Commun. math. Phys. 25, 87—126 (1972)
© by Springer-Verlag 1972

Surface Tension in the Ising Model

G. GALLAVOTTI
Istituto di Matematica dell'Università di Roma, Italia

A. MARTIN-LÖF
Rockefeller University, Department of Mathematics New York, New York, USA

Received August 16, 1971

Abstract. We investigate the problem of a microscopic definition of the surface of separation between two phases in the special case of the 2-dimensional Ising model. We show how this leads to a definition of the surface tension which appears, in this context, as the logarithm of a partition function over a set of random surfaces. We also discuss the more general problem of defining the surface tension in an Ising ferromagnet with arbitrarily extended attractive interaction.

1. Introduction

The problem of giving a statistical-mechanical definition of surface tension does not seem to have been even posed in a completely satisfactory way mainly as a consequence of our inability to give precise meaning to the surface of separation of two pure phases and, even worse, to the very concept of coexisting phases. It is well known that we even lack a proof of the existence of a phase transition for a continuous system. However, for lattice systems one can rigorously show that phase-transitions occur under certain conditions, and the phenomenon of phase separation has recently been so deeply investigated that, as we are going to show, it is much more hopeful to investigate the problems connected with the surface tension in these systems.

In fact in a fundamental paper Minlos and Sinai [7, 8], hereafter referred as MS, have considered a ν-dimensional Ising ferromagnet enclosed in a box Ω surrounded by a layer of spins up and with a fixed total magnetization:

$$M = (\alpha m^* + (1-\alpha)(-m^*))\,|\Omega| \qquad 0<\alpha<1 \tag{1.1}$$

where m^* is the spontaneous magnetization. They have proved that if the temperature is very low and one picks up at random a configuration of spins out of the canonical ensemble defined by fixing the total magnetization as in (1.1), then with very large probability (tending to 1 as $|\Omega| \to \infty$) this configuration will consist of a "drop", roughly square in shape, with volume $\sim(1-\alpha)|\Omega|$ and with average magnetization $\sim -\mathrm{m}^*$ surrounded by a complementary region with volume $\sim\alpha|\Omega|$ and with average magnetization $\sim\mathrm{m}^*$. Furthermore the boundaries separating the spins up from the spins down are very "short" (i.e. their lengths do not exceed $c_0 \log|\Omega|$ with c_0 a certain positive constant) except one which is the boundary of the drop with down spins. They have also shown that the average correlation (suitably defined [3]) inside the drop is the correlation of the pure phase with magnetization $-\mathrm{m}^*$, and that the average correlation outside the drop is the correlation of the pure phase with magnetization m^*.

As a consequence of the MS result it is quite clear that one has to interpret the large boundary of the drop with volume $\sim(1-\alpha)|\Omega|$ as the surface of separation between the two phases. One also has to conclude that this separation into pure phases is due to the fact that all the spins on the boundary of Ω are up (i.e. that there is an infinite magnetic field acting on the boundary of Ω which favors the formation of the up-magnetized region near the boundary).

The MS result also makes it possible to give a very natural definition of surface tension which seems to be quite general, at least for lattice models, since the MS result holds for a much larger class of models on a lattice. This definition is the following (see also [6]): "compute" the partition function $Z^+(\Omega, \mathrm{m}, \beta)$ of the system enclosed in a box Ω surrounded by spins up and with magnetization $\mathrm{m}=(2\alpha-1)\mathrm{m}^*$, and suppose that one can show the following asymptotic form of Z^+ (we denote by $f(\beta)$ the free energy of the system):

$$Z^+(\Omega, \mathrm{m}, \beta) = \exp\left(\beta f(\beta)|\Omega| + 2\nu\tau_B|\Omega|^{\frac{\nu-1}{\nu}} + 2\nu\tau((1-\alpha)|\Omega|)^{\frac{\nu-1}{\nu}} + o\left(|\Omega|^{\frac{\nu-1}{\nu}}\right)\right) \tag{1.2}$$

for all $0<\alpha<1$. Then in view of the MS result the quantity $2\nu\tau_B$ would correspond to the effect of the external boundary while $2\nu\tau$ would in a natural way correspond to a phase separation effect (since it is α-independent).

Though (1.2) provides a very natural definition of surface tension we shall however follow a slightly different approach based on the same philosophy which allows an easier solution of some mathematical problems.

The essential idea of the above construction was the use of a boundary condition which produces a separation of the two phases into two disjoint regions with a "well defined" boundary. One can expect that this phase separation is produced also by the following boundary conditions: consider for simplicity $\gamma=2$ and let Ω be a cylinder with N columns and H rows, and suppose that on the upper base of Ω there is a layer of spins *up* while on the lower base there is a layer of spins *down*. We may then expect that in the two phase region a phase separation occurs in the following sense: a configuration randomly chosen out of the canonical ensemble with magnetization given by (1.1) will consist of two distinct pieces separated by a closed line λ going around Ω and such that the region above λ has volume $\sim \alpha|\Omega|$, and the average magnetization in these two regions is $\sim m^*$ and $\sim -m^*$ respectively. The lines of separation between the regions with spin up and the ones with spin down above and below the separating line should be very small (i.e. should have length not exceeding $c_0 \log|\Omega|$).

If we "compute" the partition function of the canonical ensemble with the magnetization and the boundary conditions we are considering, we should find an expression of the form:

$$Z^{+-}(\Omega, m, \beta) = \exp(\beta f(\beta)\,|\Omega| + 2\tau_B N + \tau N + o(N)) \tag{1.3}$$

where $2\tau_B$ should be the contribution of the bases, and τ should be the contribution coming from the (random) line λ separating the two phases. It should be m-independent (as τ_B).

One can wonder how to distinguish between τ_B and τ since now there is no α-dependence in front of τ. This point will become clear later: by now we only remark that a possible criterion to distinguish between τ_B and τ is to compare (1.3) with the partition function $Z^{++}(\Omega, m^*, \beta)$ with magnetization m^* and boundary spins all up ("partition function of a pure phase"). If the above picture of the phenomenon of phase separation is correct $Z^{++}(\Omega, m^*, \beta)$ should be:

$$Z^{++}(\Omega, m^*, \beta) = \exp(\beta f(\beta)\,|\Omega| + 2\tau_B N + o(N)) \tag{1.4}$$

with the same τ_B as in (1.3), and so (1.3) together with (1.4) would provide the definition of τ as:

$$\tau = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z^{+-}(\Omega, m, \beta)}{Z^{++}(\Omega, m^*, \beta)}\,. \tag{1.5}$$

We mention that the surface tension in the two dimensional Ising model with nearest neighbour interactions has been calculated by Onsager, but his definition is not a priori equivalent to ours because he works with a boundary condition for which it has not been proved that there is a strict phase separation [4, 10], so, a priori his τ could contain some extra contribution coming from the fact that the two phases might be mixed. We shall not be able to solve the interesting problem of comparing (1.5) with Onsager's value. This is as frustrating as the fact that there is no proof that Onsager's value for the spontaneous magnetization is the correct one and not a lower bound to it. Finally we remark that Onsager's way of defining the surface tension seems to be "ad hoc" for the nearest neighbour Ising model, while (1.5) can clearly be generalized to other models (not necessarily with nearest neighbour interactions) [1].

In this paper we carry out the program of showing that there is a phase separation if we use the above described boundary conditions on the 2-dimensional cylinder and if β is large (Section 3). In Section 4 we show that the limit (1.5) exists and that (1.3), (1.4) hold. In Section 5 we give some concluding remarks about how the results could be generalized to Ising models with interactions with range larger than 1.

As a consequence of the rather heavy technical work to be done and to distinguish between the physics and the mathematics involved we have preferred to write the proofs of most of the statements in appendices so one can get a clear idea of the meaning and the motivation of the various steps and results by just reading the text without the appendices. To them we refer the interested readers and the sceptical ones.

2. Notations

Let Ω be a $N \times (H+2)$ square lattice. We shall think of Ω as wrapped on a cylinder with base N along the rows.

Suppose the spins on the two bases are fixed and identical: i.e. all the spins on the upper base are $+$ and all the ones on the lower base are $-$, or else all the upper base spins are $+$ and the lower ones are $+$ etc. We shall refer to these boundary conditions as $(+, -)$, $(+, +)$, etc.

The Ising model is defined through the energy assigned to each spin configuration. This energy is obtained as a sum of terms of the form $\pm J$ each coming from a bond in Ω. A bond will contribute $+J$ if the two spins at its extremes are parallel and will contribute $-J$ if these two spins are antiparallel.

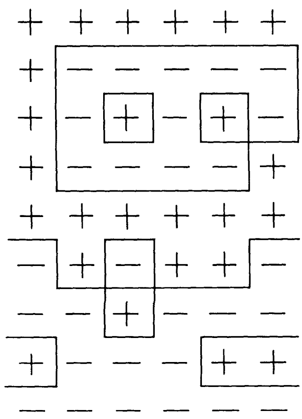

Fig. 1. A configuration with boundary conditions $(+, -)$

For a fixed boundary condition and a fixed spin configuration we draw at the midpoint of every bond contributing $-\mathrm{J}$ to the energy a perpendicular segment of length 1 (see Fig. 1). We obtain in this way a family of lines lying on a lattice shifted by (1/2, 1/2) from the original one.

A vertex of this lattice will belong to either 0, 2 or 4 segments of the lines drawn. The last case can happen when the four spins around it are arranged as:

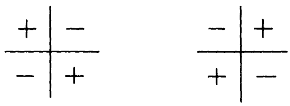

Fig. 2a

Suppose we deform the lines around each of the four-fold vertices as follows:

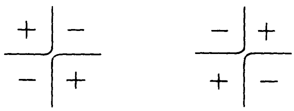

Fig. 2b

It is easy to realize that after this operation the set of lines splits into a set of several separated *closed* self-avoiding contours $\gamma_1, \ldots, \gamma_n$ (see Fig. 2b). These lines separate the spins up from the spins down and, if $|\gamma_i|$ denotes the perimeter of γ_i the number of bonds which contribute $-\mathrm{J}$ to the total energy is simply $\sum_i |\gamma_i|$. (Here and in the following a self-avoiding walk is allowed to touch itself or an other walk as in Fig. 2b, but not in any other way.)

Therefore the energy of a configuration to which is associated a set of contours $\gamma_1, \ldots, \gamma_n$ is:

$$E(\gamma_1, \ldots, \gamma_n) = \mathrm{J}(\#\text{ of bonds in } \Omega) - 2\mathrm{J} \sum_i |\gamma_i| \,. \tag{2.1}$$

From now on we put $2\mathrm{J} = -1$.

We observe that, given the boundary condition, the set of contours uniquely determines the configuration from which they come.

However not all the sets of contours on Ω are compatible with a given boundary condition. In fact if we use the $(+,-)$ boundary condition there must be an odd number of contours encircling the cylinder, while the $(+,+)$ boundary condition implies that there must be an even number of such contours. These are the only restrictions to be imposed on the allowed sets of contours. The correspondence between the configurations and the allowed contours is one-to-one, so we shall use the two concepts in an interchangeable way.

Let $\mathfrak{M}^{xy}(\Omega)$, $x, y = \pm$, be the set of contour configurations which are allowed by the boundary condition (x, y). We shall regard $\mathfrak{M}^{xy}(\Omega)$ as an "ensemble" in which the relative probability of a configuration $(\gamma_1, \ldots, \gamma_n)$ is given by the Boltzmann factor $e^{-\beta \sum\limits_i |\gamma_i|}$. More generally, if $\mathfrak{M}$ is a set of configurations compatible with a given boundary condition we shall consider it as an "ensemble" in which the relative probabilities are defined by $e^{-\beta \sum\limits_i |\gamma_i|}$.

We define the average magnetization of a configuration $\mathrm{X} \in \mathfrak{M}^{xy}(\Omega)$ as

$$\mathrm{m}(\mathrm{X}) = ((\#\text{ of spins} + \text{in X}) - (\#\text{ of spins} - \text{in X}))/\mathrm{NH} \tag{2.2}$$

where we do not count the spins on the bases of Ω (which are thus considered as external fields acting on the boundary spins). NH is the volume of Ω which we shall also denote by $|\Omega|$.

The ensemble $\mathfrak{M}^{xy}(\Omega, \mathrm{m})$ will denote, for $|\mathrm{m}| \leqq 1$, the set of configurations X in $\mathfrak{M}^{xy}(\Omega)$ such that $\mathrm{m}(\mathrm{X}) = \mathrm{m}$. $\mathfrak{M}^{xy}(\Omega)$ and $\mathfrak{M}^{xy}(\Omega, \mathrm{m})$ are, respectively, to be thought as the grand-canonical and the canonical ensemble with boundary condition (x, y).

If $\mathfrak{M}$ is an ensemble of configurations $\mathfrak{M} \subset \mathfrak{M}^{xy}(\Omega)$ we shall denote $Z(\mathfrak{M}, \beta)$ the "partition function" relative to $\mathfrak{M}$, the quantity:

$$Z(\mathfrak{M}, \beta) = \sum_{\mathrm{X} \in \mathfrak{M}} e^{-\beta \sum\limits_i |\gamma_c|}. \tag{2.3}$$

To complete the set of basic notations we divide the contours into different classes. We call "big" a contour γ which goes around the cylinder. We call c-small the contours γ such that $|\gamma| \leqq c \log|\Omega|$ which are not big, and we call c-large the other contours.

In the paper MS a particular role is played by the value $c = c_0 = 1/333$ (Moser's constant). In this paper whenever we shall talk about large and small contours without a c in front we will mean c_0-large or c_0-small where $c_0 = 1/333.3333\ldots$ (modified Moser's constant).

3. The Phase Separation

Consider the ensemble

$$\mathfrak{M}^{+-}(\Omega, \mathrm{m}), \qquad \mathrm{m} = \alpha \mathrm{m}^* + (1-\alpha)(-\mathrm{m}^*) = (2\alpha - 1)\mathrm{m}^*,$$

where m^* is the spontaneous magnetization as defined in Appendix A and $0 < \alpha < 1$. (Here and in the following we write $\mathrm{m} = (2\alpha - 1)\mathrm{m}^*$ instead of $\mathrm{m} = |\Omega|^{-1} \times$ the integer having the same parity as $|\Omega|$ which is closest to $(2\alpha - 1)\mathrm{m}^*|\Omega|$. Of course we want m to be a possible value of the average magnetization. Likewise we are tacitly going to ignore some unimportant rounding off effects due to this inaccuracy in order not to burden the presentation too heavily.)

Suppose Ω is a $N \times (H+2)$ cylinder such that $H = N^{\delta}$, $\delta > 1$. We have mentioned in the introduction that we hope that one could say that there is a phase separation in $\mathfrak{M}^{+-}(\Omega, m)$, i.e. that in some sense one could think that the ensemble $\mathfrak{M}^{+-}(\Omega, m)$ describes the coexistence of two phases with magnetization $\sim \pm m^*$ respectively the first being on top of the other and each occupying a volume roughly of the order of $\alpha|\Omega|$ and $(1-\alpha)|\Omega|$ respectively. We have taken H to grow faster than N in order to avoid the possibility that the surface of separation comes too near to the bases of the cylinder thus causing further boundary effects which are spurious as far as the surface tension is concerned.

Here and below we shall have to consider functions of β which will have a different asymptotic behaviour as $\beta \to \infty$ (i.e. as $T \to 0$). We shall consistently denote $\delta(\beta)$, $\eta(\beta)$, $\xi(\beta)$ etc. functions of β which approach zero exponentially as $\beta \to \infty$ (i.e. functions which are bounded above by $\exp - C\beta$ for some $C > 0$). We shall denote by $a(\beta)$, $b(\beta)$, $\ell(\beta)$ etc. functions which approach zero as $\beta \to \infty$ only as a power. Finally we will denote by $A(\beta)$, $B(\beta)$, $D(\beta)$ etc. functions for which we are not interested in emphasizing the behaviour as $\beta \to \infty$.

Let us now discuss in detail in what sense we have phase separation in the ensemble $\mathfrak{M}^{+-}(\Omega, m)$.

