Brazilian Journal of Probability and Statistics
2020, Vol. 34, No. 2, 345–369
https://doi.org/10.1214/18-BJPS426

Reliability estimation in a multicomponent stress-strength model for Burr XII distribution under progressive censoring

Raj Kamal Maurya[a] **and Yogesh Mani Tripathi**[a]

[a]*Indian Institute of Technology Patna*

Abstract. We consider estimation of the multicomponent stress-strength reliability under progressive Type II censoring under the assumption that stress and strength variables follow Burr XII distributions with a common shape parameter. Maximum likelihood estimates of the reliability are obtained along with asymptotic intervals when common shape parameter may be known or unknown. Bayes estimates are also derived under the squared error loss function using different approximation methods. Further, we obtain exact Bayes and uniformly minimum variance unbiased estimates of the reliability for the case common shape parameter is known. The highest posterior density intervals are also obtained. We perform Monte Carlo simulations to compare the performance of proposed estimates and present a discussion based on this study. Finally, two real data sets are analyzed for illustration purposes.

1 Introduction

At recent past, the problem of making inference upon stress-strength reliability has gained special attention among researchers. If X and Y respectively, denote strength and stress variables then the probability $R = P(X > Y)$ denotes the stress-strength reliability. Thus, a system operates successfully provided the corresponding strength remains more than the stress applied on it. Initial attempts to study estimation problems related to stress-strength reliability were made by Birnbaum et al. (1956) and Church and Harris (1970). One may also refer to Nadar, Kızılaslan and Papadopoulos (2014), Kızılaslan (2017), Najarzadegan et al. (2016), Gunasekera (2015), Basirat, Baratpour and Ahmadi (2015) and Dey, Mazucheli and Anis (2017) for recent developments on stress-strength parameter estimation under the complete sampling situations. Note that a multicomponent system having k independent and identically distributed strength variables $X_1, \dots, X_k$ and a common random stress Y imposed on each unit functions properly provided at least s $(\le k)$ out of k strength variables exceed the random stress Y. This is known as the s-out-of-$k : M$ model. Inference upon multicomponent reliability is quite useful in many practical studies of interest such as bridge structures, communication systems, military operations etc. A motor vehicle with ten tires has, in general, two additional tires assembled for the replacement purposes. Such motor vehicles can run properly on roads as long as 10-out-of-12 : M tires work well. We mention that examples with similar nature abound in many industrial experiments where efficient estimation of stress-strength reliability is required, see, for instance, Dey, Mazucheli and Anis (2017), Kızılaslan (2017), Nadar, Kızılaslan and Papadopoulos (2014), among others. The problem of estimating reliability for various lifetime distributions has been studied by several researchers by assuming single component stress-strength systems, see, for example, Enis and Geisser (1971), Downtown (1973), Awad and Gharraf (1986). In multicomponent situation, Bhattacharyya and Johnson (1974) initially made inference upon the system reliability assuming exponential distributions. Authors obtained maximum likelihood and uniformly minimum variance

Key words and phrases. Bayes estimate, maximum likelihood estimate, multicomponent reliability, progressive censoring, uniformly minimum variance unbiased estimator.

Received March 2018; accepted November 2018.

unbiased estimates and compared their performance numerically. The related Bayesian inference for this problem is discussed in Draper and Guttman (1978). A rich literature exists on the estimation problems related to multicomponent stress-strength reliability. However, such problems have not been investigated much in presence of some censoring. Kohansal (2017) discussed that deriving inference for the multicomponent reliability in presence of some prespecified censoring may not be unrealistic in many practical studies of life testing experiments. Study of an excessive drought scenario in a particular region is an example of such situation. If the water capacity of a reservoir in a particular area in May, at least one year out of the next 5 years, is more than the amount of water achieved in November of the previous year, then claim can be that there will be no excessive drought afterwards. In many such studies, it is not unreasonable to assume that observed data from both the populations may be censored in nature. In many life testing experiments, censoring plays an important role in making inference on various unknown quantities such as the system reliability. Kohansal (2017) considered estimation of multicomponent stress strength reliability for Kumarswamy distribution under progressive censoring. The author applied different methods of estimation to estimate the reliability. A numerical study is performed to compare the performance of proposed estimates and illustrative discussions are presented based on this study.

In literature, various types of censoring have been introduced in which Type I and Type II are the most widely used censoring schemes. In Type I censoring, a test stops at a prespecified time point and observations are not recorded after this time. On the other hand in Type II censoring, a test stops when a prefixed number of observations has been recorded. Note that in these two censoring schemes in between removal of live test units from the experiment is not allowed. Progressive Type II censoring is kind of a generalization of Type II censoring in which in between removal of live units is allowed. In this censoring, a total of N test units is subjected to some life testing experiment and each item contains K components. A progressive Type II censored sample of size n can be observed as follows. After the first failure, R_1 number of live units from the remaining $N-1$ surviving units are removed from the experiment at random and also S_1 number of components are removed from the remaining $K-1$ components. At the second failure, R_2 number of live units from the remaining $N-R_1-2$ units are removed from the experiment again at random and at the same time S_2 number of components from the remaining $K-S_1-2$ components are removed as well. The test stops when nth failure time is observed along with k prefixed components and remaining surviving units $R_n = N-n-R_1-\cdots-R_{n-1}$ are removed along with $S_k = K-k-S_1-\cdots-S_{k-1}$ components. We refer to Balakrishnan and Aggarwala (2000) for further details and applications of progressive censoring in life testing experiments.

In this paper, we obtain various inference upon multicomponent stress-strength reliability assuming Burr XII distributions under progressive Type II censoring. In Section 2, we briefly introduce the Burr XII distribution. The maximum likelihood estimation and asymptotic confidence interval of reliability system are discussed in Section 3 under progressive Type II censoring when the common shape parameter is unknown. The corresponding Bayes estimates and HPD intervals are also obtained in this section. Further in Section 4, the uniformly minimum variance unbiased estimator and Bayes estimator of the stress-strength reliability are derived when the common shape parameter is known. In Section 5, we perform Monte Carlo simulations to compare the performance of studied estimators. Two real data sets are analyzed in Section 6 for illustration purposes. Finally, we present some concluding remarks in Section 7.

2 Model description

Burr (1942) studied different forms of cumulative distribution functions which are useful in various reliability and life testing experiments. The Burr XII distribution is one of the most widely applied model in reliability analysis and related areas of studies including financial, mortality and industrial experiments, forest ecology, income distributions etc. One may refer to Rastogi and Tripathi (2012), Zimmer, Keats and Wang (1998), Wingo (1993), Aslam, Azam and Jun (2016), Maurya et al. (2017), Raqab and Kundu (2005) and Surles and Padgett (1998) for some further applications of this distribution in many practical studies of interest. Probability density function of this distribution is given by

$$f(x;\alpha,\beta)=\alpha\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(\alpha+1)},\quad x>0,\alpha>0,\beta>0, \tag{1}$$

and the distribution function is

$$F_X(x;\alpha,\beta)=1-\left(1+x^{\beta}\right)^{-\alpha},\quad x>0,\alpha>0,\beta>0, \tag{2}$$

where α and β both are shape parameters. We denote this distribution by Burr(α,β). Different shapes of density and hazard rate functions can be obtained for varying values of parameters. In fact, the hazard function is either monotonically decreasing or remains unimodal depending on parameter values. The stochastic behavior of Burr XII distribution is quite similar to some known models like lognormal, gamma, and Weibull. Burr XII model is also widely used for deriving inference under acceptance sapling plans and quality controls. Several authors have studied this distribution assuming different sampling situations. Papadopoulos (1978) obtained Bayes estimates of unknown parameters and reliability characteristics under the squared error loss function (see also, Moore and Papadopoulos (2000)). A rich collection of inferential results exists for the Burr XII model assuming single component system. Rao, Aslam and Kundu (2015) analyzed the Burr XII model for multicomponent system under the complete sample case. We consider estimation of the reliability when observed data are progressive Type II censored. In this paper, we obtain various estimates of multicomponent stress-strength reliability $R_{s,k}=P[\text{at least } s \text{ of } (X_1,\dots,X_k) \text{ exceeds } Y]$ under the considered censoring scheme. Suppose that $(X_{i1},\dots,X_{ik})$, $i=1,\dots,n$, denotes a progressively censored sample taken from the Burr(α_1,β) distribution using the censoring scheme $(K,k,R_1,\dots,R_k)$ and also $(Y_1,\dots,Y_n)$ be another such sample drawn from the Burr(α_2,β) distribution using the censoring scheme $(N,n,S_1,\dots,S_n)$. The reliability in a multicomponent stress-strength model is then obtained as

$$\begin{aligned} R_{s,k}&=P\big(\text{at least } s \text{ of the } (X_1,\dots,X_k) \text{ exceed } Y\big)\\ &=\sum_{i=s}^{k}\binom{k}{i}\int_{-\infty}^{\infty}[1-F_X(y)]^i[F_X(y)]^{k-i}\,dF_Y(y), \end{aligned} \tag{3}$$

where $X_1,X_2,\dots,X_k$ are independent and identically distributed random variables with distribution function $F_X(\cdot)$. We next consider estimation of $R_{s,k}$ when the common shape parameter β is unknown.

