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Some Remarks about the Localization of States in a Quantum Field Theory*

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Abstract. For the case of a field theory with a nuclear space of test functions (for instance, the space of strongly decreasing test functions) compact sets of states are constructed; these correspond to sets of localized states. Only such states are considered which are elements of a fixed subspace of the entire Hilbert space. This subspace belongs to the m-point functions of order less than a certain fixed $2n$.

1. Introduction

If one compares the possibilities of a pure scattering theory with those of a quantum field theory, one is inclined to suppose that the methods of a quantum field theory are more adapted to express localized quantities. It is not quite obvious that a quantum field theory can indeed perform this. It is well known that one can construct in a relativistic quantum field theory dense subspaces of the whole Hilbert space by applying only very restricted subalgebras generated by the field operators to the cyclic vacuum-state. For example, the algebra with test functions all lying in a small neighborhood of a fixed point of four-dimensional space leads to a dense subspace of $\mathfrak{H}$.

It is the intention of the following remarks to get a preliminary idea about the possibilities of the localization of states in a quantum field theory. In this we are guided, roughly speaking, by the concept that two states which are localized at a certain time in two non-overlapping regions of three-dimensional space should be orthogonal or nearly orthogonal to each other [1]. Another approach to the problem of localization is that of KNIGHT [2] and LICHT [3] ("strict-localization"), who begin with the concept of localized observables.

As it may be more interesting to clarify the problem in the relativistic case, we will make the approach in a theory which essentially fulfills WIGHTMAN's axioms [4]. But not all the axioms are needed for the proof

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of the statements made below. It will be noted in the special cases which of the axioms are used, if it is not obvious by itself.

The results depend upon relations which exist between the space of test functions and the generated Hilbert space [4]. These have been presented in several papers [5—7]; in the second section one finds a short survey applying these relations to the special case we treat in this paper.

For the following remarks results are used which are valid for the asymptotic properties of matrix-elements with respect to the translation of the states in Minkowski space. One can find these results in papers chiefly written in Zürich [6—10]. For simplicity the following remarks are developed for the case of a real scalar field $A(x)$. Some concepts and theorems are taken from topology. These are explained in the text and references are given where necessary.

2. The topological space of test functions and the Hilbert space of states in a special case

One can show with rather general assumptions that field operators $A(x)$ defined at the point x make no sense [11]. One has to treat them as operator-valued distributions, i.e., multiply them by certain test functions and then integrate (see, however [12]):

$$\int A(x)\, \varphi(x)\, dx\,.$$

In this manner one gets well-defined operators in Hilbert space. Thereby use is frequently made of the space of strongly decreasing test functions $\mathscr{S}$. The following remarks are valid for a more extended class of test functions. For instance the countably-normed nuclear Hilbert spaces belong to this class [13] (compare also [14]). Let Φ be such a space and let the countable sequence of norms defined by means of a countable number of scalar products be denoted by

$$\|\ \|_0,\quad \|\ \|_1, \ldots, \|\ \|_n, \ldots$$

One introduces the scalar-products in such a manner that

$$\|\ \|_n \leqq \|\ \|_{n+1}$$

holds. For example, for $\mathscr{S}(E^1)$ (the strongly decreasing test functions in one variable) one defines the norm $\|\ \|_n$ by means of the scalar product

$$(\varphi, \chi)_n = \int (1+x^2)^{2n} \sum_{k=0}^{n} \bar{\varphi}^{(k)}(x)\, \chi^{(k)}(x)\, dx\,.$$

One has to extend the sum up to the n-th. derivative. By using this sequence of increasing norms for the definition of a fundamental system of neighborhoods of zero, one gets a topology which becomes more and more refined. It can be proven that the final topology is the same as that obtained from the usual definition of norms by L. SCHWARTZ [15].

In the case of several variables, which is treated later on, proof will be given in the Appendix.

Changing the point of view somewhat, one can introduce a sequence of Hilbert spaces

$$\Phi_0, \Phi_1, \ldots, \Phi_n, \ldots$$

Each space of this sequence is the completion of Φ with respect to the corresponding norm; it follows that

$$\Phi_0 \supset \Phi_1 \supset \cdots \supset \Phi_n \supset \cdots .$$

One gets Φ back again if one performs the intersection of all the spaces:

$$\Phi = \bigcap_{k=0}^{\infty} \Phi_k .$$

What property must be added so that this countably-normed Hilbert space is a nuclear one? To answer this question one should initially look at the concept of a nuclear operator. A nuclear operator B which, for example, maps one Hilbert space $\mathfrak{H}_1$ into a second Hilbert space $\mathfrak{H}_2$ can be written in its polar-decomposition $B = JA$, where J is an isometric mapping from $\mathfrak{H}_1$ into $\mathfrak{H}_2$, and A is a bounded Hermitian operator in $\mathfrak{H}_1$ with a positive spectrum. A then has the property that its spectrum is discrete and the sum over its positive eigen-values converges: $\sum_{n=0}^{\infty} \lambda_n < \infty$.

The nuclear operators therefore form a special subclass of the class of completely-continuous operators.

For the sequence of space $\Phi_0, \Phi_1, \ldots, \Phi_n, \ldots$ one can define a system of mappings $g_{qr}: \Phi_r \to \Phi_q$, $r > q$: each element of Φ is an element of Φ_r and of Φ_q. The symbol g_{qr} is defined as the unique continuation of the identity mapping for the elements of Φ into the whole space Φ_r. The property of nuclearity of Φ is defined in the following way:

(A) To each space Φ_q one can find a certain space Φ_r so that the mapping g_{qr} from Φ_r into Φ_q is performed by a nuclear operator.

