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Asymptotics for heavy-tailed renewal–reward processes and applications to risk processes and heavy traffic networks

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Abstract. Consider a renewal–reward process $S_{N(t)} = \sum_{k=1}^{N(t)} X_k$ and let $\{\tau_n\}$ be the interarrival times. It is well known that, under regularity conditions, $S_{N(t)}$ is asymptotically Gaussian provided X_n and τ_n have finite second moment. However, in modelling risk processes or heavy traffic networks, the assumption of the finiteness of the second moment may not be compatible. Also, the independency of the processes $\{S_n\}$ and $\{N(t)\}$ might be not realistic. In this situation, heavy-tailed distributions arise as a proper alternative and dependency between τ_n and the reward X_n should be allowed. By making use of the Mallows–Wasserstein distance we derive CLT type results for heavy-tailed renewal–reward dependent processes. Applications to risk processes and heavy traffic networks are exhibited.

1 Introduction

Consider a renewal–reward process $S_{N(t)} = \sum_{k=1}^{N(t)} X_k$ with the renewal process $\{N(t)\}$ given by $N(t) = \sup\{n : \sum_{k=1}^{n} \tau_k \le t\}$. We assume that the rewards $\{X_n\}$ and inter-renewal times $\{\tau_n\}$ are sequences of independent and identically distributed (i.i.d.) random variables (r.v.'s). It is well known that if $0 < E\{\tau_n^2\} < \infty$ and $0 < E\{X_n^2\} < \infty$ then the independence of the processes $\{N(t)\}$ and $\{X_n\}$ assures that $S_{N(t)}$ is asymptotic Gaussian distribution.

However, in modelling risk processes or heavy traffic networks the assumption of the finiteness of the second moment or the independence of $\{N(t)\}$ and $\{X_n\}$ may not be compatible. In this case, heavy-tailed distributions arise as a suitable alternative and the dependency structures should be allowed. This leads us to consider the most important class of heavy-tailed distributions, namely, the α-stable laws (see Definition 2) that possess finite mean and infinite variance (G_α, $1 < \alpha < 2$). Due to their infinite divisibility property, the stable laws play a central role in the study of asymptotic behavior of normalized partial sums, a similar role normal distribution ($\alpha = 2$) plays among distributions with finite second moment.

A useful tool to handle stable laws is provided by the Mallows–Wasserstein metric on the space of distributions.

Definition 1. The r-Mallows–Wasserstein metric between distributions F and G is

$$d_r(F, G) = \inf_{(X,Y)} \{E(|X - Y|^r)\}^{1/r}, \quad r \ge 1, \tag{1.1}$$

where the infimum is taken over all random vectors (X, Y) with marginal distributions F and G, that is, $X \overset{d}{=} F$ and $Y \overset{d}{=} G$ ($X \overset{d}{=} F$: equality in distribution, in the sense that X possesses distribution F).

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Its close connection with the weak convergence was established by Bickel and Freedman (1981). For $r \geq 1$ and for distribution functions (d.f.) $G \in \mathcal{L}_r$ and $\{F_n\}_{n\geq 1} \subset \mathcal{L}_r$ we have

$$d_r(F_n, G) \underset{n}{\to} 0 \quad \Leftrightarrow \quad F_n \xrightarrow{d} G \quad \text{and} \quad \int |x|^r \, dF_n(x) \underset{n}{\to} \int |x|^r \, dG(x), \tag{1.2}$$

where

$$\mathcal{L}_r = \left\{ F : \int |x|^r \, dF(x) < \infty \right\} \tag{1.3}$$

and $\xrightarrow{d}$ stands for convergence in distribution of random variables with distributions F_n and G.

By exploring this connection interesting results establishing Central Limit type asymptotic for stable laws can be found in Johnson and Samworth (2005) and Barbosa and Dorea (2009). In this note, we derive CLT type results by showing that $S_{N(t)}$ is asymptotically stable in the sense that

$$\frac{S_{N(t)} - b_{N(t)}}{a_{N(t)}} \xrightarrow{d} G_\alpha, \quad 1 < \alpha < 2,$$

$\{a_{N(t)}\}$ and $\{b_{N(t)}\}$ are random constants sequences with $a_{N(t)} > 0$ almost surely.