3 Estimation of $R_{s,k}$ when β is unknown

3.1 Maximum likelihood estimation

In this section, we obtain maximum likelihood estimator of the stress-strength reliability $R_{s,k}$ under progressive censoring. Let $X\sim$ Burr(α_1,β) and $Y\sim$ Burr(α_2,β) be independently distributed random variable with unknown shape parameters α_1, α_2 and a common shape parameter β. Then using equations (1) and (2) in (3), the reliability of multicomponent stress-strength model for the Burr XII distribution is obtained as

$$R_{s,k}=\sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}\frac{(-1)^j\alpha_2}{\alpha_1(i+j)+\alpha_2}. \tag{4}$$

We first obtain MLEs of parameters α_1, α_2 and β using progressive Type II censored samples. Suppose that N systems are put on a test each with K components. Then n systems each with k components are observed. Thus, strength and stress samples are respectively, observed as

$$\begin{pmatrix} X_{11} & X_{12} & \dots & X_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \dots & X_{nk} \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}.$$

The likelihood function α_1, α_2 and β is now given by

$$L(\alpha_1, \alpha_2, \beta \mid \text{data}) = c_1 \prod_{i=1}^{n} \left(c_2 \prod_{j=1}^{k} f(x_{ij})[1 - F(x_{ij})]^{R_j} \right) f(y_i)[1 - F(y_i)]^{S_i}, \tag{5}$$

where

$$c_1 = N(N - S_1 - 1) \cdots (N - S_1 - \cdots - S_{n-1} - n + 1),$$
$$c_2 = K(K - R_1 - 1) \cdots (K - R_1 - \cdots - K_{k-1} - k + 1).$$

The corresponding log-likelihood function is

$$\begin{aligned} l(\alpha_1, \alpha_2, \beta \mid \text{data}) = {} & nk \log \alpha_1 + n \log \alpha_2 + n(k+1) \log \beta \\ & + (\beta - 1) \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\sum_{i=1}^{n} \log y_i + \log x_{ij} \right) \\ & - (\alpha_2 + 1) \sum_{i=1}^{n} \log(1 + y_i^{\beta}) - (\alpha_1 + 1) \sum_{i=1}^{n} \sum_{j=1}^{k} \log(1 + x_{ij}^{\beta}) \\ & - \alpha_2 \sum_{i=1}^{n} S_i \log(1 + y_i^{\beta}) - \alpha_1 \sum_{i=1}^{n} \sum_{j=1}^{k} R_j \log(1 + x_{ij}^{\beta}) + L, \end{aligned} \tag{6}$$

where L is the constant term. By partially differentiating (6) with respect to α_1, α_2 and β, we obtain likelihood equations as:

$$\frac{\partial l}{\partial \alpha_1} = \frac{nk}{\alpha_1} - \sum_{i=1}^{n} \sum_{j=1}^{k} (R_j + 1) \log(1 + x_{ij}^{\beta}) = 0, \tag{7}$$

$$\frac{\partial l}{\partial \alpha_2} = \frac{n}{\alpha_2} - \sum_{i=1}^{n} (S_i + 1) \log(1 + y_i^{\beta}) = 0, \tag{8}$$

$$\begin{aligned} \frac{\partial l}{\partial \beta} = {} & \frac{n(k+1)}{\beta} + \left(\sum_{i=1}^{n} \log y_i + \sum_{i=1}^{n} \sum_{j=1}^{k} \log x_{ij} \right) - \sum_{i=1}^{n} \frac{(\alpha_2(S_i + 1) + 1) y_i^{\beta} \log y_i}{(1 + y_i^{\beta})} \\ & - \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{(\alpha_1(R_j + 1) + 1) x_{ij}^{\beta} \log x_{ij}}{(1 + x_{ij}^{\beta})} = 0. \end{aligned} \tag{9}$$

We have from (7) and (8), $\hat{\alpha}_1(\beta) = \frac{nk}{\sum_{i=1}^{n} (R_j + 1) \log(1 + x_{ij}^{\beta})}$ and $\hat{\alpha}_2(\beta) = \frac{n}{\sum_{i=1}^{n} (S_i + 1) \log(1 + y_i^{\beta})}$. Further for $n > 2$ and $k > 2$, the MLE of β is the solution to the equation $H(\beta) = 0$, where

$$H(\beta) = \frac{n(k+1)}{\beta} + \sum_{i=1}^{n} \log(y_i) - n \frac{\sum_{i=1}^{n} (S_i + 1) \frac{y_i^{\beta} \log y_i}{(1 + y_i^{\beta})}}{\sum_{i=1}^{n} (S_i + 1) \log(1 + y_i^{\beta})} - \sum_{i=1}^{n} \frac{y_i^{\beta} \log y_i}{(1 + y_i^{\beta})}$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{k}\log(x_{ij}) - nk\frac{\sum_{i=1}^{n}\sum_{j=1}^{k}(R_j+1)\frac{x_{ij}^{\beta}\log x_{ij}}{(1+x_{ij}^{\beta})}}{\sum_{i=1}^{n}\sum_{j=1}^{k}(R_j+1)\log(1+x_{ij}^{\beta})} - \sum_{i=1}^{n}\sum_{j=1}^{k}\frac{x_{ij}^{\beta}\log x_{ij}}{(1+x_{ij}^{\beta})}.$$

From Soliman (2005) and Maurya et al. (2017), we further have

$$\lim_{\beta\to 0} H(\beta) = +\infty, \qquad \lim_{\beta\to+\infty} H(\beta) < 0,$$

and

$$H'(\beta) < 0,$$

for all $\beta > 0$. Now following Maurya et al. (2017), we see that maximum likelihood estimator of β satisfying $H(\beta) = 0$ exists and is unique as well. Thus, respective maximum likelihood estimates $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\beta}$ of α_1, α_2 and β can be computed from above equations using some numerical method. The corresponding maximum likelihood estimate $\hat{R}_{s,k}$ of $R_{s,k}$ is now obtained as, by using the invariant property of MLE,

$$\hat{R}_{s,k} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}\frac{(-1)^j\hat{\alpha}_2}{\hat{\alpha}_1(i+j)+\hat{\alpha}_2}.$$

3.1.1 *Asymptotic confidence interval.* In this section, asymptotic interval of $\hat{R}_{s,k}$ is obtained using the asymptotic distribution of MLE $\hat{\theta} = (\alpha_1, \alpha_2, \beta)$. The expected Fisher information matrix is given by $M(\theta) = E(I(\theta))$, where $I(\theta) = [I_{ij}] = [-\frac{\partial^2 l}{\partial\theta_i\,\partial\theta_j}]$ with elements being

$$I_{11} = \frac{nk}{\alpha_1^2}, \qquad I_{22} = \frac{n}{\alpha_2^2}, \qquad I_{12} = 0 = I_{21},$$

$$I_{13} = \sum_{i=1}^{k}\sum_{j=1}^{n}\frac{(R_j+1)x_{ij}^{\beta}\log x_{ij}}{(1+x_{ij}^{\beta})} = I_{31}, \qquad I_{23} = \sum_{i=1}^{n}\frac{(S_i+1)y_i^{\beta}\log y_i}{(1+y_i^{\beta})} = I_{32},$$

$$I_{33} = \frac{n(k+1)}{\beta^2} + \sum_{i=1}^{n}\frac{(\alpha_2(S_i+1)+1)y_i^{\beta}(\log y_i)^2}{(1+y_i^{\beta})^2} + \sum_{i=1}^{n}\sum_{j=1}^{k}\frac{(\alpha_1(R_j+1)+1)x_{ij}^{\beta}(\log x_{ij})^2}{(1+x_{ij}^{\beta})^2}.$$

