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Analyticity and Uniqueness of the Invariant Equilibrium States for General Spin $\frac{1}{2}$ Classical Lattice Systems

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Abstract. Asano-Ruelle-Slawny method is generalized to discuss analyticity and uniqueness of the correlation functions in terms of the group structure associated with any lattice systems. The use of Poisson formula for abelian groups gives a simple method to obtain explicit domains where the above properties are verified.

I. Introduction

The analysis of the zeroes of the partition function Z is one of the standard methods of statistical mechanics used to derive general properties of lattice systems. A powerful technique, based on Asano's contraction [1], was given by Ruelle to study domains of zeroes of Z in the complex variable $z = e^{-2\beta h}$ [2]; the main idea is to reduce the study of the zeros of the partition function to the study of smaller polynomials associated with the *local structure of the lattice.*

More recently the Asano-Ruelle technique was generalized by Slawny [3] to discuss domains of zeroes in all the complex variables $z_B = e^{-2\beta J(B)}$ where $J(B)$ is the interaction associated with the bond B; the idea of Slawny is to start with the Low Temperature (L. T.) expansion of the partition function expressed as a summ over a certain group Γ, called the L. T. group. He then makes use of the group structure associated with lattice systems to give general conditions for the partition function to be the Asano contraction of the partition function of small subsystems. Using then a theorem due to Ruelle [4] this extension of the Asano-Ruelle technique yields new results concerning the analyticity properties of the free energy and the uniqueness of the equilibrium state for ferromagnetic systems at low temperatures.

In the following we shall extend Slawny's results to *arbitrary lattice systems* and give a general method to obtain explicit domains where Z is different from zero. These results then imply the analyticity and uniqueness properties of the free energy and the correlation functions, and they follow from a generalization of the Asano-Ruelle-Slawny method.

We consider an arbitrary spin $\frac{1}{2}$ lattice system $\{\Lambda, \mathscr{B}\}$, defined by a set of lattice sites Λ and a set $\mathscr{B}$ of bonds, $\mathscr{B} \subset \mathscr{P}(\Lambda)$; we then discuss the zeroes of polynomials of the form $M(z_{\mathscr{B}}) = \sum_{\beta \in G} \prod_{B \in \beta} z_B$ where G *is any* *subgroup of* $\mathscr{P}(\mathscr{B})$ and z_B is a complex variable associated each bond B of $\mathscr{B}$. Using the group structure of the lattice we give a necessary and sufficient condition for $M(z_{\mathscr{B}})$ to be the Asano Contraction of smaller polynomials $M(z_{\mathscr{B}_i})$ reflecting the local group structure of the lattice. It then follows immediately from Poisson Formula for abelian groups that the zeroes of $M(z_{\mathscr{B}_i})$ can be discussed in terms of the zeroes of polynomials of the form:

$$\tilde{M}(\tilde{z}_{\mathscr{B}_i}) = 1 + \prod_{B \in \mathscr{B}_i} \tilde{z}_B \quad \text{where} \quad \tilde{z}_B = \frac{1 - z_B}{1 + z_B}.$$

Since domains of zeroes of such simple polynomials are most easily obtained we then have directly domains where $M(z_{\mathscr{B}})$ is different from zero. In fact in the following discussion we have restricted our attention only to the "trivial" domains defined by $|\tilde{z}_B| < 1$ for all B, $|\tilde{z}_B| > 1$ for all B and $\left|\sum_{B \in \mathscr{B}_i} \arg \tilde{z}_B\right| < \pi$.

For the application to statistical mechanics the groups G of special interest are the "High temperature (H.T.) group $\mathscr{K}$" and the "Low Temperature (L.T.) group Γ"; it should be remarked that for $G = \mathscr{K}$ the small polynomial is *not* the partition function of the small subsystem as it is for the case $G = \Gamma$. Applying the above simple results to the special group $\mathscr{K}$ (resp. Γ) we obtain immediately explicit domains D_B such that the partition function Z is different from zero if $z_B \in D_B$ for all B in $\mathscr{B}$ and $z_B = \tanh K(B)$ (resp. $z_B = e^{-2K(B)}$). The analyticity and uniqueness property of the pressure and the correlation functions follow then immediately.

As direct consequences of these simple results, we recover in particular

– Lee-Yang circle theorem for ferromagnetic systems with two body forces [5].

– Ruelle's result of unicity of the equilibrium state for ferromagnetic systems with two body forces and external field [4].

– Sarbach and Rys' result concerning the antiferromagnetic Ising system with two body forces [6].

– The uniqueness of the *equilibrium state* ω *invariant under translation* for any lattice system at high temperature [7],

we extend Slawny's result to conclude that for any lattice system there exists at low temperature a *unique state* ω *invariant under the full symmetry group*

we obtain explicit domains where analyticity and uniqueness properties of the pressure and correlation functions hold.

To conclude this analysis we have applied the general results to the study of some specific examples with many body forces. In particular we have shown in one example the usefulness of the duality transformation to improve the domains of analyticity.

