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A Remark on a Theorem of B. MISRA

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Abstract. The two sided ideals of the C^*-algebra generated by local v. Neumann algebras are investigated.

I. Introduction

B. MISRA [1] has shown that the algebra of all local observables is simple when the following conditions are fulfilled:

1. The algebra is given as a concrete C^*-algebra in a Hilbert space fulfilling the usual assumptions of local ring systems.
2. The rings associated with bounded open regions are v. Neumann algebras.
3. For any bounded open region $\mathcal{O}$ exists another bounded open region $\mathcal{O}_1$ containing $\mathcal{O}$ such that the ring associated $\mathcal{O}_1$ is a factor.

The third condition, however, has not been derived from the other two assumptions even when we assume that the von Neumann algebra generated by the global C^*-algebra is a factor. Since in recent years different representations of the C^*-algebra of all local observables have been discussed [2], [3], [4] it is desirable to have a characterization of all two-sided ideals in the general case where 3. is not assumed. We will show that the theorem of Misra stays true without assuming 3., i.e. the C^*-algebra generated by all local observables is simple if it contains no center. For later use we will also consider some more general algebras.

II. Assumptions and notations

We denote by $\mathcal{O}$ open bounded regions in the Minkowski-space and write:

$\mathcal{O}_1 \times \mathcal{O}_2$ if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated.

$\mathcal{O}_1 < \mathcal{O}_2$ if $\mathcal{O}_1 \subset \mathcal{O}_2$ and there exists an $\mathcal{O}_3 \subset \mathcal{O}_2$ with $\mathcal{O}_1 \times \mathcal{O}_3$.

$\mathcal{O}_1 \leqslant \mathcal{O}_2$ if there exists a neighbourhood $\mathcal{N}$ of the origin such that $\mathcal{O}_1 + x < \mathcal{O}_2$ for all $x \in \mathcal{N}$.

We denote by a local ring system $\{\mathcal{R}(\mathcal{O})\}$ a family of rings of operators in a fixed Hilbert space $\mathcal{H}$ submitted to the following conditions:

1. $\mathscr{R}(\mathcal{O})$ is a v. Neumann algebra for all $\mathcal{O}$ and

a) $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathscr{R}(\mathcal{O}_1) \subset \mathscr{R}(\mathcal{O}_2)$

b) $\mathscr{R}_\infty = \{\bigcup_{\mathcal{O}} \mathscr{R}(\mathcal{O})\}''$

c) $\mathfrak{R}$ = smallest C^*-algebra containing $\{\bigcup_{\mathcal{O}} \mathscr{R}(\mathcal{O})\}$.

2. In $\mathscr{H}$ exists a unitary representation $U(x)$ of the translation groups with

a) $\mathscr{R}(\mathcal{O} + x) = U(x)\, \mathscr{R}(\mathcal{O})\, U^{-1}(x)$

b) The spectrum of $U(x)$ is contained in the closure of the future lightcone.

c) $U(x) \in \mathscr{R}_\infty$, which can be assumed without loss of generality by [5].

3. If $\mathcal{O}_1 \times \mathcal{O}_2$ then $\mathscr{R}(\mathcal{O}_1) \subset \mathscr{R}(\mathcal{O}_2)'$ (local commutativity).

4. For any $\mathcal{O}$ we have $\mathscr{R}_\infty = \{\bigcup_x \mathscr{R}(\mathcal{O} + x)\}''$ (weak additivity).

We denote by a generalized local ring system $\{\mathscr{S}(\mathcal{O})\}$ a family of rings of operators in a fixed Hilbert space $\mathscr{H}$ submitted to the following conditions:

1. $\mathscr{S}(\mathcal{O})$ is a von Neumann algebra for all $\mathcal{O}$ with

a) $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathscr{S}(\mathcal{O}_1) \subset \mathscr{S}(\mathcal{O}_2)$

b) $\mathscr{S}_\infty = \{\bigcup_{\mathcal{O}} \mathscr{S}(\mathcal{O})\}''$

c) $\mathfrak{S}$ = smallest C^*-algebra containing $\{\bigcup_{\mathcal{O}} \mathscr{S}(\mathcal{O})\}$.

