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Exponential ergodicity for a class of non-Markovian stochastic processes

Laure Pédèches

Institut de Mathématiques de Toulouse and Institut für Mathematik Potsdam

Abstract. The existence of an invariant probability measure is proven for a class of solutions of stochastic differential equations with finite delay. This is done, in this non-Markovian setting, using the cluster expansion method, from Gibbs field theory. It holds for small perturbations of ergodic diffusions.

1 Introduction

The aim of this paper is to prove the existence of an invariant probability measure, for which there is exponential ergodicity, for a class of solutions of stochastic differential equations with finite delay and non regular drift.

We will consider $\mathbb{R}^d$-valued stochastic differential equations of the form:

$$dX_t = \big(g(X_t) + \beta b\big(t, (X)_{t-t_0}^t\big)\big)\, dt + dW_t,$$

where $(X)_s^t := \{X_u | u \in [s,t]\}$ is the path of the process X between times s and t, and t_0 is a fixed positive number. We will make certain assumptions on the underlying semi-group of the reference process, weak solution of

$$dX_t = g(X_t)\, dt + dW_t.$$

The additional drift term b, will only be required to be time-local, measurable and bounded by 1.

Our interest in those equations comes from possible applications for stochastic Cucker–Smale type models in $(\mathbb{R}^d)^N$—such as the one presented by Ha, Lee and Levy in Ha, Lee and Levy (2009). It is a N-particle mean-field system in $\mathbb{R}^d$, whose velocity $v(t) = (v_1(t), \dots, v_N(t))$ satisfies, for all $t \geq 0$, the stochastic differential equation

$$dv(t) = -\frac{\lambda}{N} F\big(t, (v)_0^t\big)\, dt + dW(t),$$

where for all $i \in \{1, \dots, N\}$, $F_i(t, (v)_0^t) = \sum_{j=1}^N \psi((v_j)_0^t, (v_i)_0^t)(v_i(t) - v_j(t))$ and W is a standard dN-dimensional Brownian motion. The function ψ, supposed to be non-negative and symmetric, is called communication rate and quantifies the interaction between each pair of particles.

Various results about the existence of invariant probability measures for stochastic differential equations with delay can be found in the literature, going back to the paper of Itô and Nisio (1964), where is proven that, when the drift and diffusion coefficients are continuous, there exist stationary solutions for delayed processes. Since then, one can mention, among many others, the papers by Mohammed (1986), Bakhtin and Mattingly (2005), or the book of Da Prato and Zabczyk (1996), especially Chapter 10. General results on stochastic differential delay equations up to 2003 are gathered in a survey by Ivanov, Kazmerchuk and Swishchuk (Ivanov, Kazmerchuk and Swishchuk (2003)).

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However, they are mainly valid under strong regularity assumptions on the coefficients, despite the fact that non-regular coefficients appear in various fields, such as finance (see for instance Arriojas et al. (2007), about pricing options) or physics (with bistable systems, in Tsimring and Pikovsky (2001)); stability of non-regular processes is also a fixture in Mao (2007).

One notable exception is Scheutzow (1984), where the author considers the equation

$$dx(t) = F\big((x)_{t-1}^t\big)\,dt + dW_t$$

for a function F measurable and locally bounded. Assuming the existence and uniqueness of a weak solution, and a restrictive recurrence condition (holding if some condition on certain Lyapunov functionals are met), the existence of—and the convergence in total variation distance towards—an invariant probability measure is proven, using the strong Markov property satisfied by $((x)_{t-1}^t)_{t\in\mathbb{R}_+}$. Nothing is said, however, about the rate of convergence for such processes. The true novelty of our work is the exponential rate of the ergodicity.

As we are dealing with non-Markovian processes, most standard methods of stochastic ananlysis are not available. Thus, our main tool here will be the so-called cluster expansion method, mainly used in statistical mechanics, in particular in Gibbs field theory. As a consequence, our results will hold for irregular but small (albeit not insignificant) perturbations of the reference process. Technical results for the adaptation of the cluster expansion methods to Gibbs random fields can be found in the book by Malyshev and Minlos (1991). Subsequent papers have implemented those methods for stochastic processes, for example interacting diffusions systems or one-dimensional non-Markovian diffusions. It was done in Ignatyuk, Malyshev and Sidoravicius (1992), and, more recently, amongst others in Dai Pra and Roelly (2004), Dai Pra, Roelly and Zessin (2002) or Minlos, Roelly and Zessin (2000).

Our main result is the exponential ergodicity of the process. Moreover, with the same technique, one obtains that the decay of correlations is exponentially quick. It follows that a central limit theorem can be derived from the mixing properties implied by this inequality.

Contrary to what was done in Dai Pra and Roelly (2004), we do not require for the semigroup associated with the reference process to be ultracontractive, but we only need some strong form of hypercontractivity. One instance of a well-known process which is not ultracontractive but verifies our assumptions is the Ornstein–Uhlenbeck process. We will present some explicit results in this particular setting. Actually, the stochastic Cucker–Smale model can be seen as a mean-field perturbation of the Ornstein–Uhlenbeck process, and this led to this work. The lack of ultracontractive bounds for the underlying reference process introduces several new technical difficulties. We will therefore present a detailed proof for the cluster estimates. The use of these estimates to get to the final main theorem follows the lines of, for instance, what was done in Dai Pra and Roelly (2004) and Minlos, Roelly and Zessin (2000) and we will go rather quickly over this part.

We start by introducing our framework, the objects we will encounter and the assumptions that will be needed, before giving our main result. Then, we obtain, in Section 3, a cluster representation for the partition function defined in the first part. In Section 4, we study the cluster estimates and show that they tend to 0 when β does. In Section 5, we conclude the proof and present a few consequences of our convergence theorem. Finally, in Section 6, we explicitly compute some of the bounds in the Ornstein–Uhlenbeck setting.

2 Framework and main theorem

We introduce here the process which will serve as reference in our work: a stochastic process sufficiently regular to be exponentially ergodic with respect to its invariant (and even reversible) probability measure. We present all the assumptions that will be necessary to extend this ergodicity to small perturbations of this process.

2.1 The reference process

First, we introduce the framework in which we are considering such a stochastic process:

- $\Omega = \mathcal{C}(\mathbb{R}, \mathbb{R}^d)$ shall be the canonical continuous $\mathbb{R}^d$-valued path space, for some $d \geq 1$, and $\mathcal{F}$ the canonical Borel σ-field on Ω. $(X_t)_{t\in\mathbb{R}}$ shall be, as usual, the canonical process.
- $\mathbb{W}$ shall be the Wiener measure on $(\Omega, \mathcal{F})$, the law of a standard d-dimensional Brownian motion $(W_t)_{t\in\mathbb{R}}$.

We consider the following stochastic differential equation, for any $t \in \mathbb{R}$,

$$dX_t = g(X_t)\,dt + dW_t \tag{2.1}$$

with $g : \mathbb{R}^d \to \mathbb{R}^d$ a smooth function (say $\mathcal{C}^k$, for a certain $k \geq 2$) and (W_t) a standard d-dimensional Brownian motion. We suppose that there exists a reversible probability measure, μ, for this process. We will soon give conditions to ensure that (2.1) has a unique stationary weak solution.

Let L be its associated infinitesimal generator, defined by

$$L = \frac{1}{2}\sum_{i,j} \partial_{ij}^2 + \sum_i g_i \partial_i .$$

L is uniformly elliptic, and, as μ is reversible, symmetric in $L^2(\mu)$: for all f and g smooth enough, $\int fLg\,d\mu = \int gLf\,d\mu$. It is known that μ is then absolutely continuous with respect to the Lebesgue measure, with a positive density. Thus, μ is of the form $d\mu(x) = Ce^{-V(x)}\,dx$; in addition, $V : \mathbb{R}^d \to \mathbb{R}$ is smooth (at least $\mathcal{C}^k$).

According to Kolmogorov's characterization of reversible diffusions, in Kolmogorov (1937), this even implies that g can be written as a gradient function of a potential function V: $g = -\frac{1}{2}\nabla V$. Thus, equation (2.1) becomes:

$$dX_t = -\frac{1}{2}\nabla V(X_t)\,dt + dW_t. \tag{2.2}$$

The probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ shall denote the weak stationary solution of (2.2), with marginal law the invariant probability measure μ.

To ensure that the equation (2.2) indeed admits a stationary weak solution on $\mathbb{R}$, and in particular, that there is non-explosion in finite time, we further assume (see, e.g., Royer (1999)) that one of the two following assertions on the potential V is true:

(1) $\quad V(x) \underset{|x|\to\infty}{\longrightarrow} +\infty$ and $|\nabla V|^2 - \Delta V$ is bounded from below

(2) $\quad$ There exist $a, b \in \mathbb{R}$ such that, for all x, $x^*\nabla V(x) \geq -a|x|^2 - b$,

where x^* is the transpose of x.

In this case, there even exists a unique strong solution (Theorem 2.2.19 in Royer (1999)). The semi-group (P_t) admits a smooth transition density with respect to μ, denoted by $p(t, x, y)$. As the probability measure μ is reversible, $p(t, \cdot, \cdot)$ is symmetric:

$$\forall t, x, y, \quad p(t, x, y) = p(t, y, x).$$

We now introduce two assumptions which will be essential in the following:

(H1): μ satisfies a Poincaré inequality: there exists a constant C_P such that for all smooth functions f in $L^2(\mu)$,

$$\left\| f - \int f\,d\mu \right\|_{L^2(\mu)}^2 \leq C_P \int |\nabla f|^2\,d\mu.$$

(H2): There exists $\delta \geq 0$ such that

$$\sup_{t\geq\delta} \|p(t,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)} < \infty.$$

Remark 1. It is well known (see, for example, Ané et al. (2000)) that hypothesis (H1) is equivalent to the exponential convergence of the semi-group towards μ, that is, there exists a constant C_S such that for all $f \in L^2(\mu)$, for all $t \geq 0$,

$$\left\| P_t f - \int f\,d\mu \right\|_{L^2(\mu)} \leq e^{-C_S t} \left\| f - \int f\,d\mu \right\|_{L^2(\mu)}$$

and, moreover, $C_S = 1/C_P$ (see Cattiaux, Guillin and Zitt (2013) for a more general statement).

In particular, if $\int f\,d\mu = 0$, then (H1) implies that for every $t \geq 0$,

$$\|P_t f\|_{L^2(\mu)} \leq e^{-t/C_P} \|f\|_{L^2(\mu)}. \tag{2.3}$$

Remark 2. Using Cauchy–Schwarz's and Jensen's inequalities,

$$\begin{aligned}
\|P_\delta f\|_{L^8(\mu)}^8 &= \int \left(\int p(\delta,x,y) f(y)\mu(dy) \right)^8 \mu(dx) \\
&\leq \int \left(\int |f(y)|^2 \mu(dy) \right)^4 \left(\int p(\delta,x,y)^2 \mu(dy) \right)^4 \mu(dx) \\
&\leq \|f\|_{L^2(\mu)}^8 \iint p(\delta,x,y)^8 \mu(dy)\mu(dx)
\end{aligned}$$

This means that:

$$\|P_\delta f\|_{L^8(\mu)} \leq \|p(\delta,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)} \|f\|_{L^2(\mu)}. \tag{2.4}$$

Thus, if (H2) is satisfied, $P_\delta : L^2(\mu) \to L^8(\mu)$ is a bounded operator.

If, in addition, (H1) is satisfied, for every $k \geq 2$, $P_t : L^2(\mu) \to L^k(\mu)$ is bounded by 1 when t is large enough.

Example 1. (H1) is satisfied, for instance, if V is uniformly convex outside of a compact set, that is if the Hessian matrix of V is a non-degenerate quadratic form outside of a compact set.

