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# An Exact Solution for Uniformly Accelerated Particles in General Relativity

## III. Singular Expanding (Contracting) Null Surface

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**Abstract.** Recently an exact solution of Einstein's empty-space equations referring to four uniformly accelerated particles was given. The relation of this to static axially symmetric metrics of the Weyl and Einstein-Rosen classes is investigated in the present paper. A physical interpretation of the singularity along half of the axis of symmetry of the uniformly accelerated metric in Weyl's form is given.
An exact solution corresponding to an expanding (contracting) singular null surface is obtained by a limiting process from that for uniformly accelerated particles.

## § 1. Introduction

In a recent paper prepared with Professor W. B. Bonnor [1, 2], I gave an exact solution of Einstein's field equations

$$R_{ik} = 0\,, \tag{1.1}$$

referring to a number of uniformly accelerated particles. In this paper I investigate the relation of this metric to the metrics of Weyl and Einstein-Rosen.

An exact solution of (1.1) corresponding to a singular null surface moving with speed of light is obtained by a limiting process from a solution for two uniformly accelerated particles. The singularities of (1.1) are expected to represent mass or stresses. Hence the physical interpretation becomes difficult in the case of a singular null surface.

The plan of the paper is as follows. In § 2 I consider some transformations of flat space-time, and give a physical interpretation of the singularity along half the axis of symmetry of a uniformly accelerated metric in the Weyl form. In § 3 the derivations of the metric from Weyl and Einstein-Rosen metrics are given. The solution corresponding to a singular null surface is obtained in § 4 and the paper ends with a summary of the results.

## § 2. Some transformations of flat space-time

Let us start with flat space-time in cylindrical co-ordinates

$$ds^2 = -dr^2 - r^2\, d\theta^2 - dz^2 + dt^2\,, \tag{2.1}$$

and transform by

$$r = r, \quad \theta = \theta, \quad z = \zeta \operatorname{Cosh}\tau, \quad t = |\zeta| \operatorname{Sinh}\tau\,, \tag{A}$$

to the uniformly accelerated metric [3, 4, 5]

$$ds^2 = -dr^2 - r^2\, d\theta^2 - d\zeta^2 + \zeta^2\, d\tau^2\,. \tag{2.2}$$

From (A) it is clear that (2.2) covers only that part of the space-time for which

$$z^2 > t^2\,. \tag{2.3}$$

The part of the space-time covered by (2.2) is seperated from the rest of (2.1) by the null surface

$$\zeta^2 = z^2 - t^2 = 0\,. \tag{2.4}$$

The metric (2.2) has the following property: a test particle whose world-line in (2.1) is

$$r = 0, \quad z = (2H)^{1/2} \operatorname{Cosh}(2H)^{-1/2} s, \quad t = (2H)^{1/2} \operatorname{Sinh}(2H)^{-1/2} s\,, \tag{2.5}$$

where $H$ is a constant, has in (2.2) the world-line

$$r = 0, \quad \zeta = (2H)^{1/2}, \quad \tau = (2H)^{-1/2} s\,. \tag{2.6}$$

Now the world-line (2.5), which satisfies

$$z^2 - t^2 = 2H \tag{2.7}$$

is that of a particle moving with uniform proper acceleration (i.e. one measured in the rest frame of the particle) of magnitude $(2H)^{-1/2}$ in Minkowski space-time (2.1). It is evident from (2.6) that *all the uniformly accelerated particles have constant* $\zeta$. For this reason (2.2) may be called the uniformly accelerated metric. In the limit of $H$ tending to zero i.e. the magnitude of the uniform acceleration becoming infinite (2.7) reduces to (2.4).

We next recall the Weyl metric:

$$ds^2 = -e^{\bar{\lambda}} (d\bar{r}^2 + d\bar{z}^2) - \bar{r}^2\, e^{-\bar{\varrho}}\, d\bar{\theta}^2 + e^{\bar{\varrho}}\, d\bar{t}^2\,, \tag{2.8}$$

where $\bar{\varrho}$ and $\bar{\lambda}$ are functions of $\bar{r}$ and $\bar{z}$ only. This is sufficiently general for the description of all axially symmetric static fields in vacuo. $\bar{\varrho}$ satisfies

$$\nabla^2 \bar{\varrho} \overset{\text{def}}{=} \frac{\partial^2 \bar{\varrho}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{\varrho}}{\partial \bar{r}} + \frac{\partial^2 \bar{\varrho}}{\partial \bar{z}^2} = 0\,, \tag{2.9}$$

and once $\bar{\varrho}$ is known $\bar{\lambda}$ is determined, apart from an additive constant, by the remaining field equations ([6], Chap. VIII).