2. There exists a local ring system $\{\mathscr{R}(\mathcal{O})\}$ with

a) $\mathscr{R}(\mathcal{O}) \subset \mathscr{S}(\mathcal{O})$ for all $\mathcal{O}$.

b) If $U(x)$ is the representation of the translation group given by $\{\mathscr{R}(\mathcal{O})\}$ then

$$\mathscr{S}(\mathcal{O} + x) = U(x)\, \mathscr{S}(\mathcal{O})\, U^{-1}(x)\,.$$

3. If $\mathcal{O}_1 \times \mathcal{O}_2$ then $\mathscr{R}(\mathcal{O}_1) \subset \mathscr{S}(\mathcal{O}_2)'$.

Let $\psi \in \mathscr{H}$ and P_0 be the energy operator. We say ψ is analytic for the energy if ψ is in the domain of every power P_0^n and the sum $\sum_{n=0}^{\infty} \|P_0^n \psi\| \cdot \frac{z^n}{n!}$ has a nonzero radius of convergence.

III. Some properties of local rings

For the investigation of the ideals of local ring systems we need certain properties of local rings which we study first.

III.1 Theorem. Assume we have a continuous representation $U(t)$ of a one-parametric group with semi-bounded spectrum. Moreover assume we have two projections E, F such that

$$U(t)\, E\, U^{-1}(t) F = F\, U(t)\, E\, U^{-1}(t)$$

for $|t| < 1$. If we have $E \cdot F = 0$ then follows $U(t)\, E\, U^{-1}(t) F = 0$ for all t.

Proof. In order to make the proof transparent we make first a *special assumption*, namely, that the spectrum of $U(t)$ is bounded. In this case $U(t) = \exp\{itP\}$ with P a bounded self-adjoint operator and hence $\frac{d^n}{dt^n} U(t)\, E\, U^{-1}(t)$ is also a bounded self-adjoint operator and

$$\frac{d^n}{dt^n} U(t)\, E\, U^{-1}(t) = U(t) \left\{\frac{d^n}{d\tau^n} U(\tau)\, E\, U^{-1}(\tau)\right\}_{\tau=0} U^{-1}(t)$$

can be written as $U(t)\{A_n^+ - A_n^-\}\, U^{-1}(t)$ where A_n^+ resp. A_n^- are the positive resp. negative parts of $\left\{\frac{d^n}{d\tau^n} U(\tau)\, E\, U^{-1}(\tau)\right\}_{\tau=0}$ which are also bounded. Assume we have already proven $F(A_n^+ - A_n^-) = 0$ for $n = 0, 1, \ldots, N$. We want to show that this holds also for $n = N+1$. Now $F(A_N^+ - A_N^-) = 0$ implies $FA_N^+ = FA_N^- = 0$. $F\,U(t)\, A_N^+ U^{-1}(t)$ is a positive operator for $|t| < 1$ and since for arbitrary $\psi \in \mathscr{H}$ the function $(\psi, F\,U(t)\, A_N^+ U^{-1}(t)\psi)$ is analytic in t, positive for real t with $|t| < 1$ and zero at $t = 0$, we see that this function must have a zero of second order and hence by Schwartz inequality

$$\begin{aligned} &|(\psi, F\,U(t)\, A_N^+\, U(-t)\psi)| \leqq |t|^2\, \|\psi\|^2\, \|A_N^+\|\, e^{\|P\|} \\ \text{and}\quad &|(\psi, F\,U(t)\, A_N^-\, U(-t)\psi)| \leqq |t|^2\, \|\psi\|^2\, \|A_N^-\|\, e^{\|P\|}\,. \end{aligned} \qquad |t| < 1$$

But this implies $F \frac{d^N}{dt^N} U(t)\, E\, U^{-1}(t)$ has a zero of second order at $t = 0$ and hence $F \frac{d^{N+1}}{dt^{N+1}} U(t)\, E\, U^{-1}(t)$ is zero at $t = 0$. Since $F\,U(t)\, E\, U^{-1}(t)$ is zero at $t = 0$ by assumption, $F \frac{d^n}{dt^n} U(t)\, E\, U^{-1}(t)$ is zero at $t = 0$ by induction for all n. Since P was a bounded operator we see that $F\,U(t)\, E\, U^{-1}(t)$ is an entire analytic function and hence identically zero.