It is, however, difficult to obtain a generic condition on the potential V for hypothesis (H2) to hold; one can look at Section 3 of Cattiaux, Dai Pra and Roelly (2008) to understand the underlying difficulties: in particular, condition (A4), introduced at the beginning of Section 2 in Cattiaux, Dai Pra and Roelly (2008) is fairly close to our hypothesis (H2). In the special case of an Ornstein–Uhlenbeck reference process, discussed in Section 6, (H2) is proven thanks to the known explicit expression of the density function p.

We now prove a proposition, taking into account hypotheses (H1) and (H2) and yielding the assumption we will use in practice, rather than (H1) itself.

Proposition 1. *Under hypotheses* (H1) *and* (H2), *for* $t \geq 2\delta$,

$$\|p(t,\cdot,\cdot) - 1\|_{L^8(\mu\otimes\mu)} \leq \gamma_\delta(t)\big(\|p(\delta,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)} \vee 1\big),$$

where $\gamma_\delta(t) = 2M_\delta e^{-(t-2\delta)/C_P}$ *with* $M_\delta = \sup_{a\geq\delta} \|p(a,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)} \vee 1$.

In particular, $\gamma_\delta(t)$ *goes to* 0 *exponentially fast when* t *goes to infinity.*

Proof. The following lemma is essential for the proof of the proposition.

Lemma 1. *Set* $\delta > 0$. *Suppose that* (H1) *holds true.*
Then for all smooth f *such that* $\int f\,d\mu = 0$,

$$\forall t \geq \delta, \quad \|P_t f\|_{L^8(\mu)} \leq e^{-(t-\delta)/C_P}\|p(\delta,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)}\|f\|_{L^8(\mu)}.$$

Remark 3. It is possible for both sides of the above inequality to be infinite.

Proof. Let t be a positive number with $t \geq \delta$ and f a smooth function such that $\int f\,d\mu = 0$.
For any $t \geq \delta$, thanks to the inequality (2.4) proven in Remark 2,

$$\|P_t f\|_{L^8(\mu)} = \|P_\delta(P_{t-\delta}f)\|_{L^8(\mu)} \leq \|p(\delta,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)}\|P_{t-\delta}f\|_{L^2(\mu)}.$$

As (H1) is supposed to be satisfied, so is (2.3); hence the conclusion of this proof:

$$\begin{aligned}\|P_t f\|_{L^8(\mu)} &\leq e^{-(t-\delta)/C_P}\|p(\delta,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)}\|f\|_{L^2(\mu)} \\ &\leq e^{-(t-\delta)/C_P}\|p(\delta,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)}\|f\|_{L^8(\mu)}.\end{aligned}$$

□

We prove the proposition and start by expressing $p(t,x,y)-1$ using the semi-group:

$$\begin{aligned}p(t,x,y)-1 &= \int \big(p(t-\delta,x,z)p(\delta,z,y)-1\big)\mu(dz) \\ &= P_{t-\delta}\big(p(\delta,\cdot,y)\big)(x) - 1 = P_{t-\delta}\big(p(\delta,\cdot,y)-1\big)(x).\end{aligned}$$

Thus, applying Lemma 1 for $f = p(\delta,\cdot,y)-1$ at time $t-\delta$ ($\geq \delta$ as $t \geq 2\delta$),

$$\begin{aligned}&\int \big(p(t,x,y)-1\big)^8\mu(dx) \\ &\quad = \int P_{t-\delta}^8\big(p(\delta,\cdot,y)-1\big)(x)\mu(dx) \\ &\quad \leq e^{-8(t-2\delta)/C_P}\|p(\delta,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)}^8\|p(\delta,\cdot,y)-1\|_{L^8(\mu)}^8 \\ &\quad \leq e^{-8(t-2\delta)/C_P}2^8\|p(\delta,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)}^8\big(\|p(\delta,\cdot,y)\|_{L^8(\mu)}^8 \vee 1\big),\end{aligned}$$

which leads to:

$$\begin{aligned}&\int \big(p(t,x,y)-1\big)^8\mu(dx)\mu(dy) \\ &\quad \leq e^{-8(t-2\delta)/C_P}2^8\|p(\delta,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)}^8\big(\|p(\delta,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)}^8 \vee 1\big).\end{aligned}$$

Hypothesis (H2) ensures that $\|p(u,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)}$ is bounded uniformly in u for $u \geq \delta$, hence the result. □

Remark 4. As can be noticed in the proof, M_δ is a priori not the optimal bound (although it corresponds with $\|p(\delta,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)}$ as will be seen in the Ornstein–Uhlenbeck example in Section 6) but will be good enough for our needs (and will simplify later computations).

2.2 The perturbed stochastic differential equation

We turn our attention to the stochastic differential equation with finite delay t_0, for all $t \in \mathbb{R}$,

$$dX_t = \Big(-\frac{1}{2}\nabla V(X_t) + \beta b\big(t,(X)_{t-t_0}^t\big)\Big)dt + dW_t, \tag{2.5}$$

where the potential V and the Brownian motion (W_t) are as previously defined, β is a positive constant, which shall be small enough for the result to hold.

The perturbation drift, b shall satisfy the assumption (H3) detailed below:

(H3): $b : \mathbb{R} \times \Omega \to \mathbb{R}^d$ is a measurable function, bounded by 1, and local, in the sense that there exists a delay $t_0 > 0$ such that, for any $u \in \Omega$, $b(t, u) = b(t, (u)_{t-t_0}^t)$.

Example 2. We give here a few examples for perturbation drifts b satisfying (H3):

- we can consider b of the form $b(t, (u)_{t-t_0}^t) = g(t) \int_{t-t_0}^t f(u_t, u_s)\, ds$ for any trajectory $u \in \Omega$ with f and g bounded by 1 and measurable; for instance, $f(x, y) = \operatorname{sign}(x - y)$ or $f(x, y) = \mathbb{1}_{y \in A}$ with A a subset of $\mathbb{R}^d$ (thus obtaining an occupation time);
- we may have a dependence on the past depending on a single time, of the form $b(t, (u)_{t-t_0}^t) = g(u_{t-t_0})$ for a certain function g measurable and bounded by 1, but not necessarily continuous;
- one can also consider a drift function with jumps, such as $b((u)_{t-t_0}^t) = \mathbb{1}_{(u)_{t-t_0}^t \in A}$ with A a subset of $\mathcal{C}([-t_0, 0], \mathbb{R}^d)$.

One of the main advantages of our method is that we only require from b that it satisfies (H3), without any stronger condition on its regularity.

Recall that the probability measure Q on Ω is said to be a weak solution of the stochastic differential system (2.5) if the process

$$\left(X_t - \int_0^t \left(-\frac{1}{2} \nabla V(X_s) + \beta b\big(s, (X)_{s-t_0}^s\big) \right) ds \right)$$

is a Q-Brownian motion.

2.3 The main result

Our main theorem is the following convergence result for the stochastic differential equation with delay, for $t \in \mathbb{R}$,

$$dX_t = \left(-\frac{1}{2} \nabla V(X_t) + \beta b\big(t, (X)_{t-t_0}^t\big) \right) dt + dW_t, \tag{2.5}$$

considered as a perturbation of the reference process

$$dX_t = -\frac{1}{2} \nabla V(X_t)\, dt + dW_t. \tag{2.2}$$

Theorem 2.1. *Assume that the assumptions* (H1) *and* (H2) *are satisfied by the reference stochastic differential equation* (2.2). *Assume also that the perturbation drift b of equation* (2.5) *verifies* (H3).

Then, for β small enough,

(i) *The stochastic differential equation* (2.5) *admits a weak stationary solution Q, and thus an invariant probability measure ν.*

(ii) *There is exponential ergodicity: there exist $\theta > 0$ and $C : \mathbb{R}^d \to \mathbb{R}_+$ such that for $|t|$ and $|t'|$ large enough, for μ a.s.-$x \in \mathbb{R}^d$, for every bounded measurable function f,*

$$\big|\mathbb{E}_Q\big[f(X_t)|X_0 = x\big] - \mathbb{E}_Q\big[f(X_{t'})|X_0 = x\big]\big| \le C(x) e^{-\theta|t-t'|}.$$

(iii) *The decay of correlations is exponentially quick: there exist two positive constants θ_1 and θ_2, such that for $|t|$ and $|t'|$ large enough, for all f and g measurable and bounded by* 1, *it holds:*

$$\big|\mathbb{E}_Q\big[f(X_t)g(X_{t'})\big] - \mathbb{E}_Q\big[f(X_t)\big]\mathbb{E}_Q\big[g(X_{t'})\big]\big| \le \theta_1 e^{-\theta_2|t-t'|}.$$

The rest of the paper is devoted to the proof and the consequences of this theorem.

Remark 5. This theorem is an existence result, and does not provide any uniqueness either for the weak stationary solution or the invariant probability measure.

3 Approximation on finite-time windows and cluster representation

The main idea behind the proof is to build approximations on finite-time windows that will converge towards what will be the weak stationary solution of (2.5); the properties of these approximations will be then be inherited by this limit.

3.1 Approximations

We set the following notations:

- a, a fixed positive number, destined to become quite large;
- for every j in $\mathbb{Z}$, $I_j = [ja, (j+1)a]$;
- for every N in $\mathbb{N}^*$, $I(N) = [-Na, Na] = \bigcup_{j=-N}^{N-1} I_j$;
- for every u in Ω, $u^{(N)}(t) = u(Na)$ if $t \geq Na$, $u^{(N)}(t) = u(t)$ if $-Na \leq t \leq Na$, and $u^{(N)}(t) = u(-Na)$ if $t \leq -Na$. That is: $u^{(N)}$ is equal to u frozen outside of the interval $I(N)$.

Using Girsanov theorem (see, e.g., Lipster and Shiryayev (1977)), we can show that the restriction to any finite time interval I of the law of the perturbed process is absolutely continuous with respect to the law of the reference process, $\mathbb{P}$, and that its density is of the form $\exp(-H_I(u))\,du$ where the associated Hamiltonian H_I is defined by

$$H_I(u) = -\int_I \beta b\big(t, (u)_{t-t_0}^t\big)^* \, dW_t + \frac{\beta^2}{2}\int_I \big|b\big(t, (u)_{t-t_0}^t\big)\big|^2 \, dt, \tag{3.1}$$

for every trajectory u in the path space Ω. We will denote $H_N = H_{I(N)}$.

To obtain the theorem, our main objective is to prove the convergence of the sequence of probability measures $(Q_N)_{N\in\mathbb{N}^*}$, defined on Ω by

$$Q_N(du) = \frac{1}{Z_N}\exp\big(-H_N\big(u^{(N)}\big)\big)\mathbb{P}(du), \tag{3.2}$$

towards a weak solution of the equation (2.5) that will be time stationary. From this point, classical results of Gibbs theory shall lead to Theorem 2.1.

Remark 6. Under Q_N, the canonical process (X_t) is a weak solution of the stochastic differential system (2.5), for $t \in I(N)$, but not a stationary one.

3.2 The partition function and its cluster representation

The renormalization constant in (3.2), also called partition function, is given by

$$Z_N = \int_\Omega \exp\big(-H_N\big(u^{(N)}\big)\big)\mathbb{P}(du).$$

The aim of our next section will be to expand Z_N with respect to β uniformly in N.

The bulk of the proof shall then be to control the different terms involved in this series expansion, to show that they are smaller than a certain function of β that vanishes when β goes to 0.

The cluster expansion method, very useful in statistical mechanics, shall then lead us first to the convergence of the sequence $(Q_N)_N$ towards a weak stationary solution Q of equation (2.5) and the existence of a invariant probability measure, second to the exponential ergodicity and Theorem 2.1.

First, however, we aim to expand the partition function into clusters, that is to obtain an expression of Z_N of the form:

$$Z_N = 1 + \sum_\tau \prod_i \Gamma_{\tau_i},$$

with the meaning and nature of each of τ, i and Γ_{τ_i} to be determined.