II. Asano Contraction and Zeroes of the Partition Function

With any spin $\frac{1}{2}$ lattice system $\{\Lambda, \mathscr{B}\}$, defined by a finite set Λ of lattice sites together with a family $\mathscr{B}$ of bonds where $\mathscr{B} \subset \mathscr{P}(\Lambda)$, we can associate a group structure defined by means of the following abelian groups [8]:

$\mathscr{P}(\Lambda)=\{X; X\subset\Lambda\}$; $\quad X'\cdot X''=X'\cup X''-X'\cap X''$
"Group of configurations",

$\mathscr{P}(\mathscr{B})=\{\beta;\beta\subset\mathscr{B}\}$; $\quad \beta'\cdot\beta''=\beta'\cup\beta''-\beta'\cap\beta''$
"Group of Graphs",

$\mathscr{K}=\left\{\beta\subset\mathscr{B}; \prod_{B\in\beta} B=\phi\right\}$ "*High Temperature group*",

$\Gamma=\{\beta\subset\mathscr{B}; \exists X\subset\Lambda \text{ such that } (-1)^{|B\cap X|}=-1 \text{ iff } B\in\beta\}$
"*Low Temperature group*",

The "*High Temperature*" *expansion* of the partition function is given by

$$Z=2^{|\Lambda|}\prod_{B\in\mathscr{B}}\cosh K(B)\sum_{\beta\in\mathscr{K}}\prod_{B\in\beta}\tanh K(B) \tag{1.1}$$

and we are interested in domains which do not contain the zeroes of Z as function of the complex interactions $\{K(B)\}$. More generally with each bond B in $\mathscr{B}$ we associate a complex variable z_B and we consider the polynomial in the complex variables $z_B=\{z_B\}_{B\in\mathscr{B}}$ defined by

$$M(z_B)=\sum_{\beta\in G} z^{\beta} \tag{1.2}$$

where G is *any* arbitrary subgroup of $\mathscr{P}(\mathscr{B})$ and for any β in $\mathscr{P}(\mathscr{B})$ $z^{\beta}=\prod_{B\in\beta} z_B$. Using the identity (Poisson formula for abelian group [9])

$$\sum_{\beta\in G} f(\beta)=|G|\sum_{\bar\beta\in G^{\perp}}\tilde f(\bar\beta)$$

where

$$G^{\perp}=\{\bar\beta\subset\mathscr{B}; \sigma_{\beta}(\bar\beta)=(-1)^{|\beta\cap\bar\beta|}=+1\ \forall\beta\in G\}$$

and

$$\tilde f(\bar\beta)=2^{-|\mathscr{B}|}\sum_{\beta\in\mathscr{P}(\mathscr{B})} f(\beta)\sigma_{\beta}(\bar\beta)$$

we then have:

$$M(z_{\mathscr{B}})=\sum_{\beta\in G} z^{\beta}=|G|2^{-|\mathscr{B}|}\prod_{B\in\mathscr{B}}(1+z_B)\sum_{\bar\beta\in G^{\perp}}\tilde z^{\bar\beta} \tag{1.3}$$

where

$$\tilde z_B=\frac{1-z_B}{1+z_B}\qquad z_B=\frac{1-\tilde z_B}{1+\tilde z_B}. \tag{1.4}$$

Proposition 1. *For any subgroup G of $\mathscr{P}(\mathscr{B})$ the zeroes of the polynomial $M(z_{\mathscr{B}})=\sum_{\beta\in G} z^{\beta}$ such that $z_B\neq -1$ for all B in $\mathscr{B}$, are related to the zeroes of the polynomial*

$$\tilde M(\tilde z_{\mathscr{B}})=\sum_{\bar\beta\in G^{\perp}}\tilde z^{\bar\beta}$$

by the transformation Eq. (1.4).

The H.T. and L.T. groups defined above have the property that

$$\Gamma^{\perp}=\mathscr{K}\qquad \mathscr{K}^{\perp}=\Gamma \tag{1.5}$$

moreover $z_B = \tanh K(B)$ implies $\tilde{z}_B = e^{-2K(B)}$ and the Poisson Formula applied to the H.T. expansion of the partition function Z yields the well known "*Low Temperature*" *expansion* [8]:

$$Z = |\mathscr{K}| 2^{|\Lambda| - |\mathscr{B}|} \prod_{B \in \mathscr{B}} e^{K(B)} \sum_{\beta \in \Gamma} \prod_{B \in \beta} e^{-2K(B)}. \tag{1.6}$$

Let $\mathscr{B} = \bigcup_{i=1}^{n} \mathscr{B}_i$ be a finite covering of $\mathscr{B}$; the idea of the Asano Contraction combined with Ruelle's theorem [2] is to obtain information on domains of zeroes of $M(z_{\mathscr{B}})$, from a knowledge on domains of zeroes of the "smaller" polynomials:

$$M(z_{\mathscr{B}_i}) = \sum_{\beta_i \in G_i} z^{\beta_i} \quad \text{resp.} \quad M(\tilde{z}_{\mathscr{B}_i}) = \sum_{\bar{\beta}_i \in G_i^{\perp}} \tilde{z}^{\bar{\beta}_i} \tag{1.7}$$

where G_i and $G_i^{\perp}$ are the subgroups of $\mathscr{P}(\mathscr{B}_i) \subset \mathscr{P}(\mathscr{B})$ defined by:

$$G_i = \{\beta \cap \mathscr{B}_i; \beta \in G\} \qquad G_i^{\perp} = \{\bar{\beta}_i \subset \mathscr{B}_i; \sigma_\beta(\bar{\beta}_i) = 1 \ \forall \beta \in G_i\}$$

which yields:

$$G_i^{\perp} = G^{\perp} \cap \mathscr{P}(\mathscr{B}_i)$$

i.e. the elements $\bar{\beta}_i$ of $G_i^{\perp}$ are precisely the elements of $G^{\perp}$ which are subsets of $\mathscr{B}_i$.

