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Asymptotic behavior of the maximum of multivariate order statistics in a norm sense

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Abstract. In this work, we investigate the asymptotic behavior of the extremes of a multivariate data by using the Reduced Ordering Principle (R-ordering). When, the sup-norm is used, we reveal the interrelation between the R-ordering principle and Marginal Ordering Principle (M-ordering). The asymptotic behavior of the maximum sup-norms corresponding to the bivariate data is completely determined. Finally, an application to real data illustrates and corroborates the theoretical results.

1 Introduction

Multivariate extreme value theory is perhaps the only known toolbox for analyzing several extremal events simultaneously. Generally, the order multivariate data subject is an active field of research in theoretical and applied statistics. The ordered data may belong to the usual model of order statistics (see, e.g., Galambos, 1987, David and Nagaraja, 2003) or its extensions such as the model of generalized order statistics (see, e.g., Kamps, 1995, Burkschat, Cramer and Kamps, 2003). Moreover, the ordered data may arise from a common distribution function (DF) or it may be dependent on non-identical multivariate data (e.g., Barakat, 2009).

In this paper, we are focusing on the study of the model of multivariate order statistics. The study will extend to distributional theory and the asymptotic distributional theory. It is known that there is no any natural basis for ordering multivariate data. Therefore, the first obstacle that encounters the researchers in studying the subject of ordered multivariate data is to extend the univariate order concepts to the higher dimensional situation. Actually, a substantial effort has been directed to define some sorts of higher dimensional analogous of univariate order concepts, and much of statistical method employs various types of sub-ordering principle. Barnet (1976) presented a fourfold classification of sub-ordering principles for multivariate random vectors. These principles can be classified as follows:

1. Marginal Ordering (M-ordering). As the name suggests, ordering here takes place within one or more of the marginal samples. Most of the researchers who work on ordered multivariate data adopt this principle, among them are Finkelshtein (1953); Galambos (1975); Barakat (1997, 2001); Barakat, Nigm and Al-Awady (2015); Falk and Wisheckel (2018); Falk (2019).
2. Reduced (Aggregate) Ordering (R-ordering). With this type of ordering, each multivariate observation is reduced to a single value by means of some combinations of the component sample values. One of the most effective ways to apply this principle is by ordering the random vectors in the norm sense (see Bairamov and Gebizlioglu, 1997, Arnold, Castillo and Sarabia, 2009, Bairamov, 2016).

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3. Conditional (Sequential) Ordering (C-ordering). The final sub-ordering principle for multivariate data is one in which ordering or ranking is conducted on one of the marginal sets of observations conditional on selection, or ordering or ranking, within the data in terms of other marginal sets of observations (see Arnold, Castillo and Sarabia, 2009, Barakat and El-Shandidy, 2004, Balakrishnan and Stepanov, 2015, Barakat, Nigm and Syam, 2019).
4. Partial Ordering (P-ordering). The emphasis here moves away from consideration of the marginal samples or individual multivariate observations to consider overall interrelations properties in the total deployment of the sample. The way in which observations fall into different regions of the sample space, where such partitioning may be based on one of several possible principles, is used to distinguish between groups of observations with regard to order, rank or extremeness (see, e.g., Chaudhuri, 1996, Zani, Riani and Corbellin, 1999).

In this work, we are concerned with ordered multivariate data based on R-ordering principle. The seminal work of the R-ordering principle is Bairamov and Gebizlioglu (1997), where the authors introduced a so-called norm-ordering in multivariate observations. Namely, let $\mathbb{R}^m$, $m \geq 1$, be the real Euclidean space. Consider a probability space $(\Omega, \mathbb{F}, P)$ and $\mathbb{R}^m$ valued random vectors $\underline{Z}_i = (X_{i1}(\omega), X_{i2}(\omega), \dots, X_{im}(\omega))$, $i = 1, 2, \dots, n$, $\omega \in \Omega$, defined on the space $(\Omega, \mathbb{F}, P)$. Let the m-dimensional DF $F(x_1, \dots, x_m)$ be the common DF of the random vectors $\underline{Z}_1, \dots, \underline{Z}_n$. Denote by $\| \cdot \|$ a norm defined in $\mathbb{R}^m$. Clearly, due to an elementary result in probability theory, $\|\underline{Z}_1\|, \|\underline{Z}_2\|, \dots, \|\underline{Z}_n\|$ are i.i.d. random variables (RVs) with the DF $P(\|\underline{Z}_i\| \leq z) = \mathcal{F}(z)$, $z \in \mathbb{R}$. Moreover, if $F(x_1, \dots, x_m)$ is assumed to be continuous, the probability of any two or more of these RVs assuming equal magnitudes is zero. Therefore, there exists a unique ordered arrangement within the RVs $\|\underline{Z}_1\|, \|\underline{Z}_2\|, \dots, \|\underline{Z}_n\|$. According to the definition of norm-ordering due to Bairamov and Gebizlioglu (1997), if $\|\underline{Z}_i\| < \|\underline{Z}_j\|$, $i, j = 1, 2, \dots, n$, then $\underline{Z}_i$ is said to be less than $\underline{Z}_j$ in a norm sense and this is shown as $\underline{Z}_i \prec \underline{Z}_j$. Suppose $\underline{Z}_{1:n}$ denotes the smallest of the set $\underline{Z}_1, \underline{Z}_2, \dots, \underline{Z}_n$; $\underline{Z}_{2:n}$ denotes the second smallest, etc.; and $\underline{Z}_{n:n}$ denotes the largest in a norm sense, then, $\underline{Z}_{1:n} \prec \underline{Z}_{2:n} \prec \cdots \prec \underline{Z}_{n:n}$. However, this definition brings us to the following important question that was not tackled by Bairamov and Gebizlioglu (1997). Is any norm defined on $\mathbb{R}$ fit to define such ordered relation? To answer on this question, we first consider the case $\underline{Z}_i <_c \underline{Z}_j$, where $<_c$ is meant componentwise, that is, $X_{it}(\omega) < X_{jt}(\omega)$, $t = 1, 2, \dots, m$, and the vectors $\underline{Z}_i$, $i = 1, 2, \dots, n$, are non-negative, that is, $\underline{Z}_i \geq 0$. Then it is not acceptable to use any non-monotone norm $\| \cdot \|$, for which $\underline{Z}_j \prec \underline{Z}_i$ (i.e., $\|\underline{Z}_j\| < \|\underline{Z}_i\|$). Example of non-monotone norm (in the literature, the definition of monotone norm is a norm, for which $\|\underline{Z}_j\| < \|\underline{Z}_i\|$, whenever $|\underline{Z}_j| <_c |\underline{Z}_i|$) is the quadratic form norm $\|\underline{Z}_i\|_A = \sqrt{(\underline{Z}_i^T A \underline{Z}_i)}$, $i = 1, 2, \dots, n$, defined on $\mathbb{R}^2$ (say), where $A = A^T = (a_{ji})_{1 \leq i, j \leq 2}$ is a positive definite 2×2 matrix, such that $a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = \delta$, $\delta \in (-1, 0)$ (cf. Falk, 2019). Therefore, we should consider only the monotone norms, especially for non-negative RVs. On the other hand, it is natural that the employed norm is radial symmetric, that is, changing the sign of arbitrary components of $\underline{Z}_i \in \mathbb{R}^m$ does not alter the value of this norm. This means that the values of the considered norm are completely determined by its values on the subset $\{\underline{Z}_i \in \mathbb{R}^m : \underline{Z}_i \geq 0\}$. Also, the norm $\|\underline{Z}_t\|_A$ is not radial symmetric. However, there is a norm defined on $\mathbb{R}^m$, which is strongly related to the asymptotic behavior of the multivariate extreme theory (cf. Falk and Wisheckel, 2018, see also Remark 2.2) and above all it is monotone and radial symmetric. This norm is known as D-norm and defined by $\|\underline{Z}_i\|_D = \mathrm{E}_\eta(\max_{1 \leq t \leq m} |X_{it}(\omega)\eta_t|)$, where η_t are non-negative RVs, with $\mathrm{E}_\eta(\eta_t) = \int \cdots \int x_t \, dF_{\eta_1, \dots, \eta_t, \dots, \eta_m}(x_1, \dots, x_t, \dots, x_m) = 1$, $t = 1, 2, \dots, m$. The random vector $\eta = (\eta_1, \eta_2, \dots, \eta_m)$ is called the generator of the D-norm. Moreover, this norm contains the Logistic norm family $\|\underline{Z}_i\|_p = (\sum_{t=1}^m |X_{it}(\omega)|^p)^{\frac{1}{p}}$

and the sup-norm $\|\underline{Z}_i\|_\infty = \max_{1\le t\le m}|X_{it}(\omega)|$ (it is known that $\lim_{p\to\infty}\|\underline{Z}_i\|_p = \|\underline{Z}_i\|_\infty$). In this paper, we Consider the sup-norm and in the sequel we write $\|\cdot\|$ instead of $\|\cdot\|_\infty$. Namely, we study the asymptotic behavior of the DF

$$\mathcal{F}_{n:n}(z) = P\big(\max(\|\underline{Z}_1\|, \|\underline{Z}_2\|, \dots, \|\underline{Z}_n\|) \le z\big) = P\big(\|\underline{Z}\|_{n:n} \le z\big) = \mathcal{F}^n(z).$$

For seeking the ease, we consider $m = 2$. However, in Section 3.2, we will show that the main result Theorem 3.1, which is a two-dimensional theorem concerning the asymptotic behavior of the maximum vector $\underline{Z}_{n:n}$ in a norm sense, can be applied, without new formulation, to deal with the cases of more than two dimensions. Moreover, in this paper we consider only the case $\underline{Z}_i \ge 0$. Later, we will show that we can get rid the last restriction in many cases (see Remark 3.2). Everywhere in what follows the symbols ($\xrightarrow[n]{}$) and ($\xrightarrow[n]{w}$) stand for convergence and the weak convergence, as $n \to \infty$.