We start by conditioning the reference probability $\mathbb{P}$ on Ω with respect to the values of its marginals at times $-Na, -(N-1)a, \dots, 0, a, \dots, Na$:

$$\begin{aligned} Z_N &= \int_\Omega \exp(-H_N(u^{(N)}))\mathbb{P}(du) \\ &= \int_{\mathbb{R}^{(2N+1)d}} \int_\Omega \exp(-H_N(u^{(N)}))\mathbb{P}(du|X_{ja} = y_j, j = -N, \dots, N) \\ &\quad \otimes \mathbb{P}_{X_{-Na}}(dy_{-N}) \\ &\quad \times \bigotimes_{j=-N}^{N-1} \mathbb{P}_{X_{(j+1)a}}(dy_{j+1}|X_{ja} = y_j). \end{aligned}$$

Let $\mathbb{P}_I^{a,b}$ denote the law of the stochastic bridge over I obtained by conditioning $\mathbb{P}$ so that $X_{\inf I} = a$ and $X_{\sup I} = b$. Then, on the interval $I(N)$,

$$\mathbb{P}(\cdot|X_{ja} = y_j, j = -N, \dots, N) = \bigotimes_{j=-N}^{N-1} \mathbb{P}_{I_j}^{y_j, y_{j+1}}(\cdot). \tag{3.3}$$

Recall that by definition of the transition density p,

$$\mathbb{P}_{X_{(j+1)a}}(dy_{j+1}|X_{ja} = y_j) = p(a, y_j, y_{j+1})\mu(dy_{j+1}). \tag{3.4}$$

Combining (3.3) and (3.4), one obtains

$$\begin{aligned} Z_N = \int_{\mathbb{R}^{(2N+1)d}} \int_\Omega \exp(-H_N(u^{(N)})) \bigotimes_{j=-N}^{N-1} \mathbb{P}_{I_j}^{y_j, y_{j+1}}(du) \\ \times \bigotimes_{j=-N}^{N-1} p(a, y_j, y_{j+1}) \bigotimes_{j=-N}^{N-1} \mu(dy_j). \end{aligned}$$

Next, we re-order the terms in a convenient way:

$$\begin{aligned} Z_N = \int_{\mathbb{R}^{(2N+1)d}} \int_\Omega \prod_{j=-N}^{N-1} (\exp(-H_{I_j}(u^{(N)}))p(a, y_j, y_{j+1})) \\ \times \bigotimes_{j=-N}^{N-1} \mathbb{P}_{I_j}^{y_j, y_{j+1}}(du) \bigotimes_{j=-N}^{N-1} \mu(dy_j). \end{aligned}$$

Contrary to what was done, by mistake, between equations (13) and (14) in Dai Pra and Roelly (2004), we cannot exchange the product and the integral over Ω. This can be corrected in a way by a different decomposition:

$$Z_N = \int_{\mathbb{R}^{(2N+1)d}} \int_\Omega \prod_{j=-N}^{N-1} \alpha_j(a, y, u) \bigotimes_{j=-N}^{N-1} \mathbb{P}_{I_j}^{y_j, y_{j+1}}(du) \bigotimes_{j=-N}^{N-1} \mu(dy_j),$$

where the coefficients α_j are defined, for $j \in \{-N+1, \dots, N-2\}$, by

$$\alpha_j(a, y, u) = \exp(-H_{I_j}(u^{(N)}))\sqrt{p(a, y_{j-1}, y_j)p(a, y_j, y_{j+1})}$$

with the extremal cases $j = -N$ and $j = N-1$ as follows

$$\begin{aligned} \alpha_{-N}(a, y, u) &= \exp(-H_{I_{-N}}(u^{(N)}))\sqrt{p(a, y_{-N}, y_{-N+1})}, \\ \alpha_{N-1}(a, y, u) &= \exp(-H_{I_{N-1}}(u^{(N)}))\sqrt{p(a, y_{N-2}, y_{N-1})}\,p(a, y_{N-1}, y_N). \end{aligned}$$

In order to obtain a sum of a product of terms that are "temporally independent" from each other, we rewrite differently the product of the α_j:

$$\prod_{j=-N}^{N-1} \alpha_j(a,y,u) = \prod_{j=-N}^{N-1} \bigl(1+\alpha_j(a,y,u)-1\bigr) = 1+\sum_S \prod_{j\in S} \bigl(\alpha_j(a,y,u)-1\bigr),$$

where the sum is taken on all non-empty subsets S of $\{-N,\dots,N-1\}$.

Thus,

$$\prod_{j=-N}^{N-1} \alpha_j(a,y,u) = 1+\sum_{p\in\mathbb{N}^*} \sum_{\tau_1\sqcup\cdots\sqcup\tau_p} \prod_{i=1}^{p} \prod_{j\in\tau_i} \bigl(\alpha_j(a,y,u)-1\bigr),$$

with τ of the form $\tau=\{c,c+1,\dots,c+r\}$, with $r\ge 0$, $|c|\le N$, $c+r\le N$, and $d(\tau_i,\tau_j)\ge 2$ if $i\ne j$.

More precisely, these sets, called clusters, satisfy three conditions:

- $a\tau_i \subset I(N)$, in the sense that if $j\in\tau_i$, then $j\in\{-N,\dots,N\}$;
- if $j_1, j_2\in\tau_i$, with $j_1<j_2$, and $j_1\le j_3\le j_2$, then $j_3\in\tau_i$ (in some way, they are "connected sets", as subsets of $\mathbb{Z}$);
- if $j_1\in\tau_{i_1}$ and $j_2\in\tau_{i_2}$, with $i_1\ne i_2$, then $|j_1-j_2|\ge 2$ (they are "disjoint sets").

Notice that the sum over p is actually finite: according to the properties of the sets (τ_i), there are less than $2+Na$ of them, thus $p\le 2+Na$.

Coming back to the expression of the partition function,

$$\begin{aligned} Z_N &= \int_{\mathbb{R}^{(2N+1)d}} \int_\Omega \Biggl(1+\sum_{p\in\mathbb{N}^*} \sum_{\tau_1\sqcup\cdots\sqcup\tau_p} \prod_{i=1}^{p} \prod_{j\in\tau_i} \bigl(\alpha_j(a,y,u)-1\bigr)\Biggr) \\ &\quad\times \bigotimes_{j=-N}^{N-1} \mathbb{P}_{I_j}^{y_j,y_{j+1}}(du) \bigotimes_{j=-N}^{N-1} \mu(dy_j) \\ &= 1+\sum_{p\in\mathbb{N}^*} \sum_{\tau_1\sqcup\cdots\sqcup\tau_p} \int_{\mathbb{R}^{(2N+1)d}} \int_\Omega \prod_{i=1}^{p} \prod_{j\in\tau_i} \bigl(\alpha_j(a,y,u)-1\bigr) \\ &\quad\times \bigotimes_{j=-N}^{N-1} \mathbb{P}_{I_j}^{y_j,y_{j+1}}(du) \bigotimes_{j=-N}^{N-1} \mu(dy_j). \end{aligned} \tag{3.5}$$

The decomposition of the product of the α_j was done to be able to invert the product for i from 1 to p and both integrals in the expression (3.5) just above. This is indeed now possible:

- Take a cluster $\tau_i=\{c_i,\dots,c_i+r_i\}$.

  As $\prod_{j\in\tau_i}(\alpha_j(a,y,u)-1)$ only depends on $u(t)$ for

  $$\begin{aligned} t &\in \bigcup_{j\in\tau_i} \bigl[ja-t_0,(j+1)a\bigr] \\ &= \bigl[(c_ia-t_0))\wedge(-Na),(c_i+r_i+1)a\bigr] \subset I_{c_i-1}\cup\cdots\cup I_{c_i+r_i} \end{aligned}$$

  and $(c_{i_1}+r_{i_1}+1)a < c_{i_2}a-t_0$ for $i_1,i_2\in\{1,\dots,p\}$, $i_1\ne i_2$ and a large enough, we have

  $$(I_{c_{i_1}-1}\cup\cdots\cup I_{c_{i_1}+r_{i_1}}) \cap (I_{c_{i_2}-1}\cup\cdots\cup I_{c_{i_2}+r_{i_2}}) = \varnothing.$$

This allows us to invert the product of the α_j with the integral over Ω: we thus have

$$Z_N = 1 + \sum_{p\in\mathbb{N}^*} \sum_{\tau_1\sqcup\cdots\sqcup\tau_p} \int_{\mathbb{R}^{(2N+1)d}} \prod_{i=1}^{p} \Bigg[\int_\Omega \prod_{j\in\tau_i} (\alpha_j(a,y,u)-1) \times \bigotimes_{k=c_i-1}^{c_i+r_i-1} \mathbb{P}_{I_k}^{y_k,y_{k+1}}(du) \Bigg] \bigotimes_{j=-N}^{N-1} \mu(dy_j).$$

- Moreover, notice that the expression between the square brackets only depends on $y_{c_i-1}, y_{c_i}, \ldots, y_{c_i+r_i}$. As a consequence, we can interchange the integral over $\mathbb{R}^{2N+1}$ and the product in i.

Thus, we obtain the following cluster representation of the partition function Z_N:

$$Z_N = 1 + \sum_{p\in\mathbb{N}^*} \sum_{\tau_1\sqcup\cdots\sqcup\tau_p\subset I(N)} \prod_{i=1}^{p} \Gamma_{\tau_i}, \tag{3.6}$$

where

$$\Gamma_\tau = \int_{\mathbb{R}^{(|\tau|+1)d}} \int_\Omega \prod_{j\in\tau} (\alpha_j(a,y,u)-1) \bigotimes_{k=\min(\tau)-1}^{\max(\tau)-1} \mathbb{P}_{I_k}^{y_k,y_{k+1}} \bigotimes_{l=\min(\tau)-1}^{\max(\tau)} \mu(dy_l), \tag{3.7}$$

with $|\tau|$ the cardinal of τ.

4 Cluster estimates

Having obtained the quantities Γ_τ associated to a cluster τ, we now wish to control them. More specifically, we will show that, when the perturbation coefficient β is sufficiently small, there exists a positive function $\eta(\beta)$, which goes to 0 when β goes to 0, such that for a large enough,

$$|\Gamma_\tau| \le \eta(\beta)^{|\tau|}. \tag{4.1}$$

4.1 First upper-bound for the clusters

In order to estimate this coefficient Γ_τ, we commute the integrals and the remaining product (over the elements of τ), to obtain the following inequality.

Proposition 2. *Setting*

$$A_j(a) = \int_{\mathbb{R}^{3d}} \int_\Omega (\alpha_j(a,y,u)-1)^4 \mathbb{P}_{I_{j-1}}^{y_{j-1},y_j}(du) \times \mathbb{P}_{I_j}^{y_j,y_{j+1}}(du)\mu(dy_{j-1})\mu(dy_j)\mu(dy_{j+1}),$$

we have, for every cluster τ involved in the decomposition (3.6),

$$\Gamma_\tau \le \prod_{j\in\tau} A_j(a)^{1/4}.$$

Proof. The following lemma, taken from Minlos, Verbeure and Zagrebnov (2000), is the main ingredient of the proof.