In the next section we wish to discuss the special case of the Hilbert space of states given by the $2n$-point distribution:

$$(\Omega, A(x_1)\, A(x_2) \ldots A(x_{2n})\, \Omega) \quad (\Omega \text{ the vacuum-state}) .$$

To this distribution belongs a linear space of states of the form

$$\{A(\varphi_1)\, A(\varphi_2) \ldots A(\varphi_m)\, \Omega\}, \quad \varphi_j \in \mathscr{S}(E^4),$$

which becomes a certain Hilbert space via completion. For simplicity this will be denoted by $\mathfrak{H}$ in this paper. From the nuclear theorem of L. Schwartz [17], valid for nuclear space of test functions, one knows that $\mathfrak{H}$ contains the more general states

$$\{\int A(x_1), \ldots, A(x_m)\, \varphi(x_1, \ldots, x_m)\, dx_1, \ldots, dx_m \cdot \Omega\} \quad \varphi \in \mathscr{S}(E^{4m}) .$$

Therefore, one can look upon the construction of $\mathfrak{H}$ as a mapping from the space of test functions $\mathcal{S}(E^{4m})$, now identified with Φ, into the Hilbert space $\mathfrak{H}$. The properties of this mapping have been investigated in the more general case of the construction of the whole Hilbert space by several authors [5—7]. From these papers the following properties of Φ, $\mathfrak{H}$, and the mapping j from Φ into $\mathfrak{H}$ are known and will be used in the following sections:

(B) the mapping j is continuous,

(C) the spaces Φ and $\mathfrak{H}$ are separable.

Besides Φ and j there is given a group of symmetry G (here it is assumed that G is the Poincaré group L_i). The group G induces linear continuous transformations $T(a, \Lambda)$ in Φ and is represented by unitary operators $U(a, \Lambda)$ in $\mathfrak{H}$. The transformations of G commute with j:

$$j\{T(a, \Lambda)\,\varphi\} = U(a, \Lambda)\,\{j(\varphi)\}\,. \tag{1}$$

3. Sets of Φ which are mapped into compact sets of $\mathfrak{H}$

One can imagine the following situation:

a) The physical system is enclosed in a box with potential walls of infinite height.

b) The number of kinds of particles enclosed in the box is finite and the rest-masses of the particles are not equal to zero. Thereby one assumes that the box is large enough that one can describe a state of the system as a superposition of free particle states.

If one regards the states with energy between zero (only the given vacuum-state has energy zero) and a certain upper limit for the energy, then one gets a finite-dimensional subspace of the Hilber space. As one cannot expect to get such a simple situation in the case of a quantum field theory, HAAG has proposed that in this case one should try to describe sets of localized states with an upper limit for the energy by the concept of compact sets of states in the Hilbert space*.

In a metric space the compactness of a set is usually defined by a property of sequences. (A subset M of a metric space is called relative compact if each sequence with an infinite number of different elements in M has at least one point of accumulation, and is called compact if the points of accumulation lie in M.) However, in connection with the situation described at the beginning of this section, another definition can be used. In a metric space each compact set M has a finite ε-net (see e.g. [16], p. 56). This means for instance, in the case of a Hilbert space: given an arbitrary fixed $\varepsilon > 0$ one can introduce a finite number

* I am very grateful to Professor HAAG for giving me in a private discussion the suggestion of the physical situation described at the beginning of this section.

of states $y_k \in M$ in such a way that there can be found for each $x \in M$ at least one y_k with $\|x - y_k\| \leqq \varepsilon$.

Therefore with respect to the limitations of physical observations a compact set of states is closely related to a finite set.

For the space $\mathscr{S}(E^{4m})$ one has the countable set of norms

$$\|\varphi\|_q^2 = \int (1+\varrho^2)^{2q} \sum_{k=0}^{q} \bar{\varphi}^{(k)} \cdot \varphi^{(k)}\, dx \tag{2}$$

where

$$\varrho^2 = \sum_{j=1}^{m} x_j^2\,, \quad x_j^2 = (x_j^0)^2 + \mathrm{x}_j^2 \tag{3}$$

and

$$\varphi^{(k)} \cdot \varphi^{(k)} = \sum_{\nu_1,\ldots,\nu_k = x_1^0}^{x_m^3} \partial_{\nu_1}, \ldots, \partial_{\nu_k} \bar{\varphi}(x_1, \ldots, x_m) \times \tag{4}$$
$$\times\, \partial_{\nu_1}, \ldots, \partial_{\nu_k} \varphi(x_1, \ldots, x_m)\,.$$

If one regards sets of the form $\|\varphi\|_q^2 \leqq c^2$, then the function $(1+\varrho^2)^{2q}$ under the integral sign implies a restriction with respect to the properties of the function in the x-space, while the sum restricts the properties in the momentum-space. For a more refined investigation it would therefore be appropriate to introduce norms

$$\|\varphi\|_{q_1,q_2}^2 = \int (1+\varrho^2)^{2q_1} \sum_{k=0}^{q_2} \bar{\varphi}^{(k)} \cdot \varphi^{(k)}\, dx$$

which loosen the connection between both properties. For simplicity we are at the moment not interested in doing this. From the topological point of view it is sufficient to know that the special norms given above, define by the inequalities $\|\varphi\|_q \leqq \frac{1}{n_q}$, $n_q = 1, 2, 3, \ldots$ a countable fundamental system of neighborhoods of zero for the topology in $\mathscr{S}$ resp. in Φ. This statement is proven in the Appendix.

The linear continuous mapping j from Φ into $\mathfrak{H}$ must be bounded with respect to a certain norm in Φ, for example, with respect to $\|\;\|_q$:

$$\|j(\varphi)\|_H \leqq C\|\varphi\|_q;\ \varphi \in \mathscr{S}(E^{4m})\,.$$

Therefore one can continue j from Φ to the Hilbertspace Φ_q, the completion of Φ with respect to $\|\;\|_q$ with the same bound C. If j_q is the symbol for this continuation, then $j_q(\Phi_q)$ still lies in $\mathfrak{H}$. If appropriately restricted the mapping j_q involves linear continuous mappings j_t from $\Phi_t \to \mathfrak{H}$ for $t \geqq q$. Using the property (A) of nuclear spaces a special space Φ_r, within the family $\{\Phi_t\}$, can be chosen in such a way that g_{qr} is nuclear. So one gets:

Lemma 1. A bounded set of Φ_r, e.g., $W_r : \|\varphi\|_r \leqq c$ is mapped by j_r into a compact* set K of $\mathfrak{H}$.