Now the *general case*. Without loss of generality we can assume $U(t) = \exp\{itP\}$ with P a positive operator. Consider the operator $e^{-P} F\, U(t)\, E\, U^{-1}(t) e^{-P}$ which is the boundary-value of an analytic function holomorphic in $0 < \operatorname{Im} t < 1$ and bounded by 1 in this strip. The operator $e^{-P}\, U(t)\, E\, U^{-1}(t) F e^{-P}$ is holomorphic in $-1 < \operatorname{Im} t < 0$ and bounded by 1. Since now

$$e^{-P} F\, U(t)\, E\, U^{-1}(t)\, e^{-P} = e^{-P}\, U(t)\, E\, U^{-1}(t)\, F e^{-P}$$

for real t, $-1 < t < 1$, we see that $e^{-P} F\, U(t)\, E\, U^{-1}(t) e^{-P}$ is holomorphic in the unit circle and bounded by 1. Since it is a positive operator for real t, $|t| < 1$ and zero at $t = 0$, it must have a zero of second order or $\|e^{-P} F\, U(t)\, E\, U^{-1}(t)\, e^{-P}\| \leqq |t|^2$ for $|t| < 1$. But this implies

$$\|e^{-P} F \frac{d}{dt}\; U(t)\, E\, U^{-1}(t) e^{-P}\| \leqq \frac{|t|}{1 - |t|} \quad \text{for} \quad |t| < 1\,.$$

Let h be real, then $U(h)\, E\, U^{-1}(h) - E$ is a self-adjoint operator and let G_h^+, resp. G_h^- be the projections onto the positive resp. negative part. G_h^+ and G_h^- commute with F for sufficiently small h.

Let now t be real then we get

$$0 \leqq F\,U(t)\,G_h^+\,(U(h)\,E\,U(-h) - E)\,U(-t)$$
$$= F\,U(t)\,G_h^+\,U(h)\,E\,U(-h)\,G_h^+\,U(-t) - F\,U(t)\,G_h^+\,E\,G_h^+\,U(-t)$$

and hence

$$F\,U(-h)\,G_h^+\,E\,G_h^+\,U(h) = 0.$$

This implies

$$F\,U(-h)\,G_h^+\,E\,U(h) = 0.$$

In the same manner we get:

$$F\,G_h^-\,U(h)\,E\,U(-h) = 0.$$

From this follows:

$$\left\|\frac{1}{2}\,e^{-P}\,F\,U(t)\,(G^+E + E\,G^+)\,U(-t)\,e^{-P}\right\| \leqq c\cdot|t|^2\,|t+h|^2,$$

since the positiv and negativ part have a zero at $t = 0$ and $t = h$.

In the same manner we find:

$$\left\|\frac{1}{2}\,e^{-P}\,F\,U(t)\,(G^-\,U(h)\,E\,U(-h) + U(h)\,E\,U(-h)\,G^-)\,U(-t)\,e^{-P}\right\| \leqq$$
$$\leqq c'\,|t|^2\,|t+h|^2.$$

Adding both equations we have

$$\left\|\frac{1}{2}\,e^{-P}F\,U(t)\{G^-\,U(h)\,E\,U(-h) + U(h)\,E\,U(-h) + G^+E + E\,G^+\}\,U(-t)\,e^{-P}\right\| \leqq$$
$$\leqq c''\,|t|^2\,|t+h|^2.$$

But this gives:

$$\left\|e^{-P}\,F\,U(t)\,\frac{E + U(h)\,E\,U(-h)}{2}\,U(-t)\,e^{-P}\right\| \leqq c''\,|t|^2\,|t+h|^2 +$$
$$+\frac{1}{4}\,\|e^{-P}\,F\,U(t)\,\{(G_h^+ - G_h^-)\,(E - U(h)\,E\,U(-h)) +$$
$$+ (E - U(h)\,E\,U(-h)\,(G_h^+ - G_h^-)\}\,U(-t)\,e^{-P}\|.$$

Since the last term converges weakly to zero for h going to zero we see that the remainder has a zero of fourth order.

Hence:

$$\|e^{-P}F\,U(t)\,E\,U(-t)\,e^{-P}\| \leqq |t|^4\,.$$

Assume now we have shown that $e^{-P}F\,U(t)\,E\,U(-t)\,e^{-P}$ has a zero of order $2n$. Then $\frac{1}{t^{2n-2}}\,\frac{d}{dt}\,e^{-P}F\,U(t)\,E\,U(-t)\,e^{-P}$ has a zero of first order. Repeating the same argument we find $\frac{1}{t^{2n-2}}\,\frac{d}{dt}\,e^{-P}F\,U(t)\,E\,U(-t)\,e^{-P}$ has a zero of second order or $e^{-P}F\,U(t)\,E\,U(-t)\,e^{-P}$ has a zero of order $2n+2$ and by induction it has a zero of all orders. But this implies $e^{-P}F\,U(t)\,E\,U(-t)\,e^{-P}$ is identically zero for $|t| < 1$. Since it is for arbitrary real t the boundary-value of an analytic function holomorphic in $0 < \operatorname{Im} t < 1$, it follows by analytic continuation that

$$e^{-P} F U(t) E U^{-1}(t) e^{-P} = 0$$

for all real t. Since e^{-P} is an invertible operator we get

$$F U(t) E U^{-1}(t) \equiv 0 \quad \text{qed.}$$

As a next step we have to generalize a lemma proved in an earlier paper ([4] corollary 7) for our generalized situation. This lemma tells us that every operator belonging to a bounded region which maps one vector analytic for the energy onto another vector also analytic for the energy commutes with all translations.