In the context of renewal–reward processes and under heavy-tailed setting there is a vast bibliography on the matter. Regarding the CLT type results, Levy and Taqqu (2000) studied the case where both the inter-renewal times $\{\tau_n\}$ and rewards $\{X_n\}$ were heavy-tailed with stability index α' and α, respectively. By assuming $1 < \alpha < \alpha' < 2$ and the independence between $\{\tau_n\}$ and $\{X_n\}$ they proved that the process, suitably normalized, converges in distribution to a symmetric β-stable process which possesses stationary increments and is self-similar. More recently, Owada and Samorodnitsky (2015), established a new class of functional CLT type theorems for partial sums of symmetric stationary infinitely divisible processes with regularly varying Levy measures. For results on randomly indexed partial sums under independence of $\{\tau_n\}$ and $\{X_n\}$ we may cite Klesov (1995), Ng et al. (2004) and Pipiras, Taqqu and Levy (2004).

Dependence structures among heavy-tailed rewards X_n's have been considered in a large amount of works. For related results on asymptotics for risk processes and traffic networks we can mention Nyrhinen (2001), Tang and Tsitsiashvili (2003), Goovaerts et al. (2005), Laeven, Goovaerts and Hoedemakers (2005), Tang and Vernic (2007), Chen and Ng (2007), Zhang, Shen and Weng (2009), Yang and Wang (2013) and Sun and Wei (2014). Somehow related to our work Cheng (2015) discussed the situation when τ_n has heavier tail than X_n and $\{X_n\}$'s are widely orthant dependent.

Theorem 1 shows that if there is independence between $\{\tau_n\}$ and $\{X_n\}$, CLT type results can be derived for stable laws. In fact, we obtain convergence in Mallows–Wasserstein distance and as a consequence convergence in distribution, without the hypothesis $1 < \alpha < \alpha' < 2$. And this extends Levy and Taqqu's (2000) result. Theorem 2 shows that these results still hold when the independence between $\{X_n\}$ and $\{\tau_n\}$ is dropped. Applications for surplus process associated to risk processes and for cumulative heavy-traffic load in the context of data traffic are included.

2 Preliminary results

Definition 2. For $0 < \alpha \leq 2$, we say that $S_\alpha(\sigma, \beta, \mu)$ is an α-stable distribution with scale parameter $\sigma > 0$, skewness parameter $|\beta| \leq 1$ and shift parameter $\mu \in R$, if for any $n \geq 2$, there are real numbers $d_n = d_n(\sigma, \beta, \mu)$ such that

$$Y_1 + Y_2 + \cdots + Y_n \overset{d}{=} n^{1/\alpha} Y + d_n, \qquad Y \text{ has distribution function } S_\alpha(\sigma, \beta, \mu), \tag{2.1}$$

where $Y_1, Y_2, \dots, Y_n$ are independent copies of Y. The normal distribution corresponds to $\mathcal{N}(\mu, 2\sigma^2) = S_2(\sigma, 0, \mu)$.

Proposition 1. *Assume that Y is $S_\alpha(\sigma, \beta, \mu)$ then*

(a) *If $0 < \alpha' < \alpha < 2$, then $E(|Y|^{\alpha'}) < \infty$ and $E(|Y|^\alpha) = \infty$.*
(b) *If $\alpha > 1$, then for $\mu = E(Y)$ we have $d_n = \mu(n - n^{1/\alpha})$.*
(c) *If $\alpha \neq 1$, then $aY + b$ is $S_\alpha(|a|\sigma, \operatorname{sign}(a)\beta, a\mu + b)$ for all $a \in R^*$ and $b \in R$.*

As for the Mallows–Wasserstein distance (1.1), it satisfies the metric relation

$$d_r(F, G) \le d_r(F, H) + d_r(H, G), \quad r \ge 1 \tag{2.2}$$

(see, for example, Samorodnitsky and Taqqu, 1994).

The following representation theorem will be helpful for its evaluation. Noting that the finiteness of the r-moment of F and G can be dropped which differentiates it from previous results as Bickel and Freedman (1981) or Dorea and Ferreira (2011).

Lemma 1 (Dorea and Ferreira, 2011). *For $r \ge 1$ we have*

$$d_r^r(F, G) = E\{|X^* - Y^*|^r\} = \int |x - y|^r \, d(F(x) \wedge G(y)), \tag{2.3}$$

where X^* and Y^* has distribution F and G, respectively, and (X^*, Y^*) has distribution $F \wedge G$, that is,

$$P(X^* \le x, Y^* \le y) = F(x) \wedge G(y) = \min\{F(x), G(y)\}.$$

A key point to our proofs is the use of moment bounds. In Proposition 2, we gather some moment inequalities that include von Bahr-Esseen inequality for independent random variables and martingale inequalities (cf. Hall and Heyde, 1980).