Lemma 2. *Let $(\mu_x)_{x\in\mathcal{X}}$ be a family of probability measures, each one defined on a space E_x, where the elements x belong to some finite set $\mathcal{X}$. Let us also define a finite family $(f_i)_i$ of functions on $E_{\mathcal{X}} = \bigtimes_{x\in\mathcal{X}} E_x$ such that each f_i is $\mathcal{X}_i$-local for a certain $\mathcal{X}_i \subset \mathcal{X}$, in the sense that*

$$f_i(e) = f_i(e_{|\mathcal{X}_i}), \quad \textit{for } e = (e_x)_{x\in\mathcal{X}} \in E_{\mathcal{X}}.$$

Let $\rho_i > 0$ be numbers satisfying the following conditions:

$$\forall x \in \mathcal{X}, \quad \sum_{\mathcal{X}_i \ni x} \frac{1}{\rho_i} \le 1.$$

Then

$$\left| \int_{E_{\mathcal{X}}} \prod_i f_i \bigotimes_{x\in\mathcal{X}} d\mu_x \right| \le \prod_i \left(\int_{E_{\mathcal{X}_i}} |f_i|^{\rho_i} \bigotimes_{x\in\mathcal{X}_i} d\mu_x \right)^{1/\rho_i}.$$

We apply Lemma 2 twice consecutively, first with respect to the integral over Ω, then with respect to the integral over $\mathbb{R}^{(|\tau|+1)d}$.

- For $\tau = \{c, \dots, c+r\}$, set

$$I_\tau(y) = \int_\Omega \prod_{j=c}^{c+r} \big(\alpha_j(a, y, u) - 1\big) \bigotimes_{k=c-1}^{c+r-1} \mathbb{P}_{I_k}^{y_k, y_{k+1}}(du).$$

  Taking $\mathcal{X} = \{c-1, \dots, c+r-1\}$, $\mathcal{X}_i = \{i-1, i\}$, $E_{\mathcal{X}} = \Omega$, $E_k = \mathcal{C}(I_k, \mathbb{R}^d)$ and $d\mu_k = \mathbb{P}_{I_k}^{y_k, y_{k+1}}$, for $(\rho_j)_{j\in\tau}$ such that $\rho_j > 1$ and $\frac{1}{\rho_j} + \frac{1}{\rho_{j+1}} \le 1$, by Lemma 2,

$$I_\tau(y) \le \prod_{j=c}^{c+r} g_j(y_{j-1}, y_j, y_{j+1})^{1/\rho_j},$$

  where $g_j(y_{j-1}, y_j, y_{j+1}) = \int_\Omega |\alpha_j(a, y, u) - 1|^{\rho_j} \mathbb{P}_{I_{j-1}}^{y_{j-1}, y_j}(du) \mathbb{P}_{I_j}^{y_j, y_{j+1}}(du)$.
- Set now

$$\tilde{\Gamma}_\tau = \int_{\mathbb{R}^{(r+2)d}} \prod_{j=c}^{c+r} g_j^{1/\rho_j} \bigotimes_{l=c-1}^{c+r} \mu(dy_l).$$

  Here we choose $\mathcal{X} = \{c-1, \dots, c+r\}$, $\mathcal{X}_i = \{i-1, i, i+1\}$, $E_{\mathcal{X}} = \mathbb{R}^{(r+2)d}$, $E_x = \mathbb{R}^d$ and $d\mu_x = \mu(y_x)$, for $(\gamma_j)_{j\in\{-N,\dots,N\}}$ such that $\gamma_j > 1$ and $\frac{1}{\gamma_{j-1}} + \frac{1}{\gamma_j} + \frac{1}{\gamma_{j+1}} \le 1$, Lemma 2 ensures that

$$\tilde{\Gamma}_\tau \le \prod_{j=c}^{c+r} \left(\int_{\mathbb{R}^{3d}} |g_j|^{\gamma_j/\rho_j} \mu(dy_{j-1}) \mu(dy_j) \mu(dy_{j+1}) \right)^{1/\gamma_j}.$$

For every $i \in \tau$, every $j \in \{-N, \dots, N\}$, we take $\rho_i = \gamma_j = 4$, and this concludes the proof. □

We now control this quantity and prove that it goes to 0, uniformly in j, and even independently of N, for a large enough time-scale a.

4.2 Decomposition of $A_j(a)$

Using that

$$xy - 1 = (x-1)y + (y-1) \quad \text{and} \quad (xy-1)^4 \le 8\big((x-1)^4 y^4 + (y-1)^4\big)$$

for non-negative x and y, and coming back to the expression of the α_j, we can decompose $A_j(a)$ in two parts that will be dealt with separately:

$$A_j(a) \le 8B_j(a) + 8C_j(a);$$

- if $j \in \{-N+1, \dots, N-2\}$,

$$\begin{aligned} B_j(a) &:= \int_{\mathbb{R}^{3d}} \int_\Omega \big(e^{-H_{I_j}(u^{(N)})} - 1\big)^4 p(a, y_{j-1}, y_j)^2 p(a, y_j, y_{j+1})^2 \\ &\qquad \times \mathbb{P}_{I_{j-1}}^{y_{j-1}, y_j}(du) \mathbb{P}_{I_j}^{y_j, y_{j+1}}(du) \mu(dy_{j-1}) \mu(dy_j) \mu(dy_{j+1}) \\ &= \int_\Omega \big(e^{-H_{I_j}(u^{(N)})} - 1\big)^4 p\big(a, u((j-1)a), u(ja)\big) \\ &\qquad \times p\big(a, u(ja), u((j+1)a)\big) \mathbb{P}(du); \\ C_j(a) &:= \int_{\mathbb{R}^{3d}} \int_\Omega \Big(\sqrt{p(a, y_{j-1}, y_j) p(a, y_j, y_{j+1})} - 1\Big)^4 \\ &\qquad \times \mathbb{P}_{I_{j-1}}^{y_{j-1}, y_j}(du) \mathbb{P}_{I_j}^{y_j, y_{j+1}}(du) \mu(dy_{j-1}) \mu(dy_j) \mu(dy_{j+1}) \\ &= \int_{\mathbb{R}^{3d}} \Big(\sqrt{p(a, x, y) p(a, y, z)} - 1\Big)^4 \mu(dx) \mu(dy) \mu(dz); \end{aligned}$$

- if $j = -N$,

$$\begin{aligned} B_{-N}(a) &:= \int_\Omega \big(e^{-H_{I_{-N}}(u^{(N)})} - 1\big)^4 p\big(a, u(-Na), u((-N+1)a)\big) \mathbb{P}(du); \\ C_{-N}(a) &:= \int_{\mathbb{R}^{2d}} \Big(\sqrt{p(a, x, y)} - 1\Big)^4 \mu(dx) \mu(dy); \end{aligned}$$

- if $j = N-1$,

$$\begin{aligned} B_{N-1}(a) &:= \int_\Omega \big(e^{-H_{I_{N-1}}(u^{(N)})} - 1\big)^4 p\big(a, u((N-2)a), u((N-1)a)\big) \\ &\qquad \times p\big(a, u((N-1)a), u(Na)\big)^2 \mathbb{P}(du); \\ C_{N-1}(a) &:= \int_{\mathbb{R}^{3d}} \Big(\sqrt{p(a, x, y)} p(a, y, z) - 1\Big)^4 \mu(dx) \mu(dy) \mu(dz). \end{aligned}$$

We will now study separately the B_j and the C_j, without omitting the two boundary cases, especially the one when $j = N-1$, which will turn out to be the most troublesome.

4.3 Study of $B_j(a)$

4.3.1 *Case* $j \in \{-N+1, \dots, N-2\}$. Using Cauchy–Schwarz's inequality, we again decompose the integral in two parts:

$$\begin{aligned} B_j(a) &= \int_\Omega \big(e^{-H_{I_j}(u^{(N)})} - 1\big)^4 p\big(a, u((j-1)a), u(ja)\big) \\ &\qquad \times p\big(a, u(ja), u((j+1)a)\big) \mathbb{P}(du) \\ &\le \tilde{B}_j(a) K_j(a) \end{aligned}$$

with

$$\tilde{B}_j(a) = \Big(\int_\Omega p\big(a, u((j-1)a), u(ja)\big)^2 p\big(a, u(ja), u((j+1)a)\big)^2 \mathbb{P}(du)\Big)^{1/2}$$

and

$$K_j(a) = \Big(\int_\Omega \big(e^{-H_{I_j}(u^{(N)})} - 1\big)^8 \mathbb{P}(du)\Big)^{1/2}.$$

Notice that $K_j(a)$ is bounded uniformly in j: indeed,

$$\begin{aligned}
K_j(a)^4 &= \Big(\int_\Omega p\big(a, u((j-1)a), u(ja)\big)^2 p\big(a, u(ja), u((j+1)a)\big)^2 \mathbb{P}(du)\Big)^2 \\
&\le \int_\Omega p\big(a, u((j-1)a), u(ja)\big)^4 \mathbb{P}(du) \int_\Omega p\big(a, u(ja), u((j+1)a)\big)^4 \mathbb{P}(du) \\
&= \mathbb{E}\big[p\big(a, y((j-1)a), y(ja)\big)^4\big]\mathbb{E}\big[p\big(a, y(ja), y((j+1)a)\big)^4\big] \\
&= \Big(\int_{\mathbb{R}^{2d}} p(a,x,y)^4 p(a,x,y)\mu(dx)\mu(dy)\Big)^2 = \|p(a,\cdot,\cdot)\|^{10}_{L^5(\mu\otimes\mu)}.
\end{aligned}$$

The main goal of this subsection is to find an upper bound for

$$\tilde{B}_j(a) = \Big(\int_\Omega \big(e^{-H_{I_j}(u^{(N)})} - 1\big)^8 \mathbb{P}(du)\Big)^{1/2},$$

depending on a and going to 0 as soon as a goes to infinity.

What follows is a direct adaptation of what was done in Roelly and Ruszel (2014) and Dai Pra and Roelly (2004).

We start by noticing that for every $x \in \mathbb{R}$,

$$\big(e^{-x} - 1\big)^8 = x^8 \Big(\int_0^1 e^{-tx}\, dt\Big)^8 = x^8 \int_{[0,1]^8} e^{-(t_1+\cdots+t_8)x}\, dt_1 \cdots dt_8,$$

and thus

$$\tilde{B}_j(a)^2 = \int_{[0,1]^8} \int_\Omega H_{I_j}\big(u^{(N)}\big)^8 e^{-(t_1+\cdots+t_8)H_{I_j}(u^{(N)})} \mathbb{P}(du)\, dt_1 \cdots dt_8.$$

Set $L(z) = \int_\Omega e^{-zH_{I_j}(u^{(N)})} \mathbb{P}(du)$.

Then, L is an holomorphic function, and its eighth derivative is

$$\frac{\partial}{\partial z^8} L(z) = \int_\Omega H_{I_j}\big(u^{(N)}\big)^8 e^{-zH_{I_j}(u^{(N)})} \mathbb{P}(du),$$

which means we can rewrite $\tilde{B}_j$ as

$$\tilde{B}_j(a)^2 = \int_{[0,1]^8} \frac{\partial}{\partial z^8} L(z)|_{z=t_1+\cdots+t_8}\, dt_1 \cdots dt_8. \tag{4.2}$$

Notice that

$$\big|L(z)\big| \le \int_\Omega \big|e^{-zH_{I_j}(u^{(N)})}\big| \mathbb{P}(du) = \int_\Omega e^{-\mathfrak{Re}(z)H_{I_j}(u^{(N)})} \mathbb{P}(du) = L\big(\mathfrak{Re}(z)\big).$$

Recall that the expression of the Hamiltonian H is given by equation (3.1).