III.2. Lemma. Let $\{\mathscr{S}(\mathscr{O})\}$ be a generalized local ring system and $\{\mathscr{R}(0)\}$ the local ring system contained in $\{\mathscr{S}(\mathscr{O})\}$. Let $A \in \mathscr{S}(\mathscr{O})$ for some $\mathscr{O}$, $\psi \in \mathscr{H}$ be a vector analytic for the energy, and assume $A\psi$ is also analytic for the energy. Then for every $\mathscr{O}_1 \gg \mathscr{O}$ exists a projection in $\bigcap_x \mathscr{S}(\mathscr{O}_1 + x) \cap \mathscr{R}'_\infty$ such that $E\psi = \psi$ and $A \cdot E \in \bigcap_x \mathscr{S}(\mathscr{O}_1 + x) \cap \mathscr{R}'_\infty$.

Proof. Let $B_1 \ldots B_n \in \mathscr{S}'(\mathscr{O}_1)$, $x_1 \ldots x_n \in \mathscr{N}$, $B_i(x) = U(x) B_i U^{-1}(x)$ then we have

$$B_1(x_1) \ldots B_n(x_n) A = A B_1(x_1) \ldots B_n(x_n) \quad \text{for} \quad x_1 \ldots x_n \in \mathscr{N}.$$

Now $B_1(x_1) \ldots B_n(x_n) A$ and $A B_1(x_1) \ldots B_n(x_n)$ are both boundary-values of holomorphic functions since ψ and $A\psi$ are analytic for the energy. Since these functions coincide for $x_1 \ldots x_n \in \mathscr{N}$ they coincide everywhere. Hence we get for $B \in \{\bigcup_x \mathscr{S}'(\mathscr{O}_1 + x)\}''$ the relation $BA\psi = AB\psi$. Let now E be the projection onto the closure of the vector space $\{\bigcup_x \mathscr{S}'(\mathscr{O}_1 + x)\}''\psi$ then we get $BAE = AE \cdot B$ or $AE \in \{\bigcup_x \mathscr{S}'(\mathscr{O}_1 + x)\}'$. But also $E \in \{\bigcup_x \mathscr{S}'(\mathscr{O}_1 + x)\}'$ and E has the property $E\psi = \psi$. Since $\mathscr{S}'(\mathscr{O}_1) \supset \mathscr{R}(\mathscr{O}_2)$ for $\mathscr{O}_1 \times \mathscr{O}_2$ we have

$$\{\bigcup_x \mathscr{S}'(\mathscr{O}_1 + x)\}'' \supset \{\bigcup_x \mathscr{R}(\mathscr{O}_2 + x)\}'' = \mathscr{R}_\infty .$$

Hence E and AE are elements from $\bigcap_x \mathscr{S}(\mathscr{O}_1 + x) \cap \mathscr{R}'_\infty$ qed.

In the following argument we have to consider equivalent projections. We say two projections E_1, E_2 from a fixed von Neumann algebra R are equivalent when we can find in R a partially isometric operator V with $E_1 = VV^*$, $E_2 = V^*V$. If E_1 is equivalent to E_2 then we write $E_1 \sim E_2 \bmod R$. (For a detailed discussion see [7] chap. III.) With this notation we get

III.3. Theorem. Let $\{\mathscr{S}(\mathscr{O})\}$ and $\{\mathscr{R}(\mathscr{O})\}$ be as in Lemma III.2. Let E be a projection in $\mathscr{S}(\mathscr{O})$. Assume moreover that there exists a vector ψ analytic for the energy such that $\mathscr{R}_\infty \psi$ is dense in the Hilbert space.

a) If $\mathscr{O}_1 > \mathscr{O}$ and F is the smallest projection in the center $\mathfrak{Z}(\mathscr{S}(\mathscr{O}_1))$ of $\mathscr{S}(\mathscr{O}_1)$ with $FE = E$ then $E \sim F \bmod \mathscr{S}(\mathscr{O}_1)$.

b) If $\mathcal{O}_1 \gg \mathcal{O}$ then there exists an $F \in \mathcal{S}(\mathcal{O}_1) \cap \mathcal{R}'_\infty$ with $F \sim E$ and $FE = E$.

