Commun. math. Phys. 26, 259—270 (1972)

Stability of Homogeneous Universes*

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Received November 19, 1971; in revised form January 24, 1972

Abstract. The stability of a class of homogeneous cosmological models is investigated. It is shown that the perturbation problem for six such universes can be reduced to a system of ordinary differential equations. The time development of the perturbations is such that they remain finite at all times for which the unperturbed metric is non-singular.

I. Introduction

In a previous paper [1] the stability of the Taub universe was analyzed and shown to reduce to a system of ordinary differential equations. In this paper we generalize that result to universes whose 3-surfaces of homogeneity admit a simply transitive, 3-parameter group of motions of Bianchi types I, II, VII_0, VIII, and IX, as well as to the Kantowski-Sachs universe [2]. Except for the Kantowski-Sachs case, which is discussed in Appendix B, all these universes belong to "class A" in the classification of homogeneous cosmological models given by Ellis and McCallum [3]. These are non-rotating universes with the flow vector of matter (assumed perfect fluid) orthogonal to the surfaces of homogeneity. The group structure is of the form

$$[X_1, X_2] = N_3 X_3\,, \quad [X_2, X_3] = N_1 X_1\,, \quad [X_3, X_1] = N_2 X_2 \qquad (1.1)$$

where the N_α[1] can be chosen to equal 0 or ± 1.

If we define the 1-forms ω^α by $\langle \omega^\alpha, X_\beta \rangle = \delta^\alpha_\beta$, we can write the metric for these universes as

$$ds^2 = dt^2 - A(\omega^1)^2 - C(\omega^2)^2 - B(\omega^3)^2 \qquad (1.2)$$

where A, C, and B are functions of the time, t, to be determined by Einstein's field equations (see [3], Section 4, for justification of this form of the metric). We will further restrict our universes by requiring *local rotational symmetry* [3, 4]. This is equivalent to demanding $A = C$ and $N_1 = N_2$ and is made in order that the "Laplacian operator" $g^{\alpha\beta} X_\alpha X_\beta$ separates in the coordinates used to express the X_α (see Appendix A). We thus consider groups satisfying

$$[X_1, X_2] = N X_3\,, \quad [X_2, X_3] = n X_1\,, \quad [X_3, X_1] = n X_2 \qquad (1.3)$$

* Work supported in part by the National Science Foundation.

[1] Greek indices have the range 1, 2, 3; Latin 0, 1, 2, 3.

and, correspondingly, 1-forms ω^α satisfying

$$d\omega^1 = -n\omega^2 \wedge \omega^3\,, \qquad d\omega^2 = -n\omega^3 \wedge \omega^1\,, \qquad d\omega^3 = -N\omega^1 \wedge \omega^2\,. \tag{1.4}$$

The Bianchi types corresponding to different choices of n and N are given in the following table[2]:

n	N	Bianchi type
0	0	I
0	± 1	II
± 1	0	VII_0
± 1	∓ 1	VIII
± 1	± 1	IX

II. The Field Equations

The gauge of the perturbation is chosen so that the metric takes the form:

$$g_{ij} + \delta g_{ij} = \begin{pmatrix} 1-\beta & 0 & 0 & 0 \\ 0 & -A(1+\alpha+\gamma) & -\kappa & -\lambda \\ 0 & -\kappa & -A(1+\alpha-\gamma) & -\mu \\ 0 & -\lambda & -\mu & -B(1+\beta) \end{pmatrix} \tag{2.1}$$

where A and B are functions of t, while α, β, γ, κ, λ, μ are functions of t, x^1, x^2, x^3. This choice is made because it simplifies the equations. It does not affect the fact that a solution by separation of variables is possible. Indeed, from the symmetry of the problem, one expects[3] that $(\delta g_{01}, \delta g_{02})$ and $(\delta g_{31}, \delta g_{32})$ will transform as vectors under rotations in the space spanned by X_1 and X_2; $\delta g_{00}, \delta g_{03}, \delta g_{33}$ and the trace of δg_{AB} $(A, B = 1, 2)$ will transform as scalars; while the traceless part of δg_{AB} will transform as a tensor. (See also Regge and Wheeler [6] for similar considerations for the Schwarzschild metric.)

The exact equations determining A and B are

$$T_0^0 = \varrho = \frac{1}{2}\frac{\dot{A}\dot{B}}{AB} + \frac{1}{4}\frac{\dot{A}^2}{A^2} + \frac{nN}{A} - \frac{N^2 B}{4A^2}\,, \tag{2.2}$$

$$T_1^1 = T_2^2 = -p = \frac{1}{2}\frac{\ddot{A}}{A} + \frac{1}{2}\frac{\ddot{B}}{B} + \frac{1}{4}\frac{\dot{A}\dot{B}}{AB} - \frac{1}{4}\frac{\dot{A}^2}{A^2} - \frac{1}{4}\frac{\dot{B}^2}{B^2} + \frac{N^2 B}{4A^2}\,, \tag{2.3}$$

$$T_3^3 = -p = \frac{\ddot{A}}{A} - \frac{1}{4}\frac{\dot{A}^2}{A^2} + \frac{nN}{A} - \frac{3}{4}\frac{N^2 B}{A^2}\,, \tag{2.4}$$

[2] Only the relative sign of n, N is significant.