For any real number x, we can obtain an alternative expression of L, using Cauchy–Schwarz inequality and the martingale property of $\exp(-H)$:

$$\begin{aligned}
L(x) &= \int_\Omega \exp\Big(x \int_{I_j} \beta b\big(t, (u)_{t-t_0}^t\big)^* \, dW_t - \frac{x}{2}\int_{I_j} \beta^2 \big|b\big(t, (u)_{t-t_0}^t\big)\big|^2 \, dt\Big)\mathbb{P}(du) \\
&= \int_\Omega \exp\Big(x \int_{ja}^{(j+1)a} \beta b\big(t, (u)_{t-t_0}^t\big)^* \, dW_t - x^2 \int_{ja}^{(j+1)a} \beta^2 \big|b\big(t, (u)_{t-t_0}^t\big)\big|^2 \, dt\Big) \\
&\quad \times \exp\Big(\frac{2x^2 - x}{2} \int_{ja}^{(j+1)a} \beta^2 \big|b\big(t, (u)_{t-t_0}^t\big)\big|^2 \, dt\Big)\mathbb{P}(du) \\
&\le \Big(\int_\Omega \exp\Big(2x \int_{ja}^{(j+1)a} \beta b\big(t, (u)_{t-t_0}^t\big)^* \, dW_t \\
&\quad - 2x^2 \int_{ja}^{(j+1)a} \beta^2 \big|b\big(t, (u)_{t-t_0}^t\big)\big|^2 \, dt\Big)\mathbb{P}(du)\Big)^{1/2} \\
&\quad \times \Big(\int_\Omega \exp\Big(x(2x-1) \int_{ja}^{(j+1)a} \beta^2 \big|b\big(t, (u)_{t-t_0}^t\big)\big|^2 \, dt\Big)\mathbb{P}(du)\Big)^{1/2} \\
&= \Big(\int_\Omega \exp\Big(x(2x-1) \int_{ja}^{(j+1)a} \beta^2 \big|b\big(t, (u)_{t-t_0}^t\big)\big|^2 \, dt\Big)\mathbb{P}(du)\Big)^{1/2}.
\end{aligned}$$

We now apply Cauchy's inequality to L, for ρ such that L is well defined on $B(z, \rho) = \{v \in \mathbb{C}, |v - z| \le \rho\}$:

$$\Big|\frac{\partial}{\partial z^8} L(z)\Big| \le \frac{8!}{\rho^8} \sup_{v \in B(z,\rho)} \big|L(v)\big|. \tag{4.3}$$

Thanks to the above expression of L,

$$\big|L(v)\big|^2 \le \int_\Omega \exp\Big(\mathfrak{Re}(v)\big(2\mathfrak{Re}(v) - 1\big) \int_{ja}^{(j+1)a} \beta^2 \big|b\big(t, (u)_{t-t_0}^t\big)\big|^2 \, dt\Big)\mathbb{P}(du).$$

As $|v - z|^2 = \rho^2$, we have $(\mathfrak{Re}(v) - z) \le \rho^2$ for $z = t_1 + \cdots + t_8$, it follows that $\mathfrak{Re}(v) \in \{x \in \mathbb{R} : |z - x| < \rho\}$, hence $\mathfrak{Re}(v) \le z + \rho$, which implies $\mathfrak{Re}(v)(2\mathfrak{Re}(v) - 1) \le 2(z + \rho)^2$.

Subsequently,

$$\big|L(v)\big|^2 \le \int_\Omega \exp(2(\rho + z)^2) a\beta^2)\mathbb{P}(du),$$

and thus,

$$\sup_{v \in \mathcal{C}(z,\rho)} \big|L(v)\big| \le \exp\big((\rho + z)^2 a\beta^2)\big), \tag{4.4}$$

where $\mathcal{C}(z, \rho) = \{v \in \mathbb{C}, |v - z| = \rho\}$.

Combining (4.2), (4.3) and (4.4),

$$\begin{aligned}
\tilde{B}_j(a)^2 &\le \int_{[0,1]^8} \frac{8!}{\rho^8} \exp\big((\rho + t_1 + \cdots + t_8)^2 a\beta^2\big) \, dt_1 \cdots dt_8 \\
&\le \frac{8!}{\rho^8} \exp\big((\rho + 8)^2 a\beta^2)\big).
\end{aligned}$$

It implies that, for every $\rho \ge 8$,

$$\tilde{B}_j(a)^2 \le \frac{8!}{\rho^8} e^{4\rho^2 a\beta^2}. \tag{4.5}$$

We want to determine which $\rho \geq 8$ will minimize the right-hand side of this last inequality.

Let f be the function given by $f(\rho) = \frac{8!}{\rho^8} e^{4\rho^2 a\beta^2}$. Then, $f'(\rho) = (-\frac{8}{\rho} + 8\rho a\beta^2) f(\rho)$.

Thus, $f'(\rho) = 0$ if and only if $\rho^2 = \frac{1}{a\beta^2}$, which is larger than 8 if and only if

$$a\beta^2 \leq \frac{1}{8}, \tag{4.6}$$

and the optimal inequality for (4.5) is

$$\tilde{B}_j(a)^2 \leq 8! e^4 (a\beta^2)^4. \tag{4.7}$$

Finally coming back to the expression of B_j, we have obtained, under condition (4.6),

$$B_j(a) \leq \sqrt{8!} e^2 \| p(a, \cdot, \cdot) \|_{L^5(\mu\otimes\mu)}^{5/2} (a\beta^2)^2. \tag{4.8}$$

4.3.2 *Boundary cases*, $j \in \{-N, N-1\}$. Remember that

$$B_{-N}(a) = \int_\Omega (e^{-H_{I_{-N}}(u^{(N)})} - 1)^4 p\big(a, u(-Na), u\big((-N+1)a\big)\big) \mathbb{P}(du).$$

As in the previous case, we can write

$$B_{-N}(a) \leq K_{-N}(a) \Big(\int_\Omega (e^{-H_{I_{-N}}(u^{(N)})} - 1)^8 \mathbb{P}(du) \Big)^{1/2},$$

where

$$K_{-N}(a)^2 = \int_\Omega p\big(a, u(-Na), u\big((-N+1)a\big)\big)^2 \mathbb{P}(du).$$

This square root can be dealt with in exactly the same fashion as is done above.

Furthermore,

$$\begin{aligned} K_{-N}(a)^2 &= \mathbb{E}\big[p\big(a, y(-Na), y\big((-N+1)a\big)\big)^2\big] \\ &= \int_{\mathbb{R}^{2d}} p(a, x, y)^2 p(a, x, y) \mu(dx)\mu(dy) = \| p(a, \cdot, \cdot) \|_{L^3(\mu\otimes\mu)}^3. \end{aligned}$$

Hence the following result:

$$B_{-N}(a) \leq \sqrt{8!} e^2 \| p(a, \cdot, \cdot) \|_{L^3(\mu\otimes\mu)}^{3/2} (a\beta^2)^2. \tag{4.9}$$

We now turn our attention to

$$\begin{aligned} B_{N-1}(a) = &\int_\Omega (e^{-H_{I_{N-1}}(u^{(N)})} - 1)^4 p\big(a, u\big((N-2)a\big), u\big((N-1)a\big)\big) \\ &\times p\big(a, u\big((N-1)a\big), u(Na)\big)^2 \mathbb{P}(du). \end{aligned}$$

We proceed in a similar way to decompose $B_{N-1}(a)$ into the product of two terms and we have to study the quantity:

$$\begin{aligned} &K_{N-1}(a) \\ &= \sqrt{\int_\Omega p\big(a, u\big((N-2)a\big), u\big((N-1)a\big)\big)^2 p\big(a, u\big((N-1)a\big), u(Na)\big)^4 \mathbb{P}(du)}. \end{aligned}$$

In order to obtain an upper bound for a moment of $p(a,\cdot,\cdot)$, with respect to $\mu\otimes\mu$, smaller than 8, Cauchy–Schwarz's inequality will not suffice: we have to apply Hölder's inequality. We choose the conjugated numbers 3 and $3/2$:

$$K_{N-1}(a)^2 \le \left(\int_\Omega p\big(a, u\big((N-2)a\big), u\big((N-1)a\big)\big)^6 \mathbb{P}(du)\right)^{1/3} \times \left(\int_\Omega p\big(a, u\big((N-1)a\big), u(Na)\big)^6 \mathbb{P}(du)\right)^{2/3},$$

which leads to

$$K_{N-1}(a) \le \|p(a,\cdot,\cdot)\|^7_{L^7(\mu\otimes\mu)},$$

and subsequently to

$$B_{N-1}(a) \le \sqrt{8!}e^2 \|p(a,\cdot,\cdot)\|^{7/2}_{L^7(\mu\otimes\mu)} (a\beta^2)^2. \tag{4.10}$$

4.3.3 *A bound for $B_j(a)$, uniform in N.* From (4.8), (4.9) and (4.10), we deduce that, for every $j\in\{-N,\dots,N-1\}$,

$$B_j(a) \le \sqrt{8!}e^2 \|p(a,\cdot,\cdot)\|^{7/2}_{L^7(\mu\otimes\mu)} (a\beta^2)^2. \tag{4.11}$$

4.4 Study of $C_j(a)$

4.4.1 *General case*, $j\in\{-N+1,\dots,N-2\}$. We remind that

$$C_j(a) = \int_{\mathbb{R}^{3d}} \big(\sqrt{p(a,x,y)p(a,y,z)}-1\big)^4 \mu(dx)\mu(dy)\mu(dz).$$

Again, we seek an upper bound for $C_j(a)$ which vanishes when a goes to infinity.

It can be easily checked that for every positive real number U,

$$(\sqrt{1+U}-1)^4 \le \frac{1}{16}U^4, \tag{4.12}$$

that for positive x and y,

$$xy-1 = (x-1)(y-1)+(x-1)+(y-1),$$

and that, thanks to the convexity of $u\mapsto u^4$, for any a, b and c,

$$(a+b+c)^4 \le 27(a^4+b^4+c^4).$$

Subsequently,

$$\begin{aligned} C_j(a) &\le \frac{1}{16}\int_{\mathbb{R}^{3d}} \big(p(a,x,y)p(a,y,z)-1\big)^4 \mu(dx)\mu(dy)\mu(dz) \\ &\le \frac{1}{16}\int_{\mathbb{R}^{3d}} \big[\big(p(a,x,y)-1\big)\big(p(a,y,z)-1\big)+\big(p(a,x,y)-1\big) \\ &\quad + \big(p(a,y,z)-1\big)\big]^4 \mu(dx)\mu(dy)\mu(dz) \\ &\le \frac{27}{16}\int_{\mathbb{R}^{3d}} \big(\big(p(a,x,y)-1\big)\big(p(a,y,z)-1\big)\big)^4 \mu(dx)\mu(dy)\mu(dz) \\ &\quad + \frac{27}{8}\int_{\mathbb{R}^{2d}} \big(p(a,x,y)-1\big)^4 \mu(dx)\mu(dy) \\ &\le \frac{27}{16}\int_{\mathbb{R}^{2d}} \big(p(a,x,y)-1\big)^8 \mu(dx)\mu(dy) \\ &\quad + \frac{27}{8}\int_{\mathbb{R}^{2d}} \big(p(a,x,y)-1\big)^4 \mu(dx)\mu(dy), \end{aligned}$$

using, once more, Cauchy–Schwarz's inequality to obtain the final line.

Furthermore,

$$\int_{\mathbb{R}^{2d}} \big(p(a,x,y)-1\big)^4 \mu(dx)\mu(dy) \le \sqrt{\int_{\mathbb{R}^{2d}} \big(p(a,x,y)-1\big)^8 \mu(dx)\mu(dy)}.$$

Thus, according to Proposition 1,

$$C_j(a) \le \frac{27}{16}\gamma_\delta(a)^8\big(\|p(\delta,\cdot,\cdot)\|^8_{L^8(\mu\otimes\mu)} \vee 1\big) + \frac{27}{8}\gamma_\delta(a)^4\big(\|p(\delta,\cdot,\cdot)\|^4_{L^8(\mu\otimes\mu)} \vee 1\big).$$

4.4.2 *Boundary cases*, $j \in \{-N, N-1\}$. We can check that both boundary cases exhibit an analogous behaviour.