This theorem enlightens the well-known result that the local rings are not finite [8], [9], [10] by showing explicitly some projections which are not finite.

Proof. a) Let ψ be the cyclic vector analytic for the energy. Then by the Reeh-Schlieder theorem we have for any $\mathcal{R}(\mathcal{O})$, $\overline{\mathcal{R}(\mathcal{O})\psi} = \mathcal{H}$ ([11], [5] Lemma 5). Now define the projection F by $F\mathcal{H} = \overline{\mathcal{S}(\mathcal{O}_1)E\psi}$. We have $F \in \mathcal{S}(\mathcal{O}_1)'$. But since $\mathcal{O}_1 > \mathcal{O}$ there exists an $\mathcal{O}_2 \times \mathcal{O}$ and $\mathcal{O}_2 \subset \mathcal{O}_1$ hence

$$F\mathcal{H} = \overline{\mathcal{S}(\mathcal{O}_1)E\psi} = \overline{\mathcal{S}(\mathcal{O}_1)\,\mathcal{R}(\mathcal{O}_2)E\psi} = \overline{\mathcal{S}(\mathcal{O}_1)\,ER(\mathcal{O}_2)\psi} = \overline{\mathcal{S}(\mathcal{O}_1)E\mathcal{H}}\,.$$

Therefore we get

$$\begin{aligned}\mathcal{S}(\mathcal{O}_1)'\,F\mathcal{H} = \mathcal{S}(\mathcal{O}_1)'\,\overline{\mathcal{S}(\mathcal{O}_1)\,E\mathcal{H}} &= \overline{\mathcal{S}(\mathcal{O}_1)'\,\mathcal{S}(\mathcal{O}_1)\,E\mathcal{H}}\\ &= \overline{\mathcal{S}(\mathcal{O}_1)\,E\,\mathcal{S}(\mathcal{O}_1)'\,\mathcal{H}} = \overline{\mathcal{S}(\mathcal{O}_1)\,E\mathcal{H}}\,.\end{aligned}$$

This means $F\mathcal{H}$ is invariant under $\mathcal{S}(\mathcal{O}_1)'$ or $F \in \mathcal{S}(\mathcal{O}_1)$ hence $F \in \mathfrak{Z}(\mathcal{S}(\mathcal{O}_1))$. Since we have $\overline{\mathcal{S}(\mathcal{O}_1)F\psi} = \overline{\mathcal{S}(\mathcal{O}_1)E\psi} = F\mathcal{H}$ follows $F\mathcal{H} = \overline{\mathcal{S}(\mathcal{O}_1)'F\psi} \sim E\mathcal{H} = \overline{S(\mathcal{O}_1)'E\psi} \bmod \mathcal{S}(\mathcal{O}_1)$ ([7] chap. III § 1 corollaire de Théorème 2). It is easy to see that F is the smallest projection in $\mathfrak{Z}(\mathcal{S}(\mathcal{O}_1))$ with the property $FE = E$.

b) If now $\mathcal{O}_1$ m $\mathcal{O}$ then from $(1 - F)E = 0$ follows by theorem III.1. that $(1 - F)\,U(x)\,E\,U^{-1}(x) = 0$ for all x which are timelike. Since $(1 - F)\,U(x)\,E\,U^{-1}(x)e^{-P_0}$ with P_0 the energy-operator is the boundary-value of an analytic function, it follows that $(1 - F)\,U(x)\,E\,U^{-1}(x)e^{-P_0}$ vanishes for all x and hence $(1 - F)\,U(x)\,E\,U^{-1}(x) = 0$ for all x, which is equivalent to $F\,U(x)\,E\,U^{-1}(x) = U(x)\,E\,U^{-1}(x)$ for all x. If now Φ is any vector analytic for the energy and $g(x)$ a function with compact support in momentumspace then $\int dx\, g(x)\,U(x)\,E\,U^{-1}(x)\Phi$ is again analytic for the energy and we have the relation

$$F\int dx\, g(x)\,U(x)\,E\,U^{-1}(x)\Phi = \int dx\, g(x)\,U(x)\,E\,U^{-1}(x)\Phi\,.$$