The transformation

$$\left.\begin{aligned} r &= +\{+(\bar{r}^2+\bar{z}^2)^{1/2}-\bar{z}\}^{1/2}, \\ \theta &= \bar{\theta}, \\ \zeta &= \pm\{+(\bar{r}^2+\bar{z}^2)^{1/2}+\bar{z}\}^{1/2}, \\ \tau &= \bar{t}, \end{aligned}\right\} \tag{B}$$

where the $+$ or $-$ signs are to be chosen according as $\zeta$ is positive or negative, takes (2.2) into Weyl's form, namely

$$ds^2 = -e^{\bar{\lambda}_{(0)}}(d\bar{r}^2+d\bar{z}^2) - \bar{r}^2\, e^{-\bar{\varrho}_{(0)}}\, d\bar{\theta}^2 + e^{\bar{\varrho}_{(0)}}\, d\bar{t}^2\,, \tag{2.10}$$

where

$$\bar{\varrho}_{(0)} = \log(R+\bar{z})\,, \tag{2.11}$$

$$\bar{\lambda}_{(0)} = \log\left(\frac{1}{2R}\right), \tag{2.12}$$

$$R = +(\bar{r}^2+\bar{z}^2)^{1/2}\,. \tag{2.13}$$

The world-line (2.6) now becomes

$$\bar{r} = 0, \quad \bar{z} = H, \quad \bar{t} = (2H)^{-1/2}s\,. \tag{2.14}$$

As $H$ tends to zero the proper length $s$ along the trajectory of the test particle vanishes. Rindler (1960) has given a covariant definition of uniformly accelerated motion for test particles and shown that the trajectory of the test particle becomes a null geodesic, when the magnitude of its acceleration becomes infinite (i.e. when $H$ tends zero). Thus a test particle (photon!) at rest in (2.10) at $\bar{r}=0$, $\bar{z}=0$ will have the speed of light relative to (2.1). We may regard this point $\bar{r}=0$, $\bar{z}=0$, as a point dividing the $\bar{z}$-axis into two parts. A test particle at rest on the part with $\bar{z}>0$, will have speed less than that of light with respect to (2.1), while if there be a test particle at rest at any point of the part with $\bar{z}<0$, it would be expected to acquire speed greater than that of light relative to (2.1). Thus the entire negative $\bar{z}$-axis is a forbidden line from a physical point of view, and indeed it is singular. However this singularity is only due to the peculiar choice of the co-ordinate system, and can ultimately be traced back to the null surface (2.4), which forms the boundary of the region of validity of (2.2). We ought to remember that (2.10) represents flat space-time, since it was obtained from (2.1) by transformations. In fact (2.10) can be transformed back into (2.1) by means of

$$\left.\begin{aligned}\bar{r} &= +(z^2 - t^2)^{1/2} r\,,\\ \bar{\theta} &= \theta\,,\\ \bar{z} &= \frac{1}{2}(z^2 - t^2 - r^2)\,,\\ \bar{t} &= \tanh^{-1}\frac{t}{|z|}\,.\end{aligned}\right\}\qquad (C)$$

The transformation (C) maps the whole Weyl space-time onto that half of the space-time (2.1) for which $z^2 \geqq t^2$. In the mapping the point $(\bar{r}_1, \bar{\theta}_1, \bar{z}_1, \bar{t}_1)$ passes into the points $(r_1, \theta_1, z_1, t_1)$ and $(r_1, \theta_1, -z_1, t_1)$. The region $R > 0$ formally corresponds to the region $r^2 + \zeta^2 > 0$ of (2.2), which is the same as the region $r^2 + z^2 > t^2$ of (2.1).

