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Qualitative Magnetic Cosmology

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Abstract. The technique of phase plane analysis, which was used in a previous paper [4] to study the behaviour of a class of perfect-fluid anisotropic cosmological models, is applied to some simple anisotropic models that contain a uniform magnetic field. A formal correspondence is established between these magnetic models (of Bianchi type I) and certain *perfect fluid models (of Bianchi type II), and new exact solutions are consequently discovered.*

1. Introduction

Non-linear differential equations, which are capable of describing interactions between different components of a physical system, and are thus generally more realistic than linear equations, are notoriously difficult to analyse. In the simplest of cases, however, it is possible to depict qualitatively the behaviour of the solutions, and then the information that is frequently required (asymptotic behaviour, special solutions, stability properties, etc.) can be obtained immediately. Such an analysis can be performed with ordinary differential equations of the form

$$\begin{aligned} \dot{x} &= X(x, y) \\ \dot{y} &= Y(x, y) \end{aligned} \tag{1.1}$$

where a dot ($\dot{}$) denotes differentiation with respect to an independent variable, t, which is often a measure of time. The system (1.1) is known as a plane autonomous system, since there are only two dependent variables (and so trajectories of the motion can be drawn in a "phase plane"), and because the variable t is not explicitly present on the right-hand sides. The reasons why plane autonomous systems can be so successfully treated stem from the Jordan curve theorem, which essentially states that any closed curve in a plane will divide that plane into two distinct parts. If for some reason we were confined at some time to the inside of a given closed curve in the phase plane, then we would be confined to that region for all future time. The apparent falsity of this statement in higher dimensions has such profound consequences that all attempts to generalize to the case of three or more dependent variables have met with little success, although the simplest types of critical points (where the right-hand sides disappear simultaneously) in three dimensions have recently been explored ([1, 5]). One further qualification to the above remarks is that higher order systems can be examined if they possess a sufficiently high degree of symmetry. For instance, an autonomous system containing three variables (equivalently, a non-autonomous system of two variables, or one non-autonomous second-order differential equation) may be examined in a "Lie plane" if it admits a one-parameter continuous group of transformations (for some interesting applications of this method, the reader is referred to a paper by Dresner [7]).

The field equations of general relativity form a complicated system of ten second-order partial differential equations. For cosmological purposes, spatial homogeneity is postulated, and this simplifies the problem to a considerable extent, since ordinary differential equations will result, with the independent variable being temporal. But since the equations are so numerous, one would not expect to be able to use a phase plane analysis in any but the simplest of circumstances. It is therefore rather surprising that a moderately large subclass can be found for which there are essentially only two dependent variables, and for which phase plane techniques play an illuminative role. In a previous paper [4], several non-rotating spatially homogeneous models containing a perfect fluid source were examined in this way, and it is the purpose of the present paper to attempt to generalize the results by introducing a uniform magnetic field. This does not seem possible in most cases, but if the models admit a continuous isometry group of Bianchi type I (which is the "simplest" anisotropic case; see the specialization diagram of MacCallum [14]) some measure of success is encountered, as we shall see in the following section.

2. Magnetic Fields in Homogeneous Cosmologies

The discussion of magnetic fields in spatially homogeneous universe models appears to have been most extensively covered by Jacobs and Hughston and by Ellis and MacCallum ([27, 28] and private communications). Interest in this stems from the observations of an intergalactic magnetic field (see, e.g. [13], where an upper limit of order $2 \cdot 10^{-8}$ gauss is given). Hoyle [10] seems to have been the first to suggest that the origin of this field could be primeval. Brecher and Blumenthal [2] (see also [9]) claim that any possible sources of a primordial magnetic field cannot account adequately for the observed field strength (but compare the comments of Reinhardt and Thiel [20]). There is some controversy over whether or not the field could become infinite at the big bang singularity. Thorne [25] has suggested that the field could not exceed approximately $4.4 \cdot 10^{13}$ gauss, since otherwise it would have been quantized in early times, and this would indicate that no large-scale field could them emerge (cf. Chiu and Canuto [3], who consider the properties of matter in such intense fields).