Consider the set of configurations $\tilde{\mathfrak{M}}_0^{+-}(\Omega, m) \subset \mathfrak{M}^{+-}(\Omega, m)$ consisting of the configurations $X \in \mathfrak{M}^{+-}(\Omega, m)$ such that the conditions 1–4 below are verified:

1. X contains just one big contour λ and λ is such that:

$$|\lambda| \leqq N(1 + \ell(\beta)). \tag{3.1}$$

2. Calling Ω_λ the region above the big contour λ associated to X we have: (for some p with $3/4 < p < 1$)

$$\begin{aligned} \big||\Omega_\lambda| - \alpha|\Omega|\big| &\leqq \varkappa(\beta)\,|\Omega|^p \\ \big||\Omega - \Omega_\lambda| - (1-\alpha)\,|\Omega|\big| &\leqq \varkappa(\beta)\,|\Omega|^p . \end{aligned} \tag{3.2}$$

3. Calling m^+ the average magnetization of the configuration in Ω_λ and m^- the average magnetization in $\Omega - \Omega_\lambda$ we have:

$$\begin{aligned} \big|m^+|\Omega_\lambda| - m^*\alpha|\Omega|\big| &\leqq \varkappa(\beta)\,|\Omega|^p \\ \big|m^-|\Omega - \Omega_\lambda| + m^*(1-\alpha)\,|\Omega|\big| &\leqq \varkappa(\beta)\,|\Omega|^p . \end{aligned} \tag{3.3}$$

4. The set of the c_0-large contours in X has a total length which does not exceed $N\,d(\beta)$.

In view of the mentioned results of MS [7] one can hope that the set of configurations $\tilde{\mathfrak{M}}_0^{+-}(\Omega, m)$ has probability very close to 1 if N is large enough (and β is fixed to be large enough). In Appendix B we prove in fact that:

$$\lim_{N\to\infty} \frac{Z(\tilde{\mathfrak{M}}_0^{+-}(\Omega, \mathrm{m}), \beta)}{Z(\mathfrak{M}^{+-}(\Omega, \mathrm{m}), \beta)} = 1 \tag{3.4}$$

provided β is large enough (almost ridiculously large from a numerical point of view) and provided the functions $\ell(\beta)$ in (3.1), $\varkappa(\beta)$ in (3.2) and $d(\beta)$ in 4. and p are suitably chosen.

Formula (3.4) says that picking up at random a configuration out of the ensemble $\mathfrak{M}^{+-}(\Omega, \mathrm{m})$ we shall almost surely obtain a configuration in $\tilde{\mathfrak{M}}_0^{+-}(\Omega, \mathrm{m})$.

The above picture could be improved by showing that not only $\tilde{\mathfrak{M}}_0^{+-}(\Omega, \mathrm{m})$ has a probability in $\mathfrak{M}^{+-}(\Omega, \mathrm{m})$ tending to 1, but also that the same happens if property 4 is strengthened to:

4'. There are *no* c_0-large outer contours in X.

We shall not, however, give the proof of this stronger result since we do not need it.

From the physical point of view (3.4) says that if we look at a sample of our system which is enclosed in Ω and subject to the $(+, -)$ boundary condition, then we shall find that it looks like two "seas" of up and down spins, the first on top of the other, separated by a rather well defined surface λ at height $\sim(1-\alpha)\mathrm{H}$ (see (3.1), (3.2)). Furthermore the sea of up-spins will have a lot of small holes in it which contain down spins and these holes are in such a number as to give to the sea of up-spins an average magnetization $\sim \mathrm{m}^*$ (see (3.3)). A similar picture holds for the sea of negative spins. The dimensions of the holes are very small ($\leqq c_0 \log|\Omega|$).

In the next section we shall show how the partition function $Z(\mathfrak{M}^{+-}(\Omega, \mathrm{m}), \beta)$ can be written as in (1.3) where the surface term $2\tau_{\mathrm{B}} + \tau$ appears naturally split into two parts: one ($2\tau_{\mathrm{B}}$) coming from the bases of Ω and the other (τ) from the big contour which is present in every configuration of $\tilde{\mathfrak{M}}_0^{+-}(\Omega, \mathrm{m})$. This fact will provide further evidence to the interpretation of the phenomenon we are describing as phase separation and of λ as a (random) phase-separating surface. At the same time we shall have succeeded in providing a natural description of surface tension on purely statistical mechanical grounds.

The technique used to prove (3.4) is essentially the same as that of MS. In appendix B we report the necessary modifications (most of which are rather trivial for the reader familiar with their pioneering work).

We have written Appendix A mainly to provide some necessary definitions and also to describe a slightly different way of deriving the contour correlation functions which seems simpler than that of MS.

4. The Surface Tension

To establish the asymptotic formulas (1.3) and (1.4) and to prove that the limit (1.5) exists we are going to compare the two partition functions $Z(\mathfrak{M}^{+-}(\Omega, \mathrm{m}), \beta)$ and $Z(\mathfrak{M}^{++}(\Omega, \mathrm{m}^*), \beta)$ to that of the ensemble $\mathfrak{M}_0^{++}(\Omega)$ of configurations without big contours using a method similar to that used by Bellemans [2] to extract the surface contributions. In Appendix B, Lemma 3, we prove that

$$\lim_{\mathrm{N}\to\infty} Z(\mathfrak{M}_0^{++}(\Omega, \mathrm{m}^*), \beta)/Z(\mathfrak{M}^{++}(\Omega, \mathrm{m}^*), \beta) = 1 \tag{4.1}$$

and in Lemma 1 that

$$1 \geqq Z(\mathfrak{M}_0^{++}(\Omega, \mathrm{m}^*), \beta)/Z(\mathfrak{M}_0^{++}(\Omega), \beta) \geqq \mathrm{D}(\beta)\,|\Omega|^{-1/2} e^{-\mathrm{R}(\beta)\mathrm{N}^{1/2}}. \tag{4.2}$$

These relations imply that in (1.4) and (1.5) we can replace $\mathfrak{M}^{++}(\Omega, \mathrm{m}^*)$ by $\mathfrak{M}_0^{++}(\Omega)$ without changing τ or τ_{B}. We first show that

$$\tau = \lim_{\mathrm{N}\to\infty} \mathrm{N}^{-1} \log Z(\mathfrak{M}^{+-}(\Omega, \mathrm{m}), \beta)/Z(\mathfrak{M}_0^{++}(\Omega), \beta)$$

exists and is independent of m for $|\mathrm{m}| < \mathrm{m}^*$. The proof proceeds in several steps in each of which the partition functions are replaced by simpler objects. At the end of the process τ will appear as the limit of a partition function of a certain "one-dimensional" system. Formula (1.4) will be obtained as a byproduct in the proof of (1.5).

The reader who wants to keep in touch with the physical meaning of the various steps should keep in mind that the results of Appendix A are essentially a rigorous form of the droplet theory of phase transitions. From this point of view all the steps of the proof of (1.5) will appear very natural. In fact the droplet theory was for us the guide to understanding what results we should try to prove.

Let us first remark that we can use the ensemble $\tilde{\mathfrak{M}}_0^{+-}(\Omega, \mathrm{m})$ introduced in the previous section instead of $\mathfrak{M}^{+-}(\Omega, \mathrm{m})$ to compute the limit (1.5) because of (3.4).

If λ denotes the big contour of a configuration $\mathrm{X} \in \tilde{\mathfrak{M}}_0^{+-}(\Omega, \mathrm{m})$ we can write:

$$Z(\tilde{\mathfrak{M}}_0^{+-}(\Omega, \mathrm{m}), \beta) = \sum_{\lambda} \sum_{\gamma_1 \ldots \gamma_n} e^{-\beta|\lambda| - \beta \sum_i |\gamma_i|} \tag{4.3}$$

where the first sum runs over the set of big contours λ for which there is at least one configuration $\mathrm{X} \in \tilde{\mathfrak{M}}_0^{+-}(\Omega, \mathrm{m})$ containing λ as the big contour.

We can write (4.3) as:

$$Z(\tilde{\mathfrak{M}}_0^{+-}(\Omega, \mathrm{m}), \beta) = \sum_{\lambda} e^{-\beta|\lambda|} \sum_{\mathrm{m}^+, \mathrm{m}^-} Z(\mathfrak{M}_0^{++}(\Omega_\lambda, \mathrm{m}^+), \beta)\, Z(\mathfrak{M}_0^{--}(\Omega - \Omega_\lambda, \mathrm{m}^-), \beta) \tag{4.4}$$

where the magnetizations in the two regions are restricted as in (3.3):

$$\mathrm{m}^+ = \frac{\mathrm{m}^* \alpha |\Omega|}{|\Omega_\lambda|} + \varepsilon \frac{|\Omega|^p}{|\Omega_\lambda|}, \qquad \mathrm{m}^- = -\frac{\mathrm{m}^*(1-\alpha)|\Omega|}{|\Omega - \Omega_\lambda|} - \varepsilon \frac{|\Omega|^p}{|\Omega - \Omega_\lambda|} \tag{4.5}$$

with $|\varepsilon| \leqq \varkappa(\beta)$.

A lower bound on $Z(\tilde{\mathfrak{M}}_0^{+-}(\Omega, \mathrm{m}), \beta)$ can be obtained by restricting the sums on the r.h.s. of (4.4). For each big contour λ appearing in (4.4) consider the set of its vertical translates. If we translate λ one step in the vertical direction the area of the region Ω_λ above it changes by N. Therefore, for each big contour we can find a congruent one λ, such that:

$$0 \leqq |\Omega_\lambda| - \alpha|\Omega| < \mathrm{N}\,. \tag{4.6}$$

We construct a lower bound on (4.4) by summing only over λ : s fulfilling (4.6). We also restrict the summation over ε by requiring that

$$\mathrm{m}^- = -\mathrm{m}^* \tag{4.7}$$

or equivalently, if we define ϱ by:

$$|\Omega_\lambda| = (1+\varrho)\,\alpha|\Omega|\,, \tag{4.8}$$

$$\varepsilon = -\mathrm{m}^* \varrho \alpha |\Omega|^{1-\mathrm{p}}\,. \tag{4.9}$$

For this choice of ε we get $\mathrm{m}^+ = \mathrm{m}^* \dfrac{1+\varrho}{1-\varrho}$. This choice is allowed if $|\Omega|$ is large enough, because $|\varepsilon| \leq \mathrm{m}^* \alpha |\Omega|^{1-\mathrm{p}} \mathrm{N}(\alpha|\Omega|)^{-1} = \mathrm{m}^* \mathrm{N}^{1-\mathrm{p}(1+\delta)} \leqq \mathrm{m}^* \mathrm{N}^{-1/2}$ so $|\varepsilon| \leqq \varkappa(\beta)$ for N large. (We have used the fact that $|\Omega| = \mathrm{N}^{1+\delta}$ and $\delta \geqq 1$, $\mathrm{p} \geqq 3/4$.)

Let us compare the lower bound $\tilde{Z}(\tilde{\mathfrak{M}}_0^{+-}(\Omega, \mathrm{m}))$ on (4.4) obtained as indicated above to the quantity:

$$Z_0 = \sum_{\lambda} e^{-\beta|\lambda|} Z(\mathfrak{M}_0^{++}(\Omega_\lambda), \beta)\, Z(\mathfrak{M}_0^{--}(\Omega - \Omega_\lambda), \beta) \tag{4.10}$$

where the sum in (4.10) is also restricted by (4.6), (3.1).

Let Z_1 be defined as Z_0 but with the less stringent restrictions (3.2), (3.1). We then have:

$$
\begin{aligned}
\frac{Z_1}{Z_0} &\geqq \frac{Z_1}{Z_0}\,\frac{Z(\tilde{\mathfrak{M}}_0^{+\,-}(\Omega,\mathrm{m}),\beta)}{Z_1} \geqq \frac{\tilde{Z}(\tilde{\mathfrak{M}}_0^{+\,-}(\Omega,\mathrm{m}))}{Z_0}\\
&= \frac{\sum_\lambda e^{-\beta|\lambda|} Z\left(\mathfrak{M}_0^{+\,+}\left(\Omega_\lambda, \mathrm{m}^*\left(\frac{1+\varrho}{1-\varrho}\right)\right),\beta\right) Z(\mathfrak{M}_0^{-\,-}(\Omega-\Omega_\lambda, -\mathrm{m}^*),\beta)}{\sum_\lambda e^{-\beta|\lambda|} Z(\mathfrak{M}_0^{+\,+}(\Omega_\lambda),\beta)\, Z(\mathfrak{M}_0^{-\,-}(\Omega-\Omega_\lambda),\beta)}\\
&\geqq \inf_\lambda \frac{Z\left(\mathfrak{M}_0^{+\,+}\left(\Omega_\lambda, \mathrm{m}^*\left(\frac{1+\varrho}{1-\varrho}\right)\right),\beta\right)}{Z(\mathfrak{M}_0^{+\,+}(\Omega_\lambda),\beta)}\,\frac{Z(\mathfrak{M}_0^{-\,-}(\Omega-\Omega_\lambda, -\mathrm{m}^*),\beta)}{Z(\mathfrak{M}_0^{-\,-}(\Omega-\Omega_\lambda),\beta)}\\
&\geqq \frac{\mathrm{D}^2(\beta)}{|\Omega|}\, e^{-2\mathrm{R}(\beta)\mathrm{N}^{1/2}}
\end{aligned}
\tag{4.11}
$$

where we have used the fact that $|\varrho| \leqq \mathrm{N}/\alpha|\Omega|$ and Lemma 1 in Appendix B. Formula (4.11) shows that in the computation of (4.2) we can replace $Z(\mathfrak{M}^{+\,-}(\Omega,\mathrm{m}),\beta)$ by Z_0 if $\mathrm{N}^{-1}\log Z_1/Z_0 \to 0$ as $\mathrm{N}\to\infty$. Next we prove this and show how to compare Z_0 to $Z(\mathfrak{M}_0^{+\,+}(\Omega),\beta)$.

In Appendix A we prove that one can define a function $\varphi^T(\mathrm{X})$ defined for finite contour configurations X on the infinite cylinder such that the following "virial expansion" is valid:

$$
Z(\mathfrak{M}_0^{+\,+}(\Theta),\beta) = \exp \sum_{\mathrm{X}\subset\Theta} \varphi^T(\mathrm{X}) \tag{4.12}
$$

for any region Θ. ($\mathrm{X}\subset\Theta$ means that all the different contours in X lie in Θ.) Moreover, $\varphi^T(\mathrm{X})$ satisfies the estimates:

$$
\begin{aligned}
&\sum_{\gamma\in\mathrm{X}} |\varphi^T(\mathrm{X})| \leqq \xi(\beta) e^{-\frac{\beta}{2}|\gamma|}\\
&\sum_{\substack{\mathrm{p}\in\mathrm{X}\\ \mathrm{q}\in\mathrm{X}}} |\varphi^T(\mathrm{X})| \leqq F e^{-2\beta}\left(4e^{-\frac{\beta}{2}}\right)^{\mathrm{d}^{1/2}}
\end{aligned}
\tag{4.13}
$$

where p and q are two points at distance d (see (A.21), (A.25)). The importance of (4.12), (4.13) lies in the fact that $\varphi^T(\mathrm{X})$ does *not* depend on Θ. (Neither does it depend on the diameter of the cylinder unless X encircles it. $\varphi^T(\mathrm{X})$ is also translation invariant, see (A.20).)