Motivation of the work

Actually there are many motivations of this study. First, the asymptotic behavior of the statistics $\|\underline{Z}\|_{n:n}$ and $\underline{Z}_{n:n}$ is very strongly relevant and any modeling problem related to the extreme values starting with asymptotic behavior of the extreme values. Moreover, the study of the DF $\mathcal{F}_{n:n}(z)$ itself is important in many applications. Below, we list some of these applications.

1. As Bairamov and Gebizlioglu (1997) indicated, there are many situations in practice for which we need to investigate the distributional properties of a random vector whose elements are magnitudes of distance related characteristics of an event. For instance, in a two dimensional space, bombing on and around a target point has destructive effects on the point itself depending on its distance from the site of explosion in conjunction with some other factors (see Example 3.1). Similarly, multidimensional epidemiological processes can be analyzed in terms of the norm ordered statistics for the spread of disease analysis.
2. Consider a model in reliability theory, which has n independent components each consisting of m arbitrarily dependent elements connected by using a parallel system. Denote the life length of the ith component of the system by $\underline{Z}_i = (X_{i1}, X_{i2}, \dots, X_{im})$, $i = 1, 2, \dots, n$, where X_{it} $(t = 1, 2, \dots, m)$ denotes the life length of the tth element of the ith component. Thus, the first and last failures in the system occur at times

$$\begin{aligned}\underline{Z}_{1:n} &= \min\big\{\max\{X_{11}, X_{12}, \dots, X_{1m}\}, \max\{X_{21}, X_{22}, \dots, X_{2m}\},\\ &\quad \dots, \max\{X_{n1}, X_{n2}, \dots, X_{nm}\}\big\} = \min\big\{\|\underline{Z}_1\|, \|\underline{Z}_2\|, \dots, \|\underline{Z}_n\|\big\}\end{aligned}$$

   and $\underline{Z}_{n:n} = \max\{\|\underline{Z}_1\|, \|\underline{Z}_2\|, \dots, \|\underline{Z}_n\|\}$, respectively. Therefore, to know the time of the first failure and the time of the second failure, etc. in such a system, we need to arrange random vectors $\underline{Z}_1, \underline{Z}_2, \dots, \underline{Z}_n$ by the magnitude of sup-norm (for more details, see Bairamov, 2006).

2 Auxiliary and preliminary results

In this section, we will give some required results of extreme value theory in univariate and bivariate cases. The results concerning the bivariate extreme value theory are based on the M-ordering principle.

2.1 Some required results of univariate order statistics

Theorem 2.1 (Extremal Types Theorem, for maximum). *Let* $X_{n:n} = \max\ (X_1, X_2, \ldots, X_n)$, *where* X_i *i.i.d. RVs with univariate DF* $F(x)$. *Then, by using the elementary relation* $\lim_{x\to 1} \frac{\log x}{x-1} = 1$, *we get*

$$F_{n:n}(a_n x + b_n) = P(X_{n:n} \le a_n x + b_n) \underset{n}{\overset{w}{\longrightarrow}} H(x), \tag{2.1}$$

where $a_n > 0$, $b_n \in \mathbb{R}$ *are some suitable normalizing constants and* $H(\cdot)$ *is a non-degenerate DF, if and only if*

$$n\big(1 - F(a_n x + b_n)\big) \underset{n}{\longrightarrow} -\log H(x). \tag{2.2}$$

Moreover, the limit function $H(\cdot)$ *must have one and only one of three types* $H_{i,\beta}(x) = \exp(-u_{i,\beta}(x))$, $i = 1, 2, 3$, $\beta > 0$, *where* $H_{3,\beta}(x) = H_3(x) = \exp(-u_3(x))$ (*in which* x *may be replaced by* $ax + b$ *for any* $a > 0$, $b \in \mathbb{R}$), *where*

$$\left.\begin{array}{l} \textit{Type I (Fréchet type)}: \quad u_{1,\beta}(x) = \begin{cases} x^{-\beta}, & x > 0, \\ \infty, & x \le 0, \end{cases} \\ \textit{Type II (max-Weibull type)}: \quad u_{2,\beta}(x) = \begin{cases} (-x)^{\beta}, & x \le 0, \\ 0, & x > 0, \end{cases} \\ \textit{Type III (Gumbel type)}: \quad u_{3,\beta}(x) = u_3(x) = e^{-x}, \quad -\infty < x < \infty. \end{array}\right\} \tag{2.3}$$

Conversely, any such DF $\exp(-u_{i,\beta}(x))$, $i \in \{1, 2, 3\}$, *appears as a limit in* (2.1) *and in fact does so when* $\exp(-u_{i,\beta}(x))$ *is itself the DF of each* X_i. *In this case, we write* $F \in \mathcal{D}(H_{i,\beta})$ *and call* $H_{i,\beta}(\cdot)$, $i = 1, 2, 3$, *max-stable DFs.*

In the sequel, we write $(a_{i,n}, b_{i,n})$, $a_{i,n} > 0$, for the used normalizing constants, where the subscript i is equal 1, or 2, or 3, according to the weak convergence was took place to the type $H_{1,\beta}(x)$, or $H_{2,\beta}(x)$, or $H_3(x)$, respectively.

Corollary 2.1 (cf. Leadbetter, Lindgren and Rootzén, 1983). *Set* $x^0 = \sup\{x : F(x) < 1\}$ *and* $\gamma(t) = \inf\{x : F(x) \ge 1 - \frac{1}{t}\}$ (*clearly,* $\gamma(t) \underset{t}{\longrightarrow} x^0$). *The normalizing constants* $a_{i,n} > 0$ *and* $b_{i,n}$ *for each types in Theorem* 2.1 *can be chosen such as*

Type I: $a_{1,n} = |\gamma(n)|$ *and* $b_{1,n} = 0$, *where* $x^0 = \infty$.
Type II: $a_{2,n} = |x^0 - \gamma(n)|$ *and* $b_{2,n} = x^0$, *where* $x^0 < \infty$.
Type III: $a_{3,n} = g(b_{3,n})$ *and* $b_{3,n} = \gamma(n)$, *where* $x^0 \le \infty$ *and* $g(t) = (1 - F(t))^{-1} \int_t^{x^0} (1 - F(x))\, dx < \infty$, $t < x^0$.

Lemma 2.1 (Gnedenko, 1943). *If* $F \in \mathcal{D}(H_3)$, *then* $a_{3,n}/b_{3,n} \underset{n}{\longrightarrow} 0$.

Lemma 2.2 (Barakat, 1998). *Let* ε *be an arbitrary small positive number.*

(i) *If* $F \in \mathcal{D}(H_{1,\beta})$, *then* $a_{1,n} n^{-\beta^{-1}+\varepsilon} \underset{n}{\longrightarrow} \infty$ *and* $a_{1,n} n^{-\beta^{-1}-\varepsilon} \underset{n}{\longrightarrow} 0$;
(ii) *If* $F \in \mathcal{D}(H_{2,\beta})$, *then* $a_{2,n} n^{\beta^{-1}+\varepsilon} \underset{n}{\longrightarrow} \infty$ *and* $a_{2,n} n^{\beta^{-1}-\varepsilon} \underset{n}{\longrightarrow} 0$;
(iii) *If* $F \in \mathcal{D}(H_3)$, *then* $a_{3,n} n^{+\varepsilon} \underset{n}{\longrightarrow} \infty$ *and* $a_{3,n} n^{-\varepsilon} \underset{n}{\longrightarrow} 0$.

The following corollary is a simple consequence of Lemma 2.2.

Corollary 2.2. *Let* ε *be an arbitrary small positive number. Then, we get*

(i) $\frac{a_{1,n}}{a_{2,n}} n^{-2\beta^{-1}+2\varepsilon} \underset{n}{\longrightarrow} \infty$ *and* $\frac{a_{1,n}}{a_{2,n}} n^{-2\beta^{-1}-2\varepsilon} \underset{n}{\longrightarrow} 0$;
(ii) $\frac{a_{1,n}}{a_{3,n}} n^{-\beta^{-1}+2\varepsilon} \underset{n}{\longrightarrow} \infty$ *and* $\frac{a_{1,n}}{a_{3,n}} n^{-\beta^{-1}-2\varepsilon} \underset{n}{\longrightarrow} 0$;
(iii) $\frac{a_{2,n}}{a_{3,n}} n^{\beta^{-1}+2\varepsilon} \underset{n}{\longrightarrow} \infty$ *and* $\frac{a_{2,n}}{a_{3,n}} n^{\beta^{-1}-2\varepsilon} \underset{n}{\longrightarrow} 0$.