Hence by Lemma III.2. there exists for any $\mathcal{O}_2 \gg \mathcal{O}$, a projection G with the properties $G \in \bigcap_x \mathcal{S}(\mathcal{O}_2 + x) \cap \mathcal{R}'_\infty$,

$$G\int dx\, g(x)\,U(x)\,E\,U^{-1}(x)\Phi = \int dx\, g(x)\,U(x)\,E\,U^{-1}(x)\Phi$$

for all $g(x)$ with compact support in momentumspace and all Φ analytic for the energy such that $FG \in \bigcap_x \mathcal{S}(\mathcal{O}_2 + x) \cap \mathcal{R}'_\infty$. This implies $GE\Phi = E\Phi$. On the other hand we have for $B \in \{\bigcup_x \mathcal{S}'(\mathcal{O}_2 + x)\}''$ the relation

$$\begin{aligned}BF\int dx\, g(x)\,U(x)\,E\,U^{-1}(x)\Phi &= FB\int dx\, g(x)\,U(x)\,E\,U^{-1}(x)\Phi\\ &= B\int dx\, g(x)\,U(x)\,E\,U^{-1}(x)\Phi\end{aligned}$$

which implies $FG = G$. This means $E \sim G \bmod \mathscr{S}(\mathcal{O}_2)$ since $E \sim F \geqq G \geqq E$. Choosing $\mathcal{O}_2 \gg \mathcal{O}_1 \gg \mathcal{O}$ in an arbitrary position we get the desired result since $G \in \mathscr{S}(\mathcal{O}_2) \cap \mathscr{R}'_\infty$.

IV. The structure of two sided ideals

Now we are prepared to study the two sided ideals of local ring-systems. First we need a

IV.1. Lemma. Let $\mathscr{R}_n$ be an increasing sequence of von Neumann algebras, $\mathscr{R}_m \subset \mathscr{R}_n$ for $m < n$. Denote by $\mathfrak{R}$ the normclosure of $\bigcup_n \mathscr{R}_n$. Let $\mathfrak{J}$ be a nonzero norm-closed twosided ideal of $\mathfrak{R}$ then $\mathfrak{J} \cap \mathscr{R}_n$ contains a nonzero element for some n.

Proof. Let $A = A^* \in \mathfrak{J}$ and $\|A\| = 1$. Then for some n exists an operator $B \in \mathfrak{R}_n$ with $B = B^*$ and $\|A - B\| \leqq \frac{1}{8}$. Since $\mathscr{R}_n$ is a von Neumann algebra there exist projections E_n, $n = -4, -3, \ldots +4$, $E_n E_m = 0$ for $n \neq m$ such that $\left\| B - \sum_{n=-4}^{+4} \frac{n}{4} E_n \right\| \leqq \frac{1}{8}$. From $|A| = 1$ follows that not all $E_n = 0$ for $|n| \geqq 3$. Combining both equations we get $\left\| A - \sum \frac{n}{4} E_n \right\| \leqq \frac{1}{4}$. Denote by Π a faithful representation of $\mathfrak{R}/\mathfrak{J}$; then we have $\left\| \sum \frac{n}{4} \Pi(E_n) \right\| \leqq \frac{1}{4}$. Since we have again $\Pi(E_n)\,\Pi(E_m) = \delta_{nm} \Pi(E_n)$ follows $\Pi(E_n) = 0$ for $n > 2$ or $E_n \in \mathfrak{J}$ since Π was a faithful representation of $\mathfrak{R}/\mathfrak{J}$ qed.

IV.2. Lemma. Let $\mathscr{R}_n$ and $\mathfrak{R}$ be as in the preceding lemma, and $\mathfrak{J}$ a norm closed twosided ideal then $\mathfrak{J}$ coincides with the normclosure of $\mathfrak{J} \cap \{\bigcup_n \mathscr{R}_n\}$.

Proof. Let $A = A^* \in \mathfrak{J}$ and $\|A\| = 1$. Give $\varepsilon > 0$ then exists a $\mathscr{R}_n$ and an operator $B \in \mathfrak{R}_n \cap I$ such that $\|A - B\| \leqq 2\varepsilon$. This holds since we can find a $B_1 \in \mathscr{R}_n$ with $\|A - B_1\| \leqq \varepsilon$ and a $B \in \mathscr{R}_n \cap \mathfrak{J}$ with $\|B - B_1\| \leqq \varepsilon$ (see the proof of Lemma IV.1.). But this implies A is a norm limit of elements in $\mathfrak{J} \cap \{\bigcup_n \mathscr{R}_n\}$ qed.