Using (4.12) we can write:

$$
Z(\mathfrak{M}_0^{+\,+}(\Omega_\lambda),\beta) = \exp \sum_{\mathrm{X}\subset\Omega_\lambda} \varphi^T(\mathrm{X}), \tag{4.14}
$$

$$
Z(\mathfrak{M}_0^{-\,-}(\Omega-\Omega_\lambda),\beta) = \exp \sum_{\mathrm{X}\subset\Omega-\Omega_\lambda} \varphi^T(\mathrm{X}) \tag{4.15}
$$

hence:

$$\begin{aligned}Z(\mathfrak{M}_0^{++}(\Omega_\lambda),\beta)\,Z(\mathfrak{M}_0^{--}(\Omega-\Omega_\lambda),\beta)&\\ = Z(\mathfrak{M}_0^{++}(\Omega),\beta)\exp-\sum_{\substack{\mathrm{X}i\lambda\\ \mathrm{X}\subset\Omega}}\varphi^T(\mathrm{X})&\end{aligned}\tag{4.16}$$

(Here $\mathrm{X}i\lambda$ means that X intersects λ, and we have used the fact that $\varphi^T(\mathrm{X})=0$ if X consists of two non-overlapping pieces (A.20).) Therefore if we define:

$$\mu_\Omega(\lambda)=\sum_{\substack{\mathrm{X}i\lambda\\ \mathrm{X}\subset\Omega}}\varphi^T(\mathrm{X})\tag{4.17}$$

we have proved that:

$$Z_0=Z(\mathfrak{M}_0^{++}(\Omega),\beta)\Big(\sum_\lambda e^{-\beta|\lambda|-\mu_\Omega(\lambda)}\Big)\tag{4.18}$$

and similarly for Z_1. We now observe that in these sums the distance of any λ to the bases of Ω is larger than const N^δ because of the restrictions (3.1), (4.6) and (3.2), so that if we define:

$$\mu(\lambda)=\sum_{\mathrm{X}i\lambda}\varphi^T(\mathrm{X})\tag{4.19}$$

we find (using (4.13)) that:

$$|\mu(\lambda)-\mu_\Omega(\lambda)|=o(\mathrm{N})\tag{4.20}$$

uniformly in λ. We have therefore found that:

$$Z_0=Z(\mathfrak{M}_0^{++}(\Omega),\beta)\Big(\sum_\lambda e^{-\beta|\lambda|-\mu(\lambda)}\Big)e^{o(N)}\tag{4.21}$$

and similarly for Z_1. We now remark that $\mu(\lambda)$ depends only on the shape of λ. Therefore if we say that two contours are equivalent when they are congruent modulo a vertical translation and denote the equivalence class of λ by (λ) we find that:

$$Z_0\geqq Z(\mathfrak{M}_0^{++}(\Omega),\beta)\Big(\sum_{\substack{(\lambda)\\ |\lambda|\leqq \mathrm{N}(1+\ell(\beta))}} e^{-\beta|\lambda|-\mu(\lambda)}\Big)e^{o(\mathrm{N})},\tag{4.22}$$

$$Z_0\leqq \mathrm{H}Z(\mathfrak{M}_0^{++}(\Omega),\beta)\Big(\sum_{\substack{(\lambda)\\ |\lambda|\leqq N(1+\ell(\beta))}} e^{-\beta|\lambda|-\mu(\lambda)}\Big)e^{o(N)}\tag{4.23}$$

and similarly for Z_1. We thus see that $1\leqq Z_1/Z_0\leqq \mathrm{H}e^{o(N)}$, so that $\lim\limits_{\mathrm{N}\to\infty}\mathrm{N}^{-1}\log Z_1/Z_0=0$, and we can also conclude that in computing the limit (4.2) we can not only replace $Z(\mathfrak{M}^{+-}(\Omega,\mathrm{m}),\beta)$ by Z_0 but also by:

$$Z(\mathfrak{M}_0^{++}(\Omega),\beta)\Big(\sum_{\substack{(\lambda)\\ |\lambda|\leqq \mathrm{N}(1+\ell(\beta))}} e^{-\beta|\lambda|-\mu(\lambda)}\Big)\equiv Z(\mathfrak{M}_0^{++}(\Omega),\beta)\tilde{Z}_0\tag{4.24}$$

(4.24) proves that the limit (4.2), if it exists, is m-independent and is equal to:

$$\tau = \lim_{N\to\infty} \frac{1}{N} \log \tilde{Z}_0 . \tag{4.25}$$

To prove the existence of (4.25) we proceed as follows. Consider a contour $\lambda \in (\lambda)$. Since λ is wrapped around the cylinder Ω with base N there must be at least $N(1-\ell(\beta))$ columns of the cylinder Ω which intersect the contour in just one point. (The columns we are talking about are columns of the lattice in Ω where the spins are sitting. Therefore each column intersects the contour in the middle of a straight portion.)

For each such "regular" column consider the $N^{1/3}$ columns which are adjacent to it ($N^{1/3}/2$ to the left and $N^{1/3}/2$ to the right). Call the shortest length of λ contained in any of these strips l. If M is the maximum number of disjoint such strips then $lM \leqq$ total length in the strips $\leqq N(1+\ell(\beta))$. But a maximal family of M strips, centered at the columns $c_1, \ldots, c_M$ say, has the property that any regular column has a distance at most $N^{1/3}/2$ from it, so the union of the strips of width $2N^{1/3}$ centered at $c_1, \ldots, c_M$ contain all the regular columns, and we can conclude that $M(2N^{1/3}) \geqq \#$ of columns in this union $\geqq N(1-\ell(\beta))$. We thus see that $M \geqq N(1-\ell(\beta))/2N^{1/3}$ and

$$l \leqq N(1+\ell(\beta))/M \leqq 2N^{1/3}(1+\ell(\beta))/(1-\ell(\beta)) < N^{1/2}$$

if N is large, so we can always pick one regular column whose associated strip contains not more than a portion $N^{1/2}$ of the length of λ. This can be done for any λ.

Suppose that instead of considering the set of big contours on a cylinder we consider the set of contours C_N which walk on a planar lattice from the origin to the point $(N, 0)$ through a selfavoiding walk with $\leqq N(1+\ell(\beta))$ steps, starting horizontally and not leaving the vertical strip with base from $(0,0)$ to $(N,0)$ and such that the strips of width $N^{1/3}/2$ to the right of $(0,0)$ and to the left of $(N,0)$ do not contain more than a portion $N^{1/2}$ long of λ. To each $\lambda \in C_N$ clearly corresponds a class of big contours on the cylinder with base N. The number of elements $\lambda \in C_N$ which give rise to the same class is at most N.

Define for $\lambda \in C_N$:

$$\mu^{\sigma}(\lambda) = \sum_{\substack{X i \lambda \\ X \subset \lambda^+ \cup \lambda^-}} \varphi^T(X) \tag{4.26}$$

where λ^+ and λ^- are the regions into which λ divides the strip with base $(0,0), (N,0)$. If λ denotes also the big contour on the cylinder associated to λ we see that:

$$|\mu(\lambda)-\mu^0(\lambda)|\leqq O(\mathrm{N}^{1/2}+e^{-\beta \mathrm{N}^{1/2}}) \tag{4.27}$$

(we use here (4.13)). Hence:

$$\tilde{\tilde{Z}}_0(\mathrm{N})=\sum_{\lambda\in C_\mathrm{N}} e^{-\beta|\lambda|-\mu^0(\lambda)} \tag{4.28}$$

is such that: $e^{o(\mathrm{N})}\tilde{\tilde{Z}}_0\leqq\tilde{Z}_0\leqq \mathrm{N}e^{o(\mathrm{N})}\tilde{\tilde{Z}}_0$ and therefore to study the limit (4.25) we can use $\tilde{\tilde{Z}}_0$ instead of $\tilde{Z}_0$.

We do this by relating $\tilde{\tilde{Z}}_0(\mathrm{N}+\mathrm{M})$ to $\tilde{\tilde{Z}}_0(\mathrm{N})\,\tilde{\tilde{Z}}_0(\mathrm{M})$. This product is a sum of terms corresponding to pairs $(\lambda_\mathrm{N},\lambda_\mathrm{M})\in C_\mathrm{N}\times C_\mathrm{M}$. To each such pair we can associate a λ in $C_{\mathrm{N}+\mathrm{M}}$ by the following construction: Join λ_N and λ_M to form a big contour on a cylinder with circumference $\mathrm{N}+\mathrm{M}$. By the above argument there is a regular column on this cylinder such that the restriction on the length in its associated strip of width $(\mathrm{N}+\mathrm{M})^{1/3}$ is fulfilled. If we choose such a regular column according to some rule and open the contour at its location we get a $\lambda=\mathrm{F}(\lambda_\mathrm{N},\lambda_\mathrm{M})$ in $C_{\mathrm{N}+\mathrm{M}}$. This mapping is such that at most $\mathrm{N}+\mathrm{M}$ pairs are mapped on the same λ, and also $\mu^0(\lambda)$ is "almost" the sum of $\mu^0(\lambda_\mathrm{N})$ and $\mu^0(\lambda_\mathrm{M})$:

$$|\mu^0(\lambda_\mathrm{N})+\mu^0(\lambda_\mathrm{M})-\mu^0(\lambda)|\leqq O(\mathrm{N}+\mathrm{M})^{1/2}.$$

We then get

$$\begin{aligned}\tilde{\tilde{Z}}_0(\mathrm{N})\,\tilde{\tilde{Z}}_0(\mathrm{M})&\leqq\sum_{\lambda\in C_{\mathrm{N}+\mathrm{M}}}\#\,\mathrm{F}^{-1}(\lambda)\,e^{-\beta|\lambda|-\mu^0(\lambda)+O(\mathrm{N}+\mathrm{M})^{1/2}}\\&\leqq(\mathrm{N}+\mathrm{M})\,e^{O(\mathrm{N}+\mathrm{M})^{1/2}}\,\tilde{\tilde{Z}}_0(\mathrm{N}+\mathrm{M})\end{aligned} \tag{4.29}$$

which implies that the limit (4.25) exists using the well known techniques to prove the existence of a thermodynamic limit (using subadditivity arguments).

We also observe that if we define:

$$\mathrm{Q}(\varepsilon,\beta)=\sum_{\substack{(\lambda)\\|\lambda|=\mathrm{N}(1+\varepsilon)}} e^{-\mu(\lambda)} \tag{4.30}$$

then for small ε $(\varepsilon\leqq\ell(\beta))$

$$\mathscr{g}(\varepsilon,\beta)=\lim_{\mathrm{N}\to\infty}\frac{1}{\mathrm{N}}\log\mathrm{Q}(\varepsilon,\beta) \tag{4.31}$$

exists (as a consequence of a subadditivity argument very similar to the one discussed above), and

$$\tau(\beta)=\sup_{0\leqq\varepsilon\leqq\ell(\beta)}(\mathscr{g}(\varepsilon,\beta)-\beta\varepsilon)-\beta\,. \tag{4.32}$$

Since $\mathscr{g}(\varepsilon, \beta) \approx -\varepsilon \log \varepsilon$ for small ε (see Appendix C, and use $|\mu(\lambda)| \leqq |\lambda| \, \delta(\beta)$ following from (A.17)), we realize that the maximum is attained at $\varepsilon_0 = O(e^{-\beta})$ and one could check that in the region of β in which our estimates allow a proof of (4.25) $\ell(\beta) \gg e^{-\beta}$. This fact could be used to improve quite strongly the picture of the typical configurations by proving that in $\mathfrak{M}^{+-}(\Omega, m)$ the length of the big contour is such that $|\lambda|/N \to 1 + \varepsilon_0$ in probability.

Finally formula (1.4) for $\mathfrak{M}_0^{++}(\Omega)$ follows from (4.14) by replacing there Ω_λ with Ω and by using (4.13). We do not perform the straightforward calculation.

5. Concluding Remarks

As we have just remarked the maximum in (4.32) is obtained for $\varepsilon_0 = O(e^{-\beta})$ when β is large. Formula (4.32) therefore implies that:

$$\tau = -\beta + O(e^{-\beta}). \tag{5.1}$$

This is in agreement with Onsager's result for:

$$\tau = -\beta - \log \operatorname{th} \frac{\beta}{2}. \tag{5.2}$$

It would be interesting to find a "Mayer" expansion for $\tau(\beta)$ in powers of $e^{-\beta}$. This would allow a more precise comparison with Onsager's result.

The way Onsager defines and computes (5.2) is the following. Consider instead of a ferromagnetic Ising model with coupling $+\mathrm{J}$ an antiferromagnetic model with coupling $-\mathrm{J}$; consider the model in a periodic box. Let N and M be the length of the sides of this box and assume M is even and N is odd. Since the model is antiferromagnetic and N is odd there is going to be somewhere in Ω a mismatch in the antiferromagnetic arrangement of the spins (we are assuming that the temperature is low enough so that the system is in an antiferromagnetic state). Since the interaction is nearest neighbour the situation should be the same as the one obtained by putting together two oppositely magnetized phases in the ferromagnetic Ising model. Therefore one will be able to interpret the difference in free energy when N is even and when N is odd as due to the phase separation and therefore as a surface tension. Using this definition Onsager computed the value (5.2) for τ.

Onsager's derivation is very natural, however one cannot yet prove that when N is odd and $\mathrm{J} > 0$ the situation is really the same as when $\mathrm{J} < 0$ and two oppositely magnetized phases are in contact in the sense of this paper. A technical difficulty for such a proof is the lack of a MS type of result valid for periodic boundary conditions. A more serious difficulty seems to be the fact that in Onsager's formulation one works in the grand canonical formalism, and therefore the partition function is the sum of canonical partition functions with arbitrary magnetization. In particular there will be contributions from the magnetizations very close to $\pm m^*$ and, as we have seen in the preceding sections, this complicates a proof of the equivalence of Onsager's definition of surface tension and our definition. (We observe that we have not dealt with the phenomenon of the interference of the surface of separation and the boundaries of the box Ω, which are very close to each other when m is very close to $\pm m^*$.)

Finally we mention that the MS type of results, if appropriately reformulated, hold also for more general Ising models (for instance for finite range interactions of negative strength). It is probably possible to generalize the results of this paper to this more general situation. In *particular one can define in general the surface tension as:*

$$\tau = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z(\mathfrak{M}^{+-}(\Omega, m), \beta)}{Z(\mathfrak{M}^{++}(\Omega), \beta)} \tag{5.3}$$

where Ω is a cylinder with base N and height of the order N^δ, $\delta > 1$. The ensembles $\mathfrak{M}^{+-}(\Omega, m)$ and $\mathfrak{M}^{++}(\Omega)$ are obtained by putting all the spins above $\Omega = +$ and all the ones below $= -$. Ω has to be taken with horizontally periodic boundary conditions. Using the more general MS results one can probably prove the existence of the limit in (5.3) and the fact that it comes from a surface of separation of two phases (i.e. the results analogous to Section 3 and 4 of this paper). Although the MS results on the phase separation and the estimates for the contour correlation functions extend rather directly to the 3-dimensional case we do not think that the extension of the proof of the existence of the surface tension is as straight-forward because the "joining operation" used in the proof, which is simple for lines of separation, becomes more complicated for surfaces of separation.

Appendix A

In this appendix we shall deal with a number of technical points. Although the technique is rather standard and the results are essentially the same as the ones in Ref. [7] we give here the derivation in some detail, because the method we use seems to be considerably simpler than the one used by MS in [8] (although the starting point is their integral equation for the contour correlations) and also because it seems easier than pointing out the several slight changes needed to adapt the MS results to our case.

Consider a space F of functions defined on the finite sets of contours small or large but not big contained in a infinitely long cylinder. The sets of contours we are considering here are not subject to the restriction that the contours should not overlap; we even allow them to coincide. If $\varphi \in \mathrm{F}$ then φ associates to every set X of contours a number $\varphi(\mathrm{X})$. If $\varphi \in \mathrm{F}$ then we call $|\varphi|_n = \sup_{\gamma_1 \dots \gamma_n} |\varphi(\gamma_1, \dots, \gamma_n)|$ and we suppose that $|\varphi|_n < \infty$ for all the functions in F. We call $N(\mathrm{X})$ the number of contours in X.

If $\varphi_1, \varphi_2 \in \mathrm{F}$ we define their convolution product $\varphi_1 \cdot \varphi_2 \in \mathrm{F}$ as:

$$(\varphi_1 \cdot \varphi_2)(\mathrm{X}) = \sum_{\mathrm{X}_1 \cup X_2 = \mathrm{X}} \varphi_1(\mathrm{X}_1)\, \varphi_2(\mathrm{X}_2) \tag{A.1}$$

here X is a general set of contours and is determined by the set of different contours in it and by their multiplicities. The sum $\sum_{\mathrm{X}_1 \cup \mathrm{X}_2 = \mathrm{X}}$ is to be regarded as the sum over the ordered couples $\mathrm{X}_1, \mathrm{X}_2$ which decompose X into two sets of contours.

Let us now define the exponential of a function $\varphi \in \mathrm{F}_0$ where $\mathrm{F}_0 = \{\varphi; \varphi \in F, \varphi(\emptyset) = 0\}$:

$$(\operatorname{Exp}\varphi)(\mathrm{X}) = \sum_{n \geqq 0} \frac{\varphi^{n\cdot}(\mathrm{X})}{n!} \tag{A.2}$$

where we have put $\varphi^{0\cdot}(\mathrm{X}) = \mathbf{1}(\mathrm{X})$, $\mathbf{1}(\mathrm{X}) = 0$ if $\mathrm{X} \neq \emptyset$, $\mathbf{1}(\emptyset) = 1$. So:

$$(\operatorname{Exp}\varphi)(\mathrm{X}) = \mathbf{1}(\mathrm{X}) + \sum_{n \geqq 1} \frac{1}{n!} \sum_{\cup_\iota \mathrm{X}_\iota = \mathrm{X}} \varphi(X_1) \dots \varphi(\mathrm{X}_n)\,. \tag{A.3}$$

The exponential is well defined since the sum runs over a finite set of indices.

One can define an inverse function to the exponential over the set $\mathrm{F}_1 = \{\varphi_1; \varphi_1 \in F, \varphi_1(\emptyset) = 1\}$: if $\varphi_1 = \mathbf{1} + \varphi$, $\varphi \in F_0$

$$\begin{aligned}(\operatorname{Log}\varphi_1)(\mathrm{X}) &= \sum_{n \geqq 1} \frac{(-1)^{n+1}}{n} \varphi^{n\cdot}(\mathrm{X}) \\ &= \sum_{n \geqq 1} \frac{(-1)^{n+1}}{n} \sum_{\cup_\iota \mathrm{X}_\iota = \mathrm{X}} \varphi(X_1) \dots \varphi(\mathrm{X}_n)\end{aligned} \tag{A.4}$$

Observe that, again, the sum runs over a finite number of indices. We have also that $\operatorname{Exp}\,\operatorname{Log}\varphi_1 = \varphi_1$.