Indeed, on the one hand, recall that

$$C_{-N}(a) = \int_{\mathbb{R}^{2d}} \big(\sqrt{p(a,x,y)}-1\big)^4 \mu(dx)\mu(dy).$$

Thus, thanks to (4.12) and Proposition 1,

$$\begin{aligned} C_{-N}(a) &\le \frac{1}{16}\int_{\mathbb{R}^{2d}} \big(p(a,x,y)-1\big)^4 \mu(dx)\mu(dy) \\ &\le \frac{1}{16}\sqrt{\int_{\mathbb{R}^{2d}} \big(p(a,x,y)-1\big)^8 \mu(dx)\mu(dy)} \\ &\le \frac{1}{16}\gamma_\delta(a)^4\big(\|p(\delta,\cdot,\cdot)\|^4_{L^8(\mu\otimes\mu)} \vee 1\big). \end{aligned}$$

On the other hand,

$$C_{N-1}(a) = \int_{\mathbb{R}^{3d}} \big(\sqrt{p(a,x,y)}\,p(a,y,z)-1\big)^4 \mu(dx)\mu(dy)\mu(dz).$$

Notice that

$$\begin{aligned} &\big(\sqrt{p(a,x,y)}\,p(a,y,z)-1\big)^4 \\ &\quad \le 8\big(p(a,y,z)-1\big)^4 p(a,x,y)^2 + 8\big(\sqrt{p(a,x,y)}-1\big)^4. \end{aligned}$$

With Cauchy–Schwarz's inequality and the computation of $C_{-N}(a)$ thrown in, this leads to

$$\begin{aligned} C_{N-1}(a) &\le 8\int_{\mathbb{R}^{3d}} \big(p(a,y,z)-1\big)^4 p(a,x,y)^2 \mu(dx)\mu(dy)\mu(dz) \\ &\quad + 8\int_{\mathbb{R}^{2d}} \big(\sqrt{p(a,x,y)}-1\big)^4 \mu(dx)\mu(dy) \\ &\le 8\sqrt{\int_{\mathbb{R}^{2d}} \big(p(a,x,y)-1\big)^8 \mu(dx)\mu(dy)}\sqrt{\int_{\mathbb{R}^{2d}} p(a,x,y)^4 \mu(dx)\mu(dy)} \\ &\quad + \frac{1}{2}\gamma_\delta(a)^4\big(\|p(\delta,\cdot,\cdot)\|^4_{L^8(\mu\otimes\mu)} \vee 1\big) \\ &\le 8\gamma_\delta(a)^4\big(\|p(\delta,\cdot,\cdot)\|^4_{L^8(\mu\otimes\mu)} \vee 1\big)\|p(a,\cdot,\cdot)\|^2_{L^4(\mu\otimes\mu)} \\ &\quad + \frac{1}{2}\gamma_\delta(a)^4\big(\|p(\delta,\cdot,\cdot)\|^4_{L^8(\mu\otimes\mu)} \vee 1\big) \\ &= \left(8\|p(a,\cdot,\cdot)\|^2_{L^4(\mu\otimes\mu)} + \frac{1}{2}\right)\gamma_\delta(a)^4\big(\|p(\delta,\cdot,\cdot)\|^4_{L^8(\mu\otimes\mu)} \vee 1\big). \end{aligned}$$

4.4.3 *Global upper bound for all C_j.* Taking into account all three cases, when $a \geq 2\delta$, for every $j \in \{-N, \ldots, N-1\}$,

$$C_j(a) \leq \gamma_\delta(a)^4 (\|p(\delta,\cdot,\cdot)\|^4_{L^8(\mu\otimes\mu)} \vee 1) \times \left(\left(\frac{27}{16}\gamma_\delta(a)^4 + 8\right)(\|p(\delta,\cdot,\cdot)\|^4_{L^8(\mu\otimes\mu)} \vee 1) + 4\right). \tag{4.13}$$

To obtain the control of $|\Gamma_\tau|$ we are seeking, we still need to put all the pieces together and to determine its domain of validity with respect to a and b.

4.5 Back to the clusters

At last, the proposition below gives us the cluster estimates.

Proposition 3. *Assume that* (H1) *and* (H2) *are satisfied. Let ε be a positive number.*

There exist a minimal time-scale a_ε, defined in (4.22), *and an upper-bound β_ε, given by* (4.23), *such that if $a \geq a_\varepsilon$ and $\beta \leq \beta_\varepsilon$, then, for every cluster τ, the quantity Γ_τ defined in* (3.7) *satisfies*

$$|\Gamma_\tau| \leq \varepsilon^{|\tau|}. \tag{4.14}$$

Proof. Suppose that $a \geq 2\delta$ and that (4.6) holds, i.e. $\beta \leq \frac{1}{\sqrt{8a}}$.

Recall that

$$M_\delta = \sup_{a\geq\delta} \|p(a,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)} \vee 1. \tag{4.15}$$

Thus, we can obtain bounds for B_j and C_j easier to deal with: according to (4.11) and (4.13) respectively,

$$\begin{aligned} B_j(a) &\leq \sqrt{8!}e^2 M_\delta^{7/2}(a\beta^2)^2, \\ C_j(a) &\leq M_\delta^4(4 + 2M_\delta^4(4 + \gamma_\delta(a)^4))\gamma_\delta(a)^4. \end{aligned} \tag{4.16}$$

Remember that

$$|\Gamma_\tau(a)| \leq \prod_{j\in\tau} (8B_j(a) + 8C_j(a))^{1/4},$$

so (4.14) will be satisfied if, for a sufficiently large, both $B_j(a)$ and $C_j(a)$ are smaller than $\varepsilon^4/16$.

- One can check, by solving a second order inequality in $\gamma_\delta(a)^4$, that for all a such that

$$\gamma_\delta(a)^4 \leq \left(2 + \frac{1}{M_\delta^4}\right)\left(\sqrt{1 + \frac{\varepsilon^4}{32(1+2M_\delta^4)^2}} - 1\right), \tag{4.17}$$

  the condition $C_j(a) \leq \frac{\varepsilon^4}{16}$ is true.

  We recall that γ_δ was introduced in Proposition 1 and is defined by

$$\gamma_\delta(a) = 2M_\delta e^{-(a-2\delta)/C_P},$$

  with C_P the constant associated with the Poincaré's inequality satisfied by μ, according to hypothesis (H1).

  Using this expression, and setting

$$a_C(\varepsilon) = 2\delta - \frac{C_P}{4}\ln\left(\frac{1}{16M_\delta^4}\left(2 + \frac{1}{M_\delta^4}\right)\left(\sqrt{1 + \frac{\varepsilon^4}{32(1+2M_\delta^4)^2}} - 1\right)\right), \tag{4.18}$$

it can be shown that (4.17) is equivalent to:

$$a \geq a_C(\varepsilon).$$

Thus, for every $a \geq a_C(\varepsilon)$, $C_j(a) \leq \frac{\varepsilon^4}{16}$.
It can be noticed that

$$a_C(\varepsilon) \geq 2\delta \quad \text{if and only if} \quad \varepsilon \leq 2^{5/2} M_\delta^2 \left(8M_\delta^8 + 2M_\delta^4 + 1\right)^{1/4}. \tag{4.19}$$

- From (4.16), it can be seen that $B_j(a) \leq \frac{\varepsilon^4}{16}$ if

$$\beta \leq \frac{1}{2\sqrt{e}(8!)^{1/8} M_\delta^{7/8}} \frac{\varepsilon}{\sqrt{a}}. \tag{4.20}$$

Notice that

$$\frac{\varepsilon}{2\sqrt{e}(8!)^{1/8} M_\delta^{7/8}} \leq \frac{1}{\sqrt{8}} \quad \text{if and only if} \quad \varepsilon \leq \sqrt{\frac{e}{2}}(8!)^{1/8} M_\delta^{7/8}. \tag{4.21}$$

Thus, according to (4.6), (4.17) and (4.20), setting

$$a_\varepsilon = a_C(\varepsilon) \vee (2\delta),$$

that is

$$a_\varepsilon = 2\delta - \left[\frac{C_P}{4} \ln\left(\frac{1}{16M_\delta^4}\left(2 + \frac{1}{M_\delta^4}\right)\left(\sqrt{1 + \frac{\varepsilon^4}{32(1+2M_\delta^4)^2}} - 1\right)\right)\right]_-, \tag{4.22}$$

where $x_- = \min(x, 0)$, and

$$\beta_\varepsilon = \left(\frac{\varepsilon}{2\sqrt{e}(8!)^{1/8} M_\delta^{7/8}} \wedge \frac{1}{\sqrt{8}}\right) \frac{1}{\sqrt{a_\varepsilon}},$$

that is,

$$\beta_\varepsilon = \frac{\frac{\varepsilon}{2\sqrt{e}(8!)^{1/8} M_\delta^{7/8}} \wedge \frac{1}{\sqrt{8}}}{\sqrt{2\delta - [\frac{C_P}{4} \ln(\frac{1}{16M_\delta^4}(2 + \frac{1}{M_\delta^4})(\sqrt{1 + \frac{\varepsilon^4}{32(1+2M_\delta^4)^2}} - 1))]_-}}, \tag{4.23}$$

the proposition holds. □

Remark 7. We recall hypothesis (H2): there exists $\delta \geq 0$ such that

$$\sup_{t \geq \delta} \| p(t, \cdot, \cdot) \|_{L^8(\mu \otimes \mu)} < \infty.$$

One can notice that we only require $\| p(t, \cdot, \cdot) \|_{L^8(\mu \otimes \mu)}$ to be finite for certain values of t and it is not necessary for the supremum over t to be finite. However, the current form of (H2) simplifies the writing of the proofs, and it is satisfied by the important case of the Ornstein–Uhlenbeck process, as will be seen in Section 6.

5 Completion of the proof of Theorem 2.1

In order to connect with the cluster expansion method, and to obtain an inequality of the form of (4.1), we have to show that ε can be expressed as a function of β that will go to 0 when β goes to 0.

Suppose that

$$\varepsilon \leq \left(2^{5/2} M_\delta^2 (8M_\delta^8 + 2M_\delta^4 + 1)^{1/4}\right) \wedge \left(\sqrt{\frac{e}{2}}(8!)^{1/8} M_\delta^{7/8}\right) =: \varepsilon_0. \tag{5.1}$$

Then according to equivalences (4.19) and (4.21),

$$\beta_\varepsilon = \frac{\varepsilon}{2\sqrt{2e}(8!)^{1/8} M_\delta^{7/8} \sqrt{\delta - \frac{C_P}{8} \ln(\frac{1}{16M_\delta^4}(2 + \frac{1}{M_\delta^4})(\sqrt{1 + \frac{\varepsilon^4}{32(1+2M_\delta^4)^2}} - 1))}}. \tag{5.2}$$

One can see that $\varepsilon \mapsto \beta_\varepsilon$ is of the form

$$\frac{C_1 \varepsilon}{\sqrt{1 - C_2 \ln(C_3(\sqrt{1 + C_4 \varepsilon^4} - 1))}},$$

for some C_1, C_2, C_3 and C_4 depending only on δ and C_P, and is thus an invertible function.

Compute the derivative of β_ε with respect to ε:

$$\beta_\varepsilon' = \frac{\beta_\varepsilon}{\varepsilon}\left(1 + \frac{C_2 C_4}{C_1^2} \frac{\varepsilon^2 \beta_\varepsilon^2}{\sqrt{1 + C_4 \varepsilon^4}(\sqrt{1 + C_4 \varepsilon^4} - 1)}\right).$$

β_ε' is positive for every ε in $(0, \varepsilon_0]$; thus, $\varepsilon \mapsto \beta_\varepsilon$ is (strictly) increasing from $(0, \varepsilon_0]$ to $(0, \beta_{\varepsilon_0}]$.

Therefore, $\varepsilon \mapsto \beta_\varepsilon$ admits an inverse function on $(0, \beta_{\varepsilon_0}]$ that we will call η: η is increasing and $\eta(x)$ goes to 0 when x goes to 0. For simplicity, we denote by β_0 the bound β_{ε_0} given by (5.1) and (5.2).

We can now rewrite Proposition 3 in a more amenable way.