Proposition 2. *Let $\xi_1, \xi_2, \dots$ be a sequence of independent r.v.'s with zero-mean and let $V_n = \xi_1 + \cdots + \xi_n$. Then for $1 < r < 2$ we have*

$$E\{|V_n|^r\} \le 2 \sum_{k=1}^{n} E\{|\xi_k|^r\} \tag{2.4}$$

and

$$\lambda^r P\Big(\max_{k \le n} |V_k| > \lambda\Big) \le E\{|V_n|^r\}, \quad \forall \lambda > 0. \tag{2.5}$$

Moreover, if $\{|\xi_n|^r\}_{n \ge 1}$ is uniformly integrable then $\frac{1}{n} E\{|V_n|^r\} \underset{n}{\to} 0$.

Note that our partial sum $S_{N(t)}$ is a sequence of randomly indexed sums. Let ν_n be random indexes, questions arise whether ξ_{ν_n} will preserve the convergence in distribution of $\xi_n \xrightarrow{d} \xi$. To overcome this difficulty, we shall make use of the following.

Anscombe's Condition. Let $\{\nu_n\}_{n \ge 1}$ be a sequence of random indexes diverging to infinity. Assume that $\frac{\nu_n}{n} \xrightarrow{P} \gamma > 0$ and that given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for $n \ge N(\epsilon)$ we have

$$P\Big(\max_{|k-n| \le \delta n} |\xi_k - \xi_n| \ge \epsilon\Big) \le \epsilon.$$

Then

$$\xi_n \xrightarrow{d} \xi \quad \Rightarrow \quad \xi_{\nu_n} \xrightarrow{d} \xi$$

($\xrightarrow{p}$: convergence in probability, see more in Anscombe, 1952).

First, we show the following variant.

Lemma 2. *Let $\{\xi_n\}_{n\geq 1}$ be a sequence of i.i.d. r.v.'s with zero-mean and let $V_n = \sum_{k=1}^{n} \xi_k$. Assume that $1 < r < 2$ then given $\epsilon > 0$ we have for $1 < r' < r$ and $0 < \delta < 1/3$,*

$$P\Big(\max_{|k-n|\leq\delta n}\Big|\frac{V_k}{k^{\frac{1}{r}}} - \frac{V_n}{n^{\frac{1}{r}}}\Big| \geq \epsilon\Big) \\ \leq \frac{4\delta^{\frac{r'}{r}}}{\epsilon^{r'}} \max\Big\{\frac{1}{[2\delta n]^{\frac{r'}{r}}} E\big(|V_{[2\delta n]}|^{r'}\big), \frac{1}{[n(1+\delta)]^{\frac{r'}{r}}} E\big(|V_{[n(1+\delta)]}|^{r'}\big)\Big\}, \tag{2.6}$$

where $[\cdot]$ stands for the largest integer.

Proof. Let $\epsilon_1 = \epsilon/2$. We have

$$P\Big(\max_{|k-n|\leq\delta n}\Big|\frac{V_k}{k^{\frac{1}{r}}} - \frac{V_n}{n^{\frac{1}{r}}}\Big| \geq \epsilon\Big) \leq P(A_n \geq \epsilon_1) + P(B_n \geq \epsilon_1),$$

where

$$A_n = \max_{|k-n|\leq\delta n}\Big|\frac{V_k - V_n}{n^{\frac{1}{r}}}\Big| \quad \text{and} \quad B_n = \max_{|k-n|\leq\delta n}\Big|V_k\Big(\frac{1}{k^{\frac{1}{r}}} - \frac{1}{n^{\frac{1}{r}}}\Big)\Big|.$$

Clearly, $\{V_n, \sigma(\xi_1, \ldots, \xi_n)\}_{n\geq 1}$ forms a martingale. By stationarity and using (2.5), we have for $1 < r' < r$

$$P(A_n \geq \epsilon_1) \leq P\Big(\max_{|k-n|\leq 2\delta n} |V_k| \geq \epsilon_1 n^{\frac{1}{r}}\Big) \\ \leq \frac{1}{\epsilon_1^{r'} n^{\frac{r'}{r}}} E\{|V_{[2\delta n]}|^{r'}\} \leq \frac{2\delta^{\frac{r'}{r}}}{\epsilon_1^{r'}} \frac{1}{[2\delta n]^{\frac{r'}{r}}} E\{|V_{[2\delta n]}|^{r'}\}.$$