[3] I am indebted to Dr. James B. Hartle for this remark.

$$2\delta R^0{}_0 - \delta R = 2\delta\varrho = \dot\alpha\left(\frac{\dot A}{A} + \frac{\dot B}{B}\right) + \dot\beta\frac{\dot A}{A} + \alpha\left(\frac{N^2B}{A^2} - \frac{2nN}{A}\right) + \beta\left(\frac{\dot A\dot B}{AB} + \frac{1}{2}\frac{\dot A^2}{A^2} - \frac{1}{2}\frac{N^2B}{A^2}\right) - \frac{2\alpha_{33}}{B}$$
$$- \frac{\alpha_{11} + \alpha_{22} + \beta_{11} + \beta_{22}}{A} + \frac{A(\gamma_{11} - \gamma_{22}) + \kappa_{12} + \kappa_{21}}{A^2} + \frac{2(\lambda_1 + \mu_2)_3}{AB} + \frac{N(\mu_1 - \lambda_2)}{AB}, \tag{2.5}$$

$$2\delta R^0{}_1 = 2(\varrho + p)\delta u_1 = -\left(\dot\alpha + \dot\beta + \frac{\dot B}{B}\beta\right)_1 + \left(\gamma_1 + \frac{\kappa_2}{A}\right)^{\cdot} + \frac{A}{B}\left(\frac{\lambda_3 - n\mu}{A}\right)^{\cdot} + \frac{NB}{A}\left(\frac{\mu}{B}\right)^{\cdot}, \tag{2.6}$$

$$2\delta R^0{}_2 = 2(\varrho + p)\delta u_2 = -\left(\dot\alpha + \dot\beta + \frac{\dot B}{B}\beta\right)_2 + \left(\frac{\kappa_1}{A} - \gamma_2\right)^{\cdot} + \frac{A}{B}\left(\frac{\mu_3 + n\lambda}{A}\right)^{\cdot} - \frac{NB}{A}\left(\frac{\lambda}{B}\right)^{\cdot}, \tag{2.7}$$

$$2\delta R^0{}_3 = 2(\varrho + p)\delta u_3 = -\left\{2\dot\alpha + \left(\frac{\dot A}{A} + \frac{\dot B}{B}\right)\alpha + \frac{\dot A}{A}\beta\right\}_3 + \frac{B}{A}\left(\frac{\lambda_1 + \mu_2}{B}\right)^{\cdot}, \tag{2.8}$$

$$\delta R^1{}_1 + \delta R^2{}_2 - \delta R = 0 = \ddot\alpha + \ddot\beta + \dot\alpha\left(\frac{\dot A}{A} + \frac{1}{2}\frac{\dot B}{B}\right) + \dot\beta\left(\frac{\dot A}{A} + \frac{3}{2}\frac{\dot B}{B}\right) - \alpha\frac{N^2B}{A^2} + 2\beta T^1_1 + \frac{(\beta - \alpha)_{33}}{B} + \frac{(\lambda_1 + \mu_2)_3}{AB} - \frac{N}{A^2}(\mu_1 - \lambda_2), \tag{2.9}$$

$$2\delta R^3{}_3 - \delta R = 0 = 2\ddot\alpha + 3\dot\alpha\frac{\dot A}{A} + \dot\beta\frac{\dot A}{A} + \alpha\left(\frac{3N^2B}{A^2} - \frac{2nN}{A}\right) + 2\beta\left(\frac{\ddot A}{A} - \frac{1}{4}\frac{\dot A^2}{A^2} + \frac{1}{4}\frac{N^2B}{A^2}\right)$$
$$+ \frac{\beta_{11} + \beta_{22} - \alpha_{11} - \alpha_{22}}{A} + \frac{A(\gamma_{11} - \gamma_{22}) + \kappa_{12} + \kappa_{21}}{A^2} + \frac{3N}{A^2}(\mu_1 - \lambda_2), \tag{2.10}$$

$$\delta R^1{}_1 - \delta R^2{}_2 = 0 = -\left\{\ddot\gamma + \dot\gamma\left(\frac{\dot A}{A} + \frac{1}{2}\frac{\dot B}{B}\right) + \gamma\left(\frac{4n^2}{B} - \frac{2nN}{A}\right)\right\} + \frac{\gamma_{33}}{B} - \frac{\kappa_3}{A}\left(\frac{4n}{B} - \frac{N}{A}\right) + \frac{2n}{AB}(\mu_1 + \lambda_2) - \frac{(\lambda_1 - \mu_2)_3}{AB}, \tag{2.11}$$

$$2\delta R^1{}_2 = 0 = -\left\{\left(\frac{\kappa}{A}\right)^{\cdot\cdot} + \left(\frac{\kappa}{A}\right)^{\cdot}\left(\frac{\dot A}{A} + \frac{1}{2}\frac{\dot B}{B}\right) + \frac{\kappa}{A}\left(\frac{4n^2}{B} - \frac{2nN}{A}\right)\right\} + \frac{\kappa_{33}}{AB} + \gamma_3\left(\frac{4n}{B} - \frac{N}{A}\right) - \frac{2n}{AB}(\lambda_1 - \mu_2) - \frac{(\mu_1 + \lambda_2)_3}{AB}, \tag{2.12}$$