We define on F the operation $\mathrm{D_X}$ as:

$$(\mathrm{D_X}\varphi)(\mathrm{Y}) = \varphi(\mathrm{X} \cup \mathrm{Y})\,. \tag{A.5}$$

This operation has the property that:

$$D_\gamma(\varphi_1 \cdot \varphi_2) = (D_\gamma \varphi_1) \cdot \varphi_2 + \varphi_1 \cdot (D_\gamma \varphi_2)$$

$$D_\gamma(\operatorname{Exp}\varphi) = (D_\gamma \varphi) \cdot \operatorname{Exp}\varphi \tag{A.6}$$

$$D_{\mathrm{X}}(\operatorname{Exp}\varphi) = \left(\sum_{n \geq 1} \frac{1}{n!} \sum_{\underset{\iota}{\cup} X_\iota = X} (D_{\mathrm{X}_1}\varphi) \dots (D_{\mathrm{X}_n}\varphi) \right) \cdot \operatorname{Exp}\varphi .$$

Finally we find that the important formula below holds:

$$\sum_{\mathrm{X}} (\operatorname{Exp}\varphi)(\mathrm{X})\, \chi(\mathrm{X}) = \exp \sum_{\mathrm{X}} \varphi(\mathrm{X})\, \chi(\mathrm{X}) \tag{A.7}$$

if $\sum_{\mathrm{X}} |\varphi(\mathrm{X})\, \chi(\mathrm{X})| < \infty$ and if χ is multiplicative, i.e. if $\chi(\gamma_1, \dots, \gamma_n) = \chi(\gamma_1) \dots \chi(\gamma_n)$. Especially we are going to use (A.7) when

$$\chi(\gamma) = \begin{cases} 1 \text{ if } \gamma \text{ lies in some region } \Theta \\ 0 \text{ otherwise} \end{cases}$$

so that (A.7) becomes:

$$\sum_{\mathrm{X} \subset \Theta} (\operatorname{Exp}\varphi)(\mathrm{X}) = \exp \sum_{\mathrm{X} \subset \Theta} \varphi(\mathrm{X}) . \tag{A.8}$$

Consider now the particular function $\varphi \in F_1$:

$$\varphi(\mathrm{X}) = \begin{cases} e^{-\beta \sum_\iota |\gamma_\iota|} & \text{if X consists of mutually} \\ & \text{compatible contours, } \mathrm{X} = (\gamma_1, \dots, \gamma_n) \\ 0 & \text{otherwise} \end{cases} \tag{A.9}$$

$\varphi(\emptyset) = 1$; therefore $\varphi^{\mathrm{T}} = \operatorname{Log}\varphi$ is defined.

We now wish to investigate the X dependence of $\varphi^{\mathrm{T}}(\mathrm{X})$. Consider for this purpose the function $\varphi^{-1} \in \mathrm{F}$ and the functions:

$$\Delta_{\mathrm{X}}(\mathrm{Y}) = (\varphi^{-1} \cdot D_{\mathrm{X}} \varphi)(\mathrm{Y}) \tag{A.10}$$

where X and Y are sets of contours: φ^{-1} is the inverse in the sense of the product (A.1), i.e. $\varphi^{-1} \cdot \varphi = \varphi \cdot \varphi^{-1} = \mathbf{1}$, ($\varphi^{-1}$ is well defined if $\varphi(\emptyset) \neq 0$).

Let $\gamma \cup \mathrm{X}$ be a set of contours without overlappings. Then we can write an equation for $\Delta_{\gamma \cup \mathrm{X}}(\mathrm{Y})$ for Y arbitrary and X without overlappings along the lines of Ref. [5]:

$$\Delta_{\gamma \cup \mathrm{X}}(\mathrm{Y}) = e^{-\beta |\gamma|} \sum_{\mathrm{S} \subset \mathrm{Y}}^{*} (-1)^{\mathrm{N(S)}} \Delta_{\mathrm{X} \cup \mathrm{S}}(\mathrm{Y} \setminus \mathrm{S}) . \tag{A.11}$$

Here Y\S denotes the complement of S in Y. Σ^* indicates that the summation is extended over all subsets S of Y such that all the contours in S intersect γ and $X \cup S$ is a set of contours without intersections. ($\Delta_X(Y) = 0$ if X has intersections, which can be seen from (A.10).) The set $S = \emptyset$ has also to be included in the summation, and $\Delta_\emptyset(Y) = \mathbf{1}(Y)$.

Put:

$$I_m = \sup_{\substack{\gamma_1, \ldots, \gamma_n \\ m \geqq n \geqq 1}} \sum_{\substack{Y \\ N(Y) = m - n}} |\Delta_{\gamma_1, \ldots, \gamma_n}(Y)| \, e^{\frac{\beta}{2} \sum_i |\gamma_i|} . \tag{A.12}$$

Then we deduce from (A.11) that for some $c > 0$:

$$\begin{aligned}
&\sum_{\substack{Y \\ N(Y) + N(X) = m}} |\Delta_{\gamma \cup X}(Y)| \, e^{\frac{\beta}{2}(|\gamma| + |X|)} \\
&\leqq \sum_{\substack{Y \\ N(Y) + N(X) = m}} \sum_{S \subset Y}{}^{*} |\Delta_{X \cup S}(Y \setminus S)| \, e^{\frac{\beta}{2}(|\gamma| + |X|) - \beta|\gamma|} \\
&\leqq \sum_{S}{}^{*} I_m e^{-\frac{\beta}{2}(|\gamma| + |S|)} \leqq I_m e^{-\frac{\beta}{2}|\gamma|} \sum_{n \geqq 0} \frac{1}{n!} \Bigg(\sum_{\sigma \text{ inters.} \gamma} e^{-\frac{\beta}{2}|\sigma|} \Bigg)^n \qquad (A.13) \\
&\leqq I_m e^{-\frac{\beta}{2}|\gamma|} \exp \Bigg(\sum_{p \in \gamma} \sum_{p \in \sigma} e^{-\frac{\beta}{2}|\sigma|} \Bigg) \leqq I_m e^{-\frac{\beta}{2}|\gamma|} \exp \Bigg(|\gamma| \sum_{l=4}^{\infty} \Big(3 e^{-\frac{\beta}{2}} \Big)^l \Bigg) \\
&\leqq I_m \exp |\gamma| \left(-\frac{\beta}{2} + c e^{-\beta} \right)
\end{aligned}$$

if β is not too small. (Here and repeatedly in the following we use the fact that the number of different contours of length l that go through a point is less than 3^l.) Because $|\gamma| \geqq 4$ we can thus conclude that:

$$I_{m+1} \leqq I_m e^{-\beta} \tag{A.14}$$

if β is large enough. Since:

$$\begin{aligned}
I_1 &= \sup_{\gamma} |\Delta_\gamma(\emptyset)| \, e^{\frac{\beta}{2}|\gamma|} = \sup_{\gamma} |(\varphi^{-1} \cdot D_\gamma \varphi)(\emptyset)| \, e^{\frac{\beta}{2}|\gamma|} \\
&= \sup_{\gamma} |\varphi(\gamma)| \, e^{\frac{\beta}{2}|\gamma|} = \sup_{l \geqq 4} e^{-\frac{\beta}{2}|l|} = e^{-2\beta}
\end{aligned} \tag{A.15}$$

we deduce from (A.14) that:

$$I_m \leqq e^{-\beta(m+1)} . \tag{A.16}$$

Furthermore $(\varphi^{-1}\cdot D_\gamma\varphi)(\mathrm{X})=\varphi^{\mathrm{T}}(\gamma\cup\mathrm{X})$, which follows from (A.6), and this implies (if β is large enough):

$$\begin{aligned}\sum_{\mathrm{p}\in\mathrm{X}}|\varphi^{\mathrm{T}}(\mathrm{X})| &\leqq \sum_{\mathrm{p}\in\gamma}\sum_{\mathrm{X}}|\varphi^{\mathrm{T}}(\gamma\cup\mathrm{X})| \leqq \sum_{\mathrm{p}\in\gamma}\sum_{\mathrm{m}=1}^{\infty}\sum_{\mathrm{N(X)=m}-1}|\Delta_\gamma(\mathrm{X})| \\ &\leqq \sum_{\mathrm{p}\in\gamma}\sum_{\mathrm{m}=1}^{\infty} e^{-\beta(\mathrm{m}+1)-\frac{\beta}{2}|\gamma|} \leqq e^{-2\beta}(1-e^{-\beta})^{-1}\sum_{\mathrm{l}=4}^{\infty}\left(3e^{-\frac{\beta}{2}}\right)^{\mathrm{l}}=\delta(\beta)\end{aligned} \tag{A.17}$$

where p is any fixed point and $\delta(\beta)\to 0$ exponentially fast.

Another result which can be proved from (A.11) by induction is that:

$$(-1)^{\mathrm{N(Y)}}\Delta_{\mathrm{X}}(\mathrm{Y})\geqq 0\,. \tag{A.18}$$

Hence

$$(-1)^{\mathrm{N(X)}-1}\varphi^T(\mathrm{X})\geqq 0\,. \tag{A.19}$$

We can obtain a formula for $\varphi^T(\mathrm{X})$ from the expression for the Ursell functions in terms of connected graphs usually used in the theory of the Mayer expansion. φ can be written $\varphi(\gamma_1,\ldots,\gamma_n)=e^{-\beta\sum_i|\gamma_i|}\prod_{i<j}\Theta(\gamma_i,\gamma_j)$, with

$$\Theta(\gamma,\gamma')=\begin{cases}1 & \text{if } \gamma,\gamma' \text{ compatible}\\ 0 & \text{otherwise}.\end{cases}$$

The product can be viewed as a Boltzmann factor of a pair interaction between the $\gamma:s$. φ^T can then be expressed in terms of the functions

$$f(\gamma,\gamma')=\Theta(\gamma,\gamma')-1=\begin{cases}-1 & \text{if } \gamma,\gamma' \text{ incompatible}\\ \ \ 0 & \text{otherwise}\end{cases}$$

$$\varphi^T(\gamma_1,\ldots,\gamma_n)=e^{-\beta\sum_i|\gamma_i|}\sum_C\prod_{\{\gamma',\gamma''\}\subset C}f(\gamma',\gamma''),$$

where $\sum_C$ denotes the sum over all connected graphs C with vertices $\gamma_1,\ldots,\gamma_n$ (Ref. [11]). If we construct the graph G with edges only between incompatible $\gamma:s$ having weights -1, we can write:

$$\varphi^{\mathrm{T}}(\gamma_1,\ldots,\gamma_n)=\text{const}\ e^{-\beta\sum_i|\gamma_i|}\sum_{\substack{C\subset G\\ C\ \text{Conn.}}}(-1)^{\#\text{ of edges in } C}. \tag{A.20}$$

From this expression we see that $\varphi^T(\gamma_1,\ldots,\gamma_n)=0$ if G is not connected, i.e. if $(\gamma_1,\ldots,\gamma_n)$ consists of two groups such that every γ in one is compatible with every γ in the other. We also see that φ^{T} is translation invariant and $\varphi^{\mathrm{T}}(\mathrm{X})$ does not depend on the fact that X is situated on a cylindrical lattice and not on a planar one unless X encircles the cylinder. Formulae (A.12), (A.16) imply the next (A.21):

$$\left|\sum_{X} \varphi^T(\gamma \cup X)\right| \leqq \sum_{X} |\varphi^T(\gamma \cup X)| \leqq e^{-\frac{\beta}{2}|\gamma|} e^{-2\beta}(1-e^{-\beta})^{-1}. \quad \text{(A.21)}$$

Next we need an upper bound on:

$$\varrho(p, Q) = \sum_{\substack{p \in X \\ X i Q}} |\varphi^T(X)| \quad \text{(A.22)}$$

where p is a point and Q a set of points on the lattice and XiQ means that X intersects Q.

Let d be the distance between p and Q. We divide the sum in (A.22) into two parts according to wether $N(X) > d^{1/2}$ or $N(X) \leqq d^{1/2}$. The first part can be estimated as in (A.17) using (A.16):

$$\begin{aligned} \text{first part} &\leqq \sum_{p \in \gamma} \sum_{N(X) \geqq d^{1/2}} |\varphi^T(\gamma \cup X)| \leqq \sum_{p \in \gamma} e^{-\frac{\beta}{2}|\gamma| - 2\beta} \sum_{m \geqq d^{1/2}} e^{-m\beta} \\ &\leqq \text{const}\, e^{-2\beta - \beta d^{1/2}}. \end{aligned} \quad \text{(A.23)}$$

If $N(X) \leqq d^{1/2}$ we can conclude that the longest contour in X, $\bar{\gamma}$, has a length $l \geqq d^{1/2}$ if $\varphi^T(X) \neq 0$. This is true because if $\varphi^T(X) \neq 0$ it follows from (A.20) that the contours in X form one overlapping group so that $d \leqq \text{length of } X \leqq lN(X) \leqq ld^{1/2}$, and $l \geqq d^{1/2}$. We also have $d(p, \bar{\gamma}) \leqq lN(X) \leqq ld^{1/2}$ for the same reason, so $\bar{\gamma}$ must intersect the square with side $2ld^{1/2}$ centered at p. The second part can therefore be estimated as follows using (A.21):

$$\begin{aligned} \text{second part} &\leqq \sum_{l \geqq d^{1/2}} \sum_{\substack{|\bar{\gamma}| = l \\ d(p, \bar{\gamma}) \leqq l d^{1/2}}} \sum_{X} |\varphi^T(\bar{\gamma} \cup X)| \\ &\leqq \sum_{l \geqq d^{1/2}} \sum_{\substack{|\bar{\gamma}| = l \\ d(p, \bar{\gamma}) \leqq l d^{1/2}}} \text{const}\, e^{-2\beta - \frac{\beta}{2}|\bar{\gamma}|} \leqq \text{const}\, e^{-2\beta} \sum_{l \geqq d^{1/2}} (2l^2 d)\left(3e^{-\frac{\beta}{2}}\right)^l \\ &\leqq \text{const}\, e^{-2\beta}\left(4e^{-\frac{\beta}{2}}\right)^{d^{1/2}} \end{aligned} \quad \text{(A.24)}$$

if β is large enough, and we have proved that for some constant F:

$$\varrho(p, Q) = \sum_{\substack{p \in X \\ X i Q}} |\varphi^T(X)| \leqq F e^{-2\beta}\left(4e^{-\frac{\beta}{2}}\right)^{d(p,Q)^{1/2}} \quad \text{(A.25)}$$

if β is not too small. Formulae (A.16), (A.17), (A.20), (A.21), (A.25) embody the main results of this appendix.