For the term B_n, first we show that

$$\Big|\frac{1}{k^{\frac{1}{r}}} - \frac{1}{n^{\frac{1}{r}}}\Big| \leq \frac{2\delta^{\frac{1}{r}}}{[n(1+\delta)]^{\frac{r'}{r}}}, \quad |k-n| \leq \delta n, 0 < \delta < \frac{1}{3}. \tag{2.7}$$

Since $(1+\delta)^{\frac{1}{r}} - 1 \leq \delta^{\frac{1}{r}}$ for $\delta > 0$, we have for $k \geq n$

$$0 \leq \frac{1}{n^{\frac{1}{r}}} - \frac{1}{k^{\frac{1}{r}}} \leq \frac{1}{[n(1+\delta)]^{\frac{1}{r}}}\big((1+\delta)^{\frac{1}{r}} - 1\big) \leq \frac{\delta^{\frac{1}{r}}}{[n(1+\delta)]^{\frac{1}{r}}}.$$

If $k \leq n$, then

$$\frac{1}{k^{\frac{1}{r}}} - \frac{1}{n^{\frac{1}{r}}} \leq \frac{1}{(n(1-\delta))^{\frac{1}{r}}} - \frac{1}{n^{\frac{1}{r}}} \\ \leq \frac{1}{[n(1+\delta)]^{\frac{1}{r}}}\Big(\frac{1+\delta}{1-\delta}\Big)^{\frac{1}{r}}\big(1 - (1-\delta)^{\frac{1}{r}}\big) \leq \frac{2\delta^{\frac{1}{r}}}{[n(1+\delta)]^{\frac{1}{r}}}.$$

By assumption $0 < \delta < 1/3$, so that the last inequality follows from

$$\left(\frac{1+\delta}{1-\delta}\right)^{\frac{1}{r}} \le 2 \quad \text{and} \quad 1-(1-\delta)^{\frac{1}{r}} \le \delta^{\frac{1}{r}}.$$

Note that $\max_{|k-n|\le\delta n} |V_k| \le \max_{k\le n(1+\delta)} |V_k|$. Now, using (2.7) and the inequality (2.5) we get for $1 < r' < r$

$$\begin{aligned} P(B_n \ge \epsilon_1) &\le P\left(\max_{k\le n(1+\delta)} |V_k| \ge \epsilon_1 \frac{[n(1+\delta)]^{\frac{1}{r}}}{2\delta^{\frac{1}{r}}}\right) \\ &\le \frac{2\delta^{\frac{r'}{r}}}{\epsilon_1^{r'}} \frac{1}{[n(1+\delta)]^{\frac{r'}{r}}} E\{|V_{[n(1+\delta)]}|^{r'}\}. \end{aligned}$$

And (2.6) follows. □

Condition 1. (a) Let $\{(X_n, \tau_n)\}_{n\ge1}$ be a sequence of i.i.d. r.v.'s with X_n has distribution F_X and τ_n has F_τ. Assume that $\tau_n \ge 0$ and that $0 < \mu_\tau = E(\tau_n) < \infty$.

(b) Assume that for some $1 < \alpha < 2$ there exists Y with distribution $G_\alpha = S_\alpha(\sigma, \beta, \mu_Y)$ such that $d_\alpha(F_X, G_\alpha) < \infty$.

Consider the renewal process associated with $\{\tau_n\}$. From the classical renewal theory, we have under Condition 1 as $t \to \infty$,

$$\frac{N(t)}{t} \xrightarrow{\text{a.s.}} \frac{1}{\mu_\tau}, \quad N(t) = \sup\left\{n : \sum_{k=1}^{n} \tau_k \le t\right\} \tag{2.8}$$

(a.s.: almost sure convergence). The results from Lemma 2 can now be applied.

Lemma 3. *Assume that Condition 1 holds. Then for independent copies $Y_1, Y_2, \ldots$ of Y we have*

$$\frac{\sum_{k=1}^{N(t)} Y_k - (N(t) - N^{\frac{1}{\alpha}}(t))\mu_Y}{N^{\frac{1}{\alpha}}(t)} \xrightarrow{d} Y. \tag{2.9}$$

If, in addition, $\{\tau_n\}$ and $\{Y_n\}$ are independent then

$$\frac{\sum_{k=1}^{N(t)} Y_k - (N(t) - N^{\frac{1}{\alpha}}(t))\mu_Y}{N^{\frac{1}{\alpha}}(t)} \stackrel{d}{=} Y. \tag{2.10}$$