Observe that MLE of $\hat{R}_{s,k}$ is asymptotically normal with mean $R_{s,k}$ and the corresponding variance is given by

$$\sigma^2_{R_{s,k}} = \left(\frac{\partial R_{s,k}}{\partial\alpha_1}\right)^2 M_{11}^{-1} + \left(\frac{\partial R_{s,k}}{\partial\alpha_2}\right)^2 M_{22}^{-1} + 2\left(\frac{\partial R_{s,k}}{\partial\alpha_1}\right)\left(\frac{\partial R_{s,k}}{\partial\alpha_2}\right)M_{12}^{-1},$$

where

$$\frac{\partial R_{s,k}}{\partial\alpha_1} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^{j+1}\frac{\alpha_2(i+j)}{(\alpha_1(i+j)+\alpha_2)^2},$$

$$\frac{\partial R_{s,k}}{\partial\alpha_2} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^{j}\frac{\alpha_1(i+j)}{(\alpha_1(i+j)+\alpha_2)^2}.$$

Thus $100(1-\xi)\%$ confidence interval of $R_{s,k}$ is of the form $(\hat{R}_{s,k} \pm q_{\xi/2}\hat{\sigma}_{R_{s,k}})$ where $q_{\xi/2}$ denotes the upper $(\xi/2)$th quantile of the standard normal distribution.

3.2 Bayesian estimation

In this section, we obtain point and interval estimates of $R_{s,k}$ using the Bayesian approach. We assume that parameters α_1, α_2 and β are a priori distributed as gamma $G(a_i, b_i)$, $i = 1, 2, 3$, distributions, respectively. The corresponding joint prior density is then given by

$$g(\alpha_1, \alpha_2, \beta) \propto \alpha_1^{a_1-1} e^{-b_1\alpha_1} \alpha_2^{a_2-1} e^{-b_2\alpha_2} \beta^{a_3-1} e^{-b_3\beta}, \quad a_i, b_i > 0; i = 1, 2, 3.$$

Subsequently, the joint posterior distribution of α_1, α_2 and β given the observed data is obtained as

$$\pi(\alpha_1, \alpha_2, \beta \mid \text{data}) = \frac{L(\alpha_1, \alpha_2, \beta \mid \text{data}) g(\alpha_1, \alpha_2, \beta)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\alpha_1, \alpha_2, \beta \mid \text{data}) g(\alpha_1, \alpha_2, \beta)\, d\alpha_1\, d\alpha_2\, d\beta}. \tag{10}$$

We compute Bayes estimate of $R_{s,k}$ under the square error loss function. Note that the desired estimate turns out to be the posterior mean of $R_{s,k}$. We observe that this posterior mean involves the ratio of two integrals which cannot be simplified analytically due to intractable nature of the corresponding posterior distribution. However, in such situations one may apply the Lindley approximation method (see, Lindley (1980)), Tierney and Kadane (TK) method (see, Tierney and Kadane (1986)) and Metropolis–Hastings (MH) algorithm (see, Metropolis et al. (1953) and Hastings (1970)) to compute the Bayes estimate $\tilde{R}^B_{s,k}$ of $R_{s,k}$.

3.2.1 *Lindley method.* In this section, we obtain an explicit expression for the Bayes estimate of multicomponent reliability using the Lindley method. In this approach, we obtain Taylor series expansion of the function involved in (10) about the maximum likelihood estimator (see, Sinha (1986)). Using this technique, the approximate Bayes estimate of $\tilde{R}_{s,k}$ under the squared error loss function is obtained as

$$\begin{aligned}
\tilde{R}^{\text{LA}}_{s,k} &= u + (u_1 p_1 + u_2 p_2 + u_3 p_3 + p_4 + p_5) \\
&\quad + 0.5\big[(\sigma_{11} l_{111} + 2\sigma_{12} l_{121} + 2\sigma_{13} l_{131} + 2\sigma_{23} l_{231} \\
&\quad + \sigma_{22} l_{221} + \sigma_{33} l_{331})(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) \\
&\quad + (\sigma_{11} l_{112} + 2\sigma_{12} l_{122} + 2\sigma_{13} l_{132} + 2\sigma_{23} l_{232} \\
&\quad + \sigma_{22} l_{222} + \sigma_{33} l_{332})(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) \\
&\quad + (\sigma_{11} l_{113} + 2\sigma_{12} l_{123} + 2\sigma_{13} l_{133} + 2\sigma_{23} l_{233} \\
&\quad + \sigma_{22} l_{223} + \sigma_{33} l_{333})(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})\big],
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
p_i &= \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \quad i = 1, 2, 3, \\
p_4 &= u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23}, \\
p_5 &= 0.5(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}).
\end{aligned}$$

We also have $\rho_1 = \frac{a_1-1}{\alpha_1} - b_1$, $\rho_2 = \frac{a_2-1}{\alpha_2} - b_2$, $\rho_3 = \frac{a_3-1}{\beta} - b_3$ and σ_{ik} denotes element of the matrix $[-l_{ik}]^{-1}$, $i, k = 1, 2, 3$. Note that $u = \hat{R}_{s,k}$ and other expressions in (11) are given in Appendix A. Sometimes if the number of parameters are relatively large then finding higher order derivatives are computationally intensive. In such situations, we can use the TK method to determine the given posterior expectation.

3.2.2 *Tierney and Kadane method.* Here we obtain Bayes estimator of $R_{s,k}$ using the Tierney and Kadane (1986) method. Note that Bayes estimator is posterior expectation of the stress-strength reliability. Let $I(x, y)$ denotes the posterior expectation of a parametric function $u(\alpha_1, \alpha_2, \beta)$ with respect to the posterior distribution $\pi(\text{data} \mid \alpha_1, \alpha_2, \beta)$ such that

$$I = \frac{\int_0^\infty \int_0^\infty \int_0^\infty u(\alpha_1, \alpha_2, \beta) e^{l(\text{data}|\alpha_1,\alpha_2,\beta)+\rho(\alpha_1,\alpha_2,\beta)}\, d\alpha_1\, d\alpha_2\, d\beta}{\int_0^\infty \int_0^\infty \int_0^\infty e^{l(\text{data}|\alpha_1,\alpha_2,\beta)+\rho(\alpha_1,\alpha_2,\beta)}\, d\alpha_1\, d\alpha_2\, d\beta}. \tag{12}$$

We mention that $l(\text{data} \mid \alpha_1, \alpha_2, \beta)$ denotes the log-likelihood function and $\rho(\alpha_1, \alpha_2, \beta) = \log g(\alpha_1, \alpha_2, \beta)$. Let $(\hat{\alpha}_{1\delta}, \hat{\alpha}_{2\delta}, \hat{\beta}_\delta)$ and $(\hat{\alpha}_{1\delta^\star}, \hat{\alpha}_{2\delta^\star}, \hat{\beta}_{\delta^\star})$ respectively, maximize functions

$$\delta(\alpha_1, \alpha_2, \beta) = \frac{l(\alpha_1, \alpha_2, \beta) + \rho(\alpha_1, \alpha_2, \beta)}{n} \quad \text{and}$$

$$\delta_u^\star(\alpha_1, \alpha_2, \beta) = \delta(\alpha_1, \alpha_2, \beta) + \frac{\log u(\alpha_1, \alpha_2, \beta)}{n}.$$

Then using the TK method we express the estimate I as

$$I = \sqrt{\frac{|\Sigma_u^\star|}{|\Sigma|}} \exp\big[n\{\delta_u^\star(\hat{\alpha}_{1\delta^\star}, \hat{\alpha}_{2\delta^\star}, \hat{\beta}_{\delta^\star}) - \delta(\hat{\alpha}_{1\delta}, \hat{\alpha}_{2\delta}, \hat{\beta}_\delta)\}\big], \tag{13}$$

where $|\Sigma|$ and $|\Sigma_u^\star|$ are determinant of negatives of inverse hessian of $\delta(\alpha_1, \alpha_2, \beta)$ and $\delta_u^\star(\alpha_1, \alpha_2, \beta)$ computed at $(\hat{\alpha}_{1\delta}, \hat{\alpha}_{2\delta}, \hat{\beta}_\delta)$ and $(\hat{\alpha}_{1\delta^\star}, \hat{\alpha}_{2\delta^\star}, \hat{\beta}_{\delta^\star})$ respectively. The related computations of $|\Sigma|$ and $|\Sigma_u^\star|$ are provided in Appendix B. Bayes estimates of multicomponent reliability can be evaluated easily using both the approaches as discussed above. However, these methods are not useful in finding credible intervals of the reliability. So next we discuss Metropolis–Hastings (MH) algorithm which can be used to obtain both Bayes estimates and credible intervals.