Inequality (A.17) together with (A.8) allows us to write:

$$Z(\mathfrak{M}_0^{++}(\Theta),\beta)=\sum_{X\subset\Theta}\varphi(\mathrm{X})=\exp\sum_{X\subset\Theta}\varphi^T(\mathrm{X}). \tag{A.26}$$

An important application of the above calculation is the following estimate for the probability $\pi_\Theta(\gamma)$ in $\mathfrak{M}_0^{++}(\Theta)$ that a given contour γ is an outer contour of a configuration $\mathrm{X}\in\mathfrak{M}_0^{++}(\Theta)$. (Let Θ' and Θ'' be the regions inside and outside γ respectively, and use (A.7) with $\chi'(\gamma')=1$ if $\gamma'\subset\Theta'$, $\chi'(\gamma')=0$ otherwise and $\chi''(\gamma'')=1$ if $\gamma''\subset\Theta''$ and γ'' does not surround γ, $\chi''(\gamma'')=0$ otherwise):

$$\begin{aligned}
\pi_\Theta(\gamma)&=P(\gamma\text{ is a contour and is not surrounded by any other one})\\
&=\sum_{\mathrm{X}',\mathrm{X}''}\varphi(\gamma\cup\mathrm{X}'\cup\mathrm{X}'')\,\chi'(\mathrm{X}')\,\chi''(\mathrm{X}'')/\sum_{\mathrm{X}\subset\Theta}\varphi(\mathrm{X})\\
&=e^{-\beta|\gamma|}\Big(\sum_{\mathrm{X}'}\varphi(\mathrm{X}')\,\chi'(\mathrm{X}')\Big)\Big(\sum_{\mathrm{X}''}\varphi(\mathrm{X}'')\,\chi''(\mathrm{X}'')\Big)/\sum_{\mathrm{X}\subset\Theta}\varphi(\mathrm{X}) \qquad\qquad (\mathrm{A}.27)\\
&=\exp\Big(-\beta|\gamma|+\sum_{\mathrm{X}'\subset\Theta'}\varphi^T(\mathrm{X}')+\sum_{\mathrm{X}''\subset\Theta''}\varphi^T(\mathrm{X}'')\,\chi''(\mathrm{X}'')-\sum_{\mathrm{X}\subset\Theta}\varphi^T(\mathrm{X})\Big)\\
&=\exp\Big(-\beta|\gamma|-\sum_{\substack{\mathrm{X}i\gamma\\ \mathrm{X}\subset\Theta}}\varphi^T(X)-\sum_{\substack{\mathrm{X}s\gamma\\ \mathrm{X}\subset\Theta}}\varphi^T(\mathrm{X})\Big)
\end{aligned}$$

($\mathrm{X}s\gamma$ means that X does not intersect γ but some contour in X surrounds γ). The first sum in the exponent can be estimated using (A.17)

$$\Big|\sum_{\substack{\mathrm{X}i\gamma\\ \mathrm{X}\subset\Theta}}\varphi^T(\mathrm{X})\Big|\leqq\sum_{\mathrm{p}\in\gamma}\sum_{\mathrm{p}\in\mathrm{X}}|\varphi^T(\mathrm{X})|\leqq|\gamma|\,\delta(\beta). \tag{A.28}$$

For the second we get using (A.21)

$$\Big|\sum_{\substack{\mathrm{X}s\gamma\\ \mathrm{X}\subset\Theta}}\varphi^T(\mathrm{X})\Big|\leqq\sum_{\gamma''s\gamma}\sum_{\mathrm{X}}|\varphi^T(\gamma''\cup\mathrm{X})|\leqq\delta(\beta)\sum_{\gamma''s\gamma}e^{-\frac{\beta}{2}|\gamma''|}. \tag{A.29}$$

To estimate the last sum we make an auxiliary construction. Let l be the largest horizontal or vertical distance between any pair of points on γ (realized in the horizontal direction say). Let L be a straight line to the right from a point of γ furthest to the right. Then any γ'' surrounding γ intersects L at some point p, whose distance from γ we call d. We then have, using the bound $\mathrm{l}^2\geqq$ the area inside $\gamma\geqq\dfrac{|\gamma|}{4}$:

$$\begin{aligned}
\sum_{\gamma''s\gamma}e^{-\frac{\beta}{2}|\gamma''|}&\leqq\sum_{\mathrm{p}\in\mathrm{L}}\sum_{\substack{\mathrm{p}\in\gamma''\\ \gamma''s\gamma}}e^{-\frac{\beta}{2}|\gamma''|}\leqq\sum_{\mathrm{d}\geqq0}\sum_{n\geqq2(1+\mathrm{d})}\Big(3e^{-\frac{\beta}{2}}\Big)^n \qquad (\mathrm{A}.30)\\
&\leqq\mathrm{const}\,e^{-\beta\mathrm{l}}\leqq\mathrm{const}\,e^{-\frac{\beta}{2}|\gamma|^{1/2}}.
\end{aligned}$$

Combining these two estimates we finally get:

$$e^{-\beta|\gamma|} > \pi_\Theta(\gamma) \geqq e^{-\beta|\gamma| - \delta(\beta)|\gamma|}. \tag{A.31}$$

(The first inequality follows from the third line of (A.27) because every term in the numerator is contained in the denominator too). Several other applications of the results of this appendix will be found later.

We remark that if we had considered the ensemble $\mathfrak{M}_{0,c}^{++}(\Theta)$ containing only c-small contours, the formula corresponding to (A.27) would still be true and the estimate (A.31) would be true for the analogous probability $\pi_{\Theta,c}(\gamma)$ with the same $\delta(\beta)$. Furthermore by comparing the two expressions (A.27) one can show that

$$|\pi_{\Theta,c}(\gamma)/\pi_\Theta(\gamma) - 1| \leqq |\Theta|\, \delta(\beta)\, (3e^{-\beta})^{c \log|\Omega|} \tag{A.32}$$

if γ is c-small and β large enough. We also remark that all the above results would also be true if we had considered configurations on an infinite plane lattice instead of on an infinite cylinder.

Let Θ be a region bounded by two big contours separated by a distance of order N. We shall now show that the average magnetization in Θ converges to its expected value as $\mathrm{N} \to \infty$ in the ensemble $\mathfrak{M}_0^{--}(\Theta)$. To this end we study n, the number of spins up, and show that $|\Theta|^{-1}\langle \mathrm{n} \rangle$ has a limit as $\mathrm{N} \to \infty$ and that the variance of n, D(n), is bounded by const$|\Theta|$.

For each configuration X in $\mathfrak{M}_0^{--}(\Theta)$ we can identify the outer contours, and n(X) can be written as a sum of contributions from the regions bounded by them. If the outer contours are fixed these contributions are independent random variables. If γ is an outer contour the corresponding contribution $\mathrm{n}(\gamma)$ has a distribution determined by the ensemble $\mathfrak{M}^+(\gamma)$ of configurations in the region enclosed by γ which have the spins adhering to γ from the inside all $+$ (these spins are included in $\mathrm{n}(\gamma)$). If we write $\mathrm{n}(\mathrm{X}) = \sum_{\gamma \subset \Theta} n(\gamma)\, \chi_\gamma(\mathrm{X})$ with $\mathrm{n}(\gamma)$ distributed as just indicated and $\chi_\gamma(\mathrm{X}) = 1$ if γ is an outer contour of X and $=0$ otherwise we get the following expressions for the mean and variance of n(X):

$$\langle \mathrm{n} \rangle = \sum_{\gamma \subset \Theta} \langle \mathrm{n}(\gamma) \rangle\, \pi_\Theta(\gamma) \tag{A.33}$$

$$\begin{aligned} \mathrm{D}(\mathrm{n}) = & \sum_{\gamma \subset \Theta} \mathrm{D}(\mathrm{n}(\gamma))\, \pi_\Theta(\gamma) + \langle \mathrm{n}(\gamma) \rangle^2 (\pi_\Theta(\gamma) - \pi_\Theta^2(\gamma)) \\ & + \sum_{\substack{\gamma_1, \gamma_2 \subset \Theta \\ \gamma_1 \neq \gamma_2}} \langle \mathrm{n}(\gamma_1) \rangle \langle \mathrm{n}(\gamma_2) \rangle (\pi_\Theta(\gamma_1, \gamma_2) - \pi_\Theta(\gamma_1)\, \pi_\Theta(\gamma_2)). \end{aligned} \tag{A.34}$$

To study these quantities we need to estimate the difference between $\pi_\Theta(\gamma)$ and its limit $\pi(\gamma)$ and the difference $\pi_\Theta(\gamma_1, \gamma_2) - \pi_\Theta(\gamma_1)\, \pi_\Theta(\gamma_2)$. From the formula (A.27) we get:

$$\pi_\Theta(\gamma) = \exp\left(-\beta|\gamma| - \sum_{\substack{Xb\gamma \\ X\subset\Theta}} \varphi^T(X)\right) \tag{A.35}$$

where $Xb\gamma$ means X blocks γ, i.e. X intersects or surrounds γ, and we see that $\pi_\Theta(\gamma)$ converges to

$$\pi(\gamma) = \exp\left(-\beta|\gamma| - \sum_{Xb\gamma} \varphi^T(X)\right) \tag{A.36}$$

as $N\to\infty$. In (A.36) the summation is over all configurations X on an infinite plane lattice. The difference between $\pi_\Theta(\gamma)$ and $\pi(\gamma)$ when γ is not too long can be estimated as follows: Suppose e.g. that $|\gamma| < N$, and consider also the plane region $\tilde\Theta$ obtained by cutting the cylinder along a vertical column C as far away from γ as possible and making it plane. The distance $d(\gamma, C)$ is $> N/4$ if $|\gamma| < N$. Define $\tilde\pi(\gamma)$ by:

$$\tilde\pi(\gamma) = \exp\left(-\beta|\gamma| - \sum_{\substack{Xb\gamma \\ X\subset\tilde\Theta}} \varphi^T(X)\right). \tag{A.37}$$

We then have:

$$|\log\pi_\Theta(\gamma) - \log\tilde\pi(\gamma)| \leqq \sum_{\substack{Xb\gamma \\ XiC}} |\varphi^T(X)| \tag{A.38}$$

and

$$|\log\tilde\pi(\gamma) - \log\pi(\gamma)| \leqq \sum_{\substack{Xb\gamma \\ Xi\partial\Theta}} |\varphi^T(X)| + \sum_{\substack{Xb\gamma \\ XiC}} |\varphi^T(X)| \tag{A.39}$$

because only overlapping configurations have $\varphi^T(X) \neq 0$. To estimate the sums in (A.38), (A.39) we use (A.25). It implies that

$$\sum_{\substack{p\in X \\ Xs\gamma}} |\varphi^T(X)| \leqq F e^{-2\beta}\left(4e^{-\frac{\beta}{2}}\right)^{d^{1/2}(p,\gamma)} \tag{A.40}$$

because we can take Q to be $\gamma\cup$ (a path from γ to ∞) in (A.25) and use the fact that XiQ if $Xs\gamma$. In (A.38), (A.39) we thus get:

$$\sum_{\substack{Xi\gamma \\ Xi\partial\Theta}} |\varphi^T(X)| \leqq F e^{-2\beta}|\gamma|\left(4e^{-\frac{\beta}{2}}\right)^{d^{1/2}(\gamma,\partial\Theta)} \tag{A.41}$$

$$\sum_{\substack{Xi\gamma \\ XiC}} |\varphi^T(X)| \leqq F e^{-2\beta}|\gamma|\left(4e^{-\frac{\beta}{2}}\right)^{\frac{N^{1/2}}{2}}. \tag{A.42}$$

To estimate $\sum_{\substack{X s \gamma \\ X i \partial\Theta}} |\varphi^T(X)|$ let L be the shortest path from γ to $\partial\Theta$, $d = d(\gamma, \partial\Theta)$, and use the fact that XiL if $Xs\gamma$, so that:

$$\sum_{\substack{X s \gamma \\ X i \partial\Theta}} |\varphi^T(X)| \leqq \sum_{p \in L} \sum_{\substack{p \in X \\ X s \gamma \\ X i \partial\Theta}} |\varphi^T(X)| \leqq \sum_{\substack{p \in L \\ d(\varrho, \gamma) \leqq d/2}} \sum_{\substack{p \in X \\ X i \partial\Theta}} |\varphi^T(X)|$$

$$+ \sum_{\substack{p \in L \\ d(p, \gamma) > d/2}} \sum_{\substack{p \in X \\ X s \gamma}} |\varphi^T(X)| \tag{A.43}$$

$$\leqq 2 \sum_{l=d/2}^{d} F e^{-2\beta} \left(4e^{-\frac{\beta}{2}}\right)^{l^{1/2}} \leqq F e^{-2\beta} d \left(4e^{-\frac{\beta}{2}}\right)^{\left(\frac{d}{2}\right)^{1/2}}$$

$$\leqq \text{const}\, e^{-2\beta} \left(4e^{-\frac{\beta}{4}}\right)^{d^{1/2}}$$

if β is large. (We have used the fact that $d(p, \partial\Theta) + d(p, \gamma) = d$ if $p \in L$.) $\sum_{\substack{X s \gamma \\ X i C}} |\varphi^T(X)|$ can be estimated in the same way. Combining all these estimates we get:

$$|\log \pi_\Theta(\gamma) - \log \pi(\gamma)| \leqq \text{const}\, e^{-2\beta} |\gamma| \left(\left(4e^{-\frac{\beta}{4}}\right)^{d^{1/2}(\gamma, \partial\Theta)} + \left(4e^{-\frac{\beta}{4}}\right)^{\frac{N^{1/2}}{2}}\right) \tag{A.44}$$

and remembering that $\pi_\Theta(\gamma) \leqq e^{-\beta|\gamma|}$:

$$|\pi(\gamma) - \pi_\Theta(\gamma)|$$

$$\leqq e^{-\beta|\gamma|} \left(\exp\left(\text{const}\, e^{-2\beta} |\gamma| \left(\left(4e^{-\frac{\beta}{4}}\right)^{d^{1/2}(\gamma, \partial\Theta)} + \left(4e^{-\frac{\beta}{4}}\right)^{\frac{N^{1/2}}{2}}\right) - 1\right)\right). \tag{A.45}$$

Using this estimate one can show that if we define ϱ^* by:

$$\varrho^* = \sum_{(\gamma)} \langle n(\gamma) \rangle \, \pi(\gamma) \tag{A.46}$$

where the summation is over all possible shapes of a contour on the infinite plane lattice, then we have:

$$|\langle n \rangle - \varrho^* |\Theta|| \leqq \text{const}\, e^{-\beta} |\partial\Theta| \tag{A.47}$$

so that

$$\lim_{N \to \infty} |\Theta|^{-1} \langle n \rangle = \varrho^* . \tag{A.48}$$

To study $\pi_\Theta(\gamma_1, \gamma_2) - \pi_\Theta(\gamma_1)\, \pi_\Theta(\gamma_2)$ we use a formula analogous to (A.35):

$$\pi_\Theta(\gamma_1, \gamma_2) / \pi_\Theta(\gamma_1)\, \pi_\Theta(\gamma_2) = \exp\left(- \sum_{\substack{X b \gamma_1 \cup \gamma_2 \\ X \subset \Theta}} \varphi^T(X) + \sum_{\substack{X b \gamma_1 \\ X \subset \Theta}} \varphi^T(X) + \sum_{\substack{X b \gamma_2 \\ X \subset \Theta}} \varphi^T(X)\right)$$

$$= \exp\left(\sum_{\substack{X b \gamma_1 \\ X b \gamma_2 \\ X \subset \Theta}} \varphi^T(X)\right). \tag{A.49}$$

The sums in (A.49) can be estimated as above:

$$\sum_{\substack{X i \gamma_1 \\ X i \gamma_2}} |\varphi^T(X)| \leqq F e^{-2\beta} |\gamma_1| \left(4e^{-\frac{\beta}{2}}\right)^{d^{1/2}(\gamma_1,\gamma_2)}, \tag{A.50}$$

$$\sum_{\substack{X i \gamma_1 \\ X s \gamma_2}} |\varphi^T(X)| \leqq F e^{-2\beta} |\gamma_1| \left(4e^{-\frac{\beta}{2}}\right)^{d^{1/2}(\gamma_1,\gamma_2)}. \tag{A.51}$$

To estimate $\sum_{\substack{X s \gamma_1 \\ X s \gamma_2}} |\varphi^T(X)|$ let L be a straight line to the right from the right-most point on $\gamma_1 \cup \gamma_2$ (lying on γ_1 say). Then if $X s \gamma_1$ X intersects L at some point p at distance d from γ_1. Then because $d(p, \gamma_2) \geqq d + d(\gamma_1, \gamma_2)$ we have:

$$\begin{aligned} \sum_{\substack{X s \gamma_1 \\ X s \gamma_2}} |\varphi^T(X)| &\leqq \sum_{p \in L} \sum_{\substack{p \in X \\ X s \gamma_2}} |\varphi^T(X)| \leqq \sum_{d \geqq 0} F e^{-2\beta} \left(4e^{-\frac{\beta}{2}}\right)^{(d + d(\gamma_1,\gamma_2))^{1/2}} \\ &\leqq \text{const}\, e^{-2\beta} d^{1/2}(\gamma_1,\gamma_2) \left(4e^{-\frac{\beta}{2}}\right)^{d^{1/2}(\gamma_1,\gamma_2)}. \end{aligned} \tag{A.52}$$

We thus finally get from (A.49):

$$\begin{aligned} &|\log \pi_\Theta(\gamma_1,\gamma_2) - \log \pi_\Theta(\gamma_1) - \log \pi_\Theta(\gamma_2)| \\ &\qquad \leqq \text{const}\, e^{-\beta}(|\gamma_1| + |\gamma_2|) \left(4e^{-\frac{\beta}{2}}\right)^{d^{1/2}(\gamma_1,\gamma_2)} \end{aligned} \tag{A.53}$$

and

$$\begin{aligned} &|\pi_\Theta(\gamma_1,\gamma_2) - \pi_\Theta(\gamma_1)\, \pi_\Theta(\gamma_2)| \\ &\quad \leqq e^{-\beta(|\gamma_1| + |\gamma_2|)} \left(\exp\left(\text{const}\, e^{-\beta}(|\gamma_1| + |\gamma_2|) \left(4e^{-\frac{\beta}{2}}\right)^{d^{1/2}(\gamma_1,\gamma_2)}\right) - 1\right). \end{aligned} \tag{A.54}$$

Using (A.54) and the estimate $\pi_\Theta(\gamma) \leqq e^{-\beta|\gamma|}$ it is easy to show that

$$D(n) \leqq \text{const}\, e^{-\beta} |\Theta| \tag{A.55}$$

so that $|\Theta|^{-1} n$ converges to ϱ^* in probability. (A.47) and (A.55) imply the corresponding estimates for the average magnetization $m = 2n|\Theta|^{-1} - 1$ (in the ensemble $\mathfrak{M}_0^{--}(\Theta)$):

$$|\langle m \rangle + m^*| \leqq \text{const}\, e^{-\beta} |\partial\Theta|\, |\Theta|^{-1}, \tag{A.56}$$

$$D(m) \leqq \text{const}\, e^{-\beta} |\Theta|^{-1} \tag{A.57}$$

where $m^* = 1 - 2\varrho^*$ is the spontaneous magnetization. The corresponding result for the ensemble $\mathfrak{M}_0^{++}(\Theta)$ follows by symmetry. (We use the name spontaneous magnetization for m^* although we only use the fact that it is the limit of the average magnetization in $\mathfrak{M}_0^{++}(\Omega)$. In fact MS [9] have proved that m^* is also the limit of the magnetization in a small positive magnetic field.)

