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WELL-POSEDNESS OF INITIAL VALUE PROBLEM FOR DISCRETE NONLINEAR WAVE EQUATIONS

SERGEI BAK*
Department of Mathematics
Vinnitsa State Pedagogical University
Ukraine

GASTON M. N'GUÉRÉKATA†
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA

ALEXANDER PANKOV‡
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA

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Abstract

We consider the initial value problem for discrete nonlinear wave equations. Under natural assumptions, we prove results on global well-posedness in a wide class of weighted l^2 spaces. Admissible spaces include spaces power and exponential decaying sequences.

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1 Introduction

In this paper we consider discrete nonlinear wave equations of the form

$$\ddot{q}_n = a_n q_{n+1} + a_{n-1} q_{n-1} + b_n q_n - f_n(q_n), \quad n \in \mathbb{Z}, \tag{1.1}$$

*E-mail address: sergiy_bak@online.ua
†E-mail address: Gaston.N'Guerekata@morgan.edu
‡E-mail address: Alexander.Pankov@morgan.edu

where the coefficients a_n and b_n are sequences of real numbers, and the nonlinearity f_n is a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(0) = 0$. Here and in what follows $\dot{}$ and $\ddot{}$ stand for the first and second time derivatives respectively. The unknown $q_n(t)$ is a sequence of real functions of real variable t. We study the initial value problem for equation (1.1) with initial conditions

$$q_n(0) = q_n^{(0)}, \quad \dot{q}_n(0) = q_n^{(1)}, \quad n \in \mathbb{Z}, \tag{1.2}$$

where $q_n^{(0)}$ and $q_n^{(1)}$ are given real sequences.

In fact, (1.1) is an infinite sequence of ordinary differential equations. But a better point of view is to consider equation (1.1) as an operator differential equation

$$\ddot{q} = Aq - B(q) \tag{1.3}$$

in certain Hilbert, or even Banach, space E of sequences. Here A is the linear operator defined by

$$(Aq)_n = a_n q_{n+1} + a_{n-1} q_{n-1} + b_n q_n, \quad n \in \mathbb{Z}, \tag{1.4}$$

and B is the nonlinear operator defined by

$$(B(q))_n = f_n(q_n), \quad n \in \mathbb{Z}. \tag{1.5}$$

Within this framework, initial conditions (1.2) become

$$q(0) = q^{(0)}, \quad \dot{q}(0) = q^{(1)}, \tag{1.6}$$

where $q^{(0)}$ and $q^{(1)}$ are given elements of the space E.

The simplest choice of such space is $E = l^2$, the space of two-sided square summable sequences. In this space equation (1.1) is Hamiltonian. In [4] (see also [9, Section 1.4]) the Hamiltonian structure, together with the classical existence and uniqueness theorem for operator differential equations and a cut-off argument, is used to obtain rather general global well-posedness of the initial value problem in l^2. We review those results in Section 2. The aim of the present paper is to extend the l^2-well-posedness results to weighted l^2-spaces and, hence, provide a refined information about problem (1.1), (1.2). This is done in Section 4. Similar idea has been used in [7] to study the discrete nonlinear Schrödinger equation. In Section 3 we discuss weighted l^2-spaces l^2_Θ and operators in such spaces. Section 5 is devoted to simplest examples appearing in applications.

2 Hamiltonian Structure and l^2-theory

Throughout the paper we impose the following assumptions.

(i) *The coefficients a_n and b_n are bounded real sequences .*

(ii) *The nonlinearity f_n is a real valued function on $\mathbb{R}$ such that $f_n(0) = 0$, and f_n is locally Lipschitz continuous uniformly with respect to $n \in \mathbb{Z}$,* i.e.*, for any $R > 0$ there exists a constant $C(R) > 0$ such that*

$$|f_n(r_1) - f_n(r_2)| \le C(R)|r_1 - r_2|, \quad |r_1|, |r_2| \le R, \quad n \in \mathbb{Z}.$$