3.2.3 *Metropolis–Hastings algorithm.* We use the Metropolis–Hastings algorithm (see, Metropolis et al. (1953) and Hastings (1970)) to compute the Bayes estimate and credible intervals of $R_{s,k}$. It is seen that marginal posterior distributions of α_1 and α_2 given β and observed data turn out to be gamma distributions respectively. However, the corresponding marginal distribution of β appears in an unknown form. Thus, we have

$$\alpha_1 \mid \beta, \text{data} \sim G_{\alpha_1}\Bigg(nk + a_1, b_1 + \sum_{i=1}^{n}\sum_{j=1}^{k}(R_j + 1)\log\big(1 + x_{ij}^\beta\big)\Bigg),$$

$$\alpha_2 \mid \beta, \text{data} \sim G_{\alpha_2}\Bigg(n + a_2, b_2 + \sum_{i=1}^{n}(S_i + 1)\log\big(1 + x_{ij}^\beta\big)\Bigg),$$

and

$$\pi(\beta \mid \alpha_1, \alpha_2, \text{data}) \propto \beta^{n(k+1)+a_3-1} e^{-\beta b_3}\Bigg(\prod_{i=1}^{n} y_i^{\beta-1}\big(1 + y_i^\beta\big)^{-1}\Bigg)\Bigg(\prod_{i=1}^{n}\prod_{j=1}^{k} x_{ij}^{\beta-1}\big(1 + x_{ij}^\beta\big)^{-1}\Bigg).$$

We can simulate α_1 and α_2 from the given gamma posterior distributions, however β cannot be simulated directly. To simulate samples from the posterior distribution of β, we use the MH algorithm with a normal proposal distribution. The following steps are required to obtain samples and desired estimates.

Step 1: Choose an initial guess $(\alpha_{1_0}, \alpha_{2_0}, \beta_0)$ of $(\alpha_1, \alpha_2, \beta)$.

Step 2: Set $t = 1$.

Step 3: Simulate $\beta_{(t)}$ from $\pi(\beta \mid \alpha_{1(t-1)}, \alpha_{2(t-1)}, \text{data})$.

Step 4: Simulate $\alpha_{1(t)}$ from $G_{\alpha_1}(nk + a_1, b_1 + \sum_{i=1}^{n}\sum_{j=1}^{k}(R_j + 1)\log(1 + x_{ij}^{\beta}))$.

Step 5: Simulate $\alpha_{2(t)}$ from $G_{\alpha_2}(n + a_2, b_2 + \sum_{i=1}^{n}(S_i + 1)\log(1 + y_i^{\beta}))$.

Step 6: Compute $R_{(t)s,k} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^j \frac{\alpha_{2(t)}}{(\alpha_{1(t)}(i+j)+\alpha_{2(t)})}$.

Step 7: Set $t = t + 1$.

Step 8: Repeat steps 3–7, M times.

Bayes estimate of stress-strength reliability under the square error loss is now obtained as

$$\tilde{R}_{s,k}^{\mathrm{MH}} = \frac{1}{M}\sum_{i=1}^{M} R_{(t)s,k}.$$

We have employed the method of Chen and Shao (1999) to construct the $100(1-\xi)\%$ HPD interval of $R_{s,k}$.

4 Estimation of $R_{s,k}$ when β is known

In this section, we obtain maximum likelihood and Bayes estimators of the system reliability under the assumption that common shape parameter β is known. We also derive the exact expression for Bayes estimate and obtain uniformly minimum variance unbiased estimator as well. For comparison purposes, approximate Bayes estimates are also discussed.

4.1 Maximum likelihood estimator of $R_{s,k}$

Here we obtain maximum likelihood estimator of $R_{s,k}$ based on progressive Type II censored data. The corresponding likelihood function is defined in equation (5). The log-likelihood function of α_1 and α_2 is given by

$$\begin{aligned} l(\alpha_1, \alpha_2 \mid \beta, \text{data}) = {} & nk\log\alpha_1 + n\log\alpha_2 - (\alpha_1 + 1)\sum_{i=1}^{n}\sum_{j=1}^{k}\log\big(1 + x_{ij}^{\beta}\big) \\ & - \alpha_2\sum_{i=1}^{n} S_i\log\big(1 + y_i^{\beta}\big) - \alpha_1\sum_{i=1}^{n}\sum_{j=1}^{k} R_j\log\big(1 + x_{ij}^{\beta}\big) + C, \end{aligned} \tag{14}$$

where C is the constant term. By partially differentiating (14) with respect to α_1 and α_2, the likelihood equations are obtained as

$$\frac{\partial l}{\partial \alpha_1} = \frac{nk}{\alpha_1} - \sum_{i=1}^{n}\sum_{j=1}^{k}(R_j + 1)\log\big(1 + x_{ij}^{\beta}\big) = 0,$$

$$\frac{\partial l}{\partial \alpha_2} = \frac{n}{\alpha_2} - \sum_{i=1}^{n}(S_i + 1)\log\big(1 + y_i^{\beta}\big) = 0.$$

Subsequently, MLEs of unknown parameters turn out to be

$$\hat{\alpha}_1 = \frac{nk}{\sum_{i=1}^{n}\sum_{j=1}^{k}(R_j + 1)\log(1 + x_{ij}^{\beta})}, \qquad \hat{\alpha}_2 = \frac{n}{\sum_{i=1}^{n}(S_i + 1)\log(1 + y_i^{\beta})}.$$

In sequel, MLE of the system reliability $R_{s,k}$ is obtained as

$$\hat{R}_{s,k} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^j\frac{\hat{\alpha}_2}{(\hat{\alpha}_1(i+j) + \hat{\alpha}_2)}.$$

The asymptotic distribution of $\hat{R}_{s,k}$ is normal with mean $R_{s,k}$ and variance

$$\sigma^2_{R_{s,k}} = \left(\frac{\partial R_{s,k}}{\partial \alpha_1}\right)^2 \frac{\alpha_1^2}{nk} + \left(\frac{\partial R_{s,k}}{\partial \alpha_2}\right)^2 \frac{\alpha_2^2}{n},$$

where $(\frac{\partial R_{s,k}}{\partial \alpha_1})^2$ and $(\frac{\partial R_{s,k}}{\partial \alpha_2})^2$ are computed in Section (3.1). Thus $100(1-\xi)\%$ confidence interval of $R_{s,k}$ is obtained as $(\hat{R}_{s,k} \pm q_{\xi/2}\hat{\sigma}_{R_{s,k}})$ where $q_{\xi/2}$ is the upper $(\xi/2)$th quantile of the standard normal distribution. In next section, we derive the uniformly minimum variance unbiased estimator (UMVUE) of $R_{s,k}$.