* The set $j_r(W_r^c)$ itself is in general only relatively compact.

Proof: $j_r = j_q g_{qr}$; g_{qr} maps W_r^c into a compact set Q_c of Φ_q and j_q is continuous; therefore Q_c is mapped into a compact set of $\mathfrak{H}$.

The magnitude of the norm $\| \ \|_r$ and the mapping j_r from Φ_r into $\mathfrak{H}$ are important for the following sections. But if one tries to compare different elements of $\mathfrak{H}$ with respect to the magnitude of the r-norm of their inverse images the difficulty arises that j_r is not unique. Therefore one can proceed in the following manner: If N is the linear set of elements of Φ_r which is mapped into the zero element of $\mathfrak{H}$, it follows from the continuity of j_r that N is a closed linear subspace of Φ_r. Therefore one can introduce in the quotient-space$\Phi_r/N = \hat{\Phi}_r$ a norm by the following definition (see e.g. [16], p. 86):

$$\|\hat{\varphi}\|_r = \inf_{\varphi_k \in \alpha} \|\varphi_k\|_r; \quad \hat{\varphi} \in \hat{\Phi}_r; \quad \varphi_k \in \Phi_r. \tag{5}$$

The symbol α is a class of elements which is mapped into the same element of $\mathfrak{H}$, while $\hat{\Phi}_r$ is again a Hilbert space. It is obvious that the mapping j_r from $\hat{\Phi}_r$ into $\mathfrak{H}$ which is now unique has the property given by:

Lemma 1a. A bounded set of $\hat{\Phi}_r$, e.g., $\|\hat{\varphi}\|_r \leqq c$ denoted by $\hat{W}_r^c$ is mapped by j_r into a compact set of $\mathfrak{H}$.

The transformations of T leave N and its orthogonal complement N_c in Φ_r invariant. Therefore the operation of performing the inferior and a transformation $T(g)$ induced by an element g of G commute:

(D) if $\varphi_e \in \alpha$ has the property $\|\varphi_e\|_r \leqq \|\varphi_k\|_r$ for all $\varphi_k \in \alpha$ then $\|T\varphi_e\|_r \leqq \|\chi_k\|_r$ for $\chi_k \in \beta$ and $T\varphi_e \in \beta$. The symbols α resp. β are classes mapped resp. into the same elements of $\mathfrak{H}$.

In the following we will use the shorter notations $\mathfrak{H} = \hat{\Phi}_r$ and $f = j_r$. The linear space $f(\mathfrak{H})$ is denoted by $\mathfrak{D}$. ($\mathfrak{D}$ is dense in $\mathfrak{H}$.)

From Lemma 1 resp. Lemma 1a a simple conclusion can be drawn. By means of the property (C) it is clear that one can construct a countable orthonormal basis for $\mathfrak{H}$ with elements from $\mathfrak{D}$. If $\{h_\nu\}$ with $(h_\nu, h_\varkappa) = \delta_{\nu\varkappa}$ is such a basis and if $\{\hat{\varphi}_\nu\}$ are the inverse images for the $\{h_\nu\}$: $f(\hat{\varphi}_\nu) = h_\nu$, then the following lemma is valid.

Lemma 2: Each bounded neighborhood $\hat{W}_r^c$: $\|\hat{\varphi}\|_r \leqq c$ contains only a finite number of the inverse images $\hat{\varphi}_\nu$ for the basic vectors h_ν.

Proof: If $\hat{W}_r^c$ would contain an infinite number of the $\hat{\varphi}_\nu$ the set $f(\hat{W}_r^c)$ would contain an infinite set of orthonormal vectors; $f(\hat{W}_r^c)$ could therefore not be situated in a compact subset in $\mathfrak{H}$. This is contrary to Lemma 1.

From Lemma 2 follows that one can introduce an ordering for the elements of an orthonormal base in $\mathfrak{H}$ of the described type (with inverse images in $\mathfrak{H}$) with respect to the magnitude of the r-norm of its inverse images:

$$\|\hat{\varphi}_1\|_r \leqq \|\hat{\varphi}_2\|_r \leqq \cdots \|\hat{\varphi}_\nu\|_r \leqq \|\hat{\varphi}_{\nu+1}\|_r \leqq \cdots.$$

Only a finite number of elements can have the same magnitude of the r-norm and the values of the r-norm have no point of accumulation.

4. Definition of localized states

In this section a certain definition of the localization of states will be given. In the next two sections some properties of the states which are localized by this definition, are investigated. By these properties one gets a certain justification for the definition of localization given below.

To get compact sets of states in the Hilbert space $\mathfrak{H}$ one was led to consider a certain r-norm in the space of test functions Φ. However, this use of the r-norm is still not adequate for the definition of localization of a state at a certain point. From (2) and (3) it follows that for the countable set of norms, and therefore also for the r-norm, the point $x = 0$ is distinguished. However, the concept of localization in a certain region of a state or a set of states should not prefer one distinguished region above another one. Therefore, one could try to use a more general set of norms with $(1 + \Sigma\,(x_j - a)^2)^{2r}$ instead of the norms with the function $(1 + \Sigma\, x_j^2)^{2r}$ in the integrand. Rather than doing that, it is more appropriate to consider for each element h of $\mathfrak{H}$ the set of the elements $\{h_b\}$ obtained by applying all possible translations $U(b)$ to h.