Proof. Let $1 < \alpha' < \alpha$, by Proposition 1 we have $E\{|Y|^{\alpha'}\} < \infty$. In particular $\mu_Y < \infty$. First, we show that (2.10) holds or equivalently

$$\frac{1}{N^{\frac{1}{\alpha}}(t)} \sum_{k=1}^{N(t)} Y'_k \stackrel{d}{=} Y', \quad Y'_k = Y_k - \mu_Y, Y' = Y - \mu_Y. \tag{2.11}$$

Since $\{Y'_n\}$ are i.i.d. we have from (2.1) and Proposition 1 that for any u

$$\begin{aligned} E\left\{\exp\left(\frac{iu}{N^{\frac{1}{\alpha}}(t)} \sum_{k=1}^{N(t)} Y'_k\right)\right\} &= \sum_{n\ge1} E\left\{1(N(t)=n)\exp\left\{\left(\frac{iu}{n^{\frac{1}{\alpha}}} \sum_{k=1}^{n} Y'_k\right)\right\}\right\} \\ &= \sum_{n\ge1} P(N(t)=n)E\{\exp(iu(Y'))\} = E\{\exp(iu(Y'))\}. \end{aligned}$$

Using the notation from Lemma 2, write $V_n = \sum_{k=1}^{n} Y'_k$ and let $\alpha' > 1$. From (2.1), we have $E\{|V_n|^{\alpha'}\} = n^{\frac{\alpha'}{\alpha}} E\{|Y'|^{\alpha'}\}$. It follows that

$$\frac{1}{[2\delta n]^{\frac{\alpha'}{\alpha}}} E\{|V_{[2\delta n]}|^{\alpha'}\} = \frac{1}{[n(1+\delta)]^{\frac{\alpha'}{\alpha}}} E\{|V_{[n(1+\delta)]}|^{\alpha'}\} = E\{|Y'|^{\alpha'}\}.$$

Given $\epsilon > 0$, Lemma 2 gives for $0 < \delta < 1/3$

$$P\left(\max_{|k-n|\le\delta n}\left|\frac{V_k}{k^{\frac{1}{r}}} - \frac{V_n}{n^{\frac{1}{r}}}\right| \ge \epsilon\right) \le \frac{4\delta^{\frac{\alpha'}{\alpha}}}{\epsilon^{\alpha'}} E\{|Y'|^{\alpha'}\}.$$

Note that from Proposition 1 (b) and (2.11), we have $\frac{V_n}{n^{\frac{1}{\alpha}}} \xrightarrow{d} Y'$. Thus, Anscombe's Condition is satisfied by the sequence $\{\frac{V_n}{n^{\frac{1}{\alpha}}}\}$ provided $\delta < \min\{\frac{1}{3}, (\frac{\epsilon^{\alpha'+1}}{4E\{|Y'|^{\alpha'}\}})^{\frac{\alpha}{\alpha'}}\}$. On the other hand, from (2.8) we have $\frac{N(t_n)}{t_n} \xrightarrow{\text{a.s.}} \frac{1}{\mu_\tau} > 0$ for any sequence $t_n \to \infty$. Hence, $\frac{V_{N(t)}}{N^{\frac{1}{\alpha}}(t)} \xrightarrow{d} Y'$ and (2.9) follows. $\square$

3 CLT and applications

Let $\alpha > 1$ and Y with distribution $G_\alpha = S_\alpha(\sigma, \beta, \mu_Y)$. Assume that $Y_1, Y_2, \ldots$ are independent copies of Y and such that (X_k, Y_k) has distribution $F_X \wedge G_\alpha$ as defined by (2.3). Note that, under Condition 1, we have both μ_Y and $\mu_X = E(X_k)$ finite.

Let $Y'_k = Y_k - \mu_Y$ and $X'_k = X_k - \mu_X$ then we also have (X'_k, Y'_k) with distribution $F'_X \wedge G'_\alpha$ where Y'_k is $G'_\alpha = S_\alpha(\sigma, \beta, 0)$ and X'_k is F'_X. By Lemma 1, it follows that $d^\alpha_\alpha(F'_X, G'_\alpha) = E\{|X'_k - Y'_k|^\alpha\} < \infty$.