If $G = \mathscr{K}$, $G^{\perp} = \Gamma$ and $G_i^{\perp}$ is the subgroup of Γ defined by those $\beta_i \subset \mathscr{B}_i$.

If $G = \Gamma$, $G^{\perp} = \mathscr{K}$ and $G_i^{\perp}$ is the subgroup of $\mathscr{K}$ defined by those $\beta_i \subset \mathscr{B}_i$.

Let us remark that for $G = \Gamma$, which corresponds to the L.T. expansion and $z_B = e^{-2K(B)}$, the small polynomial $M(z_{\mathscr{B}_i})$ is related to the partition function of the subsystem $\{\Lambda_i; \mathscr{B}_i\}$, $\Lambda_i = \bigcup_{B \in \mathscr{B}_i} B$ by Eq. (1.6); the corresponding statement is however not correct for $G = \mathscr{K}$.

Let $\mathscr{B} = \cup \mathscr{B}_i$ be a finite covering of $\mathscr{B}$ and

$$P(z_{\mathscr{B}}) = \sum_{\beta \subset \mathscr{B}} c_\beta z^\beta \qquad P_i(z_{\mathscr{B}_i}) = \sum_{\beta_i \subset \mathscr{B}_i} c_{i,\beta_i} z^{\beta_i}$$

be a family of polynomials. The polynomial $P(z_{\mathscr{B}})$ is the "*Asano contraction*" of $\{P_i(z_{\mathscr{B}_i})\}$ if $c_\beta = \prod_i c_{i,\beta \cap \mathscr{B}_i}$; moreover a variable z_B is said to undergo contraction if B belongs to more than one $\mathscr{B}_i$ [3]. The interest of this definition lies in the following theorem:

Theorem (Ruelle [2]). *If $P(z_{\mathscr{B}})$ is the Asano Contraction of $\{P_i(z_{\mathscr{B}_i})\}$ and if $P_i(z_{\mathscr{B}_i}) \neq 0$ when $z_B \notin R_{i,B}$ for all B in $\mathscr{B}_i$, $i = 1, 2, \ldots, n$, then $P(z_{\mathscr{B}}) \neq 0$ when $z_B \notin -\prod_i (-R_{i,B})$ for all B in $\mathscr{B}$, where $R_{i,B}$, $i = 1, \ldots, n$ $B \in \mathscr{B}_i$, are closed subsets of the complex plane which do not contains 0 if z_B undergo contraction.*

Proposition 2. *Let G be any subgroup of $\mathscr{P}(\mathscr{B})$, $\mathscr{B} = \cup \mathscr{B}_i$ and $G_i = \{\beta \cap \mathscr{B}_i; \beta \in G\}$. $M(z_{\mathscr{B}}) = \sum_{\beta \in G} z^\beta$ is the Asano Contraction of $M(z_{\mathscr{B}_i}) = \sum_{\beta_i \in G_i} z^{\beta_i}$ if and only if the subgroup of $\mathscr{P}(\mathscr{B})$ generated by $\cup G_i^{\perp}$ coïncide with $G^{\perp}$.*

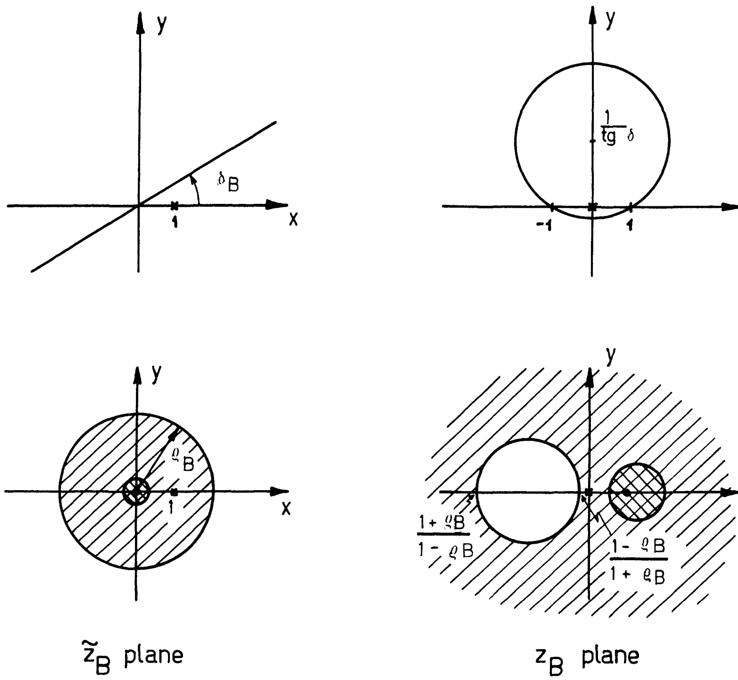

Fig. 1. The homographic mapping $Z_B = \dfrac{1-\tilde{z}_B}{1+\tilde{z}_B}$

The proof of this proposition is identical with the one given by Slawny [3] for the particular case $G = \Gamma$ and we shall not repeat it.