Definition: A normed state h of $\mathfrak{H}$ shall be called localized at the point $x = a$ (in the sense of r) in degree c if one has for the inverse image of $h_{-a} = U(-a)\,h$:

$$\|\hat{\varphi}_{-a}\|_r = c \quad \text{and} \quad \|\hat{\varphi}_{-b}\|_r \geqq c$$

with b as an arbitrary 4-vector; $\hat{\varphi}_{-b}$ is mapped into $h_{-b} = U(-b)\,h$.

From the structure of the r-norm given by (2), (3), and (4) one can deduce that an absolute minimum must exist if one applies all translations to a state h of $\mathfrak{H}$. But it is not obvious whether this absolute minimum for the r-norm is reached by several translations $a_1, a_2, \ldots$ or just by one. The following lemma shows that the point where the minimum is reached is unique.

Lemma 3: From $\|\hat{\varphi}_{-a_1}\|_r = \|\hat{\varphi}_{-a_2}\|_r \leqq \|\hat{\varphi}_{-b}\|_r$, b arbitrary, it follows $a_1 = a_2$. The symbols $\hat{\varphi}_{-a_1}$, $\hat{\varphi}_{-a_2}$ resp. $\hat{\varphi}_{-b}$ are the inverse images for

$$h_{-a_1} = U(-a_1)h\,, \; h_{-a_2} = U(-a_2)\,h \quad \text{resp.} \quad h_{-b} = U(-b)h\,.$$

Proof: Let us assume that there is a state h_1 and a corresponding inverse image $\hat{\varphi}_1$ with absolute minimum at two different points. Then one finds a second state h corresponding to an inverse image with the property that for a suitable 4-vector $c \neq 0$ the following equation and inequality are valid:

$$\|\hat{\varphi}_{+c}\|_r = \|\hat{\varphi}_{-c}\|_r \leqq \|\hat{\varphi}_{-b}\|_r \tag{6}$$

with arbitrary b. One has

$$\|\hat{\varphi}_{+c}\|_r^2 = \int [1 + \sum_{j=1}^{m} (x_j + c)^2]^{2r} p(x_1, \ldots, x_m)\, dx_1, \ldots, dx_m$$

$$= \|\hat{\varphi}_{-c}\|_r^2 = \int [1 + \sum_{j=1}^{m} (x_j - c)^2]^{2r} p(x_1, \ldots, x_m)\, dx_1, \ldots, dx_m\,,$$

where p is a positive function of its elements depending on the defining test function for h and its derivatives up to the order r. From (6a) it follows that the terms belonging to an uneven power of C do not contribute to the norm; but the terms with an even power of C have a strict minimum for $c = 0$. By obtaining this contradiction to the equation (6), the proof of the lemma is completed.

Remark: The proof was carried out regarding only one representative in of the whole class $\hat{\varphi}$ of $\mathfrak{H} = 1_r$. However, this simplification is correct by means of the property (D) in Section 3.

From Lemma 3 it is clear that each state has one and only one centre of localization in the sense of the definition given above. It is not difficult to recognize the following property:

(E) the centre of localization of a state is situated within the convex hull of the support of the test function corresponding to this state.

This has some value for the following two remarks.

a) If one has two states h_1 and h_2 localized at the same point, e.g. at the point $x = 0$, a superposition of h_1 and h_2 is in general not localized at the point $x = 0$. This feature is originated by interference phenomena of the involved states. One can find simple examples of this. Let us for instance assume that the support of $\varphi(x_1, \ldots, x_m)$ is a convex set that does not contain the origin. Then the states h_1 resp. h_2 corresponding to $\varphi(x_1, \ldots x_m) + \varphi(-x_1, \ldots, -x_m)$ resp. to $\varphi(x_1, \ldots, x_m) - \varphi(-x_1, \ldots, -x_m)$ are both localized at $x = 0$ (which can be proven by means of an argument based on symmetry relations). However, the superposition $h_1 + h_2$ and $h_1 - h_2$ are localized at different points, unequal to $x = 0$; this is obvious by property (E). On the other hand property (E) guarantees that for two states which belong to test functions with supports within the same small part of the space E^{4m}, the centre of localization cannot change its position very much if one regards arbitrary superpositions of these states.

b) The r-norms used above combine in a special manner restrictions with regard to x-space and momentum-space. Lemma 1 is clearly also valid for sets of test functions which are still more restricted, but with a looser connection between the restrictions in position and in momenta: i.e.,

$$\int (1 + \varrho^2)^{2s} \sum_{k=0}^{t} \bar{\varphi}^{(k)} \cdot \varphi^{(k)}\, dx \leqq c^2 \quad \text{with} \quad s \geqq r, t \geqq r\,.$$

These norms may also be used for the construction of compact sets in the Hilbert space $\mathfrak{H}$ and can be minimized for each state by applying a suitable translation to this state. If for a certain state and a certain norm one gets a minimum for the translation a_{st} it is not clear from the beginning how much a_{st} may differ from the value a, where the minimum of the formerly used r-norm is assumed. Then again property (E) shows that in the case of test functions with small compact supports a_{st} and a cannot be very much different from each other as both are situated in the convex hull of the support. This section should be completed by a remark which distinguishes two concepts from one another. One concept is the localization of a state (of $\mathfrak{D}$) *at* a certain point, for instance at the point $x = 0$. This idea finds its expression in the definition given in this section. One can also introduce the concept of sets of states localized *at* a certain point, e.g., at $x = 0$. Each state of the set has the property that the corresponding r-norm has its minimum exactly at $x = 0$. This concept cannot be extended to the linear hull of such a set; this follows from reasons given in a).