4.1.1 *UMVUE of $R_{s,k}$.* Here uniformly minimum variance unbiased estimator of $R_{s,k}$ is developed using progressive Type II censored data when the common parameter β is known. The corresponding likelihood function is obtained in equation (5) and the corresponding log-likelihood is similar to the function as given in equation (14) with β being known. We observe from equation (14) that

$$U = \sum_{i=1}^{n}(S_i+1)\log\left(1+y_i^{\beta}\right) \quad \text{and} \quad V = \sum_{i=1}^{n}\sum_{j=1}^{n} R_j \log\left(1+x_{ij}^{\beta}\right),$$

are the complete sufficient statistics for α_1 and α_2 when β is known. Further $Y_i^* = \log(1 + y_i^{\beta})$, $i = 1, 2, \ldots, n$, denotes a progressive Type II censored sample from the exponential distribution with mean α_2^{-1}. Next, we consider the following transformation

$$\begin{aligned}
W_1 &= NY_1^*,\\
W_2 &= (N - S_1 - 1)\left(Y_2^* - Y_1^*\right),\\
&\vdots\\
W_n &= (N - S_1 - \cdots - S_{n-1} - n + 1)\left(Y_n^* - Y_{n-1}^*\right),
\end{aligned}$$

and observe that $W_1, W_2, \ldots, W_n$ are independent and identically distributed as exponential distribution with mean α_2^{-1}. Also note that $U = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n}(S_i+1)Y_i^*$ has a gamma distribution with density function given as

$$f_U(u) = \frac{\alpha_2^n u^{n-1} e^{-\alpha_2 u}}{\Gamma n}, \quad u > 0.$$

Further considering $X_{ij}^* = -\log(1 + X_{ij}^{\beta})$, $i = 1, \ldots, n$, $j = 1, \ldots, k$ and $V = \sum_{i=1}^{n}\sum_{j=1}^{k}(R_j+1)X_{ij}^*$, we obtain the conditional distribution of Y_1^* given $U = u$ as

$$f_{Y_1*|U=u}(y) = \frac{N(n-1)(u-Ny)^{n-2}}{u^{n-1}}, \quad 0 < y < u/N,$$

and the conditional distribution of X_{11}^* given $V = v$ is given by

$$f_{X_{11}^*|V=v}(x) = \frac{K(nk-1)(v-Kx)^{nk-2}}{v^{nk-1}}, \quad 0 < x < v/K.$$

Theorem 4.1. *The UMVUE $\hat{\phi}(\alpha_1, \alpha_2)$ of $\phi(\alpha_1, \alpha_2) = \frac{\alpha_2}{\alpha_2 + (i+j)\alpha_1}$, on the basis of statistics U and V, is obtained as*

$$\hat{R}_{s,k}^{B} = \begin{cases} 1 - \displaystyle\sum_{l=0}^{n-1}(-1)^l \left(\frac{v}{u(i+j)}\right)^l \frac{\binom{n-1}{l}}{\binom{nk+l-1}{l}}, & \text{if } v < u(i+j),\\[2ex] \displaystyle\sum_{l=0}^{nk-1}(-1)^l \left(\frac{u(i+j)}{v}\right)^l \frac{\binom{nk-1}{l}}{\binom{n+l-1}{l}}, & \text{if } v > u(i+j). \end{cases}$$

Proof. Recall that Y_1^* and X_{11}^* are exponentially distributed with mean $(N\alpha_2)^{-1}$ and $(K\alpha_1)^{-1}$ respectively. Thus,

$$\psi(X_{11}^*, Y_1^*) = \begin{cases} 1, & \text{if } KX_{11}^* > (i+j)NY_1^*, \\ 0, & \text{if } KX_{11}^* < (i+j)NY_1^*, \end{cases} \tag{15}$$

is an unbiased estimator of $\phi(\alpha_1, \alpha_2)$ and so

$$\hat{\phi}(\alpha_1, \alpha_2) = E\big[\psi(X_{11}^*, Y_1^*) \mid U = u, V = v\big] = \int\int_A f_{X_{11}^*|V=v}(x) f_{Y_1{}^*|U=u}(y)\, dx\, dy \tag{16}$$

where $A = \{(x, y) : 0 < x < v/K, 0 < y < u/N, Ny(i + j) < kx\}$. For $v < u(i + j)$, we use the Lemma 1 of Basirat, Baratpour and Ahmadi (2015) and then integral (16) reduces to:

$$\begin{aligned}
\hat{\phi}(\alpha_1, \alpha_2) &= \frac{N(n-1)K(nk-1)}{u^{n-1}v^{nk-1}} \int_0^{v/k} \int_0^{Kx/(N(i+j))} (u - Ny)^{n-2}(v - Kx)^{nk-2}\, dy\, dx \\
&= 1 - \frac{K(nk-1)}{u^{n-1}v^{nk-1}} \int_0^{v/K} (v - Kx)^{nk-2}\left(u - \frac{Kx}{i+j}\right)^{n-1} dx \quad \left\{\text{put} : \frac{Kx}{v} = t\right\} \\
&= 1 - (nk-1)\int_0^1 (1-t)^{nk-2}\left(1 - \frac{vt}{u(i+j)}\right)^{n-1} dt \\
&= 1 - \sum_{l=0}^{n-1} (-1)^l \left(\frac{v}{u(i+j)}\right)^l \frac{\binom{n-1}{l}}{\binom{nk+l-1}{l}}.
\end{aligned}$$

Proceeding similarly, we have $\hat{\phi}(\alpha_1, \alpha_2) = \sum_{l=0}^{nk-1} (-1)^l (\frac{u(i+j)}{v})^l \frac{\binom{nk-1}{l}}{\binom{n+l-1}{l}}$ for $v > u(i + j)$. Finally, UMVUE of $R_{s,k}$ is obtained as

$$\tilde{R}_{s,k}^{\text{UMV}} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i}\binom{k-i}{j} (-1)^j \hat{\phi}(\alpha_1, \alpha_2).$$

□

4.2 Bayesian estimation of $R_{s,k}$

In this section, we compute Bayes estimate of the reliability using progressive Type II censored data when the parameter β is known. We assume that α_1 and α_2 are statistically independent and a priori distributed as gamma $G(a_1, b_1)$ and $G(a_2, b_2)$ distributions, respectively. The joint posterior density of α_1 and α_2 given the observed data is obtained as

$$\pi(\alpha_1, \alpha_2 \mid \beta, \text{data}) = \frac{(b_1 + V)^{nk+a_1}(b_2 + U)^{n+a_2}}{\Gamma(nk + a_1)\Gamma(n + a_2)} \alpha_1{}^{nk+a_1-1}\alpha_2{}^{n+a_2-1} e^{-\alpha_1(b_1+V)-\alpha_2(b_2+U)}.$$

Then Bayes estimate of the reliability under the square error loss function turns out to be

$$\begin{aligned}
\hat{R}_{s,k}^B &= \int_0^\infty \int_0^\infty R_{s,k}\pi(\alpha_1, \alpha_2 \mid \beta, \text{data})\, d\alpha_1\, d\alpha_2 \\
&= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i}\binom{k-i}{j} (-1)^j \int_0^\infty \int_0^\infty \frac{\alpha_2}{\alpha_2 + \alpha_1(i+j)} \pi(\alpha_1, \alpha_2 \mid \beta, \text{data})\, d\alpha_1\, d\alpha_2.
\end{aligned}$$

This can be rewritten as (see, Nadar and Kızılaslan (2016))

$$\hat{R}^{B}_{s,k} = \begin{cases} \displaystyle\sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^j \frac{(1-z)^{n+a_2}(n+a_2)}{u} {}_2F_1(w, n+a_2+1; w+1, z), \\ \quad \text{if } |z|<1, \\ \displaystyle\sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^j \frac{n+a_2}{(1-z)^{nk+a_1}w} {}_2F_1\Big(w, nk+a_1+1; w+1, \frac{z}{1-z}\Big), \\ \quad z<-1, \end{cases}$$

where $w = nk+n+a_1+a_2$ and $z = 1-\frac{(U+b_2)(i+j)}{V+b_1}$. We also note that

$$ {}_2F_1(\alpha_1, \alpha_2, \beta, z) = \frac{1}{B(\alpha_2, \beta-\alpha_2)}\int_0^1 t^{\alpha_2-1}(1-t)^{\beta-\alpha_2-1}(1-tz)^{-\alpha_1}\,dt, \quad |z|<1,$$

is the hypergeometric series.

Thus in this case closed form expression of Bayes estimate for the reliability is obtained. In general, it is relatively difficult to obtain closed form expression for such estimate. In such situations, we can use Lindley method, TK method and importance sampling procedure. For comparison purposes, we also obtain Bayes estimate of $R_{s,k}$ using these approximation procedures.