In consequence all covering $\mathscr{B} = \cup \mathscr{B}_i$ of $\mathscr{B}$ for which the Asano-Ruelle method applies are obtained in the following manner:

Given G, we first find $G^\perp$; let $G_i^\perp$, $i = 1, \ldots, n$ be a family of subgroups of $G^\perp$ which generate $G^\perp$, then $\mathscr{B}_i = \Big\{B \in \bigcup\limits_{\beta \in G^\perp} \beta\Big\}$. The simplest covering $\mathscr{B} = \cup \mathscr{B}_i$ is obtained by means of the generators $g_i^\perp$ of $G^\perp$; in this case $\mathscr{B}_i = g_i^\perp$ and, using Proposition 1, the zeroes of $M(z_{\mathscr{B}_i})$ such that $z_B \neq -1$ for all B in $\mathscr{B}$ are obtained directly from the zeroes of the polynomial

$$\tilde{M}(\tilde{z}_{\mathscr{B}_i}) = 1 + \prod_{B \in \mathscr{B}_i} \tilde{z}_B \quad \text{if} \quad \mathscr{P}(\mathscr{B}_i) \cap G^\perp = \{\phi, g_i^\perp\} \tag{1.9}$$

where z_B is related to $\tilde{z}_B$ by the transformation Eq. (1.4).

For example with $G = \mathscr{K}$, which corresponds to the H.T. expansion and $z_B = \tanh K(B)$, we have $G^\perp = \Gamma$ and we can choose the covering

$$\mathscr{B} = \bigcup_{r \in \Lambda} \gamma_r \qquad \gamma_r = \{B \ni r\}\,. \tag{1.10}$$

To study domains of zeroes of $M(z_{\mathscr{B}_i})$ for the case $\mathscr{B}_i = g_i^\perp$, we shall use the following remarks:

$$\tilde{M}(\tilde{z}_{\mathscr{B}_i}) = 1 + \prod_{B \in \mathscr{B}_i} \tilde{z}_B \qquad \text{is different from zero}$$

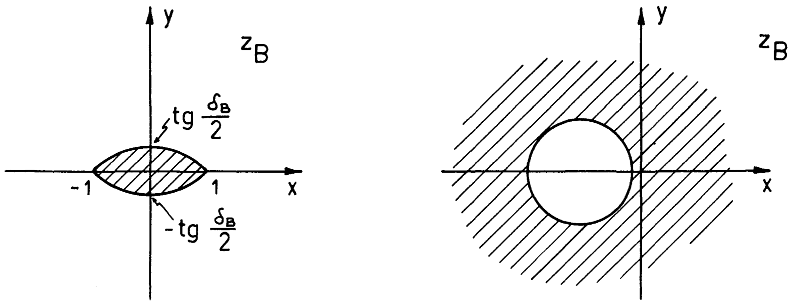

Fig. 2. Domains $D_{i,B}$ such that $M(z_{\mathscr{B}_i}) \neq 0$ if $z_B \in D_{i,B}$

i) either if $\delta_B^{\min} \leqq \arg z_B \leqq \delta_B^{\max}$ *for all* $B \in \mathscr{B}_i$
where
$$(zk-1)\pi < \Sigma\delta_B^{\min} < \Sigma\delta_B^{\max} < (2k+1)\pi \qquad k=0,1,2,\ldots,$$

ii) or if $|\tilde{z}_B| \leqq \delta_B$ *for all* $B \in \mathscr{B}_i$ where $\Pi\delta_B < 1$.

Moreover the homographic transformation $z_B = \dfrac{1-\tilde{z}_B}{1+\tilde{z}_B}$ mapps the line $\arg\tilde{z}_B = \delta$ onto the circle $x^2 + (y - \operatorname{cotg}\delta)^2 = 1 + (\operatorname{cotg}\delta)^2$, $z = x + iy$ and the circle $|\tilde{z}_B| = \varrho$ onto the circle $\left(x - \dfrac{1+\varrho^2}{1-\varrho^2}\right)^2 + y^2 = \left(\dfrac{2\varrho}{1-\varrho^2}\right)^2$ (see Fig. 1).

Proposition 3. *For the covering* $\mathscr{B} = \cup \mathscr{B}_i$ *defined by the generators of* $G^\perp$, $M(z_{\mathscr{B}_i}) \neq 0$ *if* $z_B \in D_{i,B}$ *for all* B *in* $\mathscr{B}$ *where*

i) *either* $D_{i,B}$ *is the open set intersection of the interior of the two circles*
$$x^2 + (y \mp \operatorname{cotg}\delta_B)^2 = 1 + (\operatorname{cotg}\delta_B)^2 \quad \text{where} \quad \delta_B \geqq 0 \quad \text{and} \quad \sum_{B\in\mathscr{B}_i} \delta_B = \pi$$

ii) *or* $D_{i,B}$ *is the open set outside (resp. inside) of the circle*
$$\left(x - \frac{1+\varrho_B^2}{1-\varrho_B^2}\right)^2 + y^2 = \left(\frac{2\varrho_B}{1-\varrho_B^2}\right)^2 \quad \textit{if} \quad \varrho_B > 1 \quad (\textit{resp.}\ \varrho_B < 1)$$

where
$$\prod_{B\in\mathscr{B}_i} \varrho_B = 1 \qquad \text{(see Fig. 2).}$$

It is important to notice that *in order to apply Ruelle's theorem we must take* $\delta_B > 0$ *or* $\varrho_B > 1$ *dor all B which undergo contraction* (otherwise $R_{i,B} \geqq 0$).