Appendix B

In this appendix we shall prove the basic mathematical statements used in the paper. We formulate them in a series of lemmas. The knowledge of the results of Appendix A is essential for reading this appendix. Let us start by proving the following lemma:

Lemma 1. *Let $\Theta \subset \Omega$ be a cylinder with bases not necessarily flat. Suppose* N *is the diameter of* Θ *and that the distance between the two bases of* Θ *grows as* N^{δ} *(as* $\mathrm{N} \to \infty$*). Suppose also that the length of the bases of* Θ *does not exceed* $\mathrm{N}(1+\ell(\beta))$*. Then the probability in* $\mathfrak{M}_0^{++}(\Theta)$ *of the set of configurations with magnetization exactly equal to* M, *where* M *is a number such that* $0 \leqq \mathrm{m}^*|\Theta| - \mathrm{M} \leqq \mathrm{AN}$, $\mathrm{A}>0$, *is given by* $Z(\mathfrak{M}_0^{++}(\Theta, \mathrm{M}|\Theta|^{-1}), \beta)/Z(\mathfrak{M}_0^{++}(\Theta), \beta)$ *and is such that:*

$$1 \geqq \frac{Z(\mathfrak{M}_0^{++}(\Theta, \mathrm{M}|\Theta|^{-1}), \beta)}{Z(\mathfrak{M}_0^{++}(\Theta))} \geqq \mathrm{D}(\beta)|\Theta|^{-1/2} e^{-\mathrm{R}(\beta)N^{1/2}}. \tag{B.1}$$

Proof. Let us first remark that in the ensemble $\mathfrak{M}_0^{++}(\Theta)$ the magnetization has an average value of the order of $\mathrm{m}^*|\Theta|$ within $\eta(\beta)|\Theta|^{1/2}$ and a variance $\eta(\beta)|\Theta|$, (see (A.56) (A.57)). We have to distinguish between two cases. In the first case we assume that $\mathrm{M} = \mathrm{m}^*|\Theta| - \varepsilon$ with $0 \leqq \varepsilon \leqq |\Theta|^{1/3}$ and in the second $M = \mathrm{m}^*|\Theta| - \varepsilon$ with $|\Theta|^{1/3} < \varepsilon \leqq \mathrm{AN}$.

Consider first the case $0 \leqq \varepsilon \leqq |\Theta|^{1/3}$. The proof is very similar to the proof of Lemma 3.1 in the MS paper, so we repeat it only for completness. One first calls $\mathrm{K}(\gamma)$ the number of outer contours contained in a configuration of $\mathfrak{M}_0^{++}(\Theta)$ which are congruent to a given contour γ.

The average of $\mathrm{K}(\gamma)$ is:

$$\langle \mathrm{K}(\gamma) \rangle = \sum_{\substack{\gamma' \in (\gamma) \\ \gamma' \subset \Theta}} \pi_{\Theta}(\gamma') \tag{B.2}$$

so that one can see (using (A.45)) that:

$$\langle \mathrm{K}(\gamma) \rangle - \pi(\gamma)\, |\Theta|| \leqq \mathrm{const}\, |\gamma|\, |\Theta|^{1/2} e^{-\beta|\gamma|}. \tag{B.3}$$

Similarly one can see that the variance $\mathrm{D}(\mathrm{K}(\gamma))$ is such that

$$\mathrm{D}(\mathrm{K}(\gamma)) \leqq \mathrm{const}\, |\gamma|^2 |\Theta|\, e^{-2\beta|\gamma|}. \tag{B.4}$$

The Chebyshev inequality implies then that the probability that for all γ:

$$|\mathrm{K}(\gamma) - \pi(\gamma)|\Theta|| \leqq \mathrm{B}|\gamma|\, |\Theta|^{1/2} e^{-\frac{\beta}{2}|\gamma|} \tag{B.5}$$

is larger than 1/2, for some B if β is large enough.

Let $\mathfrak{M} \subset \mathfrak{M}_0^{++}(\Theta)$ be the set of configurations for which (B.5) holds. Let X be a set of outer contours and let $\mathfrak{M}(\mathrm{X})$ be the set of configurations in $\mathfrak{M}$ which have X as the set of outer contours. The number N_0 of spins down in a configuration of $\mathfrak{M}(X)$ is a sum of independent variables and:

$$\langle N_0\rangle = \sum_{(\gamma)} \langle n(\gamma)\rangle\, K(\gamma) \tag{B.6}$$

where $\langle\cdot\rangle$ is taken in the ensemble $\mathfrak{M}(X)$ (see (A.33) (A.46)). (B.5) implies:

$$\Big|\langle N_0\rangle - |\Theta| \sum_{(\gamma)} \langle n(\gamma)\rangle\, \pi(\gamma)\Big| = \big|\langle N_0\rangle - \varrho^*|\Theta|\big| \leqq \eta(\beta)|\Theta|^{1/2} \tag{B.7}$$

and

$$D(N_0) \leqq \eta^2(\beta)|\Theta|\,. \tag{B.8}$$

We now single out among the contours in X those that are translates of a square γ_0 with side 3. (B.5) implies that there are $\sim \pi(\gamma_0)|\Theta|$ such contours ($\pi(\gamma_0) \neq 0$ see (A.31)). The number $\tilde{N}_0$ of spins enclosed in these contours is a sum of $K(\gamma_0) \sim \pi(\gamma_0)|\Theta|$ identically distributed random variables each assuming relatively prime values (i.e. 8 and 9). Therefore the local central limit theorem applies, and we can say that if $|l - \langle\tilde{N}_0\rangle| \leqq 4\eta|\Theta|^{1/2}$ e.g. then

$$\begin{aligned} P(\tilde{N}_0 = l) &= \text{const}\,(D(\tilde{N}_0))^{-1/2}\left(\exp - \frac{(l - \langle\tilde{N}_0\rangle)^2}{D(\tilde{N}_0)}\right) \\ &\quad (1 + O(K(\gamma_0)^{-1/2})) \geqq \text{const}\,|\Theta|^{-1/2} \end{aligned} \tag{B.9}$$

because $|\Theta|^{-1} D(\tilde{N}_0)$ is bounded from above and below away from 0. Denoting the remaining contribution to N_0 by $\tilde{\tilde{N}}_0$, which is independent of $\tilde{N}_0$, we then get the following estimate in the ensemble $\mathfrak{M}(X)$ if $|k - \varrho^*|\Theta|| \leqq \eta|\Theta|^{1/2}$:

$$\begin{aligned} P(N_0 = k) &\geqq P(\tilde{N}_0 + \tilde{\tilde{N}}_0 = k,\ |\tilde{N}_0 - \langle\tilde{N}_0\rangle| \leqq 4\eta|\Theta|^{1/2}) \\ &= \sum_{|l - \langle N_0\rangle| \leqq 4\eta|\Theta|^{1/2}} P(\tilde{N}_0 = l)\,P(\tilde{\tilde{N}}_0 = k - l) \\ &\geqq \text{const}\,|\Theta|^{-1/2} P(|k - \tilde{\tilde{N}}_0 - \langle\tilde{N}_0\rangle| \leqq 4\eta|\Theta|^{1/2}) \\ &= \text{const}\,|\Theta|^{-1/2} P(|k - \langle N_0\rangle + \langle\tilde{\tilde{N}}_0\rangle - \tilde{\tilde{N}}_0| \leqq 4\eta|\Theta|^{1/2}) \\ &\geqq \text{const}\,|\Theta|^{-1/2} P(|\tilde{\tilde{N}}_0 - \langle\tilde{\tilde{N}}_0\rangle| \leqq 2\eta|\Theta|^{1/2}) \\ &\geqq \text{const}\,|\Theta|^{-1/2}(1 - D(\tilde{\tilde{N}}_0)/4\eta^2|\Theta|) \geqq \text{const}\,|\Theta|^{-1/2}\,3/4 \end{aligned} \tag{B.10}$$

because if $|\tilde{\tilde{N}}_0 - \langle\tilde{\tilde{N}}_0\rangle| \leqq 2\eta|\Theta|^{1/2}$ then

$$\begin{aligned} &|k - \langle N_0\rangle + \langle\tilde{\tilde{N}}_0\rangle - \tilde{\tilde{N}}_0| \\ &\quad \leqq 2\eta|\Theta|^{1/2} + |k - \varrho^*|\Theta|| + |\varrho^*|\Theta| - \langle N_0\rangle| \leqq 4\eta|\Theta|^{1/2} \end{aligned} \tag{B.11}$$

and because

$$D(\tilde{\tilde{N}}_0) \leqq D(N_0) \leqq \eta^2|\Theta|\,. \tag{B.12}$$

The bound is independent of X. Now, if $M = m^*|\Theta| - \varepsilon$ with $0 \leqq \varepsilon \leqq |\Theta|^{1/3}$ then because $M = |\Theta| - 2N_0$ and $m^* = 1 - 2\varrho^*$ we have

$$\varepsilon = m^*|\Theta| - M = |\Theta| - 2\varrho^*|\Theta| - |\Theta| + 2N_0 = 2(N_0 - \varrho^*|\Theta|),$$

so $|k - \varrho^*|\Theta|| \leqq \eta |\Theta|^{1/2}$ for $|\Theta|$ sufficiently large, and Lemma 1 is proved because the r.h.s. of (B.10) is independent of X, and $P(\mathfrak{M}) \geqq 1/2$.

Suppose $AN \geqq \varepsilon > |\Theta|^{1/3}$. Then we can draw a little square with side $(\varepsilon/2m^*)^{1/2}$ in the middle of Θ. Since $\varepsilon > |\Theta|^{1/3}$ the side of this square tends to ∞ as $N \to \infty$, but not faster than $N^{1/2}$. Call Θ_ε the region outside this little square. Clearly in the ensemble $\mathfrak{M}_0^{++}(\Theta_\varepsilon)$ we can say that the probability to find exactly k spins down with k such that $|k - \varrho^*|\Theta_\varepsilon|| \leqq |\Theta_\varepsilon|^{1/3}$ is not smaller than const $|\Theta_\varepsilon|^{-1/2}$.

Let us consider the ensemble $\mathfrak{M}_\varepsilon$ of configurations of $\mathfrak{M}_0^{++}(\Theta)$ which contain the inner boundary of Θ_ε as an outer contour. $|\Theta_\varepsilon|$ has *been chosen so that the number of spins down* $\cong \varrho^*|\Theta_\varepsilon| + (1 - \varrho^*) \cdot |\Theta - \Theta_\varepsilon| \equiv N_d$, if $M = m^*|\Theta| - \varepsilon$. Let N_0 and N_1 be the number of down spins in Θ_ε and $\Theta - \Theta_\varepsilon$. They are independent in $\mathfrak{M}_\varepsilon$, and it is clear that $|\langle N_1 \rangle - (1 - \varrho^*)|\Theta - \Theta_\varepsilon|| \leqq \eta |\Theta - \Theta_\varepsilon|^{1/2}$, $D(N) \leqq \eta^2 |\Theta - \Theta_\varepsilon|$. The probability in $\mathfrak{M}_\varepsilon$ of having N_d spins down then has the lower bound:

$$\begin{aligned}
P(N_d) &\geqq P(N_0 + N_1 = N_d, |N_0 - \varrho^*|\Theta_\varepsilon|| \leqq |\Theta_\varepsilon|^{1/3}) \\
&= \sum_{|k - \varrho^*|\Theta_\varepsilon|| \leqq |\Theta_\varepsilon|^{1/3}} P(N_0 = k)\, P(N_1 = N_d - k) \\
&\geqq \text{const } |\Theta_\varepsilon|^{-1/2} P(|N_d - N_1 - \varrho^*|\Theta_\varepsilon|| \leqq |\Theta_\varepsilon|^{1/3}) \\
&= \text{const } |\Theta_\varepsilon|^{-1/2} P(|N_1 - (1 - \varrho^*)|\Theta - \Theta_\varepsilon|| \leqq |\Theta_\varepsilon|^{1/3}) \\
&\geqq \text{const } |\Theta_\varepsilon|^{-1/2} P\left(|N_1 - \langle N_1 \rangle| \leqq \frac{|\Theta|^{1/3}}{2}\right) \\
&\geqq \text{const } |\Theta|^{-1/2}(1 - 4D(N_1)|\Theta|^{-2/3}) \\
&\geqq \text{const } |\Theta|^{-1/2}(1 - O(\varepsilon|\Theta|^{-2/3})) \\
&= \text{const } |\Theta|^{-1/2}\left(1 - O\left(N^{1 - \frac{2(1+\delta)}{3}}\right)\right) \\
&\geqq \text{const } |\Theta|^{-1/2}
\end{aligned} \tag{B.13}$$

for N large because if $|N_1 - \langle N_1 \rangle| \leqq \frac{|\Theta|^{1/3}}{2}$ then

$$\left|N_1 - (1 - \varrho^*)|\Theta - \Theta_\varepsilon|\right| \leqq \frac{|\Theta|^{1/3}}{2} + \left|\langle N_1 \rangle - (1 - \varrho^*)|\Theta - \Theta_\varepsilon|\right|$$

$$\leqq \frac{|\Theta|^{1/3}}{2} + \eta |\Theta - \Theta_\varepsilon|^{1/2} \leqq \frac{|\Theta_\varepsilon|^{1/3}}{2}\left(1 + O\left(\frac{\varepsilon}{|\Theta_\varepsilon|}\right)^{1/3}\right) + O(\varepsilon^{1/2}) \leqq |\Theta_\varepsilon|^{1/3}$$

if N large, and because $D(N_1) \leqq O(\varepsilon)$.

The probability that the boundary of $\Theta - \Theta_\varepsilon$ is an outer contour is estimated in (A.31). So we finally get Lemma 1 with $R(\beta) = \mathrm{const}(\beta + \delta(\beta))$. Let us now prove the following lemma:

Lemma 2. *Let* m^* *be the value of the spontaneous magnetization and let* $m = \alpha m^* + (1-\alpha)(-m^*) = (2\alpha - 1) m^*$, $0 < \alpha < 1$. *Then if* β *is large enough we have:*

$$1 > \frac{Z(\mathfrak{M}^{+-}(\Omega, m), \beta)}{Z(\mathfrak{M}^{+-}(\Omega), \beta)} \geqq D(\alpha, \beta)\, N^{-(2\delta+3)} e^{-\delta(\beta) N} \tag{B.14}$$

where N *is the number of columns in* Ω.

Proof. To prove this lemma we first evaluate the probability in $\mathfrak{M}^{+-}(\Omega)$ that there is more than one big contour present in a randomly chosen configuration. Let $\lambda_1, \ldots, \lambda_{2n+1}$ be the big contours present in a configuration. The probability of this configuration is not larger than:

$$\sum_{n \geqq 1} \sum_{\lambda_1, \ldots, \lambda_{2n+1}} e^{-\beta \sum_i |\lambda_i|} \Big(\prod_i Z^{++}_{i-1,i}\Big) \Big/ Z(\mathfrak{M}^{+-}(\Omega), \beta) \leqq (\text{Same}) / Z(\mathfrak{M}_0^{+-}(\Omega), \beta) \tag{B.15}$$

where $Z^{++}_{i-1,i}$ denotes the partition function in the ensemble of contours located between λ_{i-1} and λ_i, while $\mathfrak{M}_0^{+-}(\Omega)$ denotes the ensemble of configurations in $\mathfrak{M}^{+-}(\Omega)$ containing only one big contour. From the result of Appendix A (formula (A.17)) we easily deduce that the numerator in (B.15) does not exceed:

$$Z(\mathfrak{M}_0^{++}(\Omega), \beta) \sum_{n \geqq 1} \sum_{\lambda_1, \ldots, \lambda_{2n+1}} e^{-(\beta - \delta(\beta)) \sum_i |\lambda_i|} \tag{B.16}$$

while the denominator is larger than:

$$Z(\mathfrak{M}_0^{++}(\Omega), \beta)\, e^{-(\beta + \delta(\beta)) N}. \tag{B.17}$$

The series in (B.16) is majorized by:

$$\begin{aligned} \sum_{n \geqq 1} \Big(\sum_{\lambda} e^{-(\beta - \delta(\beta)) |\lambda|}\Big)^{2n+1} &\leqq \sum_{n \geqq 1} \Big(|\Omega| \sum_{k \geqq N} \big(3 e^{-(\beta - \delta(\beta))}\big)^k\Big)^{2n+1} \\ &\leqq \sum_{n \geqq 1} \Big(|\Omega| \sum_{k \geqq N} e^{-\frac{\beta k}{2}}\Big)^{2n+1} \leqq \mathrm{const}\, |\Omega|^3 e^{-\frac{\beta}{2}(3N)} \end{aligned} \tag{B.18}$$

provided N is large enough. Hence (B.15) is not larger than:

$$\text{const}\,|\Omega|^3\,e^{-\left(\frac{\beta}{2}-\delta(\beta)\right)N}$$

and thus goes to zero as $N\to\infty$.