Lemma 2.2 determines the essential term on which the asymptotic behavior of the scale normalizing constant solely depends. For example, due to Lemma 2.2 the normalizing constants $Cn^{\frac{1}{\beta}}$, $Cn^{\frac{1}{\beta}} \log n$ and $\frac{Cn^{\frac{1}{\beta}}}{\log n}$ may fit for $a_{1,n} \underset{n}{\longrightarrow} \infty$, where $C > 0$ is a constant, while $Cn^{-\frac{1}{\beta}}$, $Cn^{-\frac{1}{\beta}} \log n$ and $\frac{Cn^{-\frac{1}{\beta}}}{\log n}$ may fit for $a_{2,n} \underset{n}{\longrightarrow} 0$, finally C, $C \log n$ and $\frac{C}{\log n}$ may fit for $a_{3,n}$.

2.2 Some required results of asymptotic behavior of bivariate maximum order statistics

In this subsection, we first give some required results of asymptotic behavior of bivariate maximum order statistics based on M-ordering principle. Then, we present a lemma that connects between the M-ordering and the R-ordering principles, when we use the sup-norm. Throughout this subsection and the rest of all the paper, let $\underline{Z}_1, \underline{Z}_2, \ldots, \underline{Z}_n \in \mathbb{R}^2$, where $\underline{Z}_i = (X_i, Y_i)$, $i = 1, 2, \ldots, n$, be i.i.d. random vectors distributed as $F(x, y) = P((X_i, Y_i) \leq_c (x, y))$. Furthermore, let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ and $Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n}$ be the two marginal ordered statistics concerning the variables $X_i, i = 1, 2, \ldots, n$, and $Y_i, i = 1, 2, \ldots, n$, respectively. Furthermore, let $G(x, y) = P((X_i, Y_i) >_c (x, y))$, where $>_c$ is meant componentwise, be the survival function of the DF $F(x, y)$. Also, let $F_1(x)$, $F_2(y)$, $G_1(x)$ and $G_2(y)$ be the marginals of $F(x, y)$ and $G(x, y)$, respectively. Moreover, write $\underline{M}_{n:n} = (X_{n:n}, Y_{n:n})$ and $F_{n:n}(x, y) = P(\underline{M}_{n:n} \leq_c (x, y))$. Finally, let $F_{1,n:n}(x)$ and $F_{2,n:n}(y)$ be the marginals of $F_{n:n}(x, y)$. Clearly,

$$F_{n:n}(x, y) = F^n(x, y). \tag{2.4}$$

Theorem 2.2 (see, Galambos 1975, 1987, Barakat, 2001). *For any suitable normalizing constants $a_n > 0$, $c_n > 0$, $b_n, d_n \in \mathbb{R}$ and some non-degenerate DF $H_n(x, y)$, we get*

$$F_{n:n}(a_n x + b_n, c_n y + d_n) \underset{n}{\overset{w}{\longrightarrow}} H(x, y), \tag{2.5}$$

if and only if

$$\left.\begin{array}{ll} (1) & nG_1(a_n x + b_n) \underset{n}{\longrightarrow} \mathcal{U}_1(x), \\ (2) & nG_2(c_n y + d_n) \underset{n}{\longrightarrow} \mathcal{U}_2(y), \\ (3) & nG(a_n x + b_n, c_n y + d_n) \underset{n}{\longrightarrow} \mathcal{U}(x, y), \end{array}\right\} \tag{2.6}$$

where each of the functions $\mathcal{U}_1(x)$ and $\mathcal{U}_2(x)$ takes one and only one of the types (2.3). *Moreover, we have $0 \leq \mathcal{U}(x, y) < \min(\mathcal{U}_1(x), \mathcal{U}_2(y)) < \infty$, or $\mathcal{U}(x, y) = \min(\mathcal{U}_1(x), \mathcal{U}_2(y)) = 0$. In this case the limit $H(x, y)$ has the form*

$$H(x, y) = \exp\bigl(-\mathcal{U}_1(x) - \mathcal{U}_2(y) + \mathcal{U}(x, y)\bigr). \tag{2.7}$$

Furthermore, the asymptotic independence of the marginals of $H(\underline{z})$ (i.e., the limit of $F_{n:n}(a_n x + b_n, c_n y + d_n)$ splits into the product of the limit marginals) occurs, if and only if $\mathcal{U}(x, y) = 0$.

Remark 2.1 (see, Barakat, 2001). It is worth mentioning that, for many known bivariate models such as Morgenstern distribution $F(x, y) = F_1(x)F_2(y)(1 + \alpha G_1(x)G_2(y))$, $-1 \leq \alpha \leq 1$; Gumbel's type 1 exponential distribution, for which $G(x, y) = \exp(-x - y - \theta xy)$, $0 \leq \theta < 1$, $x, y \geq 0$; and the usual bivariate normal distribution, the limit of $F_{n:n}(a_n x + b_n, c_n y + d_n)$ splits into the product of the limit marginals. On the other hand, the Mardia's distribution, for which $G(x, y) = \frac{G_1(x)G_2(y)}{G_1(x)+G_2(y)-G_1(x)G_2(y)}$, the limit of $F_{n:n}(a_n x + b_n, c_n y + d_n)$ does not split into the product of the limit marginals.

Remark 2.2. It is remarkable that, any bivariate DF $H(x, y)$ serves as a limit DF in (2.5) if and only if there exists a D-norm on $\mathbb{R}^2$ such that $H(x, y) = \exp(-\|(x, y)\|_D)$ (cf. Falk and Wisheckel, 2018).

We conclude this section with a lemma, which is considered an important pillar of the next our study.

Lemma 2.3. *Assuming that* $\underline{Z}_1, \underline{Z}_2, \ldots, \underline{Z}_n \geq 0$, *the exact DF of the maximum of the RVs* $\|\underline{Z}_1\|, \|\underline{Z}_2\|, \ldots, \|\underline{Z}_n\|$ *(i.e., the exact DF of* $\|\underline{Z}\|_{n:n}$*) is* $\mathcal{F}_{n:n}(z) = \mathcal{F}^n(z) = F_{n:n}(z, z)$.

Proof. Clearly, since $\underline{Z}_1, \underline{Z}_2, \ldots, \underline{Z}_n$ are i.i.d. random vectors, then by using (2.4), we get

$$\begin{aligned}
\mathcal{F}_{n:n}(z) = \mathcal{F}^n(z) &= P\big[\max\big(\|\underline{Z}_1\|, \|\underline{Z}_2\|, \ldots, \|\underline{Z}_n\|\big) \leq z\big] \\
&= P\big[\max\big(\max\big(|X_1|, |Y_1|\big), \max\big(|X_2|, |Y_2|\big), \ldots, \max\big(|X_n|, |Y_n|\big) \leq z\big)\big] \\
&= P\big[\max(X_1, Y_1) \leq z, \max(X_2, Y_2) \leq z, \ldots, \max(X_n, Y_n) \leq z\big] \\
&= P^n\big[\max(X_1, Y_1) \leq z\big] = P^n[X_1 \leq z, Y_1 \leq z] = F^n(z, z) = F_{n:n}(z, z).
\end{aligned}$$

This completes the proof. □

Remark 2.3. The main advantage of Lemma 2.3 is that if there are normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$, for which (1)–(3) in (2.6) are satisfied for some functions $\mathcal{U}_1(x)$, $\mathcal{U}_2(x)$ and $\mathcal{U}(x)$, then

$$\mathcal{F}_{n:n}(a_n z + b_n) \xrightarrow[n]{w} H(z, z),$$

where $H(z, z)$ is defined by (2.7). This strong interrelation between the M-ordering and R-ordering principles, when we use sup-norm will be emphasized in the next section.

3 The main results

In this section, the sufficient conditions for obtaining the weak limits of the DF $\mathcal{F}_{n:n}(z)$, under the conditions (1)–(3) defined in (2.6), are derived. Throughout this section, we assume that the conditions (1)–(3) are satisfied with the normalizing constants $(a_{i,n}, b_{i,n})$ and $(c_{i,n}, d_{i,n})$, $a_{i,n}, c_{i,n} > 0$, respectively, where the subscript i is equal 1, or 2, or 3, according to the weak convergence was took place to the type $H_{1,\beta}(x)$, or $H_{2,\beta}(x)$, or $H_{3,\beta}(x)$, respectively. Moreover, define the notations $x_i^0 = \sup\{x : F_i(x) < 1\}$, $i = 1, 2$, and $x^0 = \sup\{x : \mathcal{F}(x) < 1\}$, for the right-end points for the two marginals $F_1(\cdot)$, $F_2(\cdot)$ and the DF $\mathcal{F}(\cdot)$, respectively. Clearly, $x^0 = \max(x_1^0, x_2^0)$.

3.1 The case of the two marginals of $H(x, y)$ are asymptotic independent

In this subsection, we study the asymptotic behavior of the DF $\mathcal{F}_{n:n}(z)$ under the condition $\mathcal{U}(x, y) = 0$, that is, under the condition that the two marginals of $H(x, y)$ are asymptotic independent.

Theorem 3.1.