$$2\delta R^3{}_1 = 0 = \frac{(\alpha - \beta)_{31}}{B} - \frac{N(\alpha - \beta)_2}{A} - \frac{1}{B}\left(\gamma_1 + \frac{\kappa_2}{A}\right)_3 + \frac{n}{B}\left(\frac{\kappa_1}{A} - \gamma_2\right) - \frac{2N\mu_3}{AB} - \frac{(\mu_1 - \lambda_2)_2}{AB} - \frac{1}{B}\left\{\ddot\lambda - \frac{1}{2}\frac{\dot B}{B}\dot\lambda + \lambda\left(\frac{1}{2}\frac{\dot A\dot B}{AB} - \frac{\ddot A}{A} + \frac{2N^2B}{A^2}\right)\right\}, \tag{2.13}$$

$$2\delta R^3{}_2 = 0 = \frac{(\alpha - \beta)_{32}}{B} + \frac{N(\alpha - \beta)_1}{A} - \frac{1}{B}\left(\frac{\kappa_1}{A} - \gamma_2\right)_3 - \frac{n}{B}\left(\gamma_1 + \frac{\kappa_2}{A}\right) + \frac{2N}{AB}\lambda_3 + \frac{(\mu_1 - \lambda_2)_1}{AB} - \frac{1}{B}\left\{\ddot\mu - \frac{1}{2}\frac{\dot B}{B}\dot\mu + \mu\left(\frac{1}{2}\frac{\dot A\dot B}{AB} - \frac{\ddot A}{A} + \frac{2N^2B}{A^2}\right)\right\}. \tag{2.14}$$

while the linearized equations governing the perturbation functions are given in the preceding page[4]. These are best obtained using the methods of Cartan; i.e. we compute the connection 1-forms $\omega^i{}_j = g^{ik}\omega_{kj}$ by solving $dg_{ij} = \omega_{ij} + \omega_{ji}$ and $d\omega^i + \omega^i{}_k \wedge \omega^k = 0$ for the metric (2.1) and the 1-forms (1.4). Then $R_{ij} = R^s_{isj}$ where $\frac{1}{2} R^i_{jkl}\omega^k \wedge \omega^l = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$. Note that, for any function F,

$$dF = \dot{F}\,dt + (X_1 F)\omega^1 + (X_2 F)\omega^2 + (X_3 F)\omega^3$$

so that no reference to coordinates need be made. The numerical subscripts on all tensors thus refer to the tetrad frame ω^i ($\omega^0 = dt$).
In these equations the subscripts 1, 2, 3 refer to "partial derivatives" with respect to the operators X_α; for example, $(\lambda_1 + \mu_2)_3$ means $X_3(X_1\lambda + X_2\mu)$, etc. We also construct the following linear combinations of these equations:

$$\begin{aligned}
2[(\delta R^3_1)_1 + (\delta R^3_2)_2] &= \left(\frac{X_1^2 + X_2^2}{B} - \frac{N^2}{A}\right)(\alpha - \beta)_3 \\
&- \frac{1}{B}\left[(\lambda_1 + \mu_2)^{\cdot\cdot} - \frac{1}{2}\frac{\dot{B}}{B}(\lambda_1 + \mu_2)^{\cdot} + (\lambda_1 + \mu_2)\left(\frac{1}{2}\frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{A}}{A} + \frac{2N^2 B}{A^2} - \frac{2nN}{A}\right)\right] \\
&- X_3\left\{\frac{A(\gamma_{11} - \gamma_{22}) + \kappa_{12} + \kappa_{21}}{AB} + \frac{3N}{AB}(\mu_1 - \lambda_2)\right\} = 0,
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
2[(\delta R^3_2)_1 - (\delta R^3_1)_2] &= N\left(\frac{X_1^2 + X_2^2}{A} + \frac{X_3^2}{B}\right)(\alpha - \beta) + \frac{X_1^2 + X_2^2}{AB}(\mu_1 - \lambda_2) \\
&- \frac{1}{B}\left[(\mu_1 - \lambda_2)^{\cdot\cdot} - \frac{1}{2}\frac{\dot{B}}{B}(\mu_1 - \lambda_2)^{\cdot} + (\mu_1 - \lambda_2)\left(\frac{1}{2}\frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{A}}{A} + \frac{2N^2 B}{A^2} - \frac{2nN}{A}\right)\right] \\
&+ X_3\left\{\frac{A(\gamma_{12} + \gamma_{21}) + \kappa_{22} - \kappa_{11}}{AB} + \frac{2N}{AB}(\lambda_1 + \mu_2)\right\} = 0,
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\delta R^1_1 - \delta R^2_2 \pm 2i\delta R^1_2 &= -\left\{\left(\gamma \pm i\frac{\kappa}{A}\right)^{\cdot\cdot} + \left(\frac{\dot{A}}{A} + \frac{1}{2}\frac{\dot{B}}{B}\right)\left(\gamma \pm i\frac{\kappa}{A}\right)^{\cdot}\right. \\
&\left. + \left(\gamma \pm i\frac{\kappa}{A}\right)\left(\frac{4n^2}{B} - \frac{2nN}{A}\right)\right\} \\
&+ \frac{\left(\gamma \pm i\frac{\kappa}{A}\right)_{33}}{B} \pm i\left(\gamma \pm i\frac{\kappa}{A}\right)_3\left(\frac{4n}{B} - \frac{N}{A}\right) - \frac{X_3 \pm 2in}{AB}(X_1 \pm iX_2)(\lambda \pm i\mu) = 0,
\end{aligned} \tag{2.17}$$