The second concept is the localization of a set of states (of $\mathfrak{D}$) in a neighborhood of a point, for instance in a neighborhood of $x = 0$. Such a set K_0 of states is given by considering the corresponding set in $\hat{\mathfrak{H}}$ for the inverse images given by $\|\hat{\varphi}\|_r \leqq c$. The corresponding set for $x = a$ is $\|\hat{\varphi}_{-a}\| \leqq c$ where $\hat{\varphi}$ is mapped into h and $\hat{\varphi}_{-a}$ into h_{-a}. A state of the set K_0 will in general not have its minimum of the r-norm at the point $x = 0$. The set K_0 can contain elements, where the minimum of the r-norm is reached at a distant point from $x = 0$. But the definition of the r-norm shows that such a state can only be represented in $\mathfrak{H}$ by a vector of rather small length.

Clearly a set of states localized in a neighborhood of a certain point is not a linear space either. This statement even holds for the normed superposition of two normed states with inverse images in $\|\hat{\varphi}\|_r \leqq c$. While in the preceeding remarks the limitations of the ideas introduced so far became visible, it is the aim of the next section to make some constructive applications.

5. States localized at the origin

At the beginning of this section a property is listed which shows that certain orthonormal-bases in $\mathfrak{H}$ with inverse images in $\hat{\mathfrak{H}}$ may be useful.

(F) If h_1 resp. h_2 from $\mathfrak{D}$ are two orthonormal states and if $\|\hat{\varphi}_1\|_r = c_1$, $\|\hat{\varphi}_2\|_r = c_2$, $(\hat{\varphi}_1, \hat{\varphi}_2)_r = 0$, $c_1 \geqq c_2$ then a normed superposition $\alpha_1 h_1 + \alpha_2 h_2$ (with $|\alpha_1|^2 + |\alpha_2|^2 = 1$) has an inverse image with

$$c_2 \leqq \|\alpha_1 \hat{\varphi}_1 + \alpha_2 \hat{\varphi}_2\|_r \leqq c_1 .$$

Equation (F) can be easily derived from the definition of the r-norm.

To introduce a suitable orthonormal base we look for the symmetry group in $\mathfrak{H}$. The group G (where G is the Poincarégroup L_i) is represented by unitary operators in $\mathfrak{H}$; but equations (2), (3), and (4) show that the r-norm is not invariant under the transformations of this group. On the other hand one can see that the definition of each norm of the countable set of norms of Φ, especially the r-norm, is invariant with respect to the transformations of the group O_4, the orthogonal group in the 4-dimensional Euclidian space. As each element of this group maps $\hat{\mathfrak{H}}$ onto $\hat{\mathfrak{H}}$ one has a unitary representation for O_4 im $\hat{\mathfrak{H}}$; however, O_4 is not represented by unitary operators in $\mathfrak{H}$. The groups L_i and O_4 have in common the subgroup O_3, the orthogonal group in the ordinary 3-dimensional Euclidian space. In this special case it follows from property (1) that

$$f\{U_{\hat{\mathfrak{H}}}(g)\,\hat{\varphi}\} = U_{\mathfrak{H}}(g)\,\{f(\hat{\varphi})\}\,, \quad g \in O_3\,, \tag{7}$$

where $U_{\hat{\mathfrak{H}}}$ resp. $U_{\mathfrak{H}}$ are the unitary operators in $\hat{\mathfrak{H}}$ resp. $\mathfrak{H}$ corresponding to g. If $m(m+1)$ is an eigenvalue for $\mathbf{M}^2$, the square of the operator of angular momentum, and m_3 is the eigenvalue of its third component equation (7) shows that

$$f(\hat{\varphi}_{m,m_3,\varkappa}) = \sum_{\nu} h_{m,m_3,\nu}\,.$$

The elements $\hat{\varphi}_{m,m_3,\varkappa}$ and $h_{m,m_3,\nu}$ possess the same eigenvalues $m(m+1)$ resp. m_3 for $\mathbf{M}^2$ resp. M_3; the indices $\varkappa$ and ν characterise further quantum numbers.

Denoting by $\hat{\mathfrak{H}}_{m,m_3}$ the subspace of $\hat{\mathfrak{H}}$ with eigenvalues $m(m+1)$ resp. m_3, and denoting by $\mathfrak{H}_{m,m_3}$ the corresponding subspace of $\mathfrak{H}$ one has the obvious property that $f(\hat{\mathfrak{H}}_{m,m_3}) = \mathfrak{D}_{m,m_3}$ is situated within $\mathfrak{H}_{m,m_3}$ and is dense in this subspace of $\mathfrak{H}$. Therefore one can introduce in $\mathfrak{H}_{m,m_3}$ by the method of Schmidt an orthonormal base $\{h_{m,m_3,\nu}\}$ with elements from $\mathfrak{D}_{m,m_3}$. Applying to a certain $h_{m,m_3,\nu}$ the unitary transformations belonging to O_3 and performing suitable linear combinations one gets the representation space $\mathfrak{H}_{m,\nu}$ belonging to an irreducible representation of O_3. From the property of isomorphism between the representations $U_{\hat{\mathfrak{H}}}$ resp. $U_{\mathfrak{H}}$ it follows:

(G) to inverse images of the normed states of an irreducible representation $\mathfrak{H}_{m,\nu}$ belong r-norms of the same magnitude.

With respect to the localization centre one finds:

Lemma 4: A state $h_{m,\nu}$ belonging to the representation space of an irreducible representation $\mathfrak{H}_{m,\nu}$ is localized at a point x with $\mathrm{x} = 0$.

Proof: If the state were localized at a point with $\mathrm{x} = \mathfrak{c} \neq 0$ then

$$\|\hat{\varphi}_{-\mathfrak{c}}\|_r^2 = \int [1 + \sum_{j=1}^{m} (x_j + \mathfrak{c})^2]^{2r}\, p(x_1, \ldots, x_m)\, dx_1, \ldots, dx_m$$

should be minimal. The positive function $p(x_1, \ldots, x_m)$ depends on the first r derivatives of the test function $\varphi(x_1, \ldots, x_m)$; as φ belongs to the eigenvalue $m(m+1)$ of $\mathbf{M}^2$ one has $p(x_1^0, \mathrm{x}_1, \ldots, x_m^0, \mathrm{x}_m) = p(x_1^0, -\mathrm{x}_1, \ldots, x_m^0, -x_m)$, and therefore

$$\int [1 + \sum_{j=1}^{m} (x_j + \mathfrak{c})^2]^{2r} \, p(x_1, \ldots, x_m) \, dx_1, \ldots, dx_m$$

$$= \int [1 + \sum_{j=1}^{m} (x_j - \mathfrak{c})^2]^{2r} \, p(x_1, \ldots, x_m) \, dx_1, \ldots dx_{1, m}$$

This is in contradiction to the assumption made above; the conclusion is the same as in the proof of Lemma 3.