1. *Let* $F_1(a_{1,n}z + b_{1,n}) \in \mathcal{D}(H_{1,\beta_1})$ *and* $F_2(c_{1,n}z + d_{1,n}) \in \mathcal{D}(H_{1,\beta_2})$. *Then,*
   (i) *if* $\beta_1 < \beta_2$, *we get* $\mathcal{F}(a_{1,n}z + b_{1,n}) \in \mathcal{D}(H_{1,\beta_1})$;
   (ii) *if* $\beta_1 = \beta_2$, *we get* $\mathcal{F}(a_{1,n}z + b_{1,n})$, $\mathcal{F}(c_{1,n}z + d_{1,n}) \in \mathcal{D}(H_{1,\beta_1})$, *provided that* $0 < \lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}} < \infty$; $\mathcal{F}(a_{1,n}z + b_{1,n}) \in \mathcal{D}(H_{1,\beta_1})$, *provided that* $\lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}} = \infty$ *and* $\mathcal{F}(c_{1,n}z + d_{1,n}) \in \mathcal{D}(H_{1,\beta_1})$, *provided that* $\lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}} = 0$.
2. *Let* $F_1(a_{2,n}z + b_{2,n}) \in \mathcal{D}(H_{2,\beta_1})$ *and* $F_2(c_{2,n}z + d_{2,n}) \in \mathcal{D}(H_{2,\beta_2})$. *Then, if* $x_1^0 < x_2^0$, *we have* $\mathcal{F}(c_{2,n}z + d_{2,n}) \in \mathcal{D}(H_{2,\beta_2})$. *On the other hand, if* $x_2^0 < x_1^0$, *we have* $\mathcal{F}(a_{2,n}z + b_{2,n}) \in \mathcal{D}(H_{2,\beta_1})$. *Furthermore, let* $x_1^0 = x_2^0$. *Then,*
   (i) *if* $\beta_1 < \beta_2$, *or if* $\beta_1 = \beta_2$ *and* $0 \le \lim_{n\to\infty} \frac{a_{2,n}}{c_{2,n}} < \infty$, *we get* $\mathcal{F}(a_{2,n}z + b_{2,n}) \in \mathcal{D}(H_{2,\beta_1})$, *or*
   (ii) *if* $\beta_2 < \beta_1$, *or if* $\beta_1 = \beta_2$ *and* $0 \le \lim_{n\to\infty} \frac{c_{2,n}}{a_{2,n}} < \infty$, *we get* $\mathcal{F}(c_{2,n}z + d_{2,n}) \in \mathcal{D}(H_{2,\beta_2})$.
3. *Let* $F_1(a_{3,n}z + b_{3,n})$ *and* $F_2(c_{3,n}z + d_{3,n}) \in \mathcal{D}(H_3)$. *Then,* $\mathcal{F}(a_{3,n}z + b_{3,n}) \in \mathcal{D}(H_3)$, *if*

$$L_{3,3;n}(z) \equiv \left(\frac{a_{3,n}}{c_{3,n}}z + \frac{b_{3,n} - d_{3,n}}{c_{3,n}}\right) \underset{n}{\longrightarrow} \begin{cases} z + B, & or \\ \infty, \end{cases}$$

   where B is a finite constant. Moreover, $\mathcal{F}(c_{3,n}z + d_{3,n}) \in \mathcal{D}(H_3)$, *if*

$$L^{-}_{3,3;n}(z) \equiv \left(\frac{c_{3,n}}{a_{3,n}}z + \frac{d_{3,n} - b_{3,n}}{a_{3,n}}\right) \underset{n}{\longrightarrow} \begin{cases} z - B, & or \\ \infty. \end{cases}$$

4. *Let* $F_1(a_{1,n}z + b_{1,n}) \in \mathcal{D}(H_{1,\beta_1})$ *and* $F_2(c_{2,n}z + d_{2,n}) \in \mathcal{D}(H_{2,\beta_2})$. *Then,* $\mathcal{F}(a_{1,n}z + b_{1,n}) \in \mathcal{D}(H_{1,\beta_1})$.
5. *Let* $F_1(a_{1,n}z + b_{1,n}) \in \mathcal{D}(H_{1,\beta})$ *and* $F_2(c_{3,n}z + d_{3,n}) \in \mathcal{D}(H_3)$. *Then* $\mathcal{F}(c_{3,n}z + d_{3,n}) \in \mathcal{D}(H_3)$, *if* $\frac{d_{3,n}}{a_{1,n}} \underset{n}{\longrightarrow} \infty$. *Moreover,* $\mathcal{F}(a_{1,n}z + b_{1,n}) \in \mathcal{D}(H_{1,\beta})$, *if* $\frac{d_{3,n}}{a_{1,n}} \underset{n}{\longrightarrow} 0$.
6. *Let* $F_1(a_{2,n}z + b_{2,n}) \in \mathcal{D}(H_{2,\beta})$ *and* $F_2(c_{3,n}z + d_{3,n}) \in \mathcal{D}(H_3)$. *Then* $\mathcal{F}(a_{2,n}z + b_{2,n}) \in \mathcal{D}(H_{2,\beta})$, *if* $x_2^0 < x_1^0$, *or* $x_2^0 = x_1^0$ *and* $\frac{x_1^0 - d_{3,n}}{c_{3,n}} \underset{n}{\longrightarrow} \infty$. *Moreover,* $\mathcal{F}(c_{3,n}z + d_{3,n}) \in \mathcal{D}(H_3)$, *if* $x_2^0 > x_1^0$.

Remark 3.1. Clearly, in view of Theorem 2.1, any non-degenerate limit of $\mathcal{F}_{n:n}(\cdot)$ is one and only one of the max-stable DFs defined in Theorem 2.1. However, in Theorem 3.1, the limits of $\mathcal{F}_{n:n}(\cdot)$ in all cases are written in the standard form, that is, up to using scale and location constants. These scale and location constants are explicitly shown within the proof of Theorem 3.1.

The following two elementary lemmas will be needed in the proof of Theorem 3.1.

Lemma 3.1. *For any finite* $A \ge 0$ *and* B, $H(x) = H_3(x) \times H_{1,\beta}(Ax + B)$ *is not max-stable DF. Moreover,* $H(x) = H_3(x) \times H_{2,\beta}(Ax + B)$ *is max-stable DF only if* $A = B = 0$ *(i.e., in this case* $H(x) = H_3(x)$*). Finally,* $H(x) = H_3(x) \times H_3(Ax + B)$ *is max-stable DF only if* $A = 1$ *(i.e., in this case* $H(x) = H_3(x - \log(1 + e^{-B}))$*).*

Proof. First, if $H(x) = H_3(x) \times H_{1,\beta}(Ax + B)$, $A > 0$, then we get $H(x) = e^{-u_3(x)} \times e^{-u_1(Ax+B)} = e^{-e^{-x}} \times e^{-(Ax+B)^{-\beta}}$, $Ax + B > 0$, which is impossible to be a max-stable DF. Also, if $A = 0$, we get $H(x) = CH_3(x)$, where $C = H_{1,\beta}(B) =$ constant, is not a max-stable DF (actually $H(x)$ in this case is a defective DF, that is, $H(\infty) = C < 1$). On the other hand, if $H(x) = H_3(x) \times H_{2,\beta}(Ax + B)$, $A > 0$, then we get $H(x) = e^{-u_3(x)} \times e^{-u_2(Ax+B)} = e^{-e^{-x}} \times e^{-(-Ax-B)^{\beta}}$, $Ax + B \leq 0$, which is impossible to be a max-stable DF, unless $A = B = 0$ (in this case $H(x) = H(3)$). Finally, if $H(x) = H_3(x) \times H_3(Ax + B)$, we get $H(x) = e^{-u_3(x)} \times e^{-u_3(Ax+B)} = e^{-e^{-x}} \times e^{-e^{-Ax-B}}$, which is impossible to be a max-stable DF, unless $A = 1$. Moreover, in this case we have $H(x) = H_3(x - \log(1 + e^{-B}))$. This completes the proof of the lemma. □

Lemma 3.2. *Let A_n and B_n be two sequences such that $A_n \underset{n}{\longrightarrow} \infty$ and $B_n \underset{n}{\longrightarrow} \infty$, then we have*

(1) *$L_n(x) = A_n x + B_n$ has at most the following limit points ∞, $\forall x$,*

$$\Upsilon_0(x) = \begin{cases} -\infty, & x < 0, \\ \infty, & x \geq 0, \end{cases} \quad \text{and} \quad \Upsilon_1(x) = \begin{cases} -\infty, & x < x_\star, \\ B, & x = x_\star, \\ \infty, & x > x_\star. \end{cases}$$

(2) *$L_n^\star(x) = A_n x - B_n$ has at most the following limit points $-\infty$, $\forall x$, $\Upsilon_0^\star(x) = \left\{\begin{smallmatrix} -\infty, & x \leq 0, \\ \infty, & x > 0 \end{smallmatrix}\right.$ and $\Upsilon_1(x)$ (clearly, B and $x_\star$ in $\Upsilon_1(x)$ defined in* (1) *are different than those in $\Upsilon_1(x)$ defined in* (2)).