[4] The equations are written for $\delta p = 0$ for simplicity. Since $\delta p = \left(\frac{dp}{d\varrho}\right)_{\varrho = \varrho_0} \delta p$, with $\frac{dp}{d\varrho}$ given by the equation of state, the equations separate again when $\delta p \neq 0$; the only difference is that Eqs. (4.1) and (4.2) now have terms proportional to $\frac{dp}{d\varrho}$.

$$2[\delta R_1^3 \pm i\delta R_2^3] = (X_1 \pm iX_2)\left\{\left(\frac{X_3}{B} \pm i\frac{N}{A}\right)(\alpha - \beta) \pm i\frac{\mu_1 - \lambda_2}{AB}\right\}$$

$$-\frac{1}{B}\left[(\lambda \pm i\mu)^{\cdot\cdot} - \frac{1}{2}\frac{\dot{B}}{B}(\lambda \pm i\mu)^{\cdot} + (\lambda \pm i\mu)\left(\frac{1}{2}\frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{A}}{A} + \frac{2N^2B}{A^2}\right)\right] \tag{2.18}$$

$$\pm 2i\frac{N}{AB}(\lambda \pm i\mu)_3 - \frac{X_3 \pm in}{B}(X_1 \mp iX_2)\left(\gamma \pm i\frac{\kappa}{A}\right) = 0,$$

where we have used the commutation rules (1.3).

III. The Space Dependence

In analyzing the stability of the Taub universe, we showed rigorously [1] how the different equations, taken in order, implied the form of the space dependence of the perturbation. The same argument can be repeated here, but we will simply state the results. Namely, if we denote by Ψ_m^s the eigenfunctions of $X_1^2 + X_2^2$ and X_3 satisfying

and
$$\left.\begin{aligned}(X_1^2 + X_2^2)\,\Psi_m^s &= -(s - m^2)\,\Psi_m^s \\ X_3\,\Psi_m^s &= im\,\Psi_m^s,\end{aligned}\right\} \tag{3.1}$$

then the metric perturbations must satisfy the equations

$$\alpha(t, x^1, x^2, x^3) = \sum_{s,m} a_{s,m}(t)\,\Psi_m^s(x^1, x^2, x^3), \tag{3.2}$$

$$\beta(t, x^1, x^2, x^3) = \sum_{s,m} b_{s,m}(t)\,\Psi_m^s(x^1, x^2, x^3), \tag{3.3}$$

$$\lambda_1 + \mu_2 = i\sum_{s,m} P_{s,m}(t)\,\Psi_m^s, \tag{3.4}$$

$$\mu_1 - \lambda_2 = \sum_{s,m} Q_{s,m}(t)\,\Psi_m^s, \tag{3.5}$$

$$\gamma_{11} - \gamma_{22} + \frac{\kappa_{12} + \kappa_{21}}{A} = \sum_{s,m} R_{s,m}(t)\,\Psi_m^s, \tag{3.6}$$

$$\frac{\kappa_{22} - \kappa_{11}}{A} + \gamma_{12} + \gamma_{21} = i\sum_{s,m} S_{s,m}(t)\,\Psi_m^s, \tag{3.7}$$

in order that the field equations reduce to ordinary differential equations. (The ranges over which the sums extend are different for the different groups; the sums must be understood to extend over all eigenvalues of the Ψ's.)

Eqs. (3.4), (3.5), and (3.6), (3.7) can be combined as follows:

$$(3.4) \pm i\,(3.5) = (X_1 \mp i X_2)(\lambda \pm i\mu) = i \sum_{s,\,m} (P \pm Q)_{s,\,m} \Psi^s_m\,, \tag{3.8}$$

$$(3.6) \mp i\,(3.7) = (X_1 \mp i X_2)^2 \left(\gamma \pm i \frac{\kappa}{A}\right) = \sum_{s,\,m} (R \pm S)_{s,\,m} \Psi^s_m\,. \tag{3.9}$$

The integration of these equations is immediate upon noting that Ψ^s_m is an eigenfunction of the operators $(X_1 \pm i X_2)(X_1 \mp i X_2)$ and $(X_1 \pm i X_2)^2 (X_1 \mp i X_2)^2$. Indeed, using the commutation rules (1.3) and the defining equations for the Ψ^s_m's, (3.1), we find:

$$\begin{aligned}(X_1 \pm i X_2)&(X_1 \mp i X_2)\Psi^s_m \\ &= (X_1^2 + X_2^2 \mp iNX_3)\Psi^s_m = -(s - m^2 \mp mN)\Psi^s_m\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}(X_1 \pm i X_2)^2&(X_1 \mp i X_2)^2 \Psi^s_m \\ &= (X_1 \pm i X_2)(X_1^2 + X_2^2 \mp iNX_3)\;(X_1 \mp i X_2)\Psi^s_m \\ &= (X_1^2 + X_2^2 \mp 3iNX_3 + 2nN)(X_1^2 + X_2^2 \mp iNX_3)\Psi^s_m \\ &= (s - m^2 \mp 3mN - 2nN)(s - m^2 \mp mN)\Psi^s_m\,.\end{aligned} \tag{3.11}$$