Therefore the base $\{h_{m, m_3, \nu}\}$ is a suitable one for the description of states localized at a point with $\mathrm{x} = 0$.

6. Asymptotic properties under translations

The asymptotic properties of matrix elements of individual states with respect to relative translations of these states were extensively clarified in the papers [6—8]. In one of those papers the uniformity of the asymptotic convergence with respect to bounded sets in Φ is proven. The contents of the following two lemmas give some information about uniformity of convergence with respect to the r-norm in $\hat{\mathfrak{H}}$.

The asymptotic properties depend on:

a) individual features of the special theory employed, especially on the spectrum of the 4-momentum operator; b) the direction of the translation (e.g., timelike or spacelike). One can condense the results of both influences into one symbol: $\xrightarrow{Z} 0$ (convergence in the sense Z).

The first lemma of this section is a simple application of the contents of Sections 4 and 5. Regarding elements h_ν, $l_\varkappa$ in $\mathfrak{H}$ with inverse images in Φ itself, one knows that

$$|(h_\nu, [I - P_\Omega] \, U(\lambda a) \, l_\varkappa)| \xrightarrow[\lambda \to \infty]{Z} 0$$

is valid, where P_Ω is proj. operator onto Ω. (For instance in the case where Ω is an isolated eigenstate for p_u^2 and if α is spacelike, then $\xrightarrow{Z} 0$ means that the covergence is faster than each negative power of λ).

Lemma 5: For normalised elements h_ν, $l_\varkappa$ the convergence

$$|(h_\nu [I - P_\Omega] \, U(\lambda a) \, l_\varkappa)| \xrightarrow[\lambda \to \infty]{Z} 0$$

is uniform if h_ν resp. $l_\varkappa$ vary in a set which possesses inverse images in Φ and if $\|\hat{\varphi}_\nu\|_r \leqq c_1$, $\|\hat{\chi}_\varkappa\|_r \leqq c_2$; $f(\hat{\varphi}_\nu) = j(\varphi_\nu) = h_\nu$; $f(\hat{\chi}_\varkappa) = j(\chi_\varkappa) = l_\varkappa$.

The proof of Lemma 5 is obvious. Only two finite dimensional subspaces can be spanned by the normalised vectors with inverse images situated inside the two sets $\|\hat{\varphi}\|_r \leqq c_1$ resp. $\|\hat{\chi}\|_r \leqq c_2$. Therefore the convergence must be uniform.

With respect to the more general class of non-normalized states, another result can be derived.

Lemma 6: From
$$|(h_\nu, [I - P_\Omega]\, U(\lambda a)\, l_\varkappa)| \xrightarrow[\lambda\to\infty]{} 0$$
where h_ν and $l_\varkappa$ possess inverse images in Φ it follows
$$|(h_\mu, [I - P_\Omega]\, U(\lambda a)\, l_\iota)| \xrightarrow[\lambda\to\infty]{} 0 \qquad \text{uniformly}$$
if the inverse images $\hat{\varphi}_\mu$ resp. $\hat{\chi}_\iota$ of h_μ resp. l_ι vary in the bounded sets $\|\hat{\varphi}\|_r \leqq c_1$ resp. $\|\hat{\chi}\|_r \leqq c_2$ of $\hat{\mathfrak{H}}$.

Proof: From $|(h_\nu, [I - P_\Omega]\, U(\lambda a)\, l_\varkappa)| \xrightarrow[\lambda\to\infty]{} 0$ for $h_\nu, l_\varkappa$ with inverse images in Φ it follows $|(h_\mu [I - P_\Omega]\, U(\lambda a)\, l_\iota)| \xrightarrow[\lambda\to\infty]{} 0$ for h_μ, l_ι representing arbitrary states in $\mathfrak{H}$. One can derive this since $U(\lambda_p a)$ has a bound for its norm not dependent on p and $j(\Phi)$ is dense in $\mathfrak{H}$. If the convergence were not uniform one could find for a sequence $\lambda_p \to \infty$ two sequences $\{h_p\}$, $\{l_p\}$ and a certain ε such that $|(h_p, [I - P_\Omega]\, U(\lambda_p a)\, l_p)| \geqq \varepsilon$ for all p. Since the $\|h_p\|$ resp. $\|l_p\|$ are situated in two compact sets K_1 resp. K_2, one can extract from the sequences $\{h_p\}$ resp. $\{l_p\}$ two sequences $\{h_s\}$ resp. $\{l_s\}$ with $h_s \to h_0$ resp. $l_s \to l_0$. The property $|(h_s [I - P_\Omega]\, U(\lambda_s a) l_s)| \geqq \geqq \varepsilon$ for each s is still retained. With $[I - P_\Omega]\, U(\lambda_s a) = B_s$ and $\|B_s\| = 1$ it follows that $|(h_s, B_s l_s)| \leqq |(h_0, B_s, l_0)| + |(h_s - h_0, B_s l_0)| + |(h_s, B_s, l_s - l_0)|$. Since $|(h_0, B_s l_0)| \to 0$, $h_s \to h_0$, $l_s \to l_0$ and since the operators B_s are uniformly bounded $|(h_s, B_s, l_s)|$ must become arbitrarily small; this is in contradiction to the assumption.