Next transform (2.1) by

$$r = r, \quad \theta = \theta, \quad z = |\tau^*|\,\mathrm{Sinh}\,\zeta^*, \quad t = \tau^*\,\mathrm{Cosh}\,\zeta^* \qquad (A^*)$$

to the metric

$$ds^2 = -dr^2 - r^2\,d\theta^2 - \tau^{*2}\,d\zeta^{*\,2} + d\tau^{*\,2}\,. \qquad (2.15)$$

It is obvious that (2.15) covers only that part of space-time for which

$$t^2 > z^2\,. \qquad (2.16)$$

The part of space-time covered by (2.15) is separated from the rest by the null surface

$$\tau^{*2} = t^2 - z^2 = 0\,, \qquad (2.17)$$

which is the same as (2.4).

A test particle whose world-line in (2.1) is given by

$$r = 0, \quad z = |s|\,\mathrm{Sinh}\,\chi, \quad t = s\,\mathrm{Cosh}\,\chi \qquad (2.18)$$

where $\chi$ is a constant, has in (2.15) the world-line

$$r = 0, \quad \zeta^* = \chi, \quad \tau^* = s\,. \qquad (2.19)$$

Now (2.18) represents a test-particle moving with uniform velocity of magnitude $\left|\frac{dz}{dt}\right| = |\tanh\chi|$ relative to (2.1). It follows from (2.19) that *all particles at rest in* (2.15) *are in uniform rectilinear motion with respect to* (2.1), *and that the magnitude of the velocity of each particle is determined by its* $\zeta^*$ *co-ordinate in* (2.15). MARDER [7] described this property by stating that (2.15) corresponds to a uniform one-dimensional expansion of the co-ordinate system in positive and negative $z$-directions.

It is interesting to note that all particles at rest in (2.1) are *not* in uniform rectilinear motion relative to (2.15). To verify this statement let us consider a test particle at rest in (2.1). Its world-line is given by

$$r = 0, \quad z = p, \quad t = s\,, \tag{2.20}$$

where $p$ is a constant.

Only that part of (2.20) for which $s^2 > p^2$ will be mapped onto (2.15) by means of the transformation (A*). As a particular case let $p > 0$, and consider only that part of the world-line for which $s > p > 0$. The equation of the world-line in (2.15) is

$$r = 0, \quad \zeta^* = \tanh^{-1}\left(\frac{p}{s}\right), \quad \tau^* = (s^2 - p^2)^{1/2}\,. \tag{2.21}$$

Clearly (2.21) does not represent a world-line of a test particle moving with uniform velocity $d\zeta^*/d\tau^*$, relative to (2.15).

We next recall Einstein-Rosen metric

$$ds^2 = e^{\lambda^*}(dt^{*2} - dr^{*2}) - r^{*2}\,e^{-\varrho^*}\,d\theta^{*2} - e^{\varrho^*}\,dz^{*2}\,, \tag{2.22}$$

where $\varrho^*$ and $\lambda^*$ are functions of $r^*$ and $t^*$ only. One of the field equations for (2.22) is

$$\frac{\partial^2 \varrho^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial \varrho^*}{\partial r^*} - \frac{\partial^2 \varrho^*}{\partial t^{*2}} = 0\,. \tag{2.23}$$

Once $\varrho^*$ is known $\lambda^*$ is determined, apart from a constant, by the remaining field equations. There is a close formal connection between (2.22) and Weyl's metric (2.8). We obtain (2.22) from (2.8) by the substitution ([6], Chap. IX)

$$(\bar{z}, \bar{t}) \to (it^*, iz^*)\,,$$

$$(\bar{r}, \bar{\theta}) \to (r^*, \theta^*) \quad \text{and} \quad (\bar{\varrho}, \bar{\lambda}) \to (\varrho^*, \lambda^*)\,.$$

The transformation

$$\left.\begin{aligned} r &= +\,\{t^* - (t^{*2} - r^{*2})^{1/2}\}^{1/2}\,, \\ \theta &= \theta^*\,, \\ \zeta^* &= z^*\,, \\ \tau^* &= \pm\,\{t^* + (t^{*2} - r^{*2})^{1/2}\}^{1/2}\,, \end{aligned}\right\} \tag{B*}$$

where we choose the + or — sign according as $\tau^*$ is positive or negative, takes (2.15) into Einstein-Rosen form, namely

$$ds^2 = e^{\overset{*}{\lambda}_{(0)}}(dt^{*2} - dr^{*2}) - r^{*2}\,e^{-\overset{*}{\varrho}_{(0)}}\,d\theta^{*2} - e^{\overset{*}{\varrho}_{(0)}}\,dz^{*2}\,, \tag{2.24}$$