The solution of Eqs. (3.8) and (3.9) is therefore

$$\lambda \pm i\mu = -i \sum_{s,m} \frac{(P \pm Q)_{s,m}}{(s - m^2 \pm mN)} (X_1 \pm i X_2)\Psi^s_m - i c_1(t) Z_1^{\pm} + g_{\pm}(t) \tag{3.12}$$

and

$$\begin{aligned}\gamma \pm i \frac{\kappa}{A} = \sum_{s,m} &\frac{(R \pm S)_{s,m}(X_1 \pm i X_2)^2 \Psi^s_m}{(s - m^2 \pm mN)(s - m^2 \pm 3mN - 2nN)} \\ &+ c_2(t) Z_1^{\pm} + c_3(t) Z_2^{\pm} + h_{\pm}(t)\end{aligned} \tag{3.13}$$

where the functions $Z_1^{\pm}$ and $Z_2^{\pm}$ satisfy

$$(X_1 \mp X_2) Z_1^{\pm} = 0 \quad \text{and} \quad (X_1 \mp i X_2)^2 Z_2^{\pm} = 0\,. \tag{3.14}$$

The significance of these extra terms is that they provide those terms in an expansion of the form $C_{s,m}\Psi^s_m$ which may be missing from the sums in (3.12) and (3.13). To see if any terms are missing one must express $(X_1 \pm i X_2)\Psi^s_m$ and $(X_1 \pm i X_2)^2 \Psi^s_m$ in terms of $\Psi^{s'}_{m'}$ and then see if the summation over s, m includes all allowed values of s', m'. Denoting $X_1 \pm i X_2$ by $X_{\pm}$, and using the commutation rules (1.3), we find that

$$X_3 X_{\pm} \Psi^s_m = [X_{\pm} X_3 \mp in X_{\pm}]\,\Psi^s_m = i(m \mp n) X_{\pm} \Psi^s_m \tag{3.15}$$

and

$$\begin{aligned}(X_1^2 + X_2^2) X_{\pm} \Psi^s_m &= [X_{\pm}(X_1^2 + X_2^2) \pm iN(X_{\pm} X_3 + X_3 X_{\pm})]\,\Psi^s_m \\ &= -[s - m^2 - N(n \mp 2m)]\,X_{\pm}\Psi^s_m\,,\end{aligned} \tag{3.16}$$

i.e. $X_{\pm}\Psi^s_m$ is an eigenfunction of X_3 and $X_1^2 + X_2^2$; hence it is proportional to $\Psi^{s'}_{m'}$ where $m' = m \mp n$, $s' = s + (n - N)\,(n \mp 2m)$. Similarly we find

$(X_\pm)^2\,\Psi^s_m \sim \Psi^{s''}_{m''}$ where $m''=m\mp 2n$ and $s''=s+4(n-N)\,(n\mp m)$. Clearly for $n=N=0$ (Bianchi I) $s''=s'=s$, $m''=m'=m$ and the sums in (3.12) and (3.13) include all allowed values of s', m', s'', m''; the extra terms can be put equal to zero in this case. (However, this is not true in general – see [1], where the extra terms must be non-zero.)

IV. The Time Dependence

The equations determining the time dependence of a, b, P, Q, R, S are obtained by substituting Eqs. (3.2)—(3.7) in (2.9), (2.10), (2.15), and (2.16) and Eqs. (3.12), (3.13) in (2.17) and demanding that the coefficients of the different Ψ^s_m vanish. Dropping the subscripts s, m we find:

$$\begin{aligned}&\ddot a+\ddot b+\dot a\left(\frac{\dot A}{A}+\frac{1}{2}\frac{\dot B}{B}\right)+\dot b\left(\frac{\dot A}{A}+\frac{3}{2}\frac{\dot B}{B}\right)-a\frac{N^2B}{A^2}+2bT^1_1\\&\quad+\frac{m^2}{B}(a-b)-\frac{mP}{AB}-\frac{NQ}{A^2}=0\,,\end{aligned}\tag{4.1}$$

$$\begin{aligned}&2\ddot a+3\dot a\frac{\dot A}{A}+\dot b\frac{\dot A}{A}+a\left(\frac{3N^2B}{A^2}-\frac{2nN}{A}\right)+2b\left(\frac{\ddot A}{A}-\frac{1}{4}\frac{\dot A^2}{A^2}+\frac{1}{4}\frac{N^2B}{A^2}\right)\\&\quad+(s-m^2)\frac{(a-b)}{A}+\frac{3N}{A^2}Q+\frac{R}{A}=0\,,\end{aligned}\tag{4.2}$$

$$\begin{aligned}&\ddot P-\frac{1}{2}\frac{\dot B}{B}\dot P+\left(\frac{1}{2}\frac{\dot A\dot B}{AB}-\frac{\ddot A}{A}+\frac{2N^2B}{A^2}-\frac{2nN}{A}\right)P\\&\quad+m\left\{\left(s-m^2+N^2\frac{B}{A}\right)(a-b)+R+3N\frac{Q}{A}\right\}=0,\end{aligned}\tag{4.3}$$

$$\begin{aligned}&\ddot Q-\frac{1}{2}\frac{\dot B}{B}\dot Q+\left(\frac{1}{2}\frac{\dot A\dot B}{AB}-\frac{\ddot A}{A}+\frac{2N^2B}{A^2}-\frac{2nN}{A}+\frac{s-m^2}{A}\right)Q\\&\quad+m\left(S+2N\frac{P}{A}\right)+N\left[m^2+(s-m^2)\frac{B}{A}\right](a-b)=0\,,\end{aligned}\tag{4.4}$$