Proof. Clearly, the possible limit points of $L_n(x)$ are $-\infty, \infty$ and $ax + b$, $-\infty < x < \infty$, where a and b are finite constants. Besides, there may be a finite value B and a unique negative value $x_\star$, for which $L_n(x_\star) \underset{n}{\longrightarrow} B$ (e.g., if $A_n = n + 1$, $B_n = n$, then $x_\star = B = -1$). Clearly, this possibility gives $L_n(x) \underset{n}{\longrightarrow} \Upsilon_1(x)$. Now, let $L_n(x) \underset{n}{\longrightarrow} ax + b$. Put $x = 1$, we get

$$A_n + B_n \underset{n}{\longrightarrow} a + b. \tag{3.1}$$

On the other hand, by setting $x = 2$, we get

$$2A_n + B_n \underset{n}{\longrightarrow} 2a + b. \tag{3.2}$$

By subtracting (3.1) and (3.2), we get $A_n \underset{n}{\longrightarrow} a$, which contradicts the assumption of the lemma. This completes the proof of first part and the proof of the second part follows by using the same argument. □

Proof of Theorem 3.1. In view of Corollary 2.1, $b_{1,n} = d_{1,n} = 0$. Moreover in view of Lemma 2.2, Part (i), we have $\frac{a_{1,n}}{c_{1,n}} \underset{n}{\longrightarrow} \infty$, if $\beta_1 < \beta_2$. Moreover, if $\beta_1 = \beta_2$, we have either $\lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}} = \theta_1$, $0 < \theta_1 < \infty$, or $\lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}} = 0$, or $\lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}} = \infty$. Note that if $\lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}}$ does not exist, then we can choose a subsequence n_k for which the limit converges. Meanwhile, by using the Khinchin's type theorem (cf. Theorem 1.13 in Barakat, Nigm and Khaled, 2019), we can replace the normalizing constants $a_{1,n}$ and $c_{1,n}$ by a_{1,n_k} and

c_{1,n_k}, respectively. Now, consider

$$F_2^n(a_{1,n}z) = F_2^n\left[c_{1,n}\left(\frac{a_{1,n}}{c_{1,n}}z\right)\right] \xrightarrow[n]{w} \begin{cases} \in_0(z), \\ \quad \text{if } \beta_1 < \beta_2, \text{ or } \beta_1 = \beta_2, \lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}} = \infty, \\ H_{1,\beta_1}(\theta_1 z), \\ \quad \text{if } \beta_1 = \beta_2, \lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}} = \theta_1, \end{cases}$$

where $\in_0(z) = \left\{\begin{smallmatrix} 1, & z>0, \\ 0, & z \le 0 \end{smallmatrix}\right.$ is a degenerate limit DF. Therefore, by using Lemma 2.3 and in view of Remark 2.3, we get

$$\begin{aligned} \mathcal{F}_{n:n}(a_{1,n}z + b_{1,n}) &= F^n(a_{1,n}z + b_{1,n}, a_{1,n}z + b_{1,n}) \\ &\xrightarrow[n]{w} \begin{cases} \in_0(z) \times H_{1,\beta_1}(z) = H_{1,\beta_1}(z), \\ \quad \text{if } \beta_1 < \beta_2 \text{ or } \beta_1 = \beta_2, \lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}} = \infty, \\ H_{1,\beta_1}(z) \times H_{1,\beta_1}(\theta_1 z) = e^{-u_{1,\beta_1}(z) - u_{1,\beta_1}(\theta_1 z)} \\ \quad = H_{1,\beta_1}\left((1+\theta_1^{-\beta_1})^{-\frac{1}{\beta_1}} z\right), \quad \text{if } \beta_1 = \beta_2, \lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}} = \theta_1. \end{cases} \end{aligned}$$

The case $\beta_1 = \beta_2$, $\lim_{n\to\infty} \frac{a_{1,n}}{c_{1,n}} = 0$ is now obvious. In addition, the Khinchin's type theorem guarantees that the types of the obtained non-degenerate limits do not change if we use any other normalizing constants. This completes the proof of the Part (1) of Theorem 3.1.

Consider, now Part (2). In view of Corollary 2.1, we have $b_{2,n} = x_1^0 < \infty$, $d_{2,n} = x_2^0 < \infty$, $a_{2,n} \xrightarrow[n]{} 0$ and $c_{2,n} \xrightarrow[n]{} 0$. Let $x_1^0 < x_2^0$. Then, in view of (2.2), we have that $c_{2,n}z + d_{2,n} \xrightarrow[n]{} x_2^0$. Therefore, there exists n_0 such that $c_{2,n}z + d_{2,n} > x_1^0$, for $n > n_0$. That means $F_1(c_{2,n}z + d_{2,n}) = 1$, if $n > n_0$. Consequently, $F_1^n(c_{2,n}z + d_{2,n}) = 1$, if $n > n_0$. Hence, $\mathcal{F}_{n:n}(c_{2,n}z + d_{2,n}) = \mathcal{F}^n(c_{2,n}z + d_{2,n})$ converges weakly to $H_{2,\beta_2}(z)$. Similarly, of course, the same argument can be used to conclude that, if $x_2^0 < x_1^0$, then, $\mathcal{F}_{n:n}(a_{2,n}z + b_{2,n}) = \mathcal{F}^n(a_{2,n}z + b_{2,n})$ converges weakly to $H_{2,\beta_1}(z)$.

Now, consider the case $x_1^0 = x_2^0$. In view of Lemma 2.2, we have $\frac{a_{2,n}}{c_{2,n}} \xrightarrow[n]{} 0$, if $\beta_1 < \beta_2$. However, if $\beta_1 = \beta_2$, we have either $\lim_{n\to\infty} \frac{a_{2,n}}{c_{2,n}} = \theta_2$, $0 \le \theta_2 < \infty$, or $\lim_{n\to\infty} \frac{a_{2,n}}{c_{2,n}} = \infty$, or $\frac{a_{2,n}}{c_{2,n}}$ does not have a limit. Therefore, for the case $0 \le \theta_2 < \infty$, we get

$$\begin{aligned} F_2^n(a_{2,n}z + x_1^0) &= F_2^n\left[c_{2,n}\left(\frac{a_{2,n}}{c_{2,n}}z + \frac{x_1^0 - x_2^0}{c_{2,n}}\right) + x_2^0\right] \\ &\xrightarrow[n]{w} \begin{cases} 1, \quad -\infty < z \le \infty, \quad \text{if } \beta_1 < \beta_2, \text{ or } \beta_1 = \beta_2, \lim_{n\to\infty} \frac{a_{2,n}}{c_{2,n}} = 0, \\ H_{2,\beta}(\theta_2 z), \quad \text{if } \beta_1 = \beta_2 = \beta, \lim_{n\to\infty} \frac{a_{2,n}}{c_{2,n}} = \theta_2, 0 < \theta_2, \infty. \end{cases} \end{aligned}$$

Therefore, by using Lemma 2.3 and Remark 2.3, we get

$$\mathcal{F}_{n:n}(a_{2,n}z + b_{2,n}) \xrightarrow[n]{w} \begin{cases} H_{2,\beta_1}(z), \quad \text{if } \beta_1 < \beta_2, \text{ or } \beta_1 = \beta_2, \lim_{n\to\infty} \frac{a_{2,n}}{c_{2,n}} = 0, \\ H_{2,\beta}(z) \times H_{2,\beta}(\theta_2 z) = e^{-u_{2,\beta}(z) - u_{2,\beta}(\theta_2 z)} \\ \quad = H_{2,\beta}\left((1+\theta_2^{\beta})^{\frac{1}{\beta}} z\right), \quad \text{if } \beta_1 = \beta_2 = \beta, \lim_{n\to\infty} \frac{a_{2,n}}{c_{2,n}} = \theta_2. \end{cases}$$

On the other hand, if $\theta_2 = \infty$ (i.e., $\frac{c_{2,n}}{a_{2,n}} \xrightarrow[n]{} 0$) by using the above argument, clearly we have $\mathcal{F}_{n:n}(c_{2,n}z + d_{2,n})$ converges weakly to $H_{2,\beta}(z)$. Also, if $0 < \theta_2 < \infty$, we get

$\frac{c_{2,n}}{a_{2,n}} \underset{n}{\longrightarrow} \frac{1}{\theta}$. Therefore, by using the same argument, $\mathcal{F}_{n:n}(c_{2,n}z + d_{2,n})$ converges weakly to $H_{2,\beta}((1+\theta_2^{-\beta})^{\frac{1}{\beta}}z)$. Finally, if $\frac{a_{2,n}}{c_{2,n}}$ does not have a limit, then we can choose a subsequence n_k for which the limit converges. Meanwhile, by using the Khinchin's type theorem, we can replace the normalizing constants $a_{2,n}$ and $c_{2,n}$ by a_{2,n_k} and c_{2,n_k}, respectively. Moreover, the Khinchin's type theorem guarantees that the types of the obtained non-degenerate limits do not change if we use any other normalizing constants. This completes the proof of the part (2) of Theorem 3.1.