Where

$$\overset{*}{\varrho}_{(0)} = \log(t^* + T)\,, \tag{2.25}$$

$$\overset{*}{\lambda}_{(0)} = \log\left(\frac{1}{2T}\right), \tag{2.26}$$

$$T = +\,(t^{*2} - r^{*2})^{1/2}\,. \tag{2.27}$$

The world-line (2.19) is now given by

$$r^* = 0, \quad z^* = \chi, \quad t^* = \frac{1}{2} s^2 . \tag{2.28}$$

The region of validity of (2.24) is

$$t^* > r^* > 0 , \tag{2.29}$$

and is seperated from the rest of the space-time by the null surface

$$2\,(t^{*2} - r^{*2})^{1/2} = \tau^{*2} - r^2 = 0 . \tag{2.30}$$

The metric (2.24) with (2.25) and (2.26) represents flat space-time, and in fact can be transformed into (2.1) by means of

$$\left.\begin{aligned} r^* &= +(t^2 - z^2)^{1/2} r , \\ \theta^* &= \theta , \\ z^* &= \tanh^{-1} \frac{z}{|t|} , \\ t^* &= \frac{1}{2}(r^2 + t^2 - z^2) . \end{aligned}\right\} \tag{C*}$$

Each event $(r_1^*, \theta_1^*, z_1^*, t_1^*)$ of (2.24) is mapped into two events $(r_1, \theta_1, z_1, t_1)$ and $(r_1, \theta_1, z_1, -t_1)$ of (2.1). The region $t^{*2} > r^{*2}$ of validity of (2.25) corresponds to the region

$$t^2 > (r^2 + z^2) \tag{2.31}$$

of (2.1).

## § 3. Derivations of the metric

Because equation (2.9) is linear, we can simply add solutions for $\bar{\varrho}$. Let $\bar{\varrho}_{(1)}$ and $\bar{\lambda}_{(1)}$ be two functions satisfying the field equations (1.1) for the metric (2.8). We know that $\bar{\varrho}_{(0)}$ and $\bar{\lambda}_{(0)}$ as given by (2.11) and (2.12), also satisfy the same field equations. We can take

$$\bar{\varrho} = \bar{\varrho}_{(0)} + \bar{\varrho}_{(1)} \tag{3.1}$$

as a new solution. The corresponding $\bar{\lambda}$ turns out to be

$$\bar{\lambda} = \bar{\lambda}_{(0)} + \bar{\lambda}_{(1)} + \bar{\lambda}_{(2)} , \tag{3.2}$$

where $\bar{\lambda}_{(2)}$ must satisfy

$$\frac{\partial \bar{\lambda}_{(2)}}{\partial \bar{r}} = \bar{r} \left[ \frac{\partial \bar{\varrho}_{(0)}}{\partial \bar{r}} \frac{\partial \bar{\varrho}_{(1)}}{\partial \bar{r}} - \frac{\partial \bar{\varrho}_{(0)}}{\partial \bar{z}} \frac{\partial \bar{\varrho}_{(1)}}{\partial \bar{z}} \right] , \tag{3.3}$$

and

$$\frac{\partial \bar{\lambda}_{(2)}}{\partial \bar{z}} = \bar{r} \left[ \frac{\partial \bar{\varrho}_{(0)}}{\partial \bar{r}} \frac{\partial \bar{\varrho}_{(1)}}{\partial \bar{z}} + \frac{\partial \bar{\varrho}_{(1)}}{\partial \bar{r}} \frac{\partial \bar{\varrho}_{(0)}}{\partial \bar{z}} \right] . \tag{3.4}$$

These relations are compatible because of the field equations and determine $\bar{\lambda}_{(2)}$ except for an additive constant. $\bar{\lambda}_{(2)}$ is the interaction term and arises on the account of the non-linearity of the field equations other than (2.9).