The next step for the proof of Lemma 2 is the evaluation of the probability in $\mathfrak{M}_0^{+-}(\Omega)$ of the set of configurations $\mathfrak{M}$ which contain a given big contour parallel to a row of Ω. If λ is the big contour in the configurations of $\mathfrak{M}$ and if Ω_λ is the region above λ we can write:

$$Z(\mathfrak{M},\beta)=e^{-\beta N}Z(\mathfrak{M}_0^{++}(\Omega_\lambda),\beta)\,Z(\mathfrak{M}_0^{--}(\Omega-\Omega_\lambda),\beta)\,. \qquad \text{(B.19)}$$

We can also write:

$$Z(\mathfrak{M}_0^{+-}(\Omega),\beta)=\sum_\lambda e^{-\beta|\lambda|}Z(\mathfrak{M}_0^{++}(\Omega_\lambda),\beta)\,Z(\mathfrak{M}_0^{--}(\Omega-\Omega_\lambda),\beta) \qquad \text{(B.20)}$$

where $\mathfrak{M}_0^{++}$, $\mathfrak{M}_0^{--}$ denote the ensembles of configurations (with the boundary condition $(+,+)$ or $(-,-)$) which are without big contours.

Using formula (A.26) we can write:

$$Z(\mathfrak{M}_0^{++}(\Omega_\lambda),\beta)\,Z(\mathfrak{M}_0^{--}(\Omega-\Omega_\lambda),\beta)=Z(\mathfrak{M}_0^{--}(\Omega),\beta)\exp-\sum_{X i \lambda}\varphi^{T}(X)\,. \qquad \text{(B.21)}$$

Hence, using (A.17), the following bound holds:

$$\frac{Z(\mathfrak{M},\beta)}{Z(\mathfrak{M}_0^{+-}(\Omega),\beta)}\geqq\frac{e^{-(\beta+\delta(\beta))N}}{\sum\limits_\lambda e^{-(\beta-\delta(\beta))|\lambda|}}\geqq\frac{e^{-(\beta+\delta(\beta))N}}{N^\delta\sum\limits_{(\lambda)} e^{-(\beta-\delta(\beta))|\lambda|}} \qquad \text{(B.22)}$$

where (λ) denotes the equivalence class of λ, which is arbitrary provided it wraps itself around the infinite cylinder which is determined by Ω. To evaluate $\sum\limits_{(\lambda)}\xi^{|\lambda|-N}$ we call:

$$\mathscr{g}_N\left(\frac{|\lambda|-N}{N}\right)=\frac{1}{N}\log\#\quad\text{(inequivalent big contours with length }|\lambda|) \qquad \text{(B.23)}$$

It is easy to show (see appendix C) that if $\varepsilon=\dfrac{|\lambda|-N}{N}\leqq\varepsilon_0$ with ε_0 small enough:

$$\begin{aligned}&\lim_{N\to\infty}\mathscr{g}_N(\varepsilon)=\mathscr{g}(\varepsilon)<+\infty\\ &\mathscr{g}_N(\varepsilon)\leqq\mathscr{g}(\varepsilon)+\frac{\log N}{N}\\ &\mathscr{g}(\varepsilon)\leqq-3\,\varepsilon\log\varepsilon\,.\end{aligned} \qquad \text{(B.24)}$$

Using (B.24) we can write: ($\xi = e^{-\beta+\delta(\beta)}$)

$$\sum_{(\lambda)} \xi^{|\lambda|-N} = \sum_{|\lambda|-N \leqq \varepsilon_0 N} \xi^{|\lambda|-N} + \sum_{|\lambda|-N > \varepsilon_0 N} \xi^{|\lambda|-N}$$

$$\leqq \sum_{L/N \leqq 1+\varepsilon_0} \exp N\left(g_N\left(\frac{L}{N}-1\right)+\left(\frac{L}{N}-1\right)\log\xi\right) + \sum_{L/N > 1+\varepsilon_0} 3^L \xi^{L-N} \tag{B.25}$$

$$\leqq N^2 \varepsilon_0 \exp\left(N \max_{0 \leqq \varepsilon \leqq \varepsilon_0} (-3\varepsilon\log\varepsilon + \varepsilon\log\xi)\right) + \frac{(3\xi)^{N(1+\varepsilon_0)}}{(1-3\xi)\xi^N}$$

$$\leqq N^2 \varepsilon_0 \exp(N\,\xi^{1/3}) + 2(3^{1+\varepsilon_0}\xi^{\varepsilon_0})^N \leqq N^2 \varepsilon_0 e^{N\sigma(\beta)} + 2\left(3^{1+\varepsilon_0} e^{-\frac{\beta\varepsilon_0}{2}}\right)^N$$

for β large. Hence if C is a large enough constant (independent of β):

$$\sum_{(\lambda)} \xi^{|\lambda|} \leqq C N^2 e^{-(\beta-\sigma(\beta))N}. \tag{B.26}$$

Therefore from (B.22) we deduce:

$$\frac{Z(\mathfrak{M}, \beta)}{Z(\mathfrak{M}_0^{+-}(\Omega), \beta)} \geqq C^{-1} N^{-(\delta+2)} e^{-(\delta(\beta)+\sigma(\beta))N}. \tag{B.27}$$

Coming now to the proof of Lemma 2, let $m = (2\alpha - 1)m^*, 0 < \alpha < 1$ and consider the set $\mathfrak{M} \subset \mathfrak{M}_0^{+-}(\Omega)$ of the configurations which contain just one big contour λ which runs parallel to the base at height $(1-\alpha)N^\delta$. If $X \in \mathfrak{M}$ then the contour configurations X' and X'' above and below λ are in $\mathfrak{M}_0^{++}(\Omega_\lambda)$ and $\mathfrak{M}_0^{--}(\Omega - \Omega_\lambda)$ and in the ensemble $\mathfrak{M}$ they are statistically independent:

$$P_{\mathfrak{M}}(X) = P_{\mathfrak{M}_0^{++}(\Omega_\lambda)}(X')\, P_{\mathfrak{M}_0^{--}(\Omega-\Omega_\lambda)}(X''). \tag{B.28}$$

The bases of Ω_λ and $\Omega - \Omega_\lambda$ are far from each other and at a distance of the order N^δ. At this point we can use Lemma 1 to prove that in the ensemble $\mathfrak{M}$ the probability of finding a configuration with magnetization exactly equal to $m = (2\alpha - 1)m^*$ is not smaller than

$$D(\alpha, \beta)\, |\Omega|^{-1} e^{-2R(\beta)N^{1/2}}.$$

This estimate allows us to study the probability (B.14):

$$\frac{Z(\mathfrak{M}^{+-}(\Omega, m), \beta)}{Z(\mathfrak{M}_0^{+-}(\Omega), \beta)} \geqq \frac{Z(\mathfrak{M}(m), \beta)}{Z(\mathfrak{M}, \beta)} \frac{Z(\mathfrak{M}, \beta)}{Z(\mathfrak{M}_0^{+-}(\Omega); \beta)}$$

$$\geqq D(\alpha, \beta)\, |\Omega|^{-1} C^{-1} N^{-(\delta+2)} e^{-(\delta(\beta)+\sigma(\beta))N - 2R(\beta)N^{1/2}} \tag{B.29}$$

and since we already know that $Z(\mathfrak{M}_0^{+-}(\Omega), \beta)/Z(\mathfrak{M}^{+-}(\Omega), \beta) \to 1$ we have proved Lemma 2.

It is also clear that the above estimates allow us to prove:

Lemma 3. *If β is large enough the probability of $\mathfrak{M}_0^{+-}(\Omega, m)$. i.e. of the set of configurations with magnetization m and containing just one* *big contour, tends to one in the ensemble* $\mathfrak{M}^{+-}(\Omega, m)$. *Similarly, the probability of* $\mathfrak{M}_0^{++}(\Omega, m^*)$ *in* $\mathfrak{M}^{++}(\Omega, m^*)$ *tends to 1.*

Proof. In fact if P is the complement in $\mathfrak{M}^{+-}(\Omega, \mathrm{m})$ of $\mathfrak{M}_0^{+-}(\Omega, \mathrm{m})$ we have:

$$Z(\mathrm{P}, \beta)/Z(\mathfrak{M}^{+-}(\Omega), \beta) \leqq (\mathrm{B}.15) \leqq \text{const}\, |\Omega|^3\, e^{-\left(\frac{\beta}{2} - \delta(\beta)\right)\mathrm{N}} \tag{B.30}$$

and comparing the rate at which (B.30) goes to zero with (B.14) we realize that the lemma is true. The proof of the second assertion is similar using (B.1).

Lemma 4. *The probability that a configuration* $\mathrm{X} \in \mathfrak{M}^{++}(\Omega)$ *is without big contours tends to 1 as* $\mathrm{N} \to \infty$. *Similarly the probability of* $\mathfrak{M}_0^{+-}(\Omega)$ *tends to 1 in* $\mathfrak{M}^{+-}(\Omega)$.

Proof. Observe that after the proof of the preceding lemmas only the first part needs a proof. However it is also clearly implicit in the preceding ones.

We now estimate the length of the big contour λ of a configuration in $\mathfrak{M}_0^{+-}(\Omega, \mathrm{m})$:

Lemma 5. *If* C *is a large enough constant the probability that* $|\lambda| \leqq \mathrm{N}(1 + C\beta^{-1})$ *in* $\mathfrak{M}_0^{+-}(\Omega, \mathrm{m})$ *converges to 1 as* $\mathrm{N} \to \infty$. (*If* β *is large enough and* $\mathrm{m} = (2\alpha - 1)\mathrm{m}^*$ *with* $0 < \alpha < 1$.)

Proof. Using (B.19), (B.20) and (B.21) we find that the probability that $|\lambda| > \mathrm{N}(1 + \mathrm{C}\beta^{-1})$ in $\mathfrak{M}_0^{+-}(\Omega)$ is majorized by:

$$\begin{aligned} &\sum_{|\lambda| \geqq \mathrm{N}(1 + C\beta^{-1})} e^{-(\beta - \delta(\beta))\,|\lambda|} / e^{-(\beta + \delta(\beta))\mathrm{N}} \\ &\leqq |\Omega|\,(3\,e^{-\beta + \delta(\beta)})^{\mathrm{N}(1 + c\beta^{-1})}(1 - 3\,e^{-\beta + \delta(\beta)})^{-1}\, e^{(\beta + \delta(\beta))\mathrm{N}} \\ &\leqq 2\,|\Omega|\,(3^{1 + \mathrm{C}\beta^{-1}}\, e^{-\mathrm{C} + 3\delta(\beta)})^{\mathrm{N}}. \end{aligned} \tag{B.31}$$

Hence the corresponding probability in $\mathfrak{M}_0^{+-}(\Omega, \mathrm{m})$ is not larger than:

$$\begin{aligned} &(\mathrm{B}.31) \cdot Z(\mathfrak{M}_0^{+-}(\Omega), \beta)/Z(\mathfrak{M}_0^{+-}(\Omega, \mathrm{m}), \beta) \\ &\leqq 2\,|\Omega|^{-1}\, D(\alpha, \beta)\, \mathrm{N}^{2\delta + 3}(e^{3\delta(\beta) + \delta'(\beta) - \mathrm{C}}\, 3^{1 + c\beta^{-1}})^{\mathrm{N}}. \end{aligned} \tag{B.32}$$

So we see that it converges to 0 for all β large enough if C is large enough.

Let us define the small phase-boundary as follows: let $\mathrm{X} \in \mathfrak{M}_0^{+-}(\Omega)$ and let $\gamma_1, \ldots, \gamma_n$ be the c_0-large contours of X which are not enclosed in a c_0-small contour. The set $\gamma_1 \cup \ldots \cup \gamma_n$ is called the small phase boundary. The set $\gamma_1 \cup \ldots \cup \gamma_n \cup$ (big contour) is called the phase boundary.

Lemma 6. *In the ensemble* $\mathfrak{M}_0^{+-}(\Omega, m)$ *the probability that the length of the* c_0-*large contours exceeds* $\mathrm{N}\beta^{-1}$ *goes to zero as* $\mathrm{N} \to \infty$ (*If* β *is* *large enough), and hence the same is true for the length of the small phase boundary.*

Proof. Consider the set of configurations having precisely k c_0-large contours and let their total length be T. Let $\mathfrak{M}_{\mathrm{T,k}}$ be this set of configurations. The number of ways the $\gamma_1, \ldots, \gamma_k$ can be arranged is not larger than $|\Omega|^k 3^{\Sigma|\gamma_i|} = |\Omega|^k 3^{\mathrm{T}}$ and the number of ways of writing T as $\mathrm{T} = \Sigma|\gamma i|$ is not larger than 2^{T}. Furthermore the probability in $\mathfrak{M}_0^{+-}(\Omega)$ that a configuration contains given $\gamma_1, \ldots, \gamma_k$ as contours does not exceed $e^{-\beta\Sigma|\gamma_i|}$. (This follows from an argument like that proving $\pi_\Theta(\gamma) \leqq e^{-\beta|\gamma|}$ from (A.27)). So:

$$P_{\mathfrak{M}_0^{+-}(\Omega)}(\mathfrak{M}_{\mathrm{T,k}}) \leqq |\Omega|^k 2^{\mathrm{T}} 3^{\mathrm{T}} e^{-\beta \mathrm{T}}. \tag{B.33}$$

Since the contours are c_0-large the number k must be such that $\mathrm{k} \leqq T/c_0 \log|\Omega|$. Thus the probability in $\mathfrak{M}_0^{+-}(\Omega)$ that the c_0-large contours have length T does not exceed:

$$\begin{aligned} \sum_{\mathrm{k}} P_{\mathfrak{M}_0^{+-}(\Omega)}(M_{\mathrm{T,k}}) &\leqq \sum_{\mathrm{k} \leqq \mathrm{T}/c_0 \log|\Omega|} |\Omega|^k 2^{\mathrm{T}} 3^{\mathrm{T}} e^{-\beta \mathrm{T}} \\ &\leqq (6\, e^{-\beta})^{\mathrm{T}} (|\Omega|)^{\mathrm{T}/c_0 \log|\Omega|} \mathrm{T}/c_0 \log|\Omega| \leqq e^{-(\beta - 2/c_0)\mathrm{T}}. \end{aligned} \tag{B.34}$$

Therefore:

$$\begin{aligned} &P_{\mathfrak{M}_0^{+-}(\Omega)}\,(\text{length of the } c_0\text{-large contours} > \mathrm{t}) \\ &\quad \leqq e^{-(\beta - 2/c_0)\mathrm{t}} (1 - e^{-\beta + 2/c_0})^{-1} \leqq 2\, e^{-(\beta - 2/c_0)\mathrm{t}}. \end{aligned} \tag{B.35}$$

Let $\mathrm{t} = \mathrm{N}\beta^{-1}$. We find that:

$$P_{\mathfrak{M}_0^{+-}(\Omega)}\,(\text{same as (B.35)}) \leqq 2\, e^{-(\beta - 2/c_0)\mathrm{N}\beta^{-1}} = 2 e^{-\mathrm{N} + 2\mathrm{N}/\beta c_0} \tag{B.36}$$

which gives the desired result after comparison with (B.14).

Let us now consider the regions into which the phase boundary of $\mathrm{X} \in \mathfrak{M}_0^{+-}(\Omega)$ divides Ω. We shall call d-magnetized the regions having down spins adjacent to the boundary from the inside. Likewise we call u-magnetized the regions which have up spins adjacent to the boundary from the inside. Notice that as a consequence of the presence of the big contour the two notions in a sense exchange their role above and below the big contour. Let Θ be a set $\Theta \subset \Omega$ with volume $|\Theta| > \mathrm{k}|\Omega|$ (for some $\mathrm{k} > 0$). We can consider the set of contour configurations $\mathfrak{M}_c(\Theta)$ in Θ defined by the requirement that all the outer contours are c-small. We make $\mathfrak{M}_c(\Theta)$ an ensemble by introducing on it a relative probability $e^{-\beta\Sigma|\gamma_i|}$ (as usual). The region Θ is not necessarily connected and can go around the cylinder Ω.