We turn now to prove Part (3). First, we consider

$$F_2^n(a_{3,n}z + b_{3,n}) = F_2^n\left[c_{3,n}\left(\frac{a_{3,n}}{c_{3,n}}z + \frac{b_{3,n} - d_{3,n}}{c_{3,n}}\right) + d_{3,n}\right] = F_2^n\left[c_{3,n}L_{3,3;n}(z) + d_{3,n}\right].$$

Clearly, the sequence $L_{3,3;n}(z)$ has possible limit points not more than that determined by the set $\{-\infty, \Upsilon_0(z), \Upsilon_0^\star(z), \Upsilon_1(z), \infty, Az + B\}$, where $A \geq 0$ and B are finite constants. However, in view of Lemma 3.1, this set must be reduced to $\{-\infty, \Upsilon_0(z), \Upsilon_0^\star(z), \Upsilon_1(z), \infty, z + B\}$, otherwise $\mathcal{F}_{n:n}(a_{3,n}z + b_{3:n})$ may converge to a non max-stable DF $H_3(z) \times H_3(Az + B), 0 < A \neq 1$, which contradicts the extremal types theorem (Theorem 2.1). On the other hand, the limit points determined by the set $\{-\infty, \Upsilon_0(z), \Upsilon_0^\star(z), \Upsilon_1(z), B\}$ can not yield any non-degenerate limit for $\mathcal{F}_{n:n}(a_{3,n}z + b_{3:n})$. Moreover, the other two limits points ∞ and $z + B$ give the non-degenerate max-stable limits $H_3(z)$ and $H_3(z - \log(1 + e^{-B}))$, respectively. Finally, in the case that the sequence $\{L_{3,3;n}\}$ does not converge at all, we can pick out a convergent subsequence $\{L_{3,3;n_k}\}$. In this case, $\mathcal{F}_{n:n}(a_{3,n_k}z + b_{3:n_k})$ weakly converges to a non max-stable non-degenerate DF $H_3(z) \times H_3(Az + B), 0 < A \neq 1$, which in view of the Khinchin's type theorem, leads to a contradiction with the extremal types theorem. Moreover, the Khinchin's type theorem guarantees that the types of the obtained non-degenerate limits do not change if we use any other normalizing constants. This completes the proof of the part (3) of Theorem 3.1.

Consider, now Part (4). In view of Corollary 2.1, $b_{1,n} = 0$ and $d_{2,n} = x_2^0 < \infty$. Moreover in view of Lemma 2.2 and Corollary 2.2, we have

$$F_2^n(a_{1,n}z) = F_2^n\left[c_{2,n}\left(\frac{a_{1,n}}{c_{2,n}}z - \frac{x_2^0}{c_{2,n}}\right) + x_2^0\right]$$

$$= F_2^n\left[c_{2,n}\left[\frac{a_{1,n}}{c_{2,n}}\left(z - \frac{x_2^0}{a_{1,n}}\right)\right] + x_2^0\right] \underset{n}{\overset{w}{\longrightarrow}} \in_0(z).$$

Therefore $\mathcal{F}_{n:n}(a_{1,n}z + b_{1:n}) \underset{n}{\overset{w}{\longrightarrow}} H_{1,\beta_1}(z) \times \in_0(z) = H_{1,\beta_1}(z)$. The Khinchin's type theorem guarantees that the types of the obtained non-degenerate limits do not change if we use any other normalizing constants. This completes the proof of the Part (4).

Consider, now Part (5). First, set

$$F_1^n(c_{3,n}z + d_{3,n}) = F_1^n\left[a_{1,n}\left(\frac{c_{3,n}}{a_{1,n}}z + \frac{d_{3,n}}{a_{1,n}}\right)\right].$$

Actually, the sequence $\{\frac{d_{3,n}}{a_{1,n}}\}$ has limit points no more than that determined by the set $\{C, 0, \infty\}$, where C is some positive constant (note that due to Corollary 2.1, $d_{3,n} = \gamma(n) \underset{n}{\longrightarrow} x_2^0 > 0$). On the other hand, by using Lemma 3.1, we necessarily have $\frac{c_{3,n}}{a_{1,n}}z + \frac{d_{3,n}}{a_{1,n}} \to \infty$, but in view of Corollary 2.2(ii), $\frac{c_{3,n}}{a_{1,n}} \to 0$, then, it suffices that $\frac{d_{3,n}}{a_{1,n}} \to \infty$ (i.e., the first and the second limit points, C and 0, do not yield any non-degenerate limit DF for $\mathcal{F}_{n:n}(c_{3,n}z + d_{3,n})$).

Hence, $\mathcal{F}_{n:n}(c_{3,n}z+d_{3,n}) \xrightarrow[n]{w} H_3(z)$, provided that $\frac{d_{3,n}}{a_{1,n}} \xrightarrow[n]{} \infty$. Also,

$$F_2^n(a_{1,n}z) = F_2^n\Big[c_{3,n}\Big(\frac{a_{1,n}}{c_{3,n}}z-\frac{d_{3,n}}{c_{3,n}}\Big)+d_{3,n}\Big] = F_2^n\Big[c_{3,n}\Big[\frac{a_{1,n}}{c_{3,n}}\Big(z-\frac{d_{3,n}}{a_{1,n}}\Big)\Big]+d_{3,n}\Big]$$

$$\xrightarrow[n]{} \begin{cases} \in_0(z), & \text{if } \frac{d_{3,n}}{a_{1,n}} \xrightarrow[n]{} 0, \\ 0, & \text{if } \frac{d_{3,n}}{a_{1,n}} \xrightarrow[n]{} \infty \end{cases}$$

(remember that, in view of Corollary 2.2(ii), $\frac{a_{1,n}}{c_{3,n}} \to \infty$). Thus, $\mathcal{F}_{n:n}(a_{1,n}z) \xrightarrow[n]{w} H_{1,\beta}(z)$, provided that $\frac{d_{3,n}}{a_{1,n}} \xrightarrow[n]{} 0$ (note that, the second limit C yield a non max-stable DF for $\mathcal{F}_{n:n}(a_{1,n}z)$, which contradicts the extremal types theorem (i.e., Theorem 2.1). Again, the Khinchin's type theorem guarantees that the types of the obtained non-degenerate limits do not change if we use any other normalizing constants. This completes the proof of the part (5).

We turn now to prove the last part. First, remember that $x_1^0 < \infty$ and $x_2^0 \le \infty$. Thus, if we assume that $x_2^0 < x_1^0$, we have $a_{2,n}z + b_{2,n} \to x_1^0$. Therefore, there exist n_0 such that $a_{2,n}z + b_{2,n} > x_2^0$, for $n > n_0$. That means $F_2(a_{2,n}z+b_{2,n}) = 1$, if $n > n_0$. Consequently, $F_2^n(a_{2,n}z+b_{2,n}) = 1$, if $n > n_0$. Hence, $\mathcal{F}_{n:n}(a_{2,n}z+b_{2,n})$ converges weakly to $H_{2,\beta}(z)$. Similarly, by using the same argument, if we assume that $x_1^0 < x_2^0$, then $\mathcal{F}_{n:n}(c_{3,n}z+d_{3,n})$ converges weakly to $H_3(z)$.

Now, suppose $x_1^0 = x_2^0$. Consider

$$F_2^n(a_{2,n}z+x_1^0) = F_2^n\Big[c_{3,n}\Big(\frac{a_{2,n}}{c_{3,n}}z+\frac{x_1^0-d_{3,n}}{c_{3,n}}\Big)+d_{3,n}\Big].$$

Clearly, the sequence $\{\frac{x_1^0-d_{3,n}}{c_{3,n}}\} = \{\frac{d_{3,n}}{c_{3,n}}(\frac{x_1^0}{d_{3,n}}-1)\}$ has no more than limit points that determined by the set $\{\infty, 0, C\}$, where C is some positive constant (note that $d_{3,n} \uparrow x_2^0$, as $n \to \infty$, remember that $x_1^0 = x_2^0$. Thus, $\frac{x_1^0}{d_{3,n}} - 1 \downarrow 0$, as $n \to \infty$). On the other hand, the second and third limit points give $F_2^n(a_{2,n}z+x_1^0) = F_2^n[c_{3,n}(\frac{a_{2,n}}{c_{3,n}}z+\frac{x_1^0-d_{3,n}}{c_{3,n}})+d_{3,n}] \xrightarrow[n]{} e^{-1}$ and $F_2^n(a_{2,n}z+x_1^0) = F_2^n[c_{3,n}(\frac{a_{2,n}}{c_{3,n}}z+\frac{x_1^0-d_{3,n}}{c_{3,n}})+d_{3,n}] \xrightarrow[n]{} e^{-e^{-C}}$, respectively, which implies that $\mathcal{F}_{n:n}(a_{2,n}z+x_1^0)$ does not converge to a non-degenerate max-stable DF. Thus, the only limit point of the sequence $\{\frac{x_1^0-d_{3,n}}{c_{3,n}}\}$ is ∞, which yields $\mathcal{F}_{n:n}(a_{2,n}z+x_1^0) \xrightarrow[n]{w} H_2(z)$. This completes the proof of Part (6), as well as the theorem. □

Corollary 3.1. *Let F_1 and F_2 be defined as any of Parts* (1)–(6) *of Theorem* 3.1. *Then, from the proof of Theorem* 3.1, *we conclude an interesting fact that, the DFs $\mathcal{F}(\cdot)$ and $F_2(\cdot)$ belong to the same domain of attraction of a max-stable DF, if $x_1^0 < x_2^0 \le \infty$.*

Remark 3.2. Let us consider the case that the condition $\underline{Z}_i \ge 0$ is removed in such a way that $\min(x_1^0, x_2^0) > 0$, that is, we have a marginal RV (or two) that has negative and positive values. In this case, instead of this marginal we can treat with its left-truncated version at zero. In this case, Theorem 3.1 can be applied to detect the asymptotic behavior of the maximum vector $\underline{Z}_{n:n}$ in the norm sense by using the fact that (cf. Galambos, 1987, Page 73) $F(a_nx+b_n) \in \mathcal{D}(H_{i,\beta})$, $i \in \{1,2,3\}$, if and only if $F_u(a_n^\star x + b_n^\star) \in \mathcal{D}(H_{i,\beta})$, where F is any DF, F_u is the left-truncated DF of F at u and the interrelation between the normalizing constants $a_n > 0$, b_n and $a_n^\star > 0$, $b_n^\star$ can be determined by using Remark 2.2.1 in Galambos, 1987. Clearly, when

$\max(x_1^0, x_2^0) \le 0$ the last procedure is not applicable. In this case we can proceed as follows: Let the marginal F_1 (say) is such that $x_1^0 \le 0$. Then, if the left-end point of the marginal F_1 is $\alpha > -\infty$, then replace $F_1(x)$ with $F_1^\star(x) = F_1(x - \alpha)$. Clearly, $F_1^\star$ has a positive support and Theorem 3.1 can be applied now to detect the asymptotic behavior of the maximum vector $\underline{Z}_{n:n}$ in the norm sense by using the obvious fact that $F^\star \in \mathcal{D}(H_{i,\beta})$, $i \in \{1, 2, 3\}$, if and only if $F \in \mathcal{D}(H_{i,\beta})$.