Proposition 4. *There exists β_0 such that if $\beta \leq \beta_0$, then, for every cluster Γ_τ,*

$$|\Gamma_\tau| \leq \eta(\beta)^{|\tau|}, \tag{5.3}$$

where η, defined just above, is a function that goes to 0 *in* 0.

Remark 8. One can easily show that a first-order approximation of β_ε, when ε is small, is

$$\beta_\varepsilon \sim C\varepsilon\left(-\ln(\varepsilon)\right)^{-1/2},$$

for a certain, explicit, constant C, depending only on the parameters δ and C_P, given by the hypotheses (H1) and (H2). Since, for any $\alpha > 0$, when ϵ is small enough, $-\ln(\varepsilon) \leq \frac{1}{\varepsilon^{2\alpha}}$, thus $\beta_\varepsilon \geq C\varepsilon^{1+\alpha}$, and Proposition 4 holds, with $\eta : y \mapsto Cy^{1/(1+\alpha)}$.

Recall that we wish to prove the convergence of the sequence of measures $(Q_N)_N$, with

$$Q_N(du) = \frac{1}{Z_N} \exp\left(-H_N\left(u^{(N)}\right)\right)\mathbb{P}(du),$$

towards a weak stationary solution Q of the perturbed equation (2.5).

Proposition 4, just above, is the key point to prove this convergence: the cluster representation (3.6) of the partition function Z_N and the cluster estimates (5.3) are the crucial elements in order to obtain in a canonical way an expansion for the measures Q_N (see Malyshev and Minlos (1991)).

It has been explained in details in both Dai Pra and Roelly (2004) and Minlos, Roelly and Zessin (2000) (see, for instance, paragraph 4.1.4 of Minlos, Roelly and Zessin (2000), with Lemma 10 and what follows); we give here an overview of the reasoning, adapted to our framework.

For a finite subset $\mathcal{S}$ of $\mathbb{Z}$, we associate $\mathcal{I}_\mathcal{S}$ such that $\mathcal{I}_\mathcal{S} = \bigcup_{k\in\mathcal{S}} I_k$.

We define the partition function on $\mathcal{I}$ by

$$Z_\mathcal{I} = 1 + \sum_{\tau_1\sqcup\cdots\sqcup\tau_p} \prod_{i=1}^{p} \Gamma_{\tau_i},$$

where the τ_i are defined as in Section 3.2: they are connected sets, disjoint from each other, such that $a\tau_i \subset \mathcal{I}$. Notice that $\mathcal{I}_{\{-N,\dots,N-1\}} = I(N)$ and $Z_N = Z_{I(N)}$.

For $\mathcal{S}_1 \subset \mathcal{S}_2$, we define

$$f_{\mathcal{S}_1}^{\mathcal{S}_2} = \frac{Z_{\mathcal{I}_{\mathcal{S}_2}\setminus\bar{\mathcal{I}}_{\mathcal{S}_1}}}{Z_{\mathcal{I}_{\mathcal{S}_2}}},$$

where $\bar{\mathcal{I}}_S = \bigcup_{k\in\mathcal{S}} \bar{I}_k$ and $\bar{I}_k = I_{k-1} \cup I_k \cup I_{k+1}$.

The original version of the following lemma can be found in Malyshev and Minlos (1991); here it is adapted to our needs.

Lemma 3. *For β small enough,*

(i) *There exists a positive constant C_1, independent from $\mathcal{S}_1$ and $\mathcal{S}_2$, such that*

$$\bigl|f_{\mathcal{S}_1}^{\mathcal{S}_2}\bigr| < C_1 2^{|\mathcal{S}_1|}. \tag{5.4}$$

(ii) *The following assertion holds*

$$f_{\mathcal{S}_1}^{\mathcal{S}_2} = 1 + \sum_{\tau_1,\dots,\tau_p} C_{\mathcal{S}_1}(\tau_1,\dots,\tau_p) \prod_{i=1}^{p} \Gamma_{\tau_i}, \tag{5.5}$$

where the sum is over every $\tau_1,\dots,\tau_p$ for every possible integer p as defined in Section 3.2, such that $\mathcal{I}_{\mathcal{S}_1} \cap a(\tau_1\sqcup\cdots\sqcup\tau_p) \neq \varnothing$ and $a\tau_1\sqcup\cdots\sqcup a\tau_p \subset \mathcal{I}_{\mathcal{S}_2}$. $C_{\mathcal{S}_1}(\tau_1,\dots,\tau_p)$ is independent from $\mathcal{S}_2$. Furthermore, the series converges absolutely.

(iii) *The expression (5.5) admits a limit $f_{\mathcal{S}_1}$ when $\mathcal{S}_2$ tends towards $\mathbb{Z}$ and it satisfies*

$$f_{\mathcal{S}_1} = 1 + \sum_{\mathcal{I}_{\mathcal{S}_1}\cap(a\tau_1\sqcup\cdots\sqcup a\tau_p)\neq\varnothing} C_{\mathcal{S}_1}(\tau_1,\dots,\tau_p) \prod_{i=1}^{p} \Gamma_{\tau_i}. \tag{5.6}$$

(iv) *There exists a positive constant C_2 such that*

$$\bigl|f_{\mathcal{S}_1}^{\mathcal{S}_2} - f_{\mathcal{S}_1}\bigr| < C_2 2^{|\bar{\mathcal{S}}|-d(\mathcal{I}_{\mathcal{S}_1},\mathcal{I}_{\mathcal{S}_2}^c)},$$

where $\bar{\mathcal{S}}$ is such that $\mathcal{I}_{\bar{\mathcal{S}}} = \bar{\mathcal{I}}_{\mathcal{S}_1}$.

(v) *The exists a positive constant C_3 such that for any subsets $\mathcal{S}_1$ and $\hat{\mathcal{S}}_1$ of $\mathcal{S}_2$,*

$$\begin{aligned} \bigl|f_{\mathcal{S}_1\cup\hat{\mathcal{S}}_1}^{\mathcal{S}_2} - f_{\mathcal{S}_1}^{\mathcal{S}_2} f_{\hat{\mathcal{S}}_1}^{\mathcal{S}_2}\bigr| &< C_3 3^{|\mathcal{S}_1|+|\hat{\mathcal{S}}_1|} \eta(\beta)^{d(\mathcal{I}_{\mathcal{S}_1},\mathcal{I}_{\hat{\mathcal{S}}_1})}, \\ \bigl|f_{\mathcal{S}_1\cup\hat{\mathcal{S}}_1} - f_{\mathcal{S}_1} f_{\hat{\mathcal{S}}_1}\bigr| &< C_3 3^{|\mathcal{S}_1|+|\hat{\mathcal{S}}_1|} \eta(\beta)^{d(\mathcal{I}_{\mathcal{S}_1},\mathcal{I}_{\hat{\mathcal{S}}_1})}. \end{aligned} \tag{5.7}$$

Remark 9. This is where, despite the explicit bounds obtained in (4.23) and (5.1), we must renounce to an explicit expression for the required smallness of β.

Let $\mathcal{I}$ be a finite interval and N large enough such that $\mathcal{I} \subset I(N)$. Let $F_{\mathcal{I}}$ be a $\mathcal{I}$-local bounded measurable function on Ω, that is, for every u in Ω, $F_{\mathcal{I}}(u) = F(u_{\mathcal{I}})$. Our aim is to show that when β is small enough, the sequence $(\int F_{\mathcal{I}}\, dQ_N)_N$ converges.

Recall that

$$\int F_{\mathcal{I}}\, dQ_N = \frac{1}{Z_N} \int_{\Omega} F_{\mathcal{I}}(u) \exp(-H_N(u^{(N)}))\mathbb{P}(du).$$

From manipulations similar to those of Section 3.2, one can establish that

$$\int F_{\mathcal{I}}\, dQ_N = \frac{1}{Z_N} \sum_{\tau_1,\dots,\tau_p} K_{\tau_1,\dots,\tau_p}(F_{\mathcal{I}}) Z_{\mathcal{I}(N)\setminus(\mathcal{I}\cup\tau_1\sqcup\cdots\sqcup\tau_p)},$$

with $\tau_1 \sqcup \cdots \sqcup \tau_p \subset \{-N, \dots, N\}$ and where the coefficients $K_{\tau_1,\dots,\tau_p}(F_{\mathcal{I}})$ can be given explicitly, and do not depend on N.

The above expression can be written as

$$\int F_{\mathcal{I}}\, dQ_N = \sum_{\tau_1,\dots,\tau_p} K_{\tau_1,\dots,\tau_p}(F_{\mathcal{I}}) f^{I(N)}_{\mathcal{I}\cup(\tau_1\sqcup\cdots\sqcup\tau_p)}.$$

From (5.5), we have

$$\int F_{\mathcal{I}}\, dQ_N = \sum_{\tau_1,\dots\tau_p} K_{\tau_1,\dots,\tau_p}(F_{\mathcal{I}}) \left(1 + \sum_{\hat{\tau}_1,\dots,\hat{\tau}_q} C_{\mathcal{I}\cup(\tau_1\sqcup\cdots\sqcup\tau_p)}(\hat{\tau}_1, \dots, \hat{\tau}_q) \prod_{j=1}^{q} \Gamma_{\hat{\tau}_j}\right),$$

with $\tau_1 \sqcup \cdots \sqcup \tau_p \subset \{-N, \dots, N\}$, $\hat{\tau}_1 \sqcup \cdots \sqcup \hat{\tau}_q \subset \{-N, \dots, N\}$ and $(\mathcal{I} \cup (\tau_1 \sqcup \cdots \sqcup \tau_p)) \cap (\hat{\tau}_1 \sqcup \cdots \sqcup \hat{\tau}_q) \neq \varnothing$.

From (5.4) and (5.6), we can conclude that there is absolute convergence at exponential rate of the series over $\hat{\tau}_1, \dots, \hat{\tau}_q$ when $\mathcal{S}_2$ converges towards $\mathbb{Z}$, so that

$$\lim_{N\to+\infty} \int F_{\mathcal{I}}\, dQ_N = \sum_{\tau_1,\dots\tau_p} K_{\tau_1,\dots,\tau_p}(F_{\mathcal{I}}) f_{\mathcal{I}\cup(\tau_1\sqcup\cdots\sqcup\tau_p)}.$$

Setting $\int F_{\mathcal{I}}\, dQ := \lim_{N\to+\infty} \int F_{\mathcal{I}}\, dQ_N$, the following result holds.

Proposition 5. *Assume* (H1) *and* (H2). *For β small enough, there exists a stationary probability measure Q on Ω such that*:

$$Q = \lim_{N\to\infty} Q_N$$

The notion of weak limit here is understood as the limit for the topology of local convergence.

Properties that are satisfied by the approximations Q_N are inherited by the limit Q. Indeed, further classical results taken from Gibbs field theory (a combination of Proposition 2 and Lemma 4 in Dai Pra and Roelly (2004)) ensure that the probability measure Q is truly a weak stationary solution of the equation:

$$dX_t = \left(-\frac{1}{2}\nabla V(X_t) + \beta b\big(t, (X)^t_{t-t_0}\big)\right) dt + dW_t, \tag{2.5}$$

that is, under the probability measure Q, the canonical process (X_t) satisfy the stochastic system (2.5).

Hence our main result.

The property of exponential decorrelations, (iii) of Theorem 2.1,

$$\left|\int f(X_t)g(X_{t'})\,dQ-\int f(X_t)\,dQ\int g(X_{t'})\,dQ\right|\le\theta_1 e^{-\theta_2|t-t'|},$$

is a consequence of the inequality (5.7) and of a cluster representation for quantities of the form:

$$\int f(X_t)g(X_{t'})\,dQ_N-\int f(X_t)\,dQ_N\int g(X_{t'})\,dQ_N,$$

for $t,t'\in I(N)$. The fact that the bounds θ_1 and θ_2 are independent of N comes from that the convergence for the local functions is absolute and with exponential rate.