Sometimes we use the following stronger than (ii) assumption

(ii') *Assumption* (ii) *is satisfied with the constant* C *independent of* R*,* i.e.*, there exists a constant* $C > 0$ *such that*

$$|f_n(r_1) - f_n(r_2)| \leq C|r_1 - r_2|, \quad n \in \mathbb{Z}.$$

We denote by l^2 the Hilbert space of two-sided square summable sequences. The norm and inner product in this space are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Occasionally, we shall use more general spaces l^p, $1 \leq p \leq \infty$. The space l^p, $1 \leq p < \infty$, consists of two-sided real sequences $u = (u_n)$ such that the norm

$$\|u\|_{l^p} = (\sum_{n\in\mathbb{Z}} |u_n|^p)^{1/p}$$

is finite. The space l^∞ consists of all bounded sequences. The norm in this space is given by

$$\|u\|_{l^\infty} = \sup_{n\in\mathbb{Z}} |u_n|.$$

Assumption (i) guaranties that the operator A is a bounded self-adjoint operator in l^2. With this choice of the configuration space, the phase space of equation (1.1) is $l^2 \times l^2$, and the equation is a Hamiltonian system. The Hamiltonian is given by

$$H(q,p) = \frac{1}{2}[\|p\|^2 - (Aq,q)] + \sum_{n=-\infty}^{\infty} F_n(q_n),$$

where

$$F_n(r) = \int_0^r f_n(s)\, ds$$

is the primitive function of f_n. The Hamiltonian H is a C^1 functional on the phase space and, hence a conserved quantity, *i.e.*, for any solution of equation (1.1) or, equivalently, (1.3)

$$H(q,\dot{q}) = \text{const}.$$

Now we reproduce some results from [4] (see also [9, Section 1.4]). The first one is a simple straightforward consequence of classical theorems on existence and uniqueness of global solutions for operator differential equations (see, *e.g.*, [6, Chaptrer 6, Theorem 1.2] and [10, Chapter 6, Theorems 1.2 and 1.4]). This result does not use the Hamiltonian structure of equation (1.1).

Theorem 2.1. *Under assumptions* (i) *and* (ii')*, for every* $q^{(0)} \in l^2$ *and* $q^{(1)} \in l^2$ *there exists a unique solution* $q \in C^2(\mathbb{R}, l^2)$ *of problem (1.1), (1.2).*

The proof of the next theorem makes use of Theorem 2.1, the Hamiltonian structure of the equation and a cut-off argument.

Theorem 2.2. *Assume* (i) *and* (ii)*. Suppose that the operator* A *is non-positive,* i.e.*,* $(Aq,q) \leq 0$ *for all* $q \in l^2$ *and* $F_n(r) \geq 0$ *for all* $r \in \mathbb{R}$*. Then problem (1.1), (1.2) has a unique global solution* $q \in C^2(\mathbb{R}, l^2)$ *for all* $q^{(0)} \in l^2$ *and* $q^{(1)} \in l^2$.

A completely different type of nonlinearities is considered in the following

Theorem 2.3. *Assume* (i)*, and let* $f_n(r) =$ *be a positively homogeneous function of degree* $p > 1$ *such that* $|f_n(\pm 1)| \le C$ *for some positive constant* C*. Suppose that the operator* A *is negative definite,* i.e.*,*

$$(Aq, q) \le -\alpha \|q\|^2 , \tag{2.1}$$

where $\alpha > 0$*. Then there exists* $\delta > 0$ *such that for every* $q^{(0)} \in l^2$ *and* $q^{(1)} \in l^2$*, with* $\|q^{(0)}\| < \delta$ *and* $\|q^{(1)}\| < \delta$*, problem (1.1), (1.2) has a unique solution* $q \in C^2(\mathbb{R}, l^2)$*. The solution* q *is a bounded function with values in* l^2*.*

Let us point out that in [4] Theorem 2.3 is proven in the case when $f_n(r) = d_n r^2$. The general case requires only minor changes in the proof.

Now we supplement Theorem 2.2 with the following result on the boundedness of the solution.