4.2.1 *Lindley method.* In case the common shape parameter β is known, the Bayes estimate of $R_{s,k}$ using the Lindley method leads to the following expression

$$I = u + (u_1p_1 + u_2p_2 + p_3) + 0.5\big[P(u_1\sigma_{11} + u_2\sigma_{12}) + Q(u_1\sigma_{21} + u_2\sigma_{22})\big], \tag{17}$$

where

$$\begin{aligned} p_i &= \rho_1\sigma_{i1} + \rho_2\sigma_{i2}, \quad i = 1,2, \\ p_3 &= 0.5(u_{11}\sigma_{11} + u_{12}\sigma_{12} + u_{21}\sigma_{21} + u_{22}\sigma_{22}), \\ P &= l_{111}\sigma_{11} + l_{121}\sigma_{12} + l_{211}\sigma_{21} + l_{221}\sigma_{22}, \\ Q &= l_{112}\sigma_{11} + l_{122}\sigma_{12} + l_{212}\sigma_{21} + l_{222}\sigma_{22}. \end{aligned}$$

Also note that $\rho_1 = \frac{a_1-1}{\alpha_1} - b_1$, $\rho_2 = \frac{a_2-1}{\alpha_2} - b_2$ and σ_{ij} is the element of the matrix $[-l_{ij}]^{-1}$, $i, j = 1,2$. For our case $u(\alpha_1, \alpha_2) = R_{s,k}$. We mention that each expression listed above is computed at the MLE $(\hat{\alpha_1}, \hat{\alpha_2})$.

4.2.2 *Tierney and Kadane method.* In this section, Bayes estimator of the reliability is obtained using the TK method when samples are progressive Type II censored. We define following functions

$$\delta(\alpha_1, \alpha_2) = \frac{l(\alpha_1, \alpha_2 \mid \beta, \text{data}) + \rho(\alpha_1, \alpha_2)}{n} \quad \text{and} \quad \delta^\star_u(\alpha_1, \alpha_2) = \delta(\alpha_1, \alpha_2) + \frac{\log u(\alpha_1, \alpha_2)}{n}.$$

Then approximate the estimate I as

$$I = \sqrt{\frac{|\Sigma^\star_u|}{|\Sigma|}}\exp\big[n\{\delta^\star_u(\hat{\alpha}_{1\delta^\star}, \hat{\alpha}_{2\delta^\star}) - \delta(\hat{\alpha}_{1\delta}, \hat{\alpha}_{2\delta})\}\big], \tag{18}$$

where $|\Sigma|$ and $|\Sigma^\star_u|$ denote the determinant of the negatives of inverse hessian of $\delta(\alpha_1, \alpha_2)$ and $\delta^\star_u(\alpha_1, \alpha_2)$ evaluated at $(\hat{\alpha}_{1\delta}, \hat{\alpha}_{2\delta})$ and $(\hat{\alpha}_{1\delta^\star}, \hat{\alpha}_{2\delta^\star})$, respectively.

4.2.3 *MCMC method.* Note that the marginal posterior distributions of α_1 and α_2 are of the following form

$$\alpha_1 \mid \text{data} \sim \text{Gamma}(nk + a_1, b_1 + V),$$
$$\alpha_2 \mid \text{data} \sim \text{Gamma}(n + a_2, b_2 + U),$$

where U and V are defined in Section 4.1.1. We generate samples from these distributions using the Gibbs sampling. The desired steps are as follows:

Step 1: Choose an initial guess $(\alpha_{1_0}, \alpha_{2_0})$ of (α_1, α_2).

Step 2: Set $t = 1$.

Step 3: Generate $\alpha_{1(t)}$ from $G_{\alpha_1}(nk + a_1, b_1 + \sum_{i=1}^{n} \sum_{j=1}^{n} (R_j + 1) \log(1 + x_{ij}^{\beta}))$.

Step 4: Generate $\alpha_{2(t)}$ from $G_{\alpha_2}(n + a_2, b_2 + \sum_{i=1}^{n} (S_i + 1) \log(1 + y_i^{\beta}))$.

Step 5: Compute $R_{(t)s,k} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^j \frac{\alpha_{2(t)}}{(\alpha_{1(t)}(i+j)+\alpha_{2(t)})}$.

Step 6: Set $t = t + 1$.

Step 7: Repeat steps 3–6, M times.

We use these replicates to obtain estimate of $R_{s,k}$ under the squared error loss function as

$$\tilde{R}_{s,k}^{\text{MH}} = \frac{1}{M} \sum_{i=1}^{M} R_{(t)s,k}.$$

We have employed the method of Chen and Shao (1999) to compute the $100(1 - \xi)\%$ HPD interval of $R_{s,k}$.

5 Simulation study

In this section, we conduct a Monte Carlo simulation study to compare the performance of different methods such as MLE, UMVUE and Bayes estimates computed under progressive Type II censoring. We compare the performance of these estimates in terms of their mean square error (MSE) and bias values. The comparison is made on the basis of 5000 replications of each estimate. We have performed all computations on R statistical software. We arbitrarily take true parameter values of Burr XII distributions as $(\alpha_1, \alpha_2, \beta) = (1.5, 1, 2)$. Then we use the following algorithm to simulated data for various censoring schemes (see also, Balakrishnan and Sandhu (1995)).

• Generate a given number of k independent observations $(D_1, \ldots, D_k)$ from the uniform $U(0, 1)$ distribution.

• Define $E_i = D_i^{1/(i+\sum_{j=k-i+1}^{k} R_j)}$, $i = 1, 2, \ldots, k$.

• Consider $U_{i:k:K} = 1 - (E_k E_{k-1} \ldots E_{k-i+1})$ for $i = 1, 2, \ldots, k$. Then $U_{1:k:K}, U_{2:k:K}, \ldots, U_{k:k:K}$ denotes a progressive Type II censored sample from the uniform $U(0, 1)$ distribution.

• We now consider $X_{i:k:K} = F_X^{-1}(U_{i:k:K})$ where $F_X^{-1}(\cdot)$ is inverse of the Burr XII cumulative distribution function. Then $X_{1:k:K}, X_{2:k:K}, \ldots, X_{k:k:K}$ denotes a progressive censored sample for the strength variable in a single component system. In a similar manner, we can simulate multicomponent progressive censored data by considering n out of N systems coupled with k out of K components. Progressively censored data for the stress variable can be generated similarly. We also mention that mean square error of an estimator $\hat{R}$ of the reliability R is computed as $n^{-1} \sum_{i=1}^{n} (\hat{R} - R_i)^2$.

Bayes estimates of the system reliability are computed using gamma prior distributions where hyperparameters are taken as $a_1 = 3, b_1 = 2, a_2 = 2, b_2 = 2, a_3 = 2, b_3 = 1$. In Table 1, we have tabulated different censoring schemes such as $R_1, \ldots, R_8$ for strength variables and $S_1, \ldots, S_8$ for stress variables. These censoring schemes are tabulated taking different values

Table 1 *Censoring scheme for different* (N, n, K, k)

(K, k)		$C.S$	(N, n)		$C.S$
(8, 4)	R_1	$(4, 0^{*3})$	(15, 10)	S_1	$(5, 0^{*9})$
	R_2	$(0^{*3}, 4)$		S_2	$(0^{*9}, 5)$
	R_3	$(0^{*2}, 4, 0)$		S_3	$(0^{*4}, 5, 0^{*5})$
(10, 6)	R_4	$(4, 0^{*5})$	(20, 15)	S_4	$(5, 0^{*19})$
	R_5	$(0^{*5}, 4)$		S_5	$(0^{*19}, 5)$
	R_6	$(0^{*3}, 4, 0^{*3})$		S_6	$(0^{*7}, 5, 0^{*7})$
(15, 10)	R_7	$(5, 0^{*9})$	(25, 15)	S_7	$(10, 0^{*14})$
	R_8	$(0^{*5}, 5)$		S_8	$(0^{*14}, 10)$
	R_9	$(0^{*5}, 5, 0^{*4})$		S_9	$(0^{*7}, 10, 0^{*7})$