3.2 Application of Theorem 3.1 in a more than two-dimensional case

It worth mentioning that Theorem 3.1 may be formulated for multivariate order statistics (rather than bivariate order statistics), by adapting the procedure that was applied in Theorem 3.1 into more than two dimensions. Taking into account that the formulation of theory in a more than two dimensions will be confusing and impractical due to the large number of cases that should be investigated. Fortunately, we don't need to formulate Theorem 3.1 to more than two dimensions thanks to use the fact that $\max\{X_1, X_2, \ldots, X_m\} = \max\{\max\{X_1, X_2, \ldots, X_{m-1}\}, X_m\}$. Namely, by using this fact, Theorem 3.1 can be applied to deal directly with more than two-dimensional case. For example consider three-dimensional case, that is, $m = 3$. Let (X_1, X_2, X_3) be a random vector, which is distributed as a DF $F(x_1, x_2, x_3)$. Furthermore, let $F_i(x_i)$, 1, 2, 3, and $F_{i_1,i_2}(x_{i_1}, x_{i_2})$, $i_1, i_2 = 1, 2, 3$, $i_1 \neq i_2$ be the univariate and bivariate marginals of $F(x_1, x_2, x_3)$. Finally, let $a_{1,n}^{(i)} > 0$ and $b_{1,n}^{(i)}$, $i = 1, 2, 3$, be suitable normalizing constants, for which $F_1(a_{1,n}^{(1)} z + b_{1,n}^{(1)}) \in \mathcal{D}(H_{1,\beta_1})$, $F_2(a_{1,n}^{(2)} z + b_{1,n}^{(2)}) \in \mathcal{D}(H_{1,\beta_2})$, $F_3(a_{1,n}^{(3)} z + b_{1,n}^{(3)}) \in \mathcal{D}(H_{1,\beta_3})$ and $\beta_1 < \beta_2 < \beta_3$. According to the above fact, if the RVs (X_{i1}, X_{i2}, X_{i3}), $i = 1, 2, \ldots, n$, are i.i.d., we have $\mathcal{F}_{1,2,3}(z) = \mathcal{F}(z) = P(\|(X_1, X_2, X_3)\| \le z) = F(z, z, z)$ and $\mathcal{F}^n(z) = P(\max_{i=1,2,\ldots,n} \|(X_{i1}, X_{i2}, X_{i3})\| \le z) = F^n(z, z, z)$. Moreover, $\mathcal{F}_{1,2}(z) = P(\|(X_1, X_2)\| \le z) = F_{1,2}(z, z)$ (this is a univariate DF) and $\mathcal{F}_{1,2}^n(z) = P(\max_{i=1,2,\ldots,n} \|(X_{i1}, X_{i2})\| \le z) = F_{1,2}^n(z, z)$. Now, we can suggest a simple technique to investigate the asymptotic behavior of $\mathcal{F}^n(z)$ by applying Theorem 3.1 twice as follows:

1. Since $F_1(a_{1,n}^{(1)} z + b_{1,n}^{(1)}) \in \mathcal{D}(H_{1,\beta_1})$, $F_2(a_{1,n}^{(2)} z + b_{1,n}^{(2)}) \in \mathcal{D}(H_{1,\beta_2})$ and $\beta_1 < \beta_2$, then $\mathcal{F}_{1,2}(a_{1,n}^{(1)} z + b_{1,n}^{(1)}) \in \mathcal{D}(H_{1,\beta_1})$.
2. Since $\mathcal{F}_{1,2}(a_{1,n}^{(1)} z + b_{1,n}^{(1)}) \in \mathcal{D}(H_{1,\beta_1})$, $F_3(a_{1,n}^{(3)} z + b_{1,n}^{(3)}) \in \mathcal{D}(H_{1,\beta_3})$ and $\beta_1 < \beta_3$, then $\mathcal{F}(a_{1,n}^{(1)} z + b_{1,n}^{(1)}) \in \mathcal{D}(H_{1,\beta_1})$.

As another example for the application of this technique, let $F_1(a_{1,n}^{(1)} z + b_{1,n}^{(1)}) \in \mathcal{D}(H_{2,\beta_1})$, $F_2(a_{2,n}^{(2)} z + b_{2,n}^{(2)}) \in \mathcal{D}(H_{2,\beta_2})$, $F_3(a_{2,n}^{(3)} z + b_{2,n}^{(3)}) \in \mathcal{D}(H_{2,\beta_3})$ and $x_1^0 > \max(x_2^0, x_3^0)$. Then, by applying the suggested technique, we get $\mathcal{F}(a_{1,n}^{(1)} z + b_{1,n}^{(1)}) \in \mathcal{D}(H_{2,\beta_1})$. Clearly, by using this technique, Theorem 3.1 will be applied $(m - 1)$ times for the m-dimensional case.

3.3 The case of the two marginals of $H(x, y)$ are asymptotic dependent

In this subsection, we study the asymptotic behavior of the DF $\mathcal{F}_{n:n}(z)$, under the condition $\mathcal{U}(x, y) \neq 0$, i.e., under the condition that the two marginal of $H(x, y)$ are asymptotic dependent. The next theorem surprisingly reveals that the possible non-degenerate types (max-stable types) of the suitably normalized DF $\mathcal{F}_{n:n}$, when $\mathcal{U}(x, y) \neq 0$, exactly the same as those types given in Theorem 3.1, under the same conditions given in Theorem 3.1. In other words, Theorem 3.1 will be valid when $\mathcal{U}(x, y) \neq 0$.

The starting point of our study in this subsection will be Theorem 3.1, besides the fact that the normalized DF $\mathcal{F}_{n:n}$ can not converge weakly to any non-degenerate DF, unless one of the max-stable DFs. Thus, if there are suitable normalizing constants $C_n > 0$ and D_n, for which the DF $\mathcal{F}_{n:n}(C_n z + D_n)$ weakly converges to a non-degenerate limit, then, in view of (2.2), we have

$$n\big[1-\mathcal{F}(C_n z+D_n)\big] \underset{n}{\longrightarrow} u_{i,\beta^\star}(cz+d), \quad c>0, d\in\mathbb{R}, i\in\{1,2,3\}. \tag{3.3}$$

On the other hand, by using Lemma 2.3, we get

$$\begin{aligned} &n\big[1-\mathcal{F}(C_n z+D_n)\big] \\ &\quad\sim n\big[1-F(C_n z+D_n, C_n z+D_n)\big] \\ &\quad= nG_1(C_n z+D_n)+nG_2(C_n z+D_n)-nG(C_n z+D_n, C_n z+D_n). \end{aligned} \tag{3.4}$$

Now, let $C_n > 0$ and D_n be the normalizing constants used in $\mathcal{F}_{n:n}$, to get the standard max-stable laws given in Theorem 3.1 (this is clearly possible due to the Khinchin's type theorem). Meanwhile, assume that these normalizing constants still give a non-degenerate limit in (3.3). Moreover, assume that the conditions of the weak convergence in each case of Theorem 3.1 are satisfied for these normalizing constants (note that some of the limit max-stable laws defined in Theorem 3.1 have location or scale parameters. Thus, we can easily verify that all the sufficient conditions given in Theorem 3.1 will be satisfied for these modified normalizing constants). Therefore, under these assumptions and in view of the Khinchin's type theorem, we get

$$nG_1(C_n z+D_n)+nG_2(C_n z+D_n) \underset{n}{\longrightarrow} u_{j,\beta}(z), \quad i\in\{1,2,3\}, \tag{3.5}$$

where $u_{j,\beta}(z)$ is defined in each cases of Theorem 3.1. On the other hand, in view of the extremal types theorem (Theorem 2.1), the assumption that $\mathcal{F}^n(C_n z+D_n)=F^n(C_n z+D_n, C_n z+D_n)$ weakly converges to a non-degenerate DF (see, (3.3)) and Theorem 2.2 (Equations (2.4) and (2.6)), we get

$$nG(C_n z+D_n, C_n z+D_n) \underset{n}{\longrightarrow} \mathcal{U}(z,z), \tag{3.6}$$

where $\mathcal{U}(z,z)$ is defined by (2.6) (Part (3)). However, $G(C_n z+D_n, C_n z+D_n)$ is a survival function of the univariate DF, then

$$nG(C_n z+D_n, C_n z+D_n) \underset{n}{\longrightarrow} u_{t,\beta^{\star\star}}(az+b), \quad t\in\{1,2,3\}, \tag{3.7}$$

where $a > 0$ and b are some scale and location constants, respectively. Thus, (3.6) and (3.7) yield that

$$\mathcal{U}(z,z)=u_{t,\beta^{\star\star}}(az+b), \quad t\in\{1,2,3\}. \tag{3.8}$$