Theorem 2.4. *Assume that* (i) *and* (ii) *are satisfied, and* $F_n(r) \ge 0$ *for all* $n \in \mathbb{Z}$ *and* $r \in \mathbb{R}$*.*

(a) If the operator A *is non-positive and* $\lim_{r\to\pm\infty} F_n(r) = +\infty$ *uniformly with respect to* $n \in \mathbb{Z}$*, then the unique solution of problem (1.1), (1.2), with* $q^{(0)} \in l^2$ *and* $q^{(1)} \in l^2$*, is a bounded function on* $\mathbb{R}$ *with values in* l^∞*. In addition, if, for some* $s \ge 2$*, there exist* $R > 0$ *and* $c > 0$ *such that*

$$F_n(r) \ge c|r|^s , \quad \forall r \in [-R, R] , \forall n \in \mathbb{Z} , \tag{2.2}$$

then the solution is a bounded function on $\mathbb{R}$ *with values in* l^s*.*

(b) If the operator A *is negative definite, then the unique solution of problem (1.1), (1.2), with* $q^{(0)} \in l^2$ *and* $q^{(1)} \in l^2$*, is a bounded function on* $\mathbb{R}$ *with values in* l^2*.*

Proof. (a) We have that

$$H(q(t), \dot{q}(t)) = \frac{1}{2}[\|\dot{q}(t)\|^2 - (Aq(t), q(t))] + \sum_{n=-\infty}^{\infty} F_n(q_n(t)) = H(q^{(0)}, q^{(1)}) \tag{2.3}$$

because the Hamiltonian H is a conserved quantity. Since A is non-positive while F_n is non-negative, this implies that

$$F_n(q_n(t)) \le H(q^{(0)}, q^{(1)}) .$$

Therefore, there exists a constant $C > 0$ such that $|q_n(t)| \le C$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$ because F_n has infinite limit at infinity uniformly with respect to $n \in \mathbb{Z}$.

Let us prove the second part of statement (a). The assumption on the limit of F_n at infinity implies that if inequality (2.2) holds for some $R > 0$, then it holds for every $R > 0$, with the constant $c > 0$ depending on R. By the first part of the statement, there exists $R > 0$ such that $\|q(t)\|_{l^\infty} \le R$ for all $t \in \mathbb{R}$. Hence, by (2.3) and (2.2),

$$c \sum_{n=-\infty}^{\infty} |q_n(t)|^s \le H(q^{(0)}, q^{(1)})$$

for all $t \in \mathbb{R}$ which implies the required.

(b) In this case equation (2.3) and inequality (2.1) imply that

$$\alpha \|q(t)\|^2 \le H(q^{(0)}, q^{(1)})$$

for all $t \in \mathbb{R}$ and the result follows.

□

3 Weighted Spaces

Let $\Theta = (\theta_n)$ be a sequence of positive numbers (weight). The space l^2_Θ consists of all two-sided sequences of real numbers such that the norm

$$\|u\|_\Theta = (\sum_{n\in\mathbb{Z}} u_n^2 \theta_n)^{1/2}$$

is finite. This is a Hilbert space.

We always suppose that the weight Θ satisfies the following regularity assumption:

(iii) *the sequence Θ is bounded below by a positive constant and there exists a constant $c_0 > 0$ such that*

$$c_0^{-1} \le \frac{\theta_{n+1}}{\theta_n} \le c_0$$

for all $n \in \mathbb{Z}$.

A weight satisfying assumption (iii) is called *regular*.

Obviously, under this assumption l^2_Θ is densely and continuously embedded into l^2 and

$$\|u\| \le C\|u\|_\Theta\,, \quad u \in l^2_\Theta\,,$$

with some $C > 0$. Therefore, all these spaces are densely and continuously embedded into the the space l^∞ of bounded sequences, with sup-norm. If $\theta_n \equiv 1$, then $l^2_\Theta = l^2$.

From the point of view of functional analysis assumption (iii) is quite natural. It means that the space l^2_Θ is translation invariant. More precisely, let T_+ and T_- be the operators of right and left shifts, respectively, defined by

$$(T_+w)_n = w_{n-1} \quad \text{and} \quad (T_-w)_n = w_{n+1}\,.$$

Lemma 3.1. *Assumption (iii) holds if and only if both T_+ and T_- are linear bounded operators in l^2_Θ.*

Proof. Indeed, we have that

$$\|T_+w\|^2_\Theta = \sum_{n\in\mathbb{Z}} w_{n-1}^2\theta_n = \sum_{n\in\mathbb{Z}} w_n^2\theta_n\frac{\theta_{n+1}}{\theta_n}\,.$$

Hence, T_+ is bounded in l^2_Θ if and only if θ_{n+1}/θ_n is bounded. Similarly, T_- is bounded in l^2_Θ if and only if θ_{n-1}/θ_n is bounded.