The following notation will be used

$$F_{N(t)} \stackrel{d}{=} \frac{S_{N(t)} - b_{N(t)}}{N^{\frac{1}{\alpha}}(t)}, \quad b_{N(t)} = N(t)\mu_X - N^{\frac{1}{\alpha}}(t)\mu_Y$$

and

$$G_{N(t)} \stackrel{d}{=} \frac{\sum_{k=1}^{N(t)} Y_k - (N(t) - N^{\frac{1}{\alpha}}(t))\mu_Y}{N^{\frac{1}{\alpha}}(t)}.$$

Theorem 1 below can be viewed as an extension of Levy and Taqqu's (2000) results as a stronger Mallows convergence is established.

Theorem 1. *Assume that Condition 1 holds and that $\{X_n\}_{n\ge1}$ and $\{\tau_n\}_{n\ge1}$ are independent processes. Then as $t \to \infty$,*

$$d_\alpha(F_{N(t)}, G_\alpha) \underset{t}{\to} 0 \quad \textit{and} \quad \frac{S_{N(t)} - b_{N(t)}}{N^{\frac{1}{\alpha}}(t)} \xrightarrow{d} Y.$$

Moreover, for $1 < \alpha' < \alpha$ we have

$$E\left\{\left|\frac{S_{N(t)} - b_{N(t)}}{N^{\frac{1}{\alpha}}(t)}\right|^{\alpha'}\right\} \underset{t}{\to} E\{|Y|^{\alpha'}\}.$$

Proof. (i) First, we show that

$$E\left\{\left|\frac{\sum_{k=1}^{N(t)}(X'_k - Y'_k)}{N^{\frac{1}{\alpha}}(t)}\right|^{\alpha}\right\} \underset{t}{\to} 0.$$

Note that $\{(X'_n - Y'_n)\}_{n\geq 1}$ is a sequence of i.i.d. r.v.'s with zero-mean. It follows that $\{\sum_{k=1}^{n}(X'_k - Y'_k), \sigma((X'_1, Y'_1), \ldots, (X'_n, Y'_n))\}_{n\geq 1}$ forms a martingale. Since $E\{|X'_k - Y'_k|^\alpha\}$ has distribution $\alpha^\alpha(F'_X, G'_\alpha) < \infty$ the sequence $\{|X'_n - Y'_n|^\alpha\}_{n\geq 1}$ is uniformly integrable. By Proposition 2, we have

$$L_n = \frac{1}{n} E\left\{\left|\sum_{k=1}^{n}(X'_k - Y'_k)\right|^\alpha\right\} \underset{n}{\to} 0. \tag{3.1}$$

Thus, given $\epsilon > 0$ there exists $K(\epsilon)$ such that for $n \geq K(\epsilon)$ we have $L_n \leq \epsilon$. Now let $T(\epsilon)$ such that for $t \geq T(\epsilon)$ we have $P(N(t) \leq K(\epsilon)) < \epsilon$. Since we may take $\{Y'_n\}_{n\geq 1}$ independent of $\{\tau_n\}_{n\geq 1}$, it follows that

$$E\left\{\left|\frac{\sum_{k=1}^{N(t)}(X'_k - Y'_k)}{N^{\frac{1}{\alpha}}(t)}\right|^\alpha\right\} = \sum_{n\geq 1} E\left\{\frac{1}{n}\left|\sum_{k=1}^{n}(X'_k - Y'_k)\right|^\alpha\right\} P\big(N(t) = n\big) \leq 2\epsilon.$$

(ii) By Lemma 3, we have $G_{N(t)} = G_\alpha$ and by (1.1),

$$\begin{aligned} d_\alpha^\alpha(F_{N(t)}, G_\alpha) &= d_\alpha^\alpha(F_{N(t)}, G_{N(t)}) \\ &\leq E\left\{\left|\frac{\sum_{k=1}^{N(t)}(X'_k - Y'_k)}{N^{\frac{1}{\alpha}}(t)}\right|^\alpha\right\} \underset{t}{\to} 0. \end{aligned} \tag{3.2}$$

(iii) The proof will be completed by making use of (1.2). Let $1 < \alpha' < \alpha$,

$$F_{N(t)} = \frac{S_{N(t)} - b_{N(t)}}{N^{\frac{1}{\alpha}}(t)} \quad \text{and} \quad G_{N(t)} = \frac{\sum_{k=1}^{N(t)} Y_k - (N(t) - N^{\frac{1}{\alpha}}(t))\mu_Y}{N^{\frac{1}{\alpha}}(t)}.$$

Thus, $E\{|G_{N(t)}|^{\alpha'}\} < \infty$ and $G_{N(t)} \in \mathcal{L}_{\alpha'}$. Writing $F_{N(t)} = F_{N(t)} - G_{N(t)} + G_{N(t)}$, using (3.2) and Minkowski's inequality we get $F_{N(t)} \in \mathcal{L}_{\alpha'}$. Convergence of the moments and in distribution follow. □

Next, we drop the assumption of independence between $\{X_n\}$ and $\{\tau_n\}$ in the sense that for each $n \geq 1$ the random variables X_n and τ_n may be dependent.