As a solution for two particles we take [8, 9]

$$ds^2 = -e^{\bar{\lambda}_{(1)}}(d\bar{r}^2 + d\bar{z}^2) - \bar{r}^2 e^{-\bar{\varrho}_{(1)}} d\bar{\theta}^2 + e^{\bar{\varrho}_{(1)}} d\bar{t}^2 , \tag{3.5}$$

where

$$\bar{\varrho}_{(1)} = -\frac{2a_1}{R_1} - \frac{2a_2}{R_2} + K , \tag{3.6}$$

$$\bar{\lambda}_{(1)} = \frac{2a_1}{R_1} + \frac{2a_2}{R_2} - \bar{r}^2\left(\frac{a_1^2}{R_1^4} + \frac{a_2^2}{R_2^4}\right) + \frac{a_1 a_2}{(h_1 - h_2)^2} f + K , \tag{3.7}$$

$$f = 4\left[\frac{\bar{r}^2 + (\bar{z} - h_1)(\bar{z} - h_2)}{R_1 R_2} - 1\right], \tag{3.8}$$

$$R_\alpha = + \{\bar{r}^2 + (\bar{z} - h_\alpha)^2\}^{1/2} \qquad (\alpha = 1, 2) , \tag{3.9}$$

where $a_\alpha$, $K$ and $h_\alpha$ are constants and $h_1 > h_2 > 0$.

This solution represents two particles of masses $a_\alpha$ at rest on the $\bar{z}$-axis at $\bar{z} = h_\alpha$. Stresses are necessary to keep them at rest ([6], Chap. VIII); these are represented by singularities along various stretches of $0\bar{z}$ depending on the value of $K$ (for details see [9]).

Now we superimpose the two solutions (2.10) and (3.5) to obtain a new solution

$$ds^2 = - e^{\bar{\lambda}_{(0)} + \bar{\lambda}_{(1)} + \bar{\lambda}_{(2)}}(d\bar{r}^2 + d\bar{z}^2) - \bar{r}^2 e^{-(\bar{\varrho}_{(0)} + \bar{\varrho}_{(1)})} d\bar{\theta}^2 + e^{(\bar{\varrho}_{(0)} + \bar{\varrho}_{(1)})} d\bar{t}^2 , \tag{3.10}$$

where $\bar{\varrho}_{(0)}$, $\bar{\lambda}_{(0)}$ and $\bar{\varrho}_{(1)}$, $\bar{\lambda}_{(1)}$ are the same as in (2.10) and (3.5), and $\bar{\lambda}_{(2)}$ is the interaction term to be calculated from (3.3) and (3.4). We find that

$$\bar{\lambda}_{(2)} = \frac{2a_1(R - h_1)}{h_1 R_1} + \frac{2a_2(R - h_2)}{h_2 R_2} + C , \tag{3.11}$$

where

$$R^2 = (\bar{r}^2 + \bar{z}^2) , \tag{3.12}$$

and $C$ is a constant of integration.

The solution (3.10) represents two particles at rest in a uniformly accelerated co-ordinate system. Stresses along various stretches of $0\bar{z}$ are present and are represented by singularities.

It also contains an additional singularity along the entire negative $\bar{z}$-axis. Such a singularity is present even in the flat space-time (2.10). This singularity disappears on applying the transformation (C) to the metric (3.10).

The resulting metric is

$$\begin{aligned} ds^2 = -e^{\lambda} dr^2 - r^2 e^{-\varrho} d\theta^2 + (z^2 - t^2)^{-1} \{(z^2 e^{\varrho} - t^2 e^{\lambda}) dt^2 - \\ - (z^2 e^{\lambda} - t^2 e^{\varrho}) dz^2 + 2zt(e^{\lambda} - e^{\varrho}) dz\, dt\} , \end{aligned} \tag{3.13}$$

where

$$\varrho = -\frac{2a_1}{R_1} - \frac{2a_2}{R_2} + \frac{2a_1}{h_1} + \frac{2a_2}{h_2} + \log\beta\,, \tag{3.14}$$

$$\lambda = \frac{a_1 a_2}{(h_1 - h_2)^2} f - r^2(z^2 - t^2)\left(\frac{a_1^2}{R_1^4} + \frac{a_2^2}{R_2^4}\right) + \frac{2a_1 R}{h_1 R_1} + \frac{2a_2 R}{h_2 R_2} + \log\beta\,, \tag{3.15}$$

$$R = \frac{1}{2}(r^2 + z^2 - t^2)\,, \tag{3.16}$$

$$R_\alpha = |\{(R - h_\alpha)^2 + 2r^2 h_\alpha\}^{1/2}| \quad (\alpha = 1, 2)\,, \tag{3.17}$$

$$f = 4R_1^{-1}R_2^{-1}\{r^2(z^2 - t^2) + (R - r^2 - h_1)(R - r^2 - h_2) - R_1 R_2\}, \tag{3.18}$$

and $\beta$ is a new positive constant. The constants in (3.14) to (3.18) have been chosen as in the previous paper [1]. They are the same as those in (3.6) to (3.9) and (3.11) except that $K$ and $C$ are suitably chosen in terms of $\beta$ and other constants.