The analogous problem of introducing an electric field is no different from a mathematical standpoint. Since large-scale electric fields have not been observed, and since one would not expect cosmological electric currents and charge distributions [26], our discussion centres around magnetic fields.

Many authors have considered the behaviour of individual Bianchi models that contain either a pure magnetic field or a magnetic field plus fluid (see, for instance, the references cited by MacCallum [15]). The discussion is particularly straightforward for models of type I, which we shall investigate below. The notation and convention of [4] are employed, and in particular we consider the variable $x = 3\mu/\theta^2$ (where μ is the total energy density, and θ is the expansion scalar, which is a multiple of the Hubble "constant", H, and is assumed to be non-zero), which measures the dynamical importance of the matter content, and we use variables β_1' and β_2' to measure the rate of shear anisotropy in terms of the expansion, θ. The equations are expressed in terms of a time-variable, Ω, which is related to characteristic length scales, l, by the equation $l = e^{-\Omega}$, where $H = \dfrac{1}{l}\dfrac{dl}{dt} = \dfrac{1}{3}\theta$, and t measures proper time. This Ω-time was first introduced by Misner [16–19] to examine the behaviour of models near the singularity ($\Omega \to +\infty$).

For models of Bianchi type I, the metric can be written in the form

$$ds^2 = -dt^2 + X^2(t)\,dx^2 + Y^2(t)\,dy^2 + Z^2(t)\,dz^2 \tag{2.1}$$

where $X = e^{-\Omega+\beta_1}$, $Y = e^{-\Omega - \frac{\beta_1}{2} + \frac{\sqrt{3}}{2}\beta_2}$ and $Z = e^{-\Omega - \frac{\beta_1}{2} - \frac{\sqrt{3}}{2}\beta_2}$. If the matter content has an equation of state $p = (\gamma - 1)\mu$ (γ a constant, $1 \leqq \gamma \leqq 2$), the relevant equations for such models containing a uniform magnetic field aligned along the x-axis are

$$\beta_1'' = \tfrac{1}{4}\beta_1'[4 - 2(3\gamma - 4)x - \beta'^2] - (4 - 4x - \beta'^2), \tag{2.2}$$

$$\beta_2'' = \tfrac{1}{4}\beta_2'[4 - 2(3\gamma - 4)x - \beta'^2] \tag{2.3}$$

and

$$x' = \tfrac{1}{2}x[2(3\gamma - 4)(1 - x) - \beta'^2] \tag{2.4}$$

with the first integral

$$\beta'^2 := \beta_1'^2 + \beta_2'^2 = 4 - 4x - \tfrac{4}{3}Kze^{2\beta_1 + 4\Omega}, \tag{2.5}$$

where K is a positive constant and $' \equiv \frac{d}{d\Omega}$. These equations can be derived from the formulation of Jacobs [11]. Eqs. (2.2) and (2.3) are the shear propagation equations and Eq. (2.5) is the (0 0) field equation.

If we formally make the transformation $\gamma \to \frac{1}{2}(2+\gamma)$, $x \to x$, $\beta_1 \to 2\beta_1$, $\beta_2 \to 2\beta_2$, $\Omega \to 2\Omega + \frac{2}{3(2+\gamma)} \ln(4K)$, the above system becomes identical with the system for perfect fluid type II solutions (see [4]), and so information about either type can be related to the other type.

We shall consider first the case when no matter is present ($x \equiv 0$). The solutions to Eqs. (2.2) and (2.3) can be depicted in the (β_1', β_2') phase plane, and because of the above transformation properties they are qualitatively the same as the vacuum type II solutions due to Taub ([24]; see Fig. 6 of [4]). These type I solutions are all known in exact form, and are due to Rosen [21, 22]. The "locally rotationally symmetric" (LRS; see [8]) solutions, for which $\beta_2 \equiv 0$ were rediscovered by Shikin [23].