Combining now (3.3)–(3.5) and (3.8), we get

$$u_{i,\beta^\star}(cz+d)=u_{j,\beta}(z)-u_{t,\beta^{\star\star}}(az+b), \quad i,t,j\in\{1,2,3\}. \tag{3.9}$$

Now, by using an interesting fact, which is revealed by Lemma 2.2 and Theorem 2.1, that any normalizing constants (C_n, D_n) in (2.1), that is, in the extremal types theorem, determine uniquely (up to location and scale changes) the max-stable types. This, means that if, we have the two DFs $\Psi_1(\cdot)$ and $\Psi_2(\cdot)$ of different types, such that $\Psi_1^n(C_n x+D_n) \xrightarrow[n]{w} H_{i,\beta}(x)$, $i\in\{1,2,3\}$ (remember that $H_3=H_{3,\beta}$) and $\Psi_2^n(C_n x+D_n \xrightarrow[n]{w} H(x)$, where $H(x)$ is any non-degenerate limit DF, then $H(x)=H_{i,\beta}(ax+b)$, where $a>0$ and b some scale and location parameters. Thus, due to this fact, in order to get a non-degenerate limit of $\mathcal{F}_{n:n}(C_n z+D_n)$, we clearly in (3.9) should have $t=j=i$ and $\beta^{\star\star}=\beta^\star=\beta$. we now in a suitable position to state the following theorem.

Theorem 3.2. *Let $C_n > 0$ and D_n be the normalizing constants used in $\mathcal{F}_{n:n}$, to get the standard max-stable laws given in Theorem 3.1. Then,*

$$\mathcal{F}_{n:n}(C_n z + D_n) \xrightarrow[n]{w} H_{j,\beta}(cz + d), \tag{3.10}$$

where $H_{j,\beta}(z)$ is the same limits max-stable DF of $\mathcal{F}_{n:n}$, defined in each case (6 cases) of Theorem 3.1 and the convergence (3.10) occurs in each case under the same conditions given in Theorem 3.1. Moreover, the constants $a, c > 0$ and b, d in the equation (3.9) (in which $j = t = i$ and $\beta^{\star\star} = \beta^{\star} = \beta$) are determined in each case of Theorem 3.1, respectively, as follows:

1. $b = d = 0$, *and* $a^{-\beta} + c^{-\beta} = 1$.
2. $b = d = 0$, *and* $a^{\beta} + c^{\beta} = 1$.
3. $a = c = 1$ *and* $e^{-b} + e^{-d} = 1$ *(the last relation confirms that we must have $b > 0$ and $d > 0$).*
4. $b = d = 0$, *and* $a^{-\beta} + c^{-\beta} = 1$.
5. $b = d = 0$, *and* $a^{-\beta} + c^{-\beta} = 1$, *if* $j = 1$. *Moreover,* $a = c = 1$, *if* $j = 3$.
6. $b = d = 0$, *and* $a^{\beta} + c^{\beta} = 1$, *if* $j = 2$. *Moreover,* $a = c = 1$, *if* $j = 3$.

Proof. The proof of the part (1) starts with the relation $u_{1,\beta}(az + b) = u_{1,\beta 1}(z) - u_{1,\beta}(cz + d)$, which leads to

$$(az + b)^{-\beta_1} = z^{-\beta_1} - (cz + d)^{-\beta_1}, \quad \min(z, az + b, cz + d) \geq 0.$$

Now divide the equation by $z^{-\beta_1}$, to obtain

$$\left(\frac{az + b}{z}\right)^{-\beta_1} = 1 - \left(\frac{cz + d}{z}\right)^{-\beta_1}.$$

Equivalently

$$\left(a + \frac{b}{z}\right)^{-\beta_1} + \left(c + \frac{d}{z}\right)^{-\beta_1} = 1.$$

This has to be valid for all z. Therefore, the conclusion is that we must have $b = d = 0$. Immediately, we obtain $a^{-\beta} + c^{-\beta} = 1$. The proofs of the remaining parts are similar. This completes the proof of Theorem 3.2. □

Corollary 3.2. *We have $(1 - G(C_n z + D_n, C_n z + D_n))^n \xrightarrow[n]{w} H_{j,\beta}(az + b)$. Clearly, the univariate DFs $1 - G(z, z)$ and $\mathcal{F}(z)$ are related to the survival function $G(x, y)$ and the DF $F(x, y)$, respectively, but clearly, $G(z, z) \neq 1 - F(z, z)$.*

Example 3.1. In the problem of bombing on and around a target point by a fighter aircraft, consider a two dimensional space (xy-plane), where the target point is the origin point of this plane. Let (X, Y) be the x-axis and y-axis of the incident point. Clearly, X and Y are two RVs. Let the random vector (X, Y) have the standard bivariate normal distribution. Then, Theorem 3.1 and Remark 2.1 imply that the DF of the maximum distance between the target and the incidence points, in terms of sup-norm, weakly converges to H_3.

Example 3.2. In this example, we consider the extremes of a real bivariate data for air pollution from the London Air Quality Network (LAQN). Namely, data was taken from site Barking Dagenham at Rush Green square, that monitors sulphur dioxide (SO_2) (stands for the marginal X_i) and Nitrogen oxides (NO) (stands for the marginal Y_i) every hour in the period from 1-1-2010 to 31-12-2015. This data can be downloaded by any researcher in the

Table 1 *MLE's for GEV* (3.11)

The MLEs of the parameters of $G_\gamma(x;\mu,\sigma)$									
	SO_2			NO			sup-norm		
	γ	μ	σ	γ	μ	σ	γ	μ	σ
MLE's	0.221	3.36	1.655	0.978	6.98	8.119	0.952	8.02	7.97
95% C.I.	(0.21, 0.31)	(3.21, 3.51)	(1.54, 1.78)	(0.97, 1.08)	(6.59, 7.38)	(7.59, 8.68)	(0.91, 0.98)	(7.81, 8.87)	(7.78, 8.57)

form of a report every half hour, every hour or every day according to the type of study from the following site www.londonair.org.uk/london/asp/datadownload.asp. The daily maximum observations of these data sets (exactly 1949 daily maximum observations for each pollutants) are used to apply the block maxima method on the general extreme value DF (GEV) (see, Barakat, Nigm and Khaled, 2019, Khaled and Kamal, 2018)

$$G_\gamma(x;\mu,\sigma)=\exp\left\{-\left[1+\gamma\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\gamma}}\right\} \tag{3.11}$$

defined on $\{x: 1+\gamma(x-\mu)/\sigma>0\}$, where with $\gamma=0$, $\gamma=\frac{1}{\beta}>0$ and $\gamma=-\frac{1}{\beta}<0$, the GEV $G_\gamma(x;\mu,\sigma)$ corresponds to the Gumbel, max-Weibull, and Fréchet types, respectively (defined in (2.3)). On the other hand, the same method is applied on the sup-norm $\|\underline{Z}_i\|=\|(X_i,Y_i)\|$. The maximum likelihood estimates (MLEs) and the 95% asymptotic confidence intervals (95% C.I.) are obtained for the parameters μ, σ and γ, by using the MATLAB Version 7.11.0.584(R2010b). Table 1 gives the result of this study, which reveals that the limit DFs of both $X_{n:n}$ and $Y_{n:n}$, where $n=1949$, are max-Weibull with $\beta=4.53$ and $\beta=1.022$, respectively. On the other hand, the limit DF of $\|\underline{Z}\|_{n:n}$ is also a max-Weibull DF with $\beta=1.05$. This means that according to Theorem 3.1-Part (2), or Theorem 3.2 (it is assumed that the interrelation between the two pollutants is unknown for us), the limit DF of $\|\underline{Z}\|_{n:n}$ is the same as the limit DF of the marginal with smaller β (i.e., $Y_{n:n}$) with the nearly the same location and scale parameters.

4 Discussion and concluding remarks

Theorem 3.1, gives sufficient conditions for the weak convergence of $\|Z\|_{n:n}$ in terms of the weak convergence of the two marginals F_1^n and F_2^n. Meanwhile, this theorem reveals the concealed interrelation between the R-ordering and M-ordering principles, when we use the sup-norm. Moreover, the weak convergence of $\|Z\|_{n:n}$ sheds some light on the convergence of the two marginals F_1^n and F_2^n. Namely, if $\mathcal{F}(\cdot)\in\mathcal{D}(H_{j,\beta})$, $j\in\{1,2,3\}$ then at least one of the marginals F_1^n and F_2^n belongs to the domain of attraction of $H_{j,\beta}(\cdot)$. Furthermore, if $j=2$ the other marginal belongs to the domain of attraction of $H_{2,\beta}(\cdot)$, or $H_{3,\beta}(\cdot)$. Finally, Theorems 3.1 and 3.2 reveal that the weak limits of the DF of $\|Z\|_{n:n}$ are the same (up to location and scale changes), regardless the asymptotic independence of the two marginals F_1^n and F_2^n. It is worth mentioning that all the obtained results in this study can be easily switched for the minimum $\|Z\|_{1:n}$, as well as for the extremes $\|Z\|_{k:n}$, where k is constant with respect to n.

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