This solution represents four uniformly accelerated particles and stress singularities along various stretches of $z$-axis and was discussed in [1]. We may note that whereas the masses of the static particles in the metric (3.5) were $a_\alpha$, the masses of the accelerated particles are [1, 3]

$$a_\alpha(2h_\alpha)^{-1/2}\,.$$

On account of the linearity of (2.23) we can also add solutions for $\varrho^*$. Two known solutions $(\varrho^*_{(0)}, \lambda^*_{(0)})$ and $(\varrho^*_{(1)}, \lambda^*_{(1)})$ of the field equations for (2.22) can be superimposed to get a third solution $(\varrho^*_{(0)} + \varrho^*_{(1)}, \lambda^*_{(0)} + \lambda^*_{(1)} + \lambda^*_{(2)})$. $\lambda^*_{(2)}$ is the interaction term and arises on account of the non-linearity of the other field equations. It must satisfy

$$\frac{\partial \lambda^*_{(2)}}{\partial r^*} = r^*\left[\frac{\partial \varrho^*_{(0)}}{\partial r^*}\frac{\partial \varrho^*_{(1)}}{\partial r^*} + \frac{\partial \varrho^*_{(0)}}{\partial t^*}\frac{\partial \varrho^*_{(1)}}{\partial t^*}\right], \tag{3.19}$$

and

$$\frac{\partial \lambda^*_{(2)}}{\partial t^*} = r^*\left[\frac{\partial \varrho^*_{(0)}}{\partial r^*}\frac{\partial \varrho^*_{(1)}}{\partial t^*} + \frac{\partial \varrho^*_{(1)}}{\partial r^*}\frac{\partial \varrho^*_{(0)}}{\partial t^*}\right]. \tag{3.20}$$

These equations are consistent because of the field equations and determine $\lambda^*_{(2)}$ except for an additive constant.

Consider a non-flat Einstein-Rosen metric:

$$ds^2 = e^{\lambda^*}(dt^{*2} - dr^{*2}) - r^{*2}e^{-\varrho^*}d\theta^{*2} - e^{\varrho^*}dz^{*2}\,, \tag{3.21}$$

where

$$\varrho^* = -\frac{2a_1}{R_1^*}\,, \tag{3.22}$$

$$\lambda^* = \frac{2a_1}{R_1^*} + \frac{r^{*2}a_1^2}{R_1^{*4}}\,, \tag{3.23}$$

$$R_1^* = +\{(t^* + h_1)^2 - r^{*2}\}^{1/2}\,, \tag{3.24}$$

and $a_1$, $h_1$ are constants.

This solution represents an expanding singular null surface

$$r^* = t^* + h_1\,, \tag{3.25}$$

which in fact is the boundary of the region $t^* + h_1 > r^* > 0$, of validity of the metric under consideration.

Now consider the metric

$$ds^2 = e^{\overset{*}{\lambda}_{(1)}}(dt^{*2} - dr^{*2}) - r^{*2} e^{-\overset{*}{\varrho}_{(1)}}\, d\theta^{*2} - e^{\overset{*}{\varrho}_{(1)}}\, dz^{*2}\,, \tag{3.26}$$

where

$$\overset{*}{\varrho}_{(1)} = -\frac{2a_1}{R_1^*} - \frac{2a_2}{R_2^*} + K\,, \tag{3.27}$$

$$\overset{*}{\lambda}_{(1)} = \frac{2a_1}{R_1^*} + \frac{2a_2}{R_2^*} + r^{*2}\left(\frac{a_1^2}{R_1^{*4}} + \frac{a_2^2}{R_2^{*4}}\right) + \frac{a_1 a_2}{(h_1 - h_2)^2} f - K\,, \tag{3.28}$$