The above transformation for γ has a fixed point at $\gamma = 2$. Consequently, apart from multiplicative factors of 2, the magnetic type I solutions containing fluid with an equation of state $p = \mu$ are the same as the $p = \mu$ perfect fluid type II solutions discussed in [4]. They can thus be obtained in exact form as $x = Ae^{\tau}$, $\beta_1' = 4 - B_1 e^{\frac{1}{2}\tau}$ and $\beta_2' = B_2 e^{\frac{1}{2}\tau}$, where $d\tau := \frac{1}{2}[4 - 4x(\tau) - \beta'^2(\tau)]\, d\Omega$ and A, B_1 and B_2 are constants of integration. Fig. 2 (b) of [4] depicts the LRS case ($\beta_2' \equiv \beta_2 \equiv 0$); the more general case where $\beta_2' \neq 0$ can be visualized by considering the trajectories in three dimensional (x, β_1', β_2') phase space, as in Fig. 1 of the present paper. The above exact solution has been given in a different form by Jacobs [11].

Finally, we consider the LRS case where $p = (\gamma - 1)\mu$, $1 \leqq \gamma < 2$, $\mu \neq 0$ and $\beta_2' \equiv \beta_2 \equiv 0$. This divides naturally into two subcases (compare the similar division in [4] of the perfect fluid type VI_h models). These subcases are illustrated in Fig. 2 of the present paper and in Fig. 2 (a) of [4], for $1 \leqq \gamma \leqq \frac{4}{3}$ and $\frac{4}{3} < \gamma < 2$ respectively (with reference to perfect fluid type II models, this subdivision occurs for $0 \leqq \gamma \leqq \frac{2}{3}$ and $\frac{2}{3} < \gamma < 2$; the former case was not encountered in [4] since it involves an unrealistic equation of state; there is no qualitative difference between the behaviour of type II models for values of γ in the range $\frac{2}{3} < \gamma < 2$).

If $1 \leqq \gamma \leqq \frac{4}{3}$, all trajectories have essentally the same behaviour (see Fig. 2). They start (in Ω-time) at the point (1, 0) in the (x, β_1') phase plane, and for $\gamma < \frac{4}{3}$ this implies that the solutions isotropize for large t-times (since $\beta_1' \to 0$ sufficiently quickly). As $\Omega \to +\infty$, $\beta_1' \to -2$ and so the singularity is a 1-pancake: Y and Z approach non-zero constants and X tends to zero. Matter effects are unimportant in the early stages, since $x \to 0$ as $\Omega \to +\infty$, and from Eq. (2.5) we see that the effects of the magnetic field, described by the term $\frac{4}{3} Kze^{2\beta_1 + 4\Omega}$, are also negligible near the

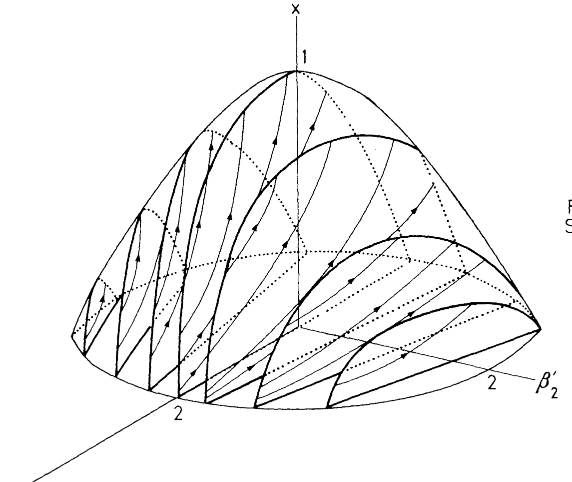

Fig. 1

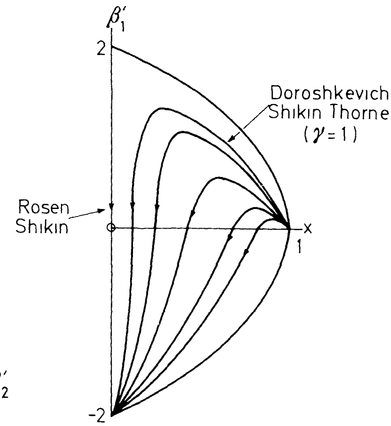

Fig. 2

Figs. 1 and 2 depict the evolution in Ω-time of some of the models discussed in the paper. The variable x measures the importance of matter, and β'_1 and β'_2 measure the importance of shear anisotropy. Fig. 1 Perfect fluid Type II solutions, or, equivalently, magnetic Type I solutions ($p = \mu$). Fig. 2. LRS magnetic Type I solutions ($p = (\gamma - 1)\mu$; $1 \leqq \gamma \leqq \frac{4}{3}$)