One can regard the uniformity of convergence as a significant property for sets of states localised in the neighborhood of certain points. If one compares the quality of convergence for different matrix elements from the view-point of physical interpretation, it is more appropriate to use only normalized states in $\mathfrak{H}$. Therefore in some respect Lemma 5 may be more applicable. However, Lemma 6 implies also some information if one wishes to compare the properties of convergence of matrix elements for normalised states in $\mathfrak{H}$ which differ with respect to the magnitude of the r-norm in $\hat{\mathfrak{H}}$. Taking for example at first a set of states with inverse images having $\|\hat{\varphi}\|_r \leqq 1$ from Lemma 6 it follows that
$$|(h_\nu [I - P_\Omega]\, U(\lambda a)\, h_\varkappa)| \leqq \varepsilon \quad \text{for} \quad \lambda \geqq \lambda_0 \tag{8}$$
and
$$\|\hat{\varphi}_\nu\|_r \leqq 1, \quad \|\hat{\varphi}_\varkappa\|_r \leqq 1, \quad f(\hat{\varphi}_\nu) = h_\nu\,, \quad f(\hat{\varphi}_\varkappa) = h_\varkappa\,.$$
If one has two normalized states l_1 resp. l_2 from $f(\hat{\mathfrak{H}}) = \mathfrak{H}$ with inverse images having $\|\hat{\varphi}_1\|_r = c_1$ resp. $\|\hat{\varphi}_2\|r = c_2$ then $\left\|\frac{1}{c_1}\,\hat{\varphi}_1\right\|_r = 1$ and $\left\|\frac{1}{c_2}\,\hat{\varphi}_2\right\|_r = 1$ Therefore for these two states one gets for $\lambda \geqq \lambda_0$ the same ε as in (7), and for the original states this gives
$$|(l_1, [I - P_\Omega]\, U(\lambda a)\, l_2)| \leqq c_1 \cdot c_2 \cdot \varepsilon \quad \text{for} \quad \lambda \geqq \lambda_0\,.$$

Since each bounded neighborhood, especially $\|\tilde{\varphi}\|_r \leqq 1$, is an absorbing* set in $\hat{\mathfrak{H}}$, it is therefore possible to give to an arbitrary pair of states of $\mathfrak{H}$ an upperlimit for their convergence properties if the uniform bound for the set with origin element in $\|\tilde{\varphi}\|_r \leqq 1$ is known.

7. Generalisation and concluding remarks

So far the special case of a linear continuous mapping from the space of test functions $\mathscr{S}(E^{4m})$ into a Hilbert space was considered. One can perform the same approach for the more general case

$$\tilde{\Phi}_n = \mathscr{S}(E^4) \oplus \mathscr{S}(E^8) \oplus \cdots \oplus \mathscr{S}(E^{4n})$$

where n is an arbitrary fixed number. Each space $\mathscr{S}(E^{4k})$ of the sum is a countably normed nuclear space and the mapping from $\tilde{\Phi}_n$ into the Hilbert space is bounded in each space with respect to a certain norm $\| \ \|_{q_k}$. For each q_k there exists an r_k such that the bounded sets with respect to $\| \ \|_{r_k}$ are mapped into compact sets of the Hilbert space. One has only a finite number of spaces $\mathscr{S}(E^{4k})$. Thus a set of elements of $\tilde{\Phi}_n$ having components in several- or all subspaces $\mathscr{S}(E^{4k})$ with the condition that each component lies in a certain fixed bounded set
$\| \ \|_{r_k} \leqq c_k$, is mapped into a compact set of the Hilbert space. This follows from a well-known topological theorem: if $K_1, K_2, \ldots, K_n$ are compact (resp. relatively compact)sets of a vector space, then $K_1 + K_s + + \cdots + K_n$ is also a compact (resp. relatively compact) set (see e.g. [17]). For a metric space like the Hilbert space, this theorem follows immediately from the property mentioned in Section 2.: Each compact (resp. relatively compact) set K_m has a finite ε — net. One can proceed through Sections 3, 4, 5, and 6, and prove the corresponding lemmas for $\tilde{\Phi}_n$. One can also give a corresponding definition of localisation of a state at a point $x = a$; also in this more general case a is unique.

However, for the passage $\tilde{\Phi}_n$ to $\tilde{\Phi}$ where $\tilde{\Phi}$ is the inductive limit (see e.g. [13]) for $n \to \infty$ some further reasoning is needed. Not until then can one judge how much the situation may change compared to the case of a finite direct sum of nuclear spaces.

With respect to the inhomogeneous Lorentz group only the subgroup of translations and the subgroup O_3 (the 3-dimensional orthogonal group) have been considered. Therefore, it would be of some interest to study the entire symmetry group, especially the one parameter noncompact subgroup of the homogeneous Lorentz group.

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* The set M of a vector space V is called absorbing if for each element $x \in V$ $\gamma x \in M$ for a certain γ.

contributed by proposing in discussions that one should use compact sets in the Hilbert space as sets of states localized in a certain region and with bounded energy. I am also indebted to Professor ARAKI and Dr. HEPP for helpful critical remarks. My gratitudes include the warm hospitality at the University of Maryland.

Appendix

Equivalence of the Systems of norms

The purpose of this remark is to show that the system of norms introduced by L. SCHWARTZ and the system of norms used in this paper, a generalisation of the system of norms used by GELFAND-WILLENKIN*, generates the same topology in $\mathcal{S}(E^{4m})$. Regarding two fundamental systems of neighborhoods of the zero-element for the two systems of norms, one has to show that in each neighborhood of the first system lies a neighborhood of the second and vice versa.

L. SCHWARTZ: *Systems of norms*:

$$\|\varphi\|'_{s,t} = \max_{0 \leqq k \leqq t} \sup_x |(1+\varrho^2)^s \varphi^{(k)}(x)| ,$$

$$x = \{x_1, x_2, \ldots, x_{4m}\}, \quad \varrho^2 = \sum_{j=1}^{4m} x_j^2, \quad \varphi^{(k)}(x) = \partial_{\nu_1}, \ldots, \partial_{\nu_k} \varphi(x)$$

The symbol max includes all partial derivatives up to the order t.