III. Applications

Let $\{\mathbb{Z}^\nu, \mathscr{B}\}$ be an infinite spin $\frac{1}{2}$ lattice system with *finite range interaction* $J(B) = kTK(B)$; moreover, to simplify the following discussion, we shall assume the system to be *invariant under translation*, i.e. $K(\tau_a B) = K(B)$ for all $a \in \mathbb{Z}^\nu$, $B \in \mathscr{B}$.

With $\mathscr{B}_0$ a fundamental family of bonds, i.e. for all $B \in \mathscr{B}$ there exists one and only one $a \in \mathbb{Z}^\nu$, $B_0 \in \mathscr{B}_0$ such that $B = \tau_a B_0$ we consider the class $[B_0]$ of bonds congruent to $B_0 \in \mathscr{B}_0$

$$[B_0] = \{\tau_a B_0 \,;\, a \in \mathbb{Z}^\nu\}\,.$$

The *free energy* of the system is defined by

$$p = \lim_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log Z(\Lambda) \tag{2.1}$$

where $\Lambda \to \mathbb{Z}^\nu$ in the sense of Van Hove and we thus have:

$$p = \log 2 + \sum_{B \in \mathscr{B}_0} \log\cosh K(B) + \lim_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \sum_{\beta \in \mathscr{K}_\Lambda} \prod_{B \in \beta} \tanh K(B) \tag{2.2}$$

$$p = \sum_{B \in \mathscr{B}_0} K(B) + \lim_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \sum_{\beta \in \Gamma_\Lambda} \prod_{B \in \beta} e^{-2K(B)} \tag{2.3}$$

where $\mathscr{K}_\Lambda$ and Γ_Λ are the groups associated with the finite system $\{\Lambda, \mathscr{B}_\Lambda\}$ and $\mathscr{B}_\Lambda = \{B \in \mathscr{B};\, B \subset \Lambda\}$.

Let us first consider the *H. T. expansion* of the finite system $\{\Lambda, \mathscr{B}_\Lambda\}$

$$G = \mathscr{K}_\Lambda\,, \qquad z_B = \tanh K(B)\,, \qquad G^\perp = \Gamma_\Lambda$$

and the covering

$$\mathscr{B}_\Lambda = \bigcup_{r \in \Lambda} \gamma_r\,, \qquad \gamma_r = \{B;\, B \in \mathscr{B}_\Lambda, B \ni r\}\,.$$

We have:

$$\tilde{M}(\tilde{z}_{\gamma_r}) = 1 + \prod_{B \ni r} \tilde{z}_B\,.$$

Since z_r does not undergo contraction while z_B undergo a contraction of order at most equal to $|B|$ if $|B| \geqq 2$, Ruelle's theorem yields the following result:

Proposition 4. 1. *The partition function $Z(\Lambda)$ of the finite system $\{\Lambda, \mathscr{B}_\Lambda\}$ is different from zero in the complex domain:*

$$|\arg e^{-2\beta h}| \leqq \delta_0\,; \qquad |\tanh K(B)| < \left(\operatorname{tg} \frac{\delta_B}{2}\right)^{|B|} \quad |B| \geqq 2 \tag{2.4}$$

where $\{\delta_B\}_{B \in \mathscr{B}}$ are arbitrary real numbers such that

$$\delta_0 \geqq 0\,, \qquad \delta_B > 0\,, \qquad \delta_{\tau_a B} = \delta_B\,, \qquad \delta_0 + \sum_{\substack{B \ni r \\ |B| \geqq 2}} \delta_B = \pi\,.$$

2. *The partition function $Z(\Lambda)$ of the finite system $\{\Lambda, \mathscr{B}_\Lambda\}$ is different from zero in the complex domain:*

$$|e^{-2\beta h}| < \varrho_0 < 1\,; \qquad z_B = \tanh K(B) \notin -\left[-\left\{z;\left|z - \frac{1+\varrho_B^2}{1-\varrho_B^2}\right| < \frac{2\varrho_B}{\varrho_B^2 - 1}\right\}\right]^{|B|} \tag{2.5}$$

where

$$\varrho_B > 1 \qquad \varrho_{\tau_a B} = \varrho_B \quad \textit{and} \quad \varrho_0 \prod_{\substack{B \ni r \\ |B| \geqq 2}} \varrho_B = 1 \,.$$

Corollary 4.1. 1. *The free energy is an analytic function of the complex variable* $z_B = \tanh K(B)$, $B \in \mathscr{B}_0$, *in the domains defined by Eq.* (2.4) *or Eq.* (2.5).

2. *At high temperature, i.e. at temperature* T *such that*

$$\left|\tanh \frac{J(B)}{kT}\right| < \left(\operatorname{tg} \frac{\delta_B}{2}\right)^{|B|}, \qquad \sum_{\substack{B \ni r \\ |B| \geqq 2}} \delta_B = \pi$$

there exists a unique equilibrium state ω invariant under translations. *Moreover the correlation functions* $\omega(\sigma_X)$ *are analytic functions of* z_B, $B \in \mathscr{B}_0$ *in the domain defined by Eq.* (2.4).

3. *For systems with non zero external magnetic field such that* $|e^{-2\beta h}| < \varrho_0 < 1$ *there exists a unique equilibrium state* ω *invariant under translation in the domain defined by Eq.* (2.5). *Moreover in this domain the correlation functions are analytic functions of* z_B.

Indeed the first part of this corollary follows from Vitali's theorem.