Theorem 2. *Assume that Condition 1 holds. Then*

$$\frac{S_{N(t)} - b_{N(t)}}{N^{\frac{1}{\alpha}}(t)} \xrightarrow{d} Y, \quad b_{N(t)} = N(t)\mu_X - N^{\frac{1}{\alpha}}(t)\mu_Y.$$

Proof. (i) Proceeding as in Theorem 1 and without loss of generality, we may assume $\mu_X = \mu_Y = 0$. Let $Z_k = X_k - Y_k$. From (3.1), we have $\frac{\sum_{k=1}^{n} Z_k}{n^{\frac{1}{\alpha}}} \xrightarrow{d} 0$. Since $Z_1, Z_2, \ldots$ is a sequence of i.i.d. r.v.'s with zero-mean, we may apply Lemma 2 with $V_n = \sum_{k=1}^{n} Z_k$, $r' = \alpha$ and $r = \alpha + 1/n$. Since (X_k, Y_k) has distribution $F_X \wedge G_\alpha$, by (2.3), we have $E\{|Z_k|^\alpha\} = d_\alpha^\alpha(F_X, G_\alpha) < \infty$. From von Bahr-Esseen inequality (2.4), we have

$$E\big\{|V_{[2\delta n]}|^\alpha\big\} \leq 2[2\delta n] d_\alpha^\alpha(F_X, G_\alpha)$$

and

$$E\big\{|V_{[n(1+\delta)]}|^\alpha\big\} \leq 2\big[n(1+\delta)\big] d_\alpha^\alpha(F_X, G_\alpha).$$

It follows that

$$P\left(\max_{|k-n|\leq\delta n}\left|\frac{V_k}{k^{\frac{1}{\alpha}}} - \frac{V_n}{n^{\frac{1}{\alpha}}}\right| \geq \epsilon\right) \leq \frac{4\delta^{\frac{\alpha}{r}}}{\epsilon^\alpha} d_\alpha^\alpha(F_X, G_\alpha) \max\left\{\frac{2[2\delta n]}{[2\delta n]^{\alpha/r}}, \frac{2[n(1+\delta)]}{[n(1+\delta)]^{\alpha/r}}\right\}.$$

Clearly $\frac{[2\delta n]}{[2\delta n]^{\alpha/r}}$ and $\frac{[n(1+\delta)]}{[n(1+\delta)]^{\alpha/r}}$ converges to 1. Then for $n \geq n_0(\epsilon)$

$$P\left(\max_{|k-n|\leq\delta n}\left|\frac{V_k}{k^{\frac{1}{\alpha}}}-\frac{V_n}{n^{\frac{1}{\alpha}}}\right|\geq\epsilon\right)\leq\frac{8\delta^{\frac{\alpha}{\alpha+1/n_0}}}{\epsilon^\alpha}(1+\epsilon)d_\alpha^\alpha(F_X,G_\alpha).$$

Anscombe's Condition and the fact that $\frac{N(t)}{t}\underset{t}{\to}\frac{1}{\mu_\tau}>0$ give us $\frac{\sum_{k=1}^{N(t)}Z_k}{N(t)^{\frac{1}{\alpha}}}\xrightarrow{d}0$.

(ii) From (2.9), we have $\frac{\sum_{k=1}^{N(t)}Y_k}{N(t)^{\frac{1}{\alpha}}}\xrightarrow{d}Y$. Using the fact: $\xi_n\xrightarrow{d}\xi$ and $\eta_n\xrightarrow{d}0$ assure $\xi_n+\eta_n\xrightarrow{d}\xi$, we complete the proof. □

These results can be directly applied to risk processes and heavy-traffic load.