Table 2 *ML and MH estimates of* $R_{s,k}$ *when the common parameter* β *is unknown*

			$s = 1$			$s = 2$	
(N, n, K, k)	$C.S$	Rt	$\hat{R}^{\mathrm{ML}}$	$\tilde{R}^{\mathrm{MH}}$	Rt	$\hat{R}^{\mathrm{ML}}$	$\tilde{R}^{\mathrm{MH}}$
(15, 10, 8, 4)	(R_1, S_1)	0.6844	0.6931	0.6844	0.4740	0.4938	0.4743
			0.0106	0.0002		0.0125	0.0001
	(R_2, S_2)		0.7559	0.6846		0.5485	0.4753
			0.0093	0.0002		0.0110	0.0002
	(R_3, S_3)		0.7367	0.6844		0.5327	0.4754
			0.0088	0.0002		0.0109	0.0003
(20, 15, 10, 6)	(R_4, S_4)	0.7493	0.7398	0.7482	0.5823	0.5773	0.5832
			0.0073	0.0003		0.0098	0.0002
	(R_5, S_5)		0.7998	0.7496		0.6455	0.5826
			0.0054	0.0002		0.0086	0.0003
	(R_6, S_6)		0.7774	0.7491		0.6142	0.5812
			0.0054	0.0003		0.0073	0.0003
(25, 15, 15, 10)	(R_7, S_7)	0.8156	0.8004	0.8191	0.6927	0.6819	0.6979
			0.0074	0.0002		0.0102	0.0002
	(R_8, S_8)		0.8341	0.8164		0.7176	0.6937
			0.0029	0.0003		0.0050	0.0002
	(R_9, S_9)		0.8216	0.8124		0.7058	0.6913
			0.0048	0.0002		0.0076	0.0001

of (N, n, K, k) where N denotes total number of systems required for the experiment, K denotes number of components in each system and n is the total number of observed system with k components observed in each system. In simulation, we have taken two different values 1 and 2 of s which means, we obtain estimate of the system reliability either when at least one component survives or at least two components survive the given stress level. In Table 2, we have tabulated MLE ($\hat{R}^{\mathrm{ML}}_{s,k}$) and MH ($\tilde{R}^{\mathrm{MH}}_{s,k}$) estimates of $R_{s,k}$ along with MSE values when the shape parameter β is unknown. In this table, Rt denotes the true value of $R_{s,k}$ corresponding to a particular censoring scheme. Also for each method, the upper value denotes the estimated value of reliability and immediate lower value is the associated MSE. Similarly in Table 3, we have computed Lindley ($\tilde{R}^{\mathrm{LA}}_{s,k}$) and TK ($\tilde{R}^{\mathrm{TK}}_{s,k}$) estimates along with their MSEs. From Tables 2–3, we observe that Bayes estimates of the reliability, in general, perform better than MLE as far as MSE and bias values are concerned. In fact, the MH procedure provides relatively better estimates followed by TK and Lindley methods, respectively. We also note that MSEs tend to decrease as the effective sample size increases.

Table 3 *Lindley and TK estimates of $R_{s,k}$ when the common parameter β is unknown*

(N, n, K, k)	$C.S$	$s=1$ Rt	$s=1$ $\tilde{R}^{\text{LA}}$	$s=1$ $\tilde{R}^{\text{TK}}$	$s=2$ Rt	$s=2$ $\tilde{R}^{\text{LA}}$	$s=2$ $\tilde{R}^{\text{TK}}$
(15, 10, 8, 4)	(R_1, S_1)	0.6844	0.6917	0.6795	0.4740	0.4632	0.4458
			1.493×10^{-5}	1.425×10^{-5}		4.665×10^{-5}	1.19×10^{-5}
	(R_2, S_2)		0.7541	0.6434		0.5588	0.4448
			6.377×10^{-5}	1.555×10^{-5}		2.748×10^{-5}	2.626×10^{-5}
	(R_3, S_3)		0.7392	0.6524		0.5312	0.4394
			6.178×10^{-5}	1.606×10^{-5}		1.837×10^{-5}	3.228×10^{-5}
(20, 15, 10, 6)	(R_4, S_4)	0.7493	0.7416	0.7407	0.5823	0.5493	0.5634
			9.075×10^{-6}	2.663×10^{-6}		5.310×10^{-6}	1.949×10^{-6}
	(R_5, S_5)		0.7989	0.7257		0.6602	0.5626
			6.147×10^{-6}	1.410×10^{-6}		4.084×10^{-6}	9.338×10^{-6}
	(R_6, S_6)		0.7746	0.7299		0.6298	0.5592
			2.185×10^{-6}	5.287×10^{-6}		6.762×10^{-6}	3.042×10^{-6}
(25, 15, 15, 10)	(R_7, S_7)	0.8156	0.8020	0.8011	0.6927	0.6844	0.6882
			3.200×10^{-7}	2.831×10^{-7}		6.739×10^{-7}	9.125×10^{-7}
	(R_8, S_8)		0.8278	0.7941		0.6891	0.6858
			2.972×10^{-7}	1.339×10^{-7}		4.112×10^{-7}	9.550×10^{-7}
	(R_9, S_9)		0.8131	0.7982		0.6883	0.6862
			3.901×10^{-7}	9.046×10^{-7}		4.948×10^{-7}	7.433×10^{-7}

Table 4 *Interval estimates of $R_{s,k}$ when the common parameter β is unknown*

(N, n, K, k)	$C.S$	$s=1$	Rt	AS	HPD	$s=2$	Rt	AS	HPD
(15, 10, 8, 4)	(R_1, S_1)		0.6844	0.4196	0.1363		0.4740	0.6750	0.1339
				0.918	0.998			0.982	0.993
	(R_2, S_2)			0.3576	0.1351			0.6164	0.1334
				0.897	0.997			0.995	0.998
	(R_3, S_3)			0.3822	0.1347			0.6502	0.1311
				0.930	0.999			0.993	0.995
(20, 15, 10, 6)	(R_4, S_4)		0.7493	0.2697	0.1275		0.5823	0.4264	0.1426
				0.867	0.996			0.944	0.997
	(R_5, S_5)			0.2395	0.1267			0.4065	0.1409
				0.841	0.998			0.966	0.998
	(R_6, S_6)			0.2584	0.1269			0.4263	0.1407
				0.886	0.994			0.976	0.997
(25, 15, 15, 10)	(R_7, S_7)		0.8156	0.1909	0.1070		0.6927	0.3027	0.1209
				0.729	0.998			0.839	0.999
	(R_8, S_8)			0.1837	0.1031			0.3016	0.1187
				0.863	0.997			0.944	0.999
	(R_9, S_9)			0.1953	0.1017			0.3169	0.1151
				0.812	0.999			0.905	0.995

We have also compared different interval estimates in terms of average length (AL) and coverage probabilities (CPs). Note that CP of an interval denotes the proportion that associated confidence interval contains the true unknown parameter. In Table 4, we have tabulated AL and CPs of asymptotic (AS) and HPD intervals. In this table for each censoring scheme and for each method, the first value denotes AL and immediate lower value denotes the corresponding CP. From this table, it is observed that average length of asymptotic confidence interval tend to remain wider than the corresponding HPD interval. Further average length of

Table 5 *UMVUE and exact Bayes estimates of $R_{s,k}$ when the common parameter β is known*

(N, n, K, k)	$C.S$	Rt	$s=1$		Rt	$s=2$	
			$\tilde{R}^{\text{UMV}}$	$\tilde{R}^{\text{EB}}$		$\tilde{R}^{\text{UMV}}$	$\tilde{R}^{\text{EB}}$
(15, 10, 8, 4)	(R_1, S_1)	0.6844	0.8256	0.6917	0.4740	0.6336	0.4913
			0.0276	0.0002		0.0390	0.0052
	(R_2, S_2)		0.5934	0.6901		0.3891	0.4850
			0.0217	0.0074		0.0186	0.0003
	(R_3, S_3)		0.7307	0.6886		0.5266	0.4707
			0.0144	0.0077		0.0169	0.0009
(20, 15, 10, 6)	(R_4, S_4)	0.7493	0.8481	0.7605	0.5823	0.7135	0.6037
			0.0361	0.0074		0.1415	0.0161
	(R_5, S_5)		0.6668	0.7540		0.4416	0.5901
			0.0973	0.0075		0.4039	0.0054
	(R_6, S_6)		0.7877	0.7524		0.6622	0.5872
			0.0494	0.0048		0.3679	0.0086
(25, 15, 15, 10)	(R_7, S_7)	0.8156	0.8912	0.8321	0.6927	0.7983	0.7190
			0.0677	0.0006		0.0246	0.0024
	(R_8, S_8)		0.7201	0.8264		0.5833	0.7174
			0.0124	0.0016		0.1217	0.0089
	(R_9, S_9)		0.8428	0.8210		0.7378	0.7013
			0.0065	0.0002		0.0144	0.0003

both the intervals tend to decrease as the effective sample size increases. The coverage probabilities of both intervals are quite satisfactory and remain in reasonable range of the nominal level.