As the correlations decay at an exponential rate, we have strong mixing properties, and, in particular, the central limit theorem below. Though this process is not Markovian, the proof of the following corollary is similar to that of the famous result obtained in Kipnis and Varadhan (1986) and expanded in Cattiaux, Chafaï and Guillin (2012).

Corollary 1. *If a smooth f is such that $\int f\,d\nu=0$, then under Q,*

$$\frac{1}{\sqrt{t}}\int_0^t f(X_s)\,ds\xrightarrow[t\to+\infty]{(d)}\mathcal{N}(0,\sigma_f^2),$$

with

$$\sigma_f^2:=2\int_{-\infty}^{+\infty}\mathbb{E}_Q\big[f(X_0)f(X_s)\big]\,ds=\int|\nabla f|^2\,d\nu.$$

6 An example: The Ornstein–Uhlenbeck dynamics as reference process

Suppose the reference drift g is a linear one.

In order to simplify the writing of the computations, we restrict ourselves to the one-dimensional situation $d=1$; the behaviour in higher dimensions is completely similar.

We are thus considering the one-dimensional Ornstein–Uhlenbeck process solution of:

$$dX_t=-\lambda X_t\,dt+dW_t,$$

where λ is a positive parameter and (W_t) is a standard one-dimensional Brownian motion.

6.1 Verification of the assumptions

It is a process whose explicit expression and general behaviour are well known; in particular, it admits the Gaussian law $\mu=\mathcal{N}(0,1/2\lambda)$, whose density is given by

$$\mu(dy)=\sqrt{\frac{\lambda}{\pi}}e^{-\lambda y^2}\,dy,$$

as its (unique) symmetric probability measure.

Furthermore, the transition density of (y_t) with respect to μ is given by

$$p(t,x,y)=\frac{1}{\sqrt{1-e^{-2\lambda t}}}\exp\left(-\frac{\lambda}{1-e^{-2\lambda t}}\big((x^2+y^2)e^{-2\lambda t}-2xye^{-\lambda t}\big)\right).$$

Thus, all the assumptions made at the beginning of Section 2.1 are satisfied, as are hypotheses (H1) and (H2):

Proposition 6. *In this setting, assumptions* (H1) *and* (H2) *are satisfied, with $C_P=\frac{1}{2\lambda}$ and any $\delta>\frac{\ln(7)}{\lambda}$. Furthermore, these bounds are optimal.*

Proof. Indeed, thanks to a well-known result (see, for instance, Ané et al. (2000)) on Poincaré's inequalities verified by Gaussian measures, for a smooth function f,

$$\mathrm{Var}_\mu(f) \le \frac{1}{2\lambda}\int (f')^2\, d\mu,$$

which implies that (H1) holds, with $C_P = 1/2\lambda$.

Moreover, (H2) follows from the lemma below:

Lemma 4. *For every positive t and for $k \in \mathbb{N}^*$,*

$$\int_{\mathbb{R}^2} p(t,x,y)^k \mu(dx)\mu(dy) \\ = \frac{1}{(1-e^{-2\lambda t})^{k/2-1}\sqrt{(1+(k-1)e^{-2\lambda t})^2 - k^2 e^{-2\lambda t}}}. \tag{6.1}$$

We thus have immediately:

Corollary 2. *For every integer k, $\|p(t,\cdot,\cdot)\|_{L^k(\mu\otimes\mu)}$ goes to 1 when t goes to infinity, and for every $K > 1$, there exists t_K such that*

$$\sup_{t\ge t_K}\|p(t,\cdot,\cdot)\|_{L^k(\mu\otimes\mu)} \le K.$$

Proof. Set $I_k(t) = \int_{\mathbb{R}^2} p(t,x,y)^k \mu(dx)\mu(dy)$.

Then, letting $K_t = \lambda(1-e^{-2\lambda t})^{-1}$, $c_t = 1 + (k-1)e^{-2\lambda t}$, and $d_t = ke^{-\lambda t}$,

$$\begin{aligned} I_k(t) &= \frac{\lambda}{\pi}(1-e^{-2\lambda t})^{-k/2} \\ &\quad\times \int_{\mathbb{R}^2} \exp\Big(-\frac{\lambda}{1-e^{-2\lambda t}}\big((1+(k-1)e^{-2\lambda t})(x^2+y^2) - 2kxye^{-\lambda t}\big)\Big)\, dx\, dy \\ &= \frac{\lambda}{\pi}(1-e^{-2\lambda t})^{-k/2}\int_{\mathbb{R}^2}\exp\Big(-K_t c_t\Big(x - \frac{d_t}{c_t}y\Big)^2\Big) \\ &\quad\times \exp\Big(-K_t c_t\Big(1-\frac{d_t^2}{c_t^2}\Big)y^2\Big)\, dx\, dy \\ &= \frac{\lambda}{\pi}(1-e^{-2\lambda t})^{-k/2}\sqrt{\frac{\pi}{K_t c_t}}\sqrt{\frac{\pi}{K_t c_t(1-\frac{d_t^2}{c_t^2})}} \\ &= \lambda(1-e^{-2\lambda t})^{-k/2}\frac{1}{K_t\sqrt{c_t^2-d_t^2}} = (1-e^{-2\lambda t})^{1-k/2}\frac{1}{\sqrt{c_t^2-d_t^2}}. \end{aligned}$$

Hence the result we were looking for. □

In particular,

$$\|p(a,\cdot,\cdot)\|^8_{L^8(\mu\otimes\mu)} = (1-e^{-2\lambda a})^{-3}(1-50e^{-2\lambda a}+49e^{-4\lambda a})^{-1/2},$$

which is finite if and only $a > \frac{\ln(7)}{\lambda}$.

Besides, a study of the function $a \mapsto (1-e^{-2\lambda a})^{-3}(1-50e^{-2\lambda a}+49e^{-4\lambda a})^{-1/2}$ shows that it is decreasing towards 1 on the open interval $(\frac{\ln(7)}{\lambda}, +\infty)$.

Thus, for every $\delta > \frac{\ln(7)}{\lambda}$,

$$\sup_{a\geq\delta}\|p(a,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)} < \infty,$$

and, furthermore,

$$\sup_{a\geq\delta}\|p(a,\cdot,\cdot)\|_{L^8(\mu\otimes\mu)} = \left(1-e^{-2\lambda\delta}\right)^{-3/8}\left(1-50e^{-2\lambda\delta}+49e^{-4\lambda\delta}\right)^{-1/16} = M_\delta,$$

where M_δ corresponds to the constant defined in (4.15). □

The perturbed equation is

$$dx_t = \left(-\lambda x_t + \beta b\left(t,(x)_{t-t_0}^t\right)\right)dt + dW_t,$$

where $b : \mathbb{R}\times\mathcal{C}([-t_0,0],\mathbb{R}) \to \mathbb{R}$ is a measurable function, bounded by 1, satisfying the assumption (H3) introduced in Section 2.2, and β is a positive number.

6.2 Numerical applications

6.2.1 *The map* $\varepsilon \mapsto \beta_\varepsilon$. We represent the function $\varepsilon \mapsto \beta_\varepsilon$, where β_ε is the bound defined in (4.23), taking $\delta = \frac{2}{\lambda}$ (as $\ln(7) \sim 1.95$).

Then $M_\delta = (1-e^{-4})^{-3/8}(1-50e^{-4}+49e^{-8})^{-1/16} \simeq 1.16$.

This map can be seen in Figure 1, for ε evolving between 0 and 1. This curve is not linear, and, as expected, non-decreasing: the smaller ε, the smaller β_ε, and it vanishes when ε vanishes.

6.2.2 *Determination of* β_ε. We seek to obtain the largest possible value β_ε, to have the largest possible window of choice for β satisfying Proposition 3. Indeed, one should note that its expression depends on the parameter δ, which appears in hypothesis (H2) and is not uniquely determined: in our case, any $\delta > \frac{\ln(7)}{\lambda}$ will do.

Consider $B_\varepsilon(\cdot,\cdot) = \beta_\varepsilon$ as a function of ε, δ and λ:

$$B_\varepsilon(\delta,\lambda) = \frac{\varepsilon}{2\sqrt{2e}(8!)^{1/8}M_\delta^{7/8}} \times\left(\delta - \frac{1}{16\lambda}\ln\left(\frac{2M_\delta^4+1}{16M_\delta^8}\left(\sqrt{1+\frac{\varepsilon^4}{32(1+2M_\delta^4)^2}}-1\right)\right)\right)^{-1/2},$$

with $M_\delta = (1-e^{-2\delta})^{-3/8}(1-50e^{-2\delta}+49e^{-4\delta})^{-1/16}$.

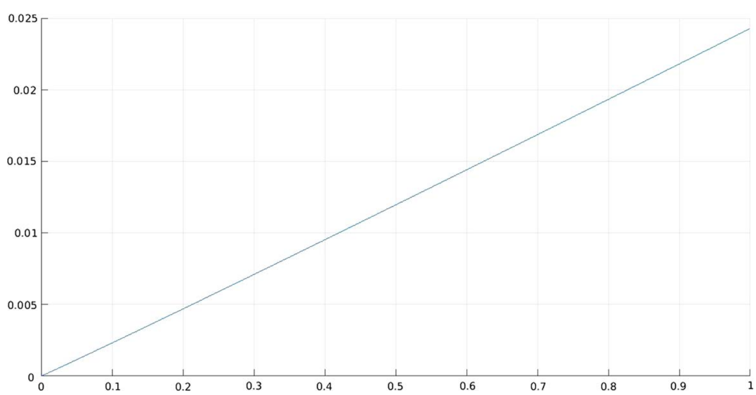

Figure 1 *The map* $\varepsilon \mapsto \bar{b}(\varepsilon)$ *between* 0 *and* 1.

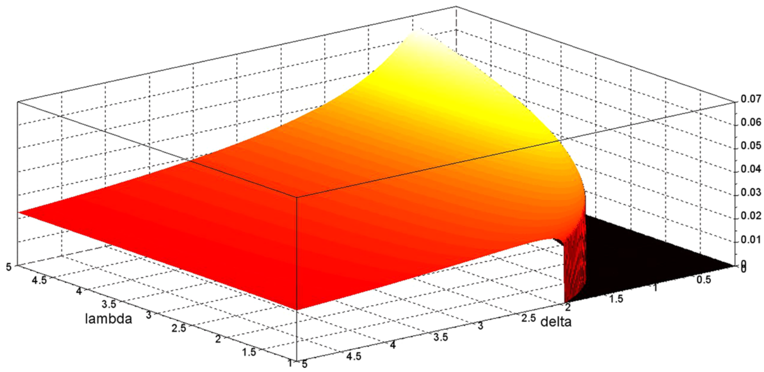

Figure 2 *The surface* $(\delta, \lambda) \mapsto B_{0.9}(\delta, \lambda)$ *for* $\delta \in [0, 10]$ *and* $\lambda \in [0, 5]$.

Set $\varepsilon = 0.9$ and $a = a_{0.9}$.

Differentiating B with respect to δ in order to find the points where the derivative vanishes, and thus the maxima of the function, looks a rather hopeless case.

We draw the map of $(\delta, \lambda) \mapsto B_{0.9}(\delta, \lambda)$ in Figure 2.

Looking closely at the relation between $\sup_\delta B_{0.9}(\delta, \lambda)$ and λ, one can conjecture that

$$\sup_{\delta > \frac{\ln(7)}{\lambda}} B_{0.9}(\delta, \lambda) \simeq 0.0291\sqrt{\lambda}.$$

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Institut de Mathématiques de Toulouse; UMR5219
Université de Toulouse; CNRS, UPS IMT
F-31062 Toulouse Cedex 9
France
E-mail: laure.pedeches@math.univ-toulouse.fr