The combination of the last two lemmas with the results of section III gives us

IV.3. Theorem. Let $\{\mathscr{S}(\mathcal{O})\}$ be a generalized local ring system and $\{\mathscr{R}(\mathcal{O})\}$ the local ring system contained in $\{\mathscr{S}(\mathcal{O})\}$. Assume we have a vector ψ analytic for the energy such that $\mathscr{R}_\infty \psi$ is dense in $\mathscr{H}$. If $\mathfrak{J}$ is a non-trivial two-sided ideal in $\mathfrak{S}$ then

a) $\mathfrak{J} \cap \{\mathscr{R}'_\infty \cap \mathfrak{S}\}$ is a non-trivial ideal;

b) $\mathfrak{J}$ is generated by $\mathfrak{J} \cap \mathscr{R}'_\infty \cap \mathfrak{S}$ i.e. I is the smallest norm-closed ideal in $\mathfrak{S}$ containing $\mathfrak{J} \cap \mathscr{R}'_\infty \cap \mathfrak{S}$.

Proof. Let $\mathfrak{J} \subset \mathfrak{S}$ be a two-sided ideal, then by IV.1. and IV.2. $\mathfrak{J} \cap \{\bigcup_n \mathscr{S}(\mathcal{O})\}$ is not empty and $\mathfrak{J}$ is the norm closure of this set. Let now $A \in \mathfrak{J} \cap \mathscr{S}(\mathcal{O})$. Then also its symmetric and skew-symmetric parts are in $\mathfrak{J} \cap \mathscr{S}(\mathcal{O})$. Hence it is sufficient to consider the self-adjoint elements.

Let $A = A^* = \int_{-M}^{+M} \lambda\, dE_\lambda \in \mathfrak{J} \cap \mathfrak{S}(\mathcal{O})$. Since $\mathscr{S}(\mathcal{O})$ is a v. Neumann algebra we find that $\int_{-M}^{-\varepsilon} + \int_{+\varepsilon}^{M} dE_\lambda$ is also in $\mathfrak{J} \cap \mathscr{S}(\mathcal{O})$. Denote by $M(\mathcal{O})$ the set of projections in $\mathfrak{J} \cap \mathscr{S}(\mathcal{O})$. Then the ideal generated by $M(\mathcal{O})$ is norm dense in $\mathfrak{J} \cap \mathscr{S}(\mathcal{O})$ because if $A = A^* = \int_{-M}^{+M} \lambda\, dE_\lambda \in \mathfrak{J} \cap \mathscr{S}(\mathcal{O})$ then $E_{-\varepsilon} + (1 - E_{+\varepsilon})$ is contained in $M(\mathcal{O})$. Hence $A\{E_{-\varepsilon} + (1 - E_{+\varepsilon})$ is in the ideal generated by $M(\mathcal{O})$. But $\|A - (E_{-\varepsilon} + (1 - E_\varepsilon))A\| \leqq \varepsilon$ which means that A is in the norm closure of the ideal generated by $M(\mathcal{O})$. Since now $\mathfrak{J}$ is a two-sided ideal and $\mathscr{S}(\mathcal{O})$ a von Neumann algebra, it follows from $E \sim F \bmod \mathscr{S}(\mathcal{O})$ and $E \in M(\mathcal{O})$ that also $F \in M(\mathcal{O})$. Now by theorem III.3. follows that for $\mathcal{O}_1 \gg \mathcal{O}$ there exists a projection F in $\mathscr{S}(\mathcal{O}_1) \cap \mathscr{R}'_\infty$ with $FE = E$ and $F \sim E \bmod \mathscr{S}(\mathcal{O}_1)$. Hence $\mathfrak{J} \cap \mathscr{R}'_\infty \cap \mathfrak{S}$ is a non-trivial ideal since $1 \notin \mathfrak{J}$. This proves a). Let now $\mathfrak{H}$ be the two-sided ideal generated by $\mathfrak{J} \cap \mathscr{R}'_\infty \cap \mathfrak{S}$ then $\mathfrak{H} \subset \mathfrak{J}$. But $\bigcup_{\mathcal{O}} M(\mathcal{O})$ generates $\mathfrak{J}$. If $E \in M(\mathcal{O})$ there exist $F \sim E \bmod \mathscr{S}(\mathcal{O}_1)$ and $FE = E$ with $F \in \mathfrak{J} \cap \cap \mathscr{R}'_\infty \cap \mathfrak{S}$. Hence $E \in \mathfrak{H}$ which implies $\bigcup_{\mathcal{O}} M(\mathcal{O}) \subset \mathfrak{H}$ or $\mathfrak{J} \subset \mathfrak{H}$ and thus $\mathfrak{H} = \mathfrak{J}$ which proves statement b) and the theorem.