$$\begin{aligned}&\ddot R+\left(\frac{\dot A}{A}+\frac{1}{2}\frac{\dot B}{B}\right)\dot R+\frac{m^2}{B}R-\frac{mN}{A}S\\&\quad+\frac{m}{AB}\{(s-m^2-2nN)P+3mNQ\}=0\,,\end{aligned}\tag{4.5}$$

$$\begin{aligned}&\ddot S+\left(\frac{\dot A}{A}+\frac{1}{2}\frac{\dot B}{B}\right)\dot S+\frac{m^2}{B}S-\frac{mN}{A}R\\&\quad+\frac{m}{AB}\{(s-m^2-2nN)Q+3mNP\}=0\,.\end{aligned}\tag{4.6}$$

Similarly the equations determining c_1, c_2, c_3, g, and h can be found by substituting (3.12) and (3.13) in (2.17) and (2.18) and using the fact that $X_{\mp}(Z_2^{\pm})$ is proportional to $Z_1^{\pm}$ which follows from (3.14). One finds that g, h, and c_2 satisfy homogeneous equations while c_1 and c_3 couple to each other. We omit these equations since we will not use them below.

The role played by the different values of n and N is best seen in these equations where they serve to couple the different functions together. One notes that when $N=0$, Q and S decouple from a, b, P, R. There are also special cases for particular values of s and m; for example, when $m=0$, P and S decouple from a, b, Q, R. Also when $s-m^2=\pm mN$, Eqs. (4.3)—(4.6) imply that $P=\pm Q \Leftrightarrow R=\pm S$ which is needed in order that (3.12) and (3.13) make sense. These special cases are best discussed for each group separately.

The initial value problem then consists in assigning $\delta\varrho$, δu_α, a, b, P, Q, R, S, $\dot{a}$, $\dot{b}$, $\dot{P}$, $\dot{Q}$, $\dot{R}$, $\dot{S}$ subject to the four constraints $\delta G_i^0 = \delta T_i^0$ (Eqs. (2.5)—(2.8)), which read

$$2\delta\varrho = \left\{ \dot{a}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right) + \dot{b}\frac{\dot{A}}{A} + a\left(\frac{N^2 B}{A^2} - \frac{2nN}{A}\right) + b\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}^2}{2A^2} - \frac{N^2 B}{2A^2}\right)\right.$$
$$\left. + \frac{s-m^2}{A}(a+b) + \frac{2m^2}{B}a + \frac{R}{A} - \frac{2mP}{AB} + \frac{NQ}{AB}\right\} \Psi_m^s , \tag{4.7}$$

$$2\varrho(\delta u_1 \pm i\delta u_2) = \left\{ -(s-m^2 \pm mN)\left(\dot{a}+\dot{b}+\frac{\dot{B}}{B}b\right) + m\frac{A}{B}\left(\frac{P\pm Q}{A}\right)^{\cdot}\right.$$
$$\left. \mp N\frac{B}{A}\left(\frac{P\pm Q}{B}\right)^{\cdot}\right\} \frac{(X_1 \pm iX_2)}{(s-m^2\pm mN)} \Psi_m^s , \tag{4.8}$$

and

$$2\varrho\delta u_3 = i\left\{\frac{B}{A}\left(\frac{P}{B}\right)^{\cdot} - m\left[2\dot{a} + \left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)a + \frac{\dot{A}}{A}b\right]\right\}\Psi_m^s . \tag{4.9}$$

These equations give $\delta\varrho$ and δu_α at subsequent times when a, b, P, Q, R, S evolve according to (4.1)—(4.6) (δu_0 is given by the requirement that $u^i u_i = +1$).

The solution of Eqs. (4.1)—(4.6) might be expected to be well behaved everywhere except near their singular points, namely, $A=0$, $B=0$, and possibly, $t=\pm\infty$. At $A=0$ or $B=0$, however, the unperturbed metric becomes singular. We can thus conclude that the instabilities of this class of universes occur at their singularities or at $t=\pm\infty$.

As was the case with Taub [1], the gauge chosen in writing (2.1) does not completely specify the coordinate system. The remaining coordinate freedom can be found as in Appendix C of [1]. The computations are lengthy but straightforward. The result is that the following expressions give a coordinate dependent solution of (4.1)—(4.6) (suppressing summation signs and indices s, m on the functions u, p, v, and c):

$$\alpha = \left[(s-m^2)v(t) - \frac{\dot{A}}{A}u(t)\right]\Psi^s_m - c(t)\,(X_1^2 - X_2^2)\,\Psi^s_m\,, \tag{4.10}$$

$$\beta = \left[2mp(t) - \frac{\dot{B}}{B}u(t)\right]\Psi^s_m\,, \tag{4.11}$$

$$\gamma \pm i\frac{\kappa}{A} = -(X_1 \pm iX_2)\,\{v(t)\,(X_1 \pm iX_2) + c(t)\,(X_1 \mp iX_2)\}\,\Psi^s_m\,, \tag{4.12}$$