□

Note that T_+ and T_- are mutually inverse operators. But let us point out that the translation invariance of the space l^2_Θ does not mean that the norm $\|\cdot\|_{l^2_\Theta}$ is translation invariant.

The most important examples of regular weights are

(a) power weight

$$\theta_n = (1+|n|)^b\,, \quad b > 0\,; \tag{3.1}$$

(b) exponential weight

$$\theta_n = \exp(\alpha|n|)\,, \quad \alpha > 0\,. \tag{3.2}$$

More generally, the weight $\theta_n = \exp(\alpha|n|^\beta)$, $\alpha > 0$, satisfies assumption (iii) if and only if $0 < \beta \le 1$.

4 Well-posedness in Weighted Spaces

We start with two simple lemmas.

Lemma 4.1. *Assume (i). Let Θ be a regular weight. Then the operator A defined by equation (1.4) acts in l^2_Θ as a bounded linear operator.*

Proof. The operator A can be represented in the form

$$A = a \circ T_- + T_+ \circ a + b\,,$$

where a and b are the operators of multiplication by the sequences (a_n) and (b_n) respectively, and $\circ$ stands for the composition of operators. The operators T_-, T_+, a and b are bounded operators in l^2_Θ by Lemma 3.1 and assumption (i) respectively. Hence, the result follows.

□

Lemma 4.2. *Under assumption (ii), the nonlinear operator B defined by equation (1.5) is a locally Lipschitz continuous operator in the space l^2_Θ,* i.e.*, for any $R > 0$ there exists a constant $C_R > 0$ such that*

$$\|B(v) - B(w)\|_{l^2_\Theta} \le C_R \|v - w\|_{l^2_\Theta} \tag{4.1}$$

for all $v \in l^2_\Theta$ and $w \in l^2_\Theta$ such that $\|v\|_{l^2_\Theta} \le R$ and $\|w\|_{l^2_\Theta} \le R$. If assumption (ii') is satisfied, then the operator B is Lipschitz continuous, i.e.*, the constant in inequality (4.1) can be chosen independent of R.*

Proof. Straightforward.

□

Our key observation is the following

Theorem 4.3. *Assume (i), (ii) and (iii). Suppose that $q \in C^2((-T,T);l^2)$ is a solution of problem (1.1), (1.2) with $q^{(0)} \in l^2_\Theta$ and $q^{(1)} \in l^2_\Theta$. Then $q \in C^2((-T,T);l^2_\Theta)$.*

Proof. Let $q \in C^2((-T,T),l^2)$ be a solution of problem 1.1), (1.2) with $q^{(0)} \in l^2_\Theta$ and $q^{(1)} \in l^2_\Theta$. Pick any $\tau \in (0,T)$ and set $R_\tau = \sup_{t\in[-\tau,\tau]} \|u(t)\|$. Let $\tilde{f}_n(r) = f_n(r)$ if $|r| \le R_\tau + 1$ and $\tilde{f}_n(r) = f_n(R_\tau + 1)$ if $|r| > R_\tau + 1$. Then on $[-\tau,\tau]$ the function $q(t)$ obviously solves the equation

$$\ddot{q}_n = a_n q_{n+1} + a_{n-1} q_{n-1} + b_n q_n - \tilde{f}_n(q_n)\,, \quad n \in \mathbb{Z}\,, \tag{4.2}$$

with the same initial data.

Obviously, the functions $\tilde{f}_n$ satisfy assumption (ii'), and, by Lemma 4.2, the corresponding operator $\tilde{B}$ is globally Lipschitz continuous in the space l^2_Θ. By Lemma 4.1, the operator A is a bounded linear operator in l^2_Θ. By the classical result [6, Chaptrer 6, Theorem 1.2] and [10, Chapter 6, Theorems 1.2 and 1.4], problem (4.2), (1.2) has a unique solution $\tilde{q} \in C^2(\mathbb{R},l^2_\Theta) \subset C^2(\mathbb{R},l^2)$. By uniqueness for the initial value problem in the space l^2, we have that $\tilde{q} = q$ on $[-\tau,\tau]$. Since $\tau \in (0,T)$ is an arbitrary point, we obtain that $q \in C^2((-T,T),l^2_\Theta)$.