$$f = 4R_1^{*-1} R_2^{*-1}\{(t^* + h_1)(t^* + h_2) - r^{*2} - R_1^* R_2^*\}\,, \tag{3.29}$$

$$R_\alpha^* = |\{(t^* + h_\alpha)^2 - r^{*2}\}^{1/2}| \qquad (\alpha = 1, 2)\,, \tag{3.30}$$

where $a_\alpha$, $K$ and $h_\alpha$ are constants and $h_1 > h_2 > 0$. The constant $K$ is introduced for latter convenience. The solution (3.26) may be obtained by combining two solutions of the type (3.21). However the surface $r^* = t^* + h_1$, lies outside the region $t^* + h_2 > r^* > 0$, of validity of the metric (3.26). Hence the physical significance of (3.26) is not clear.

Next we obtain a new solution by superimposing the solutions (2.24) and (3.26). The result is

$$\begin{aligned} ds^2 = {} & e^{\overset{*}{\lambda}_{(0)} + \overset{*}{\lambda}_{(1)} + \overset{*}{\lambda}_{(2)}}(dt^{*2} - dr^{*2}) - \\ & - r^{*2} e^{-(\overset{*}{\varrho}_{(0)} + \overset{*}{\varrho}_{(1)})}\, d\theta^{*2} - e^{(\overset{*}{\varrho}_{(0)} + \overset{*}{\varrho}_{(1)})}\, dz^{*2}\,, \end{aligned} \tag{3.31}$$

where $\overset{*}{\varrho}_{(0)}$, $\overset{*}{\lambda}_{(0)}$ and $\overset{*}{\varrho}_{(1)}$, $\overset{*}{\lambda}_{(1)}$ are given by (2.25), (2.26) and (3.27), (3.28); and $\overset{*}{\lambda}_{(2)}$ is the interaction term to be calculated from (3.19) and (3.20). It is found that

$$\overset{*}{\lambda}_{(2)} = -\frac{2a_1(T + h_1)}{h_1 R_1^*} - \frac{2a_2(T + h_2)}{h_2 R_2^*} + C \tag{3.32}$$

where

$$T = (t^{*2} - r^{*2})^{1/2}\,, \tag{3.33}$$

and $C$ is a constant of integration.

The region of validity of (3.31) is

$$t^* > r^* > 0\,, \tag{3.34}$$

and no singularity is present in this region. If we now transform (3.31) by means of (C*) we rediscover the metric (3.13) by suitable choice of the constants $K$ and $C$. The region (3.34) of the metric (3.31) corresponds to the region

$$t^2 > (r^2 + z^2) \tag{3.35}$$

of the space-time represented by (3.13).

The transformations of the last two sections can be represented schematically by the following diagram:

$$\begin{tikzcd}
(2.2) \arrow[d, "(B)"'] & (2.1) \arrow[l, "(A)"'] \arrow[r, "(A^*)"] & (2.15) \arrow[d, "(B^*)"] \\
(2.10) \arrow[ur, "(C)"'] \arrow[d, "S"'] & & (2.24) \arrow[ul, "(C^*)"] \arrow[d, "S"] \\
(3.10) \arrow[r, "(C)"'] & (3.13) & (3.31) \arrow[l, "(C^*)"]
\end{tikzcd}$$

Here $S$ denotes superimposition of the two solutions.

## § 4. Solution for a singular null surface

Consider the metric (3.13) with

$$\varrho = -\frac{2a}{R_1}, \tag{4.1}$$

$$\lambda = -\frac{r^2(z^2 - t^2)\,a^2}{R_1^4} + \frac{2a\,R}{h\,R_1} - \frac{2a}{h}, \tag{4.2}$$

$$R = \frac{1}{2}(r^2 + z^2 - t^2), \tag{4.3}$$

$$R_1 = |\{(R-h)^2 + 2r^2h\}^{1/2}|, \tag{4.4}$$

where $a$, $h$ are constants and $h > 0$.