singularity. All solutions undergo one "bounce" (where $\beta'_1 = 0$) against the "magnetic potential wall" (Jacobs and Hughston, private communication). Exact solutions are known for the dust case ($\gamma = 1$); they were discovered independently by Doroshkevich [6], Shikin [22] and Thorne [24]. As far as the present author is aware, no other exact analytic solutions are known for this class (LRS Bianchi type I with magnetic field and $1 < \gamma \leqq \frac{4}{3}$). In the particular case where $\gamma = \frac{4}{3}$, the detailed behaviour for large t-times in the neighbourhood of the point $(1, 0)$ differs from that when $\gamma < \frac{4}{3}$, because the first approximation equations are not linear there; thus although $\beta'_1 \to 0$, $\beta_1 \to -\infty$, and so isotropization does not occur. This is not treated as an extra subcase, since we are here concerned only with a schematic description of the solutions.

For $\gamma > \frac{4}{3}$ the behaviour is considerably different (see Fig. 2 (a) of [4]). All solutions start (in Ω-time) at the focus, and so none of these models isotropizes ($t \to +\infty$). There are then three distinct possibilities. In general, $\beta'_1 \to -2$ as $\Omega \to +\infty$, and then both matter and magnetic field are dynamically unimportant in the early stages, and the singularity is a 1-pancake. There is one solution which behaves like the special radiation solution of Kantowski ([12]; see the discussion in [4]) in that, as $\Omega \to +\infty$, $x \to 1$ and $\beta'_1 \to 0$. This possesses a point singularity, and matter is dynamically important at *all* times; for early t-times the effects of the magnetic field are negligible. In addition, there is a special solution at the focus, for which both matter and magnetic field are dynamically important at all times. This solution possesses a point singularity, and it is the same as one of Thorne's [25] approximate solutions near the singularity. Jacobs [11] first obtained this particular solution in exact form (his axisymmetric hard-magnetic solutions); the corresponding perfect-fluid type II solutions were obtained in [4]. Note that, just as the LRS type II exact solutions could be generalized to the case where $\beta_2' \neq 0$, we can obtain a new exact solution in this type I case by demanding that β_1' be a constant. This is given by

$$\beta_1 = \frac{1}{2}(3\gamma-4)\,\Omega\,, \qquad \beta_2'^2 = \frac{3\gamma}{4}(8-3\gamma)\,\frac{E^2 e^{6(2-\gamma)\Omega}}{1+E^2 e^{6(2-\gamma)\Omega}}$$

(E a constant, $E \geqq 0$),

$$\beta_2 = \int^{\Omega} \beta_2'\, d\Omega\,, \qquad \mu = \frac{2(4-\gamma)\,k\,e^{3\gamma\Omega}}{(3\gamma-4)(2-\gamma)}\,,$$

$$\dot{\Omega}^2 = \frac{16}{9}\,\frac{K}{(3\gamma-4)(2-\gamma)}\left[e^{3\gamma\Omega} + E^2 e^{3(4-\gamma)\Omega}\right], \quad \dot{\Omega} < 0\,, \quad t = -\int_{\Omega}^{\infty} \frac{d\Omega}{\dot{\Omega}}\,.$$

By rescaling according to $\beta_1 \to \beta_1 + \lambda$, $\Omega \to \Omega + \nu$, where $\lambda = \frac{1}{2}(3\gamma-4)\,\nu$ and $3\gamma\nu = -\ln K$ we can transform K to unity.

3. Conclusion

We have been able to depict the qualitative behaviour of a set of simple anisotropic spatially homogeneous cosmological models, which contain matter flowing orthogonally to the hypersurfaces of homogeneity and which also contain a uniform magnetic field.

An interesting relationship emerged between the models examined and those containing a perfect fluid but admitting a different isometry group; this relates two previously disconnected problems, and is essentially due to the property that a potential formalism can be developed in both cases, and that in each the potentials are exponential functions of β_1 only. In view of this rather special circumstance, it seems unlikely that any similar analysis could be performed with more complicated models.

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