The second part from Ruelle's theorem [4] and the fact that for all finite $X \subset \mathbb{Z}^\nu$ and any $\varepsilon > 0$, the free energy $p(K + \lambda K_X)$ with $K_X(Y) = 1$ if $Y \in (X)$ and zero otherwise, is analytic in the domain.

$$|\arg e^{-2\beta h}| \leqq \delta_0 \,; \qquad |\tanh K(B)| < \left(\operatorname{tg} \frac{\delta_B}{2}\right)^{|B|} ; \qquad |\tanh \lambda| < \left(\operatorname{tg} \frac{\varepsilon}{2q}\right)^{|X|}$$

$$\delta_0 + \sum_{\substack{B \ni r \\ |B| \geqq 2}} \delta_B = \pi - \varepsilon$$

$q =$ number of bonds in $[X]$ containing the site r.

Therefore $\omega(\sigma_X) = \frac{d}{d\lambda} p(K + \lambda K_X)|_{\lambda = 0}$ has same value for all translationally invariant equilibrium state.

Letting $\varepsilon \to 0$ we conclude the proof.

The last part is obtained in a similar manner taking:

$$\varrho_0 \prod_{\substack{B \ni r \\ |B| \geqq 2}} \varrho_B = \left(\frac{1}{\varrho}\right)^q \qquad |\tanh \lambda| < \frac{\varrho - 1}{\varrho + 1}$$

and letting $\varrho \xrightarrow{>} 1$

As consequence of Proposition 4 Part 2 we recover the *Lee-Yang circle theorem* for the zeroes of ferromagnetic systems with two body forces and external field: indeed if $|B| \leqq 2$ for B in $\mathscr{B}$ and q is the coordination number, we have that the partition function is different from zero in the domain:

$$|e^{-2\beta h}| < 1 - \varepsilon \qquad |\tanh K_{ij}| \notin -\left\{ \operatorname{Re} z < \frac{(1-\varepsilon)^{-1/q} - 1}{(1-\varepsilon)^{-1/q} + 1} \right\}^2$$

and therefore for all $\varepsilon \in [0, 1]$ the partition function is different from zero in the domain $|e^{-2\beta h}| < 1 - \varepsilon$ if $\tanh K_{ij} > 0$. By symmetry argument we obtain the same result for $|e^{-2\beta h}| > 1 + \varepsilon$ and $\varepsilon \to 0$ we obtain the Lee-Yang Circle theorem.

As consequence of Corollary 4.1 Part 3 we also recover Ruelle's result concerning the *unicity of the translationally invariant equilibrium state for ferromagnetic systems with two body forces and non zero external field* [4].

It should be noticed that the condition Eq. (2.5) is particularly well suited for system such that $|B| \leqq 2$ for all B in $\mathscr{B}$. Indeed it is for this case only that Eq. (2.5) yields the whole positive axis.

From Corollary 4.1 Part 2 we obtain immediately *unicity of the translationally invariant equilibrium state for* $T > T_0$ where

$$\left|\tanh \frac{J(B)}{kT_0}\right| < \left(\operatorname{tg} \frac{\pi}{2q}\right)^{|B|} \qquad q = \text{coordination number.}$$

This result which is very simple yields upper bounds on the critical temperature which are not as good as those obtained by more refined techniques [7].

Let us now consider the *L. T. expansion* of the finite system $\{\Lambda, \mathscr{B}_\Lambda\}$:

$$G = \Gamma_\Lambda, \qquad z_B = e^{-2K(B)}, \qquad G^\perp = \mathscr{K}_\Lambda$$

and the covering $\mathscr{B}_\Lambda = \bigcup_i \kappa_i$ defined by the generators κ_i of $\mathscr{K}_\Lambda$:

$$\tilde{M}(\tilde{z}_{\kappa_i}) = 1 + \prod_{B \in \kappa_i} \tilde{z}_B .$$

Using Proposition 3 combined with Ruelle's theorem we obtain the following result:

Proposition 5. *The partition function $Z(\Lambda)$ of the finite system $\{\Lambda, \mathscr{B}_\Lambda\}$ is different from zero in the domain:*

$$|\arg \tanh K(B')| \leqq \delta_{B'} \qquad \textit{if } z_{B'} \textit{ does not undergo contraction}$$

$$|z_B = e^{-2K(B)}| < \prod_{\kappa_i \ni B} \operatorname{tg} \frac{\delta_{i,B}}{2} \qquad \textit{if } z_B \textit{ undergo contraction} \tag{2.6}$$

where

$$\delta_{i,B'} \geqq 0, \qquad \delta_{i,B} > 0, \quad \delta_{i,\imath_a B} = \delta_{i,B} \sum_{B \in \kappa_i} \delta_{i,B} = \pi .$$

Definiting an *invariant equilibrium state* ω to be an equilibrium state invariant under the full symmetry group [8] (Translation group and Internal symmetry group) we then have the following result (see also Slawny [3]):

Corollary 5.1. 1. *The free energy is an analytic function of the complex variables $\{z_B = e^{-2K(B)}\}_{B \in \mathscr{B}_0}$ in the domain defined by Eq.* (2.6).

2. *If* $e^{-2\frac{J(B)}{kT}} < \prod_{\kappa_i \ni B} \operatorname{tg} \frac{\delta_{i,B}}{2}$ *with* $\sum_{B \in \kappa_i} \delta_{i,B} = \pi$, *there exists a* unique invariant equilibrium state ω.

Moreover the correlation functions $\omega(\sigma_X)$ *are analytic functions of* z_B *in the domain defined by Eq.* (2.6).