MS have proved the following lemma (Ref. [7.], Lemma 4.6).

Lemma 7. *Let* $X \in \mathfrak{M}_c(\Theta)$ *and call* $N(X)$ *the number of points in* X *which are "red" if we colour the points of* Θ *starting with white outside* Θ *and then use red after we meet the first contour and so on, alternating the colours. Then if* $c \geqq 2/3\beta$:

$$P_{\mathfrak{M}_c(\Theta)}(|N(x) - \langle N \rangle_{\mathfrak{M}_c(\Theta)}| > R\,|\Theta|^{3/4}) \leqq h_2 \exp\left(-h_1\, e^{\frac{4\beta}{3}} R^2 (k\,|\Omega|)^{1/2}\right) \tag{B.37}$$

for $1 \geqq R \geqq e^{-\beta} \cdot h_1, h_2$ *are positive numbers.*

This lemma needs not to be proved if we remark that it says something different from the corresponding lemma of MS only if Θ goes around the cylinder. In this case however the allowed contours are enclosed inside contours of length $\leqq c \log|\Omega|$. So for $|\Omega|$ large none of them goes around Θ, and so the proof goes as if Θ were open. The fact that in our case Ω is not square but elongated is of no consequence for the proof since this fact is never used in the MS paper. (It is of course necessary that the two sides of Ω go to ∞.)

Consider now the ensemble $\mathfrak{M}_0^{+-}(\Omega)$ and consider the configurations which have a fixed phase boundary X, and for which the d-magnetized region Θ_d is such that $|\Theta_d| > k|\Omega|$. Let the ensemble of such configurations be $\mathfrak{M}_X^{+-}(\Omega)$.

The contour configurations inside Θ_d will of course have only c_0-small outer contours and have probabilities determined by $\mathfrak{M}_{c_0}(\Theta_d)$. Hence:

$$\begin{aligned} &P_{\mathfrak{M}_X^{+-}(\Omega)}(|N_u(\Theta_d) - \langle N_u(\Theta_d)\rangle_{\mathfrak{M}_{c_0}(\Theta_d)}| > R\,|\Theta_d|^{3/4}) \\ &\leqq h_2 \exp\left(-h_1 e^{\frac{4\beta}{3}} R^2 (k\,|\Omega|)^{1/2}\right). \end{aligned} \tag{B.38}$$

Also:

$$\langle N_u(\Theta_d)\rangle_{\mathfrak{M}_{c_0}(\Theta_d)} = \sum_{\gamma \subset \Theta_d} \langle n(\gamma)\rangle\, \pi_{c_0,\Theta_d}(\gamma) = |\Theta_d| \sum_{(\gamma)} \langle n(\gamma)\rangle\, \pi_{c_0}(\gamma) + \mu(\beta)|\partial\Theta_d| \tag{B.39}$$

where $|\partial\Theta_d| = |\text{big contour}| + \sum_i |\gamma_i| + N \doteqdot$ (length of the phase boundary + length of a base of Ω) and $|\mu(\beta)| \leqq \text{const}\, e^{-\beta}$. Hence:

$$\begin{aligned} &P_{\mathfrak{M}_X^{+-}(\Omega)}(|N_u(\Theta_d) - \varrho_{c_0}^*|\Theta_d|\,| > R|\Theta_d|^{3/4} + \mu(\beta)|\partial\Theta_d|) \\ &\leqq h_2 \exp\left(-h_1 e^{\frac{4\beta}{3}} R^2 (k\,|\Omega|)^{1/2}\right) \end{aligned} \tag{B.40}$$

which implies

$$\begin{aligned} &P_{\mathfrak{M}_0^{+-}(\Omega)}(|N_u(\Theta_d) - \varrho_{c_0}^*|\Theta_d|| > R\,|\Theta_d|^{3/4} + \mu(\beta)|\partial\Theta_d|, |\Theta_d| > k|\Omega|) \\ &\leqq h_2 \exp\left(-h_1 e^{\frac{4\beta}{3}} R^2 (k|\Omega|)^{1/2}\right). \end{aligned} \tag{B.41}$$

If $X \in \mathfrak{M}_0^{+-}(\Omega, m)$ then, as shown in Lemma 6 and Lemma 5

$$|\partial \Theta_d| \leqq (C\beta^{-1} + 1 + \beta^{-1} + 1)\, N \leqq 3N \tag{B.42}$$

with probability $\to 1$ as $N \to 1$. Therefore:

$$\begin{aligned} &P_{\mathfrak{M}_0^{+-}(\Omega)}(\{|N_u(\Theta_d) - \varrho_{co}^*|\Theta_d|| > R|\Theta_d|^{3/4}, |\Theta_d| > k|\Omega|\} \cap \mathfrak{M}_0^{+-}(\Omega, m)) \\ &\quad \leqq h_3 \exp\left(-h_4 e^{\frac{4\beta}{3}} R^2 (k|\Omega|)^{1/2}\right) \end{aligned} \tag{B.43}$$

for suitable constants h_3, h_4 because $N/|\Omega|^{3/4} \to 0$ and $|\mu(\beta)|/R \leqq \text{const}$.
Similarly:

$$\begin{aligned} &P_{\mathfrak{M}_0^{+-}(\Omega)}(\{|N_d(\Theta_u) - \varrho_{co}^*|\Theta_u|| > R|\Theta_u|^{3/4}, |\Theta_u| > k|\Omega|\} \cap \mathfrak{M}_0^{+-}(\Omega, m)) \\ &\quad \leqq h_3 \exp\left(-h_4 e^{\frac{4\beta}{3}} R^2 (k|\Omega|)^{1/2}\right). \end{aligned} \tag{B.44}$$

From (B.43), (B.44) MS ([7], proof of Lemma 4.2 and 4.3), through purely algebraic steps which do not change in our case (though we have a different setting), deduce a result which in our language would read:

Lemma 8. *In* $\mathfrak{M}_0^{+-}(\Omega, m)$ *consider the set of configurations* $\tilde{\tilde{\mathfrak{M}}}_0^{+-}(\Omega, m)$ *such that*:

$$\begin{aligned} &||\Theta_u| - \alpha|\Omega|| \leqq \tilde{\varkappa}(\beta)|\Omega|^{3/4} \\ &||\Theta_d| - (1-\alpha)|\Omega|| \leqq \tilde{\varkappa}(\beta)|\Omega|^{3/4} \\ &|N_d(\Theta_u) - \varrho^*\alpha|\Omega|| \leqq \tilde{\varkappa}(\beta)|\Omega|^{3/4} \\ &|N_u(\Theta_d) - \varrho^*(1-\alpha)|\Omega|| \leqq \tilde{\varkappa}(\beta)|\Omega|^{3/4} \end{aligned} \tag{B.45}$$

where $N_d(\Theta_u)$ *is the number of spins down in the u-magnetized region and vice versa for* $N_u(\Theta_d)$. *The set* $\tilde{\tilde{\mathfrak{M}}}_0^{+-}(\Omega, m)$ *has a probability in* $\mathfrak{M}_0^{+-}(\Omega, m)$ *which tends to 1 as* $N \to \infty$ *if* β *is large enough.*

Since by Lemma 6 the area included inside the small phase boundary does not exceed $N^2\beta^{-2}$ it is finally easy to realize that the properties of the phase separation formulated in Section 3 are true:

Lemma 9. *Let* $\tilde{\mathfrak{M}}_0^{+-}(\Omega, m) \subset \mathfrak{M}_0^{+-}(\Omega, m)$ *be the set of configurations such that the following conditions are fulfilled:*

$$\begin{aligned} &|N^+(\Omega - \Omega_\lambda) - \varrho^*|\Omega - \Omega_\lambda|| \leqq \varkappa(\beta)|\Omega|^p \\ &|N^-(\Omega_\lambda) - \varrho^*|\Omega_\lambda|| \leqq \varkappa(\beta)|\Omega|^p \\ &||\Omega_\lambda| - \alpha|\Omega|| \leqq \varkappa(\beta)|\Omega|^p \\ &||\Omega - \Omega_\lambda| - (1-\alpha)|\Omega|| \leqq \varkappa(\beta)|\Omega|^p \\ &|\lambda| \leqq N(1 + C\beta^{-1}) \\ &|c_0\text{-}\textit{large contours}| \leqq N\beta^{-1} \end{aligned} \tag{B.46}$$

where λ is the big contour, Ω_λ the region above it and $N^-(\Omega_\lambda)$, $N^+(\Omega-\Omega_\lambda)$ the number of down spins above λ and up spins below λ respectively. Then the probability of $\tilde{\mathfrak{M}}_0^{+-}(\Omega, m)$ converges to 1 as $N \to \infty$ for all β large enough if $\varkappa(\beta) = e^{-\beta B}$ and if B^{-1} and C are chosen large enough, and p is chosen so that $N^2/|\Omega|^p = N^{2-p(1+\delta)} \to 0$ and $1 > p \geqq 3/4$. (If $2/(1+\delta) < 3/4$ we can choose $p = 3/4$, and otherwise we can e.g. choose $p = (1+2/(1+\delta))/2$).

Appendix C

In this appendix we study the number of shapes of a big contour of given length $|\lambda| = N(1+t)$ on an infinitely long cylinder. We first study how this number grows with N and then estimate it when t is small.

Consider an equivalence class (λ) of contours λ which go around the infinitely long cylinder Ω, and suppose that $|\lambda| = N(1+t)$, $t < 1$. Each column of the lattice intersects at least one horizontal segment of λ. As before we call "regular" those which intersect only one. There are at least $N(1-t)$ pairs of adjacent regular columns with no vertical segment between them. (Because for each other pair of adjacent columns at least one unit of "excess length" is needed). If we open λ between such a pair we can construct a corresponding walk on a plane lattice from $(0,0)$ to $(N,0)$ starting and ending horizontally and not leaving the strip with base $(0,0)-(N,0)$. Let the number of such walks with length $N(1+t)$ be $W(N,t)$. If we choose the cutting point according to some rule we get a mapping from the (λ): s onto the walks such that at most N (λ): s correspond to one walk. Therefore, if $\exp N g_N(t)$ is the number of (λ): s, we get the estimate:

$$W(N,t) \leqq \exp N g_N(t) \leqq N\, W(N,t)\,. \tag{C.1}$$

By a simple subadditivity argument using the fact that

$$W(N,t)\, W(M,t) \leqq W(N+M,t)$$

one can easily prove that:

$$g(t) = \lim_{N\to\infty} N^{-1} \log W(N,t) \tag{C.2}$$

exists and that $g(t) = \sup_{N \geqq 1} N^{-1} \log W(N,t)$.

We thus see that

$$g(t) = \lim_{N\to\infty} g_N(t) \tag{C.3}$$

and that

$$g_N(t) \leqq g(t) + N^{-1} \log N\,. \tag{C.4}$$

We now show that

$$g(t) = -t\log t + o(t) \tag{C.5}$$

for t small by exhibiting an upper and a lower bound for $N^{-1}\log W(N,t)$ of this magnitude.

An upper bound can be obtained by observing that $W(N,t) \leqq$ the number of unrestricted random walks from $(0,0)$ to $(N,0)$ in $N(1+t)$ steps. This number is $\binom{N(1+t)}{N(1+t/2)}^2$, which can be seen by writing each independent step (x_i, y_i) of such a walk as $(u_i + v_i - 1, u_i - v_i)$ with u_i and v_i independent Bernoulli variables. The conditions $\sum_i x_i = N$, $\sum_i y_i = 0$ then become $\sum_i u_i = \sum_i v_i = N(1+t/2)$. The binomial coefficient can easily be estimated using Stirlings formula and we get

$$\lim_{N\to\infty} N^{-1}\log W(N,t) \leqq -t\log t + o(t).$$

A lower bound can be obtained by estimating a suitable subclass of the walks. We consider those walks that never go "backwards". Such a walk consists of $N-1$ vertical jumps $x_1, \ldots, x_{N-1}$ with $\sum_i x_i = o$ and $\sum_i |x_i| = Nt$ between N horizontal jumps. Introducing $w_N(x,y) = \sum_{\substack{\sum_1^N x_i = x \\ \sum_1^N |x_i| = y}} 1$ we thus have $w_{N-1}(0, Nt) \leqq W(N,t)$. $w_N(o,y)$ can be estimated by the well known method of using a "conjugate distribution". Consider the two dimensional "canonical" probability distribution defined by

$$p(x,y) = \begin{cases} \dfrac{1-p}{1+p}\, p^y & \text{if } \; |x| = y = \text{integer} \geqq 0 \\ 0 & \text{otherwise} \end{cases} \tag{C.6}$$

depending on a parameter p, $0 < p < 1$. If (x_i, y_i) $i = 1, \ldots, N$ are independent and all have the distribution $p(x,y)$ then the sums $x = \sum_1^N x_i$, $y = \sum_1^N y_i$ have the distribution

$$p_N(x,y) = \sum_{\substack{\sum_1^N x_i = x \\ \sum_1^N |x_i| = y}} \left(\frac{1-p}{1+p}\right)^N p^{\sum_1^N |x_i|} = \left(\frac{1-p}{1+p}\right)^N p^y w_N(x,y) \tag{C.7}$$

$p_N(x, y)$ can be estimated using the local version of the central limit theorem, and thus also $\omega_N(x, y)$. We have

$$\langle x_i \rangle = 0, \langle y_i \rangle = \frac{2p}{1-p^2}, \operatorname{Cov}(x_i, y_i) = 0 \tag{C.8}$$

and put $D(x_i) = \sigma_x^2$, $D(y_i) = \sigma_y^2$.

Then:

$$p_N(x, y) = \text{const } N^{-1} \left(\exp - \frac{1}{2} \left(\frac{x^2}{N\sigma_x^2} + \frac{(y - N\langle y \rangle)^2}{N\sigma_y^2} \right) \right) (1 + O(N^{-1/2})) \tag{C.9}$$

uniformly for $x^2 + y^2 = O(N)$ in the support of P_N.

Putting $x = 0$, $y = (N+1)\langle y \rangle$ we get

$$\omega_N(0, (N+1)\langle y \rangle) = \left(\frac{1+p}{1-p} \right)^N p^{-N\langle y \rangle} O(N^{-1}). \tag{C.10}$$

If we now choose p so that $\langle y \rangle = \frac{2p}{1-p^2} = t$ we get:

$$(N+1)^{-1} \log \omega_N(0, (N+1)t) = \log \left(\frac{1+p}{1-p} \right) - t \log p + O(N^{-1} \log N). \tag{C.11}$$

For t small we get $p = t/2 + O(t^2)$ and thus finally:

$$\lim_{N \to \infty} N^{-1} \log W(N, t) \geqq -t \log t + O(t) \tag{C.12}$$

and

$$g(t) = -t \log t + O(t). \tag{C.5}$$

Acknowledgements. We wish to thank D. Abraham, M. E. Fisher, J. L. Lebowitz and E. Lieb for discussions and suggestions.

References

1. Later, however, we have proved that our surface tension coincides with Onsager's see: Abraham, D., Gallavotti, G., Martin-Löf, A.: Lettere Nuovo Cimento **2**, 143 (1971).
2. Bellemans, A.: Physica **28**, 493 (1962), Physica **28**, 617 (1962).
3. Berezin, F. A., Dobrushin, R. L., Minlos, R. A., Povzner, A. Ja., Sinai, Ja. G.: Unpublished report at the Copenhagen conference on Statistical Mechanics (1966).
4. The surface tension in the Ising model has also been studied by Fisher and Ferdinand: Fisher, M. E., Ferdinand, A. E.: Phys. Rev. Letters **19**, 169 (1967).
5. Gallavotti, G., Miracle-Sole, S.: Commun. math. Phys. **7**, 274 (1968).
6. Landau, L. D., Lifshitz, E. M.: Statistical Physics, 2. Ed. Oxford: Pergamon Press 1969.

7. Minlos, R. A., Sinai, Ja. G.: Part 1: Mat Sbornik, **73**, 115 (1967), No. 3. English translation, Math. USSR — Sbornik, **2**, 335 (1967), No. 3.
8. — — Part 2: Trudy Moskov. Mat. Obšč. **19** (1968). English translation. Trans. Moscow Math. Soc. **19**, 121 (1968).
9. — — Trudy Moskov. Mat. Obšč. **17** (1967). English translation, Trans. Moscow. Math. Soc. **17**, 237 (1967).
10. Onsager, L.: Phys. Rev. **65**, 117 (1944).
11. Ruelle, D.: Statistical Mechanics. 87—88. New York: Benjamin 1969.

G. Gallavotti
Istituto di Matematica
dell'Università di Roma
Piazzale delle Scienze
I-00187 Roma, Italy

A. Martin-Löf
Mathematics Dept.
Royal Institute of Technology
S-10044 Stockholm 70
Sweden