$$\begin{aligned}\lambda \pm i\mu = &- i\{Bp(t) + (mA \mp NB)v(t)\}\,(X_1 \pm iX_2)\,\Psi^s_m \\ &- ic(t)\,\{(m \pm 2n)A \mp NB\}\,(X_1 \mp iX_2)\,\Psi^s_m\,,\end{aligned} \tag{4.13}$$

where the functions of time u, p, v, c satisfy the equations

$$\dot{u} = mp - \frac{1}{2}\frac{\dot{B}}{B}u\,, \tag{4.14}$$

$$\dot{p} = \frac{m}{B}u\,, \tag{4.15}$$

$$\dot{v} = \frac{u}{A}\,, \qquad \dot{c} = 0\,. \tag{4.16}$$

Note that the coordinate dependent solutions form a 4-parameter family – the four initial values of u, p, v, and c needed to integrate (4.14) to (4.16). (In [1] the constant c was mistakenly taken equal to zero resulting in a 3-parameter family.)

Appendix A

The operators X_α satisfying (1.1) have been given by Ellis and McCallum [3]. Computing $g^{\alpha\beta}X_\alpha X_\beta = -\left[\frac{X_1^2}{A} + \frac{X_2^2}{C} + \frac{X_3^2}{B}\right]$ we note that this operator will have eigenfunctions of the form $\Phi_1(x^1)\,\Phi_2(x^2)\,\Phi_3(x^3)$ only if $A = C$ and $N_1 = N_2$ (except for the case $N_1 = N_2 = N_3 = 0$ where $A \neq C$ is possible). Specializing, therefore, to the "locally rotationally symmetric" case we find:

$n \neq 0$	$n = 0$	
$X_1 = \cos x^3 \dfrac{\partial}{\partial x^1} + n \sin x^3 Z$	$X_1 = \dfrac{\partial}{\partial x_1}$	(A.1)
$X_2 = -n \sin x^3 \dfrac{\partial}{\partial x^1} + \cos x^3 Z$	$X_2 = \dfrac{\partial}{\partial x^2} + N x^1 \dfrac{\partial}{\partial x^3}$	(A.2)
$X_3 = \dfrac{\partial}{\partial x^3}$	$X_3 = \dfrac{\partial}{\partial x^3}$	(A.3)

where

$$Z = [1 - nNS^2(x^1)]^{-1/2} \left\{ \frac{\partial}{\partial x^2} + NS(x^1) \frac{\partial}{\partial x^3} \right\}$$

and

$$S(x^1) = \begin{cases} \sin x^1 \\ x^1 \\ \sinh x^1 \end{cases} \quad \text{for} \quad N = \begin{cases} n \\ 0\,. \\ -n \end{cases}$$

The operator $X_1^2 + X_2^2$ takes the form:

$$\frac{\partial^2}{(\partial x^1)^2} - nN \frac{S}{S'} \frac{\partial}{\partial x^1} + \frac{1}{(S')^2} \left\{ \frac{\partial^2}{(\partial x^2)^2} + 2NS \frac{\partial^2}{\partial x^2 \partial x^3} + N^2 S^2 \frac{\partial^2}{(\partial x^3)^2} \right\} \text{ for } n \neq 0 \tag{A.4}$$

where

$$S' = \frac{d}{dx^1} S(x^1)$$

and

$$\frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + 2Nx^1 \frac{\partial^2}{\partial x^2 \partial x^3} + N^2 (x^1)^2 \frac{\partial^2}{(\partial x^3)^2} \quad \text{for } n = 0\,. \tag{A.5}$$

Thus, in all cases, the eigenfunctions of $X_1^2 + X_2^2$ and X_3 can be chosen to be eigenfunctions of $\partial/\partial x^2$ also. Letting $\Psi^s_{mk} = e^{imx^3} e^{ikx^2} \Omega^s_{mk}(x^1)$ we find that the function Ω satisfies the equation:

Bianchi type

I, VII_0:

$$\left\{ \frac{d^2}{(dx^1)^2} + s - m^2 - k^2 \right\} \Omega = 0\,, \tag{A.6}$$

II:

$$\left\{ \frac{d^2}{(dx^1)^2} + s - m^2 - (k + mNx^1)^2 \right\} \Omega = 0\,, \tag{A.7}$$

VIII:

$$\left\{ \frac{d^2}{(dx^1)^2} + \tanh x^1 \frac{d}{dx^1} - \frac{k^2 + 2mNk \sinh x^1 - m^2}{\cosh^2 x^1} + s - 2m^2 \right\} \Omega = 0\,, \tag{A.8}$$

IX:

$$\left\{ \frac{d^2}{(dx^1)^2} - \tan x^1 \frac{d}{dx^1} - \frac{k^2 + 2mNk \sin x^1 + m^2}{\cos^2 x^1} + s \right\} \Omega = 0\,. \tag{A.9}$$

All these equations are known ODE's with known eigenfunctions. (A.6) is the wave equation in one dimension, (A.7) the Schrödinger equation for a one-dimensional harmonic oscillator, (A.9) (with $x^1 = \beta + \pi/2$) is discussed in [5] (Eq. (4.7.6)), while (A.8) is related to (A.9) by a complex transformation.

In this paper we have suppressed the index k from Ψ^s_{mk} because it does not enter in our equations. However, for the expansions (3.2)—(3.7) to be complete, an additional sum over k must be understood.