Part 2 of this corollary follows from Ruelle's theorem and from the fact that for any $\bar{B} \in \bar{\mathscr{B}}$, $\bar{\mathscr{B}}$ subgroup of $\mathscr{P}(\Lambda)$ generated by $\mathscr{B}$, the group $\mathscr{K}'$ associated with $\mathscr{B}' = \mathscr{B} \cup \{\tau_a \bar{B}\}$ is generated by $\mathscr{K}_\Lambda$ and $\tau_a \bar{\kappa}$ where $\bar{\kappa} = (\bar{B}, B_1, \dots, B_m)$.

Therefore

$$M_\Lambda(z_{\mathscr{B}_\Lambda}) \neq 0 \quad \text{for} \quad |z_{\mathscr{B}}| < r_B$$

and

$$\tilde{M}_\Lambda(\tilde{z}_{\bar{\kappa}}) \neq 0 \quad \text{for} \quad |\tilde{z}_{\bar{B}}| < \bar{\varrho} < 1 \qquad |\tilde{z}_{B_i}| < \bar{\varrho}^{-\frac{1}{m}}$$

yields

$$M_\Lambda(z_{\mathscr{B}'}) \neq 0 \quad \text{for} \quad |z_B| < r_B \qquad B \notin \bigcup_{i=1}^{m} [B_i]$$

$$|z_{\mathscr{B}_i}| < \left(\frac{1 - \bar{\varrho}^{-\frac{1}{m}}}{1 + \bar{\varrho}^{-\frac{1}{m}}} \right)^{n_B} r_B$$

$$|z_{\bar{B}} - 1| < \bar{\varrho}$$

and n_B is the number of generators κ_i containing the bond B. Letting $\bar{\varrho} \overset{>}{\to} 0$ we conclude at the unicity of the equilibrium state ω invariant under translation and such that $\omega(\sigma_X) = 0$ if $X \notin \bar{\mathscr{B}}$ which is precisely the unicity of the invariant equilibrium state.

It should be remarked that in Corollary 4.1 we have the uniqueness at high temperature of the translationally invariant equilibrium state while in Corollary 5.2 we obtain the uniqueness at low temperature of the invariant equilibrium state.

In conclusion this last corollary extends to arbitrary ferromagnetic systems a similar result derived by Slawny [3] for a special class of ferromagnetic system with even bonds; moreover a lower bound on the critical temperature is immediately given by

$$\frac{J(B)}{kT} \geqq \frac{n(B)}{2} \sup_{\kappa \ni B} \log \left(\operatorname{tg} \frac{\pi}{2|\kappa|} \right).$$

To conclude this general discussion on analyticity and uniqueness properties we shall remark that the domain which we have considered were the simplest ones and therefore the result could be improved by means of a finer analysis of the domain to be considered.

IV. Examples

1. ν-dimensional Ising Model with External Field

In this example we consider the ν-dimensional Ising model with interaction $J = kTK_{ij}$ between nearest neightbour and external field $h = kTK_i$.

From Proposition 4 we have the following result:

Corollary 4.2. 1. *The partition function of the ν-dimensional Ising Model with n.n. interaction J and external field h is non zero for complex (h, J) such that* $|\arg e^{-2\beta h}| \leqq \delta_0 \quad |\tanh \beta J| < \left[\operatorname{tg}\left(\frac{\pi - \delta_0}{4\nu}\right)\right]^2$

2. *For real magnetic field h, the free energy is an analytic function of the complex variable $x = \tanh \beta J$ in the domain* $|x| < \left(\operatorname{tg}\frac{\pi}{4\nu}\right)^2$.

3. *For real interaction J such that $|\tanh \beta J| < (\operatorname{tg}\delta)^2$, the free energy is an analytic function of the complex variable $z = e^{-2\beta h}$ in the domain $|\arg z| < \pi - 4\nu\delta$.*

In particular for $\nu = 2$ and J real, $|\tanh \beta J| < (\sqrt{2} - 1)^2$ yields $e^{-2\beta J} > \frac{1}{\sqrt{2}}$ and we thus recover Sarbach and Rys result that for $e^{-2\beta J} > \frac{1}{\sqrt{2}}$ the free energy does not have singularity on the real h axis [6].

Moreover for $\nu = 2$ and $h = 0$ we recover Ruelle's result (Proposition 2.4, [10]); indeed in this case the model is selfdual and analyticity for $|\tanh \beta J| < (\sqrt{2} - 1)^2$ implies analyticity for $|e^{-2\beta J}| < (\sqrt{2} - 1)^2$ which is precisely the result of Proposition 2.4, [10].

2. Triangular Model with Three Body Forces and External Field

This model has been previously studied in the particular case $h = 0$ by means of the group structure [8, 11]; in this case using duality argument the critical point was located at $\tanh K_c = e^{-2K_c} = \sqrt{2} - 1$. This result was later confirmed by the exact solution of Baxter and Wu [12] for the free energy density at $h = 0$. However analytical properties of this model has not been investigated.

From Proposition 4 we obtain the following result.