V. Application to local ring systems

If we restrict ourselves to local ring systems then it is possible to remove the assumption about the existence of a vector which is cyclic for $\mathscr{R}_\infty$. Theorem III.3. becomes:

V.1. Theorem. Let $\{\mathscr{R}(\mathcal{O})\}$ be a local ring system and E be a projection in $\mathscr{R}(\mathcal{O})$

a) If $\mathcal{O}_1 > \mathcal{O}$ and F is the smallest projection in the center $\mathfrak{Z}(\mathscr{R}(\mathcal{O}_1))$ with $FE = E$ then $E \sim F \bmod \mathscr{R}(\mathcal{O}_1)$,

b) Is $\mathcal{O}_1 \gg \mathcal{O}$ then $F \in \mathfrak{Z}(\mathscr{R}(\mathcal{O}_1)) \cap \mathfrak{Z}(\mathfrak{R})$, where $\mathfrak{Z}(\mathfrak{R})$ denotes the center of the C^*-algebra $\mathfrak{R}$.

Proof. Let G_α be a family of projections in $\mathscr{R}^1_\infty$ such that $G_\alpha G_\beta = 0$ for $\alpha \neq \beta$, $\sum_\alpha G_\alpha = 1$ and in $G_\alpha \mathscr{H}$ exists a vector ψ_α analytic for the energy such that $\mathscr{R}_\infty \psi_\alpha = G_\alpha \mathscr{H}$. By virtue of theorem III.3. we have $F G_\alpha \sim E G_\alpha \bmod \mathscr{R}(\mathcal{O}_1) \cdot G_\alpha$. Let F_α be the smallest projection in $\mathfrak{Z}(\mathscr{R}(\mathcal{O}_1))$ with $F_\alpha G_\alpha = G_\alpha$ then we get $F_\alpha F \sim F_\alpha E \bmod \mathscr{R}(\mathcal{O}_1) \cdot F_\alpha$ ([7] chap. I § 2 prop. 2). But since now $\bigcup_\alpha F_\alpha \mathscr{H} = \mathscr{H}$ follows $F \sim E \bmod \mathscr{R}(\mathcal{O}_1)$. This proves statement a). Let now $\mathcal{O}_1 \gg \mathcal{O}$, then by theorem III.3. b) we have $F \cdot G_\alpha \in \mathscr{R}'_\infty$. Hence $F = \sum_\alpha F E_\alpha \in \mathscr{R}'_\infty$ which proves b).

This result makes it possible to generalize also theorem IV.3. we get:

V.2. Theorem. Let $\{\mathscr{R}(\mathcal{O})\}$ be a local ring system and $\mathfrak{Z}$ be the center of $\mathfrak{R}$. Denote by $\mathfrak{J}$ norm-closed two-sided ideals of $\mathfrak{R}$ then

a) $\mathfrak{J}$ is not the zero ideal if and only if $\mathfrak{J} \cap \mathfrak{Z}$ is not the zero ideal

b) $\mathfrak{J}$ is generated by $\mathfrak{J} \cap \mathfrak{Z}$.

c) The map $\mathfrak{J} \to \mathfrak{J} \cap \mathfrak{Z}$ is one-to-one mapping from the two-sided ideals of $\mathfrak{R}$ onto the ideals of $\mathfrak{Z}$.

Proof. Since we have used in the proof of theorem IV.3. only the fact that to every projection $E \in \mathscr{S}(\mathcal{O})$ and $\mathcal{O}_1 \gg \mathcal{O}$ exists a projection $F \in \mathscr{S}(\mathcal{O}_1) \cap \mathscr{R}'_\infty$ with $F \sim E$ and $FE = E$ the statements a) and b) are a simple consequence of IV.3. and V.1. Now statement c) follows from the fact that $\mathfrak{Z}$ commutes with $\mathfrak{R}$. Hence if $\mathfrak{K}$ is an ideal in $\mathfrak{Z}$ the ideal generated by $\mathfrak{K}$ is $\mathfrak{R} \cdot \mathfrak{K}$ which implies that $\mathfrak{K} = \mathfrak{Z} \cap \mathfrak{R} \cdot \mathfrak{K}$ or together with b) the map $\mathfrak{J} \to \mathfrak{J} \cap \mathfrak{Z}$ is one-to-one.

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