Appendix B

The Kantowski-Sachs universe [2] has a metric very similar to (1.2):

$$ds^2 = 4Y^2 d\eta^2 - X^2 dr^2 - Y^2[d\theta^2 + \sin^2\theta \, d\varphi^2] \tag{B.1}$$

where X and Y are functions of the parameter η:

$$X = \varepsilon(1 + \eta \tan\eta) + b \tan\eta\,, \qquad Y = a\cos^2\eta \tag{B.2}$$

(a, b, and ε are constants).

Unlike the Bianchi cases where the X_α (and hence the ω^α) were fixed by the commutation rules (1.3), here one is free to choose ω^2 and ω^3 so long as they satisfy $(\omega^2)^2 + (\omega^3)^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$. Two possible choices are $\omega^2 = d\theta$, $\omega^3 = \sin\theta \, d\varphi$ and $\omega^2 = \cos\varphi \, d\theta - \sin\varphi \sin\theta \, d\varphi$, $\omega^3 = \sin\varphi \, d\theta + \cos\varphi \sin\theta \, d\varphi$. However, neither choice makes $X_1^2 + X_2^2$ a known operator, which is desirable since the perturbation will be expanded in terms of its eigenfunctions. Moreover, the appearance of the metric on the sphere, $d\theta^2 + \sin^2\theta \, d\varphi^2$, in (B.1) would lead one to expect that an expansion in spherical harmonics should be possible. This is indeed the case if we rewrite (B.1) as

$$ds^2 = 4Y^2 d\eta^2 - X^2 dr^2 - Y^2 \sin^2\theta \left(\frac{d\theta^2}{\sin^2\theta} + d\varphi^2 \right), \tag{B.3}$$

because now we can take $\omega^2 = \dfrac{d\theta}{\sin\theta}$, $\omega^3 = d\varphi$ giving $X_2 = \sin\theta \dfrac{\partial}{\partial\theta}$, $X_3 = \dfrac{\partial}{\partial\varphi}$, so that

$$\frac{1}{\sin^2\theta}(X_2^2 + X_3^2) = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \tag{B.4}$$

becomes the total angular momentum operator in spherical polar coordinates. In addition, with $X_1 = \partial/\partial r$, we find that all the X_α commute (correspondingly $d\omega^\alpha = 0$), which simplifies the calculations. (Of course, this simplification is offset by the appearance of $\sin^2\theta$ in the unperturbed metric!)

We therefore start with (B.3) and, in analogy with (2.1), choose the gauge of the perturbation so that the perturbed metric becomes:

$$g_{ij} + \delta g_{ij} = \begin{pmatrix} 4Y^2(1-\beta) & 0 & 0 & 0 \\ 0 & -X^2(1+\beta) & -\mu & -\lambda \\ 0 & -\mu & -Y^2\sin^2\theta(1+\alpha+\gamma) & -\kappa \\ 0 & -\lambda & -\kappa & -Y^2\sin^2\theta(1+\alpha-\gamma) \end{pmatrix} \tag{B.5}$$

where, as before, α, β, γ, κ, λ, and μ are the perturbation functions. The linearized field equations, obtained as before, now contain terms in $\sin\theta$ and $\cos\theta$. The equations become more symmetrical after the substitutions $\gamma=\sigma/\sin^2\theta$ and $\kappa=Y^2\tau$ are made. Separability is achieved by expanding α, β and certain combinations of derivatives of λ, μ, σ, and τ (analogous to (3.4)—(3.7)) in terms of $e^{ikr}Y^l_m(\theta,\varphi)$. The latter equations can be combined in pairs to give (compare (3.8), (3.9)):

$$\frac{X_2 \mp iX_3}{\sin^2\theta}(\mu \pm i\lambda) = i\sum_{k,l}(P\pm Q)_{k,l}e^{ikr}Y^l_m \tag{B.6}$$

and

$$\left(\frac{X_2 \mp iX_3}{\sin^2\theta}\right)^2\left(\gamma\sin^2\theta \pm i\frac{\kappa}{Y^2}\right) = \sum_{k,l}(R\pm S)_{k,l}e^{ikr}Y^l_m \tag{B.7}$$

where P, Q, R, and S are again functions of time. The integration of these equations is less straightforward than (3.8), (3.9) because $(X_2 \pm iX_3)\,Y^l_m$ is a linear combination of Y^l_m and $Y^{l\pm1}_m$ rather than a single $Y^{l'}_{m'}$ as was the case with the Ψ^s_m's. The result is that λ and μ depend on Y^l_m and $Y^{l\pm1}_m$ while γ and κ depend on Y^l_m, $Y^{l\pm1}_m$ and $Y^{l\pm2}_m$. Clearly, no extra terms corresponding to $Z^\pm$ are needed in this case.

The equations determining the time dependence of the perturbation can be obtained from the corresponding ones for the Bianchi models (4.1)—(4.6) after we make the obvious substitutions:

$$A\to Y^2\,,\qquad B\to X^2\,,\qquad \frac{d}{dt}\to\frac{1}{2Y}\frac{d}{d\eta}\,,$$
$$s-m^2\to l(l+1)\,,\qquad m\to k\,,$$

and the not-so-obvious $nN\to1$, $N\to0$. Numerical integration of these equations leads to the expected result that the perturbations become unbounded as the singularity $Y=0$ is approached. An oscillatory behavior similar to that found in [1] is again observed here as k and l become large.

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