This is a particular case of the metric for uniformly accelerated particles. It represents two uniformly accelerated particles with the acceleration of magnitude $(2h)^{-1/2}$. Mass of each particle is $a(2h)^{-1/2}$. These particles are located at

$$r = 0, \quad z = \pm(t^2 + 2h)^{1/2}. \tag{4.5}$$

Stress singularities are present along

$$r = 0, \quad z^2 < (t^2 + 2h). \tag{4.6}$$

In the limit of $h$ tending to zero we obtain a new metric

$$\begin{aligned} ds^2 = -e^\lambda\,dr^2 - r^2\,e^{-\varrho}\,d\theta^2 + (z^2 - t^2)^{-1}\{(z^2\,e^\varrho - t^2\,e^\lambda)\,dt^2 - \\ - (z^2 e^\lambda - t^2 e^\varrho)\,dz^2 + 2zt(e^\lambda - e^\varrho)\,dz\,dt\}, \end{aligned} \tag{4.7}$$

where

$$\varrho = -\frac{2a}{R}, \tag{4.8}$$

$$\lambda = -\frac{r^2(z^2 - t^2)\,a^2}{R^4} + \frac{(z^2 - t^2 - r^2)\,a}{R^2}, \tag{4.9}$$

$$R = \frac{1}{2}(r^2 + z^2 - t^2), \tag{4.10}$$

where a is the same constant as before.

We can verify that this metric satisfies the field equations (1.1) everywhere, the only singularity being the null surface

$$r^2 + z^2 - t^2 = 0, \tag{4.11}$$

provided that $a$ is non-zero. This null surface is at infinity when $t = -\infty$, starts contracting and converges to the point $r = 0$, $z = 0$, at $t = 0$. It, then, expands and ultimately goes to infinity at $t = \infty$.

Essential singularities in the solutions for (1.1) are expected to represent mass or stress. The only singularity of this solution is a surface moving with the speed of light, so the physical interpretation becomes difficult. The singularity is of an exceptional nature as not only the $g_{ik}$ but also their determinant becomes infinite on the surface (4.11). The null surface divides the space-time into two exclusive parts in the following sense: *no test-particle (photon) can cross it and go from one part of the space-time to another.*

To examine the field in regions remote from the null surface, we consider two particular cases for which (i) $(r^2 + z^2) \gg t^2$ and (ii) $t^2 \gg (r^2 + z^2)$. We find that the leading terms in the $g_{ik}$ are exactly similar to the corresponding terms in the far-field of uniformly accelerated particles [1]. At large distances from the null surface (4.7) tends to the Minkowski metric with $t$ as time-like co-ordinate.

The Petrov type of the metric (4.7) is Type I everywhere.

## § 5. Conclusion

1. The solution (3.13) for uniformly accelerated particles can be obtained by transformation of the Weyl metric (3.10) or the Einstein-Rosen metric (3.31). Neither of these metrics cover the whole of space-time. The Weyl metric contains a co-ordinate singularity corresponding to the fact that the region covered by it is bounded by a null surface. From a physical point of view this singularity represents that part of the axis of symmetry which could be occupied only by test-particles having speed greater than that of light. The Einstein-Rosen metric contains no singularity within its region of validity, which is bounded by a null surface. Physical significance of the metric (3.31) is not clear.

2. The limiting process $h \to 0$, used to obtain (4.7) seems to be equivalent to letting the uniform acceleration of the particles tend to infinity. The resulting metric (4.7) represents a singular null surface. This is relevant to RINDLER's result that a test-particle (photon) with infinite uniform acceleration will follow a null geodesic. In the case of the solution (4.7) we cannot say anything regarding the energy (mass) carried by the singular null surface.

## References

[1] BONNOR, W. B., and N. S. SWAMINARAYAN: Z. Physik **177**, 240 (1964).
[2] SWAMINARAYAN, N. S.: Ph. D. thesis, Univ. of London (1964).
[3] BONDI, H.: Rev. Mod. Phys. **29**, 423 (1957).
[4] MARDER, L.: Proc. Cambridge Phil. Soc. **53**, 194 (1957).
[5] RINDLER, W.: Phys. Rev. **119**, 2082 (1960).
[6] SYNGE, J. L.: Relativity — The General Theory, Chapters VIII, IX. Amsterdam: North Holland Publishing Co. 1960.
[7] MARDER, L.: Proc. Roy. Soc. A **261**, 91 (1961).
[8] BONNOR, W. B.: Proc. Roy. Soc. A **251**, 55 (1959).
[9] — (to be published).