every compact set $K \subset \overline{\mathscr{C}}'_\delta \times G$ $\left|\left(\frac{\partial^2 F}{\partial \xi^2}\right)_{\underline{y}} (g, \hat{\xi})\right|$ is bounded for all $(\underline{y}, \hat{\xi}) \in K$. If we choose K suitably our corollary has been proved.

Appendix 2

Let $F(\zeta)$ denote a holomorphic function for $\zeta \in \tau_1^+ \cup \{\zeta \in \mathbb{C}^4 \mid |\zeta|^2 < \delta^2\}$ which is invariant under $\mathscr{L}_+^\uparrow$. Then there exists a function $\hat{F}(\sigma)$ holomorphic for $\sigma \in \mathbb{C} \setminus [\delta^2, \infty)$ such that $F(\zeta) = \hat{F}(\zeta^2)$. $\hat{F}(\zeta^2)$ is a holomorphic continuation of $F(\zeta)$.

Proof. 1) By a theorem of Hall and Wightman [3] there exists a function $\hat{F}(\sigma)$ holomorphic in $\mathbb{C} \setminus [0, \infty]$ such that $F(\xi) = \hat{F}(\zeta^2)$ for $\zeta \in \tau'_1$.

2) $\hat{F}(\sigma)$ can be analytically continued to $\sigma \in (0, \delta^2)$.

Proof. a) For $\sigma \in (0, \delta^2)$ let us define

$$\zeta_\sigma := \begin{pmatrix} \sqrt{\sigma} \\ \vec{0} \end{pmatrix} (\sqrt{\sigma} > 0)$$

and

$$\hat{F}(\sigma) := F(\zeta_\sigma).$$

Hence $\hat{F}(\sigma)$ is continuous for $\sigma \in (0, \delta^2)$.

b) Let $(\sigma_n^\pm)$ two sequences with limit $\sigma \in (0, \delta^2)$ but $\operatorname{Im} \sigma_n^+ > 0$ and $\operatorname{Im} \sigma_n^- < 0$. As above we define

$$\zeta \sigma_n^\pm := \begin{pmatrix} \sqrt{\sigma_n^\pm} \\ \vec{0} \end{pmatrix} \quad \text{with} \quad \arg \sqrt{\sigma_n^\pm} \in \begin{cases} \left(0, \frac{\pi}{2}\right) \\ \left(-\frac{\pi}{2}, 0\right) \end{cases}$$

$$\Rightarrow \lim_n \zeta \sigma_n^\pm = \zeta_\sigma .$$

c) By (1): $\hat{F}(\sigma_n^+) = F(\zeta \sigma_n^+)$, $\hat{F}(\sigma_n^-) = F(\zeta \sigma_n^-)$

$$\lim_n \hat{F}(\sigma_n^\pm) = \lim_n F(\zeta \sigma_n^\pm) = F\left(\lim_n \zeta \sigma_n^\pm\right) = F(\zeta\sigma) = \hat{F}(\sigma).$$

This proves (2).

3) $\hat{F}(\sigma)$ can be analytically continued to $\sigma = 0$.

Proof. By (1) and (2) $\hat{F}(\sigma)$ is holomorphic in the open disk $0<|\sigma|<\delta^2$. But for $0<|\sigma|<\frac{\delta^2}{4}$

$$|\hat{F}(\sigma)| \leqq \sup_{|\zeta|^2<\frac{\delta^2}{4}} |F(\zeta)| \leqq \max_{|\zeta|^2=\frac{\delta^2}{4}} |F(\zeta)| < \infty$$

because $F(\zeta)$ is analytic for $|\zeta|^2<\delta^2$ and the maximum principle holds for $F(\zeta)$. This means $|\hat{F}(\sigma)|$ bounded on $0<|\sigma|<\frac{\delta^2}{4}$ and therefore $\hat{F}(\sigma)$ can be analytically continued to $\sigma=0$.

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K. Baumann
Bell Laboratories
600 Mountain Avenue
Murray Hill, New Jersey 07974, USA