One gets a fundamental system of neighborhoods by the sets:

$$W'_{s,t,c} : \|\varphi\|'_{s,t} \leqq c .$$

System of norms used in this paper:

$$\|\varphi\|_p = \sqrt{\int (1+\varrho^2)^{2p} \sum_{k=0}^{p} \bar{\varphi}^{(k)} \cdot \varphi^{(k)} \, dx_1, \ldots, dx_{4m}}$$

$$\bar{\varphi}^{(k)} \cdot \varphi^{(k)} = \sum_{\nu_1, \ldots, \nu_k = 1}^{4m} \partial_{\nu_1}, \ldots, \partial_{\nu_k} \bar{\varphi}(x_1, \ldots, x_{4m}) \, \partial_{\nu_1}, \ldots, \partial_{\nu_k} \varphi(x_1, \ldots, x_{4m})$$

Fundamental system of neighborhoods:

$$W_{p,d} : \|\varphi\|_p \leqq d .$$

a) For each $W_{p,d}$ one can find a certain $W'_{s,t,c}$ with $W'_{s,t,c} \subset W_{p,d}$.
Proof:

$$\|\varphi\|_p^2 = \int (1+\varrho^2)^{2s} \left(\sum_{k=0}^{p} \bar{\varphi}^{(k)} \cdot \varphi^{(k)} \right) \frac{(1+\varrho^2)^{2p}}{(1+\varrho^2)^{2s}} \, dx_1, \ldots, dx_{4m} .$$

One chooses $s > p$ and in such a manner that $\int \frac{(1+\varrho^2)^{2p}}{(1+\varrho^2)^{2s}} dx_1, \ldots, dx_{4m} = B_{ps}^2 < \infty$.

$$\|\varphi\|_p^2 \leqq \sup_x |(1+\varrho^2)^{2s} \sum_{k=0}^{p} \bar{\varphi}^{(k)} \cdot \varphi^{(k)}| \cdot B_{ps}^2 \leqq$$

$$\leqq g(p) \cdot B_{ps}^2 \max_{0 \leqq k \leqq p} \sup_x |(1+\varrho^2)^{2s} \bar{\varphi}^{(k)} \cdot \varphi^{(k)}| \leqq g(p) \, B_{ps}^2 (\|\varphi\|'_{s,p})^2 ,$$

* GELFAND-WILLENKIN: Verallgemeinerte Funktionen, Bd. IV. S. 83

where g is a certain finite integer. Therefore if φ fulfills: $g(p)\, B_{ps}^2(\|\varphi\|'_{s,p})^2 \leqq d^2$, then φ belongs to $W_{p,d}$.

$$W'_{s,p,c} \subset W_{p,d} \quad \text{is valid for } c = \frac{d}{B_{ps}\sqrt{g(p)}}$$

b) In each $W'_{s,t,c}$ lies a certain $W_{p,d}$.

Proof: For $x_1 \leqq 0$ one has

$$|(1+\varrho^2)^s \partial_{\nu_1}, \ldots, \partial_{\nu_k} \varphi(x_1, \ldots, x_{4m})| = |(1+\varrho^2)^s \times$$

$$\times \int\limits_{u_1=-\infty}^{x_1} \partial_{u_1}\partial_{\nu_1}, \ldots, \partial_{\nu_k}\varphi(u_1, x_2, \ldots x_{4m})\; du_1| \leqq$$

$$\leqq \int\limits_{u_1=-\infty}^{x_1} (1+\varrho^2+u_1^2-x_1^2)^s\, |\partial_{u_1}\partial_{\nu_1}, \ldots, \partial_{\nu_k}\varphi(u_1, x_2, \ldots, x_{4m})|\, du_1$$

A corresponding inequality is valid for $x_1 \geqq 0$. Therefore one gets

$$\max_{x_1} |(1+\varrho^2)^s \partial_{\nu_1}, \ldots, \partial_{\nu_k} \varphi(x_1, \ldots, x_{4m}) \leqq \int\limits_{u_1=-\infty}^{+\infty} (1+u_1^2-x_1^2+\varrho^2)^s| \; \partial_{u_1}, \ldots, \partial_{\nu_k}\varphi|\, du_1$$

and more generally

$$\max_x |(1+\varrho^2)^s \partial_{\nu_1}, \ldots, \partial_{\nu_k}\varphi(x_1, \ldots, x_{4m})| \leqq \int (1+\varrho_u^2)^s|\, \partial_{u_1}, \ldots, \partial_{u_{4m}}\partial_{\nu_1}, \ldots, \partial_{\nu_k}\varphi| \; du_1, \ldots, du_{4m};$$

from the Schwartz inequality one derives

$$\sup_{0\leqq k\leqq t} \max_x |(1+\varrho^2)^s \varphi^{(k)}|^2 \leqq \int (1+\varrho_u^2)^{2s} \sum_{k=0}^{t} \bar{\varphi}^{(4m+k\,.\,(4m+k)}\, du_1, \ldots, du_{4m}$$

with

$$\varrho_u^2 = \sum_{j=1}^{4m} u_j^2\,.$$

Choosing $p \geqq \max(4m+t, s)$ one has

$$\|\;\|_{s,t} \leqq \|\;\|_p\,.$$

Therefore $W_{p,c} \subset W'_{s,t,c}$ follows for an appropriate p.

It is thus concluded from a) and b) that the two sets of norms generate the same topology in $\mathscr{S}(E^{4m})$.

Besides this it is obvious that the sets $W_{p,\frac{1}{n_p}}$ form a fundamental system of neighborhoods of zero (as pointed out in the paragraph following equation (4)).

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