□

Combining Theorem 4.3 with Theorems 2.1 – 2.3, we obtain the following corollaries.

Corollary 4.4. *Under assumptions* (i) *and* (ii'), *for every* $q^{(0)} \in l^2_\Theta$ *and* $q^{(1)} \in l^2_\Theta$ *there exists a unique solution* $q \in C^2(\mathbb{R}, l^2_\Theta)$ *of problem (1.1), (1.2).*

Corollary 4.5. *Assume* (i) *and* (ii). *Suppose that the operator A is non-positive,* i.e., $(Aq,q) \le 0$ *for all* $q \in l^2$ *and* $F_n(r) \ge 0$ *for all* $r \in \mathbb{R}$. *Then problem (1.1), (1.2) has a unique global solution* $q \in C^2(\mathbb{R}, l^2_\Theta)$ *for all* $q^{(0)} \in l^2_\Theta$ *and* $q^{(1)} \in l^2_\Theta$.

Corollary 4.6. *Assume* (i), *and let* $f_n(r)$ *be a positively homogeneous function of degree* $p > 1$ *such that* $|f_n(\pm 1)| \le C$ *for some positive constant C. Suppose that the operator A is negative definite,* i.e.,

$$(Aq,q) \le -\alpha \|q\|^2, \tag{4.3}$$

where $\alpha > 0$. *Then there exists* $\delta > 0$ *such that for every* $q^{(0)} \in l^2_\Theta$ *and* $q^{(1)} \in l^2_\Theta$, *with* $\|q^{(0)}\| < \delta$ *and* $\|q^{(1)}\| < \delta$, *problem (1.1), (1.2) has a unique solution* $q \in C^2(\mathbb{R}, l^2_\Theta)$.

Let us highlight that in Corollary 4.6 the smallness of the initial data is with respect to the l^2-norm, not in the space l^2_Θ.

5 Examples

Now we present some examples that often appear in applications (see, *e.g.*, [1, 5, 11]). In these examples Δ stands for the one-dimensional Laplacian defined by

$$(\Delta q)_n = q_{n+1} + q_{n-1} - 2q_n .$$

The first example is the well-known *Frekel-Kontorova* (FK) model. The equation reads

$$\ddot{q}_n = a(\Delta q)_n - \sin q_n , \tag{5.1}$$

where $a > 0$. This is a straightforward discretization of the sin-Gordon equation

$$u_{tt} - a u_{xx} + \sin u .$$

The last equation is a completely integrable system (see, *e.g.*, [2]). At the same time its discrete counterpart (5.1) is *not* completely integrable.

In the case of equation (5.1) the nonlinearity satisfies (ii'). Hence, Corollary 4.4 shows that the initial value problem for (5.1) is globally well-posed in every space l^2_Θ with a regular weight Θ.

Now consider the equation

$$\ddot{q}_n = a(\Delta q)_n - m^2 q_n \pm q_n^3 . \tag{5.2}$$

If the sign in front of the cubic nonlinearity is positive, this is the *repulsive discrete nonlinear Klein-Gordon* (DNKG_-) equation in case when $m^2 > 0$, and *repulsive discrete nonlinear wave* (DNW_-) equation when $m^2 = 0$. In case of negative sign, we obtain the *attractive discrete nonlinear Klein-Gordon* (DNKG_+) equation ($m^2 > 0$) and the *attractive discrete nonlinear wave* (DNW_+) equation ($m^2 = 0$) respectively.

It is easy to verify that

$$(\Delta q, q) = \sum_{n\in\mathbb{Z}} (q_n - q_{n-1})^2$$

and, hence, the operator Δ is nonnegative. By Corollary 4.5, in the attractive case the initial value problem for both DNKG_+ and DNW_+ is globally well-posed in all spaces l^2_Θ with regular weight Θ. This is because $F_n(r) = r^4/4 \geq 0$. On the other hand, in the repulsive case $F_n(r) = -r^4/4 \leq 0$. In case of DNKG_- the operator $\Delta - m^2$ is negative definite, and Corollary 4.6 guaranties the existence of global solution in l^2_Θ for all initial data in l^2_Θ that have sufficiently small l^2-norm, provided the weight Θ is regular. The case of DNW_- remains open.

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