Corollary 4.3. 1. *The partition function of the triangular model with three body forces J and external field h is non zero for complex (h, J) such that:*

$$|\arg e^{-2\beta h}| \leqq \delta_0 \qquad |\tanh \beta J| < \left(\operatorname{tg}\frac{\pi - \delta_0}{6}\right)^3 .$$

2. *For real external field the free energy is an analytic function of the complex variable $x = \tanh \beta J$ in the domain $|x| < (2 - \sqrt{3})^3$.*

3. *For $h = 0$, the free energy is an analytic function in the domain $|\tanh \beta J| < (2 - \sqrt{3})^3$ and $e^{-2\beta|J|} < (2 - \sqrt{3})^3$.*

The last statement of the above corollary follows from the fact that this model is selfdual.

To study the analyticity properties in the complex variable $z = e^{-2\beta h}$ we use Proposition 5 with the generator κ_a of $\mathscr{K}$ defined by:

$$\kappa_a = \{B_a, r_{a_1}, r_{a_2}, r_{a_3}\} \qquad |B_a| = 3 \qquad r_{a_i} \in B_a .$$

We thus obtain, since B_a does not undergo contraction.

Corollary 5.2. *For ferromagnetic interactions the free energy is an analytic function of* $z = e^{-2\beta h}$ *in the domain* $|z| < \frac{1}{3^3}$.

Moreover using the symmetry relation $Z(h, -K) = Z(-h, K)$ *we conclude that for antiferromagnetic interaction the free energy is analytic for* $|z| > \frac{1}{3^3}$.

3. Ising Model with Four Body Forces and External Field

In this last example we consider the system defined by a two dimensional square lattice with four body forces $J = kTK_4$ and magnetic field $h = kTK_1$; this model is self-dual [8] with $(K_1, K_4) \to (K_1^*, K_1^*)$ and for $h = 0$ the free energy can be easily computed.

We consider the L.T. expansion, i.e. $G = \Gamma$, $z_B = e^{-2K(B)}$ and the covering $\mathscr{B} = \cup \kappa_a$ defined by the generators κ_a of $\mathscr{K}$:

$$\kappa_a = \{B_a, r_{a_1}, \ldots, r_{a_4}\} \qquad |B_a| = 4 \qquad r_{a_j} \in B_a$$

Since B_a does not undergo contraction we obtain from Propositions:

Corollary 5.3. 1. *For real ferromagnetic interaction J the free energy is an analytic function of* $z = e^{-2\beta h}$ *in the domain* $|z| < (\sqrt{2} - 1)^4$.

2. *For real field h the free energy is an analytic function of* $x = \tanh \beta J$ *in the domain* $|x| < (\sqrt{2} - 1)^4$.

The last statement follows from the selfduality property.

To conclude this example we shall show it is possible to improve the domain of analyticity using another covering $\mathscr{B} = \cup \mathscr{B}_i$. We consider again $G = \Gamma$ and the subgroup $\mathscr{K}_i$ of $\mathscr{K}$ defined by the set of generators κ_a situated on a row (Fig. 3).

In this case $M(z_{\mathscr{B}_i})$ is related to the partition function of the one dimensional chain defined by the bonds of $\mathscr{B}_i$; using a H.T.-H.T. duality transformation [11] this partition function is proportional to the partition function of the one-dimensional Ising model with nearest neighbour interactions $K^* = K_4$ and external field h^* defined by $\tanh h_j^*$ $\tan \beta h_{i,j} \tanh \beta h_{i_1,j}$, i,e, $e^{-2h_j^*} = \tanh (h_{i,j}^* + h_{i+1,j}^*)$.

Using the Lee-Yang circle theorem we conclude that for ferromagnetic interactions $M(z_{\mathscr{B}_i})$ is different from zero if either $|e^{-2h_j^*}| < 1$ for all (i, j) or $|e^{-2h_j^*}| > 1$ for all (i, j).

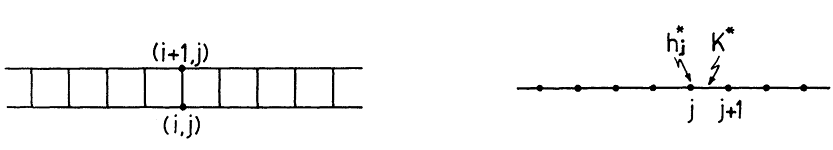

Fig. 3. Covering $\mathscr{B}_i$ and its H.T. – H.T. dual

But $|\tanh(h_{i,j}^* + h_{i+1,j}^*)| < 1$ corresponds to $|\arg e^{-2(h_{i,j}^* + h_{i+1,j}^*)}| < \frac{\pi}{2}$ therefore $M(z_{\mathscr{B}_i}) \neq 0$ if $|\arg e^{-2h_{i,j}^*}| < \frac{\pi}{4}$ for all (i,j)

$$\text{i.e.} \quad |\arg \tanh \beta h_{i,j}| < \frac{\pi}{4}$$

and thus $M(z_{\mathscr{B}i}) \neq 0$ if $|e^{-2\beta h_{ji},}| > (\sqrt{2}-1)$.

We therefore obtain the following result:

Corollary 5.4. 1. *For real ferromagnetic interactions the free energy is an analytic function of* $z = e^{-2\beta h}$ *in the domain* $|z| < (\sqrt{2}-1)^2$ *and* $|z| > (\sqrt{2}+1)^2$ *(by symmetry).*

2. *For real field the free energy is an analytic function of* $z = e^{-2\beta J}$ *in the domain* $\frac{1}{\sqrt{2}} < |z| < \sqrt{2}$.

This last statement follows from duality and from the fact that $|\tanh K_4| < (\sqrt{2}-1)^2$ is equivalent to $|e^{-2K_4}| > \frac{1}{\sqrt{2}}$.

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