3.1 Risk process

It is usually modeled by a reserve process with initial capital R_0,

$$R_t=R_0+ct-\sum_{k=1}^{N(t)}Z_k,$$

where the premiums flow in at a rate $c>0$ and $\{Z_n\}$ is the sequence of i.i.d. claims that could reach very high values resulting in EZ_n^2 not finite. If T_n is the occurrence time of the n-th claim, that is, $N(t)=\sup\{n:T_n\leq t\}$, the surplus process can be expressed as $S_{T_n}=\sum_{k=1}^n Z_k-cT_n$. By considering the inter-arrival times $\tau_1,\tau_2,\ldots$ we can write $T_n=\sum_{k=1}^n\tau_k$ and $S_{N(t)}=\sum_{k=1}^{N(t)}(Z_k-c\tau_k)$. Clearly the processes $\{\tau_n\}_{n\geq1}$ and $\{Z_n-c\tau_n\}_{n\geq1}$ are dependent. From Theorem 2, if Condition 1 is satisfied for $X_n=Z_n-c\tau_n$ and τ_n then we have the desired convergence.

Corollary 1. *If $\{(Z_n-c\tau_n,\tau_n)\}_{n\geq1}$ satisfies Condition 1, then for $\mu_Z=E(Z_n)$ we have*

$$\lim_{t\to\infty}P\left(\frac{S_{N(t)}-N(t)(\mu_Z-c\mu_\tau)-N^{\frac{1}{\alpha}}(t)\mu_Y}{N^{\frac{1}{\alpha}}(t)}\leq x\right)=G_\alpha(x),\quad\forall x.\tag{3.3}$$

Corollary 2. *Assume that $\{(Z_n,\tau_n)\}_{n\geq1}$ satisfies Condition 1 and that for some $\alpha_1>\alpha$ there exists an α_1-stable distribution $G_{\alpha_1}\overset{d}{=}S_{\alpha_1}(\sigma_1,0,\mu_1)$ such that $d_{\alpha_1}(F_\tau,G_{\alpha_1})<\infty$ then (3.3) holds.*

Proof. By Corollary 1 enough to show that $\{(Z_n-c\tau_n,\tau_n)\}_{n\geq1}$ satisfies Condition 1. Let the common d.f. of $Z_n-c\tau_n$ be denoted by $F_{Z-c\tau}$ and let Z_n has distribution F_Z. Since $\alpha>1$, by (2.2), we have

$$d_\alpha(F_{Z-c\tau},G_\alpha)\leq d_\alpha(F_{Z-c\tau},F_Z)+d_\alpha(F_Z,G_\alpha).$$

By Condition 1, we have $d_\alpha(F_Z,G_\alpha)<\infty$. Being $\alpha<\alpha_1$ and $d_{\alpha_1}(F_\tau,G_{\alpha_1})<\infty$, we have $Ee\{\tau_n^\alpha\}<\infty$. Now

$$d_\alpha^\alpha(F_{Z-c\tau},F_Z)\leq E\{|Z_n-c\tau_n-Z_n|^\alpha\}=c^\alpha E(\tau_n^\alpha)<\infty.$$

Thus, Condition 1 is satisfied. □

3.2 Heavy-traffic load

In the context of data traffic in communication networks, consider the model of an *"on/off"* source sending random loads of traffic to a network node, where there is a buffer with large capacity memory that stores the information until it is transmitted. The *"on periods"* are represented by the sequence of i.i.d. nonnegative random variables $\tau_1^{\mathrm{on}}, \tau_2^{\mathrm{on}}, \dots$ and the *"off periods"* by $\tau_1^{\mathrm{off}}, \tau_2^{\mathrm{off}}, \dots$. The inter-traffic periods τ_k are given by

$$\tau_k = \tau_k^{\mathrm{on}} + \tau_k^{\mathrm{off}}, \qquad T_n = \sum_{k=1}^{n} \tau_k$$

and for heavy-traffic situations one expects $E(\tau_n^{\mathrm{on}})^2 = \infty$. The cumulative traffic load at time t can be approximated by

$$S_{N(t)} = \sum_{k=1}^{N(t)} \tau_k^{\mathrm{on}}, \qquad N(t) = \sup\{n : T_n \le t\}. \tag{3.4}$$

Clearly, the processes $\{\tau_n^{on}\}_{n\ge1}$ and $\{\tau_n\}_{n\ge1}$ are dependent.

Corollary 3. *Assume that $\{(\tau_n^{on}, \tau_n)\}_{n\ge1}$ satisfies Condition 1. Then for $\mu_{\tau^{on}} = E(\tau_n^{on})$ we have*

$$\lim_{t\to\infty} P\left(\frac{\sum_{k=1}^{N(t)} \tau_k^{on} - N(t)\mu_{\tau^{on}} - N^{\frac{1}{\alpha}}(t)\mu_Y}{N^{\frac{1}{\alpha}}(t)} \le x\right) = G_\alpha(x).$$

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