We finally consider the case where common shape parameter β is known. We simulate stress and strength variables for (α_1, α_2) value as suggested above when $\beta = 2$. In Table 5, we have presented UMVUE and exact Bayes estimate of $R_{s,k}$ along with respective MSEs using various censoring schemes. Lindley and TK estimates along with their MSEs are listed in Table 6 and similarly in Table 7, we have presented MLE and MCMC estimates. From these tables, we observe that maximum likelihood estimates of the reliability show good behavior compared to the corresponding UMVUE estimates. However, Bayes estimates show superior performance than these two estimates. We further observe that TK estimates have an advantage over Lindley estimates. Also proposed approximate Bayes estimates remain marginally close to the Exact Bayes estimates. In fact, MH and TK procedures provide quite good results in this regard. We also note that MSEs of all the proposed estimates tend to decrease with an increase in effective sample size.

In Table 8, we have presented average lengths and coverage probabilities of asymptotic and HPD intervals for different censoring schemes. From this table, we observe that generally average length of HPD intervals tend to remain shorter than the corresponding asymptotic intervals. Further average lengths of both the intervals tend to decrease when effective sample size increases. It is also seen that coverage probabilities of both the intervals are quite satisfactory.

6 Data analysis

In this section, we discuss a pair of real data sets for illustration purposes.

6.1 Real data 1

In this example, one of our primary interest is to build a situation regarding the excessive drought. If the water capacity of a reservoir in a specific region in May, at least one year

Table 6 *Lindley and TK estimates of $R_{s,k}$ when the common parameter β is known*

(N, n, K, k)	$C.S$	Rt	$s=1$ $\tilde{R}^{LA}_{s,k}$	$\tilde{R}^{TK}_{s,k}$	Rt	$s=2$ $\tilde{R}^{LA}_{s,k}$	$\tilde{R}^{TK}_{s,k}$
(15, 10, 8, 4)	(R_1, S_1)	0.6844	0.6742	0.6902	0.4740	0.5271	0.4896
			6.679×10^{-5}	2.792×10^{-5}		4.653×10^{-5}	6.827×10^{-5}
	(R_2, S_2)		0.6762	0.6888		0.5284	0.4817
			3.426×10^{-5}	1.886×10^{-5}		1.010×10^{-5}	5.141×10^{-5}
	(R_3, S_3)		0.6782	0.6875		0.5281	0.4810
			1.866×10^{-5}	2.214×10^{-5}		2.231×10^{-5}	4.019×10^{-5}
(20, 15, 10, 6)	(R_4, S_4)	0.7494	0.7379	0.7551	0.5823	0.648	0.5947
			5.462×10^{-6}	8.585×10^{-6}		7.646×10^{-6}	2.490×10^{-5}
	(R_5, S_5)		0.7395	0.7558		0.6495	0.5917
			2.787×10^{-6}	2.486×10^{-6}		2.038×10^{-6}	5.149×10^{-6}
	(R_6, S_6)		0.7418	0.7556		0.6483	0.5903
			3.454×10^{-6}	7.965×10^{-6}		4.131×10^{-6}	2.443×10^{-6}
(25, 15, 15, 10)	(R_7, S_7)	0.8156	0.8012	0.8181	0.6927	0.8404	0.7014
			1.375×10^{-7}	3.659×10^{-7}		2.009×10^{-7}	6.890×10^{-6}
	(R_8, S_8)		0.8028	0.8177		0.8419	0.7017
			7.462×10^{-6}	1.643×10^{-7}		6.784×10^{-6}	3.633×10^{-7}
	(R_9, S_9)		0.8014	0.8165		0.8398	0.6904
			2.749×10^{-7}	1.228×10^{-7}		2.231×10^{-7}	5.543×10^{-7}

Table 7 *ML and MCMC estimates of $R_{s,k}$ when the common parameter β is known*

(N, n, K, k)	$C.S$	Rt	$s=1$ $\hat{R}^{ML}$	$\tilde{R}^{MH}$	Rt	$s=2$ $\hat{R}^{ML}$	$\tilde{R}^{MH}$
(15, 10, 8, 4)	(R_1, S_1)	0.6844	0.7323	0.6835	0.4740	0.5128	0.4736
			0.0103	1.125×10^{-5}		0.0109	1.025×10^{-5}
	(R_2, S_2)		0.7193	0.6834		0.5199	0.4733
			0.0107	1.1000×10^{-5}		0.0123	1.091×10^{-5}
	(R_3, S_3)		0.7045	0.6836		0.5083	0.4733
			0.0112	1.099×10^{-5}		0.0118	1.0623×10^{-5}
(20, 15, 10, 6)	(R_4, S_4)	0.7494	0.7856	0.7486	0.5823	0.6241	0.5817
			0.0068	1.595×10^{-5}		0.0084	1.907×10^{-5}
	(R_5, S_5)		0.7790	0.7486		0.6175	0.5818
			0.0064	1.672×10^{-5}		0.0087	2.051×10^{-5}
	(R_6, S_6)		0.7503	0.7484		0.6083	0.5817
			0.0066	1.786×10^{-5}		0.0079	2.101×10^{-5}
(25, 15, 15, 10)	(R_7, S_7)	0.8156	0.8559	0.8148	0.6927	0.7444	0.6912
			0.0047	1.753×10^{-5}		0.0080	2.684×10^{-5}
	(R_8, S_8)		0.8467	0.8144		0.7384	0.6918
			0.0048	1.776×10^{-5}		0.0076	2.617×10^{-5}
	(R_9, S_9)		0.8442	0.8147		0.7347	0.6919
			0.0053	1.742×10^{-5}		0.0078	2.654×10^{-5}

out of the next 5 years, is more than the amount of water achieved in November of the previous year, the claim will be that there will be no excessive drought afterwards. For this purpose, we try to analyze the monthly data from Shasta Reservoir in California, (see also, Kızılaslan and Nadar (2016) and Kohansal (2017)). This data set is available on the website "http://cdec.water.ca.gov/cgi-progs/queryMonthly?SHA". The data describe, from the flood

Table 8 *Interval estimates of $R_{s,k}$ when the common parameter β is known*

			$s=1$			$s=2$	
(N, n, K, k)	$C.S$	Rt	AS	HPD	Rt	AS	HPD
(15, 10, 8, 4)	(R_1, S_1)	0.6844	0.4232	0.0713	0.4740	0.6823	0.0700
			0.937	0.989		0.993	0.981
	(R_2, S_2)		0.4255	0.0715		0.6811	0.0702
			0.927	0998		0.996	0.947
	(R_3, S_3)		0.4213	0.0715		0.6873	0.0703
			0.928	0.976		0.992	0.975
(20, 15, 10, 6)	(R_4, S_4)	0.7494	0.2731	0.0685	0.5823	0.4397	0.0760
			0.876	0.983		0.968	0.968
	(R_5, S_5)		0.2768	0.0685		0.4389	0.0759
			0.879	0.985		0.970	0.991
	(R_6, S_6)		0.2749	0.0686		0.4374	0.0759
			0.892	0.993		0.977	0.985
(25, 15, 15, 10)	(R_7, S_7)	0.8156	0.1995	0.0625	0.6927	0.317	0.0762
			0.836	0.917		0.892	0.998
	(R_8, S_8)		0.1992	0.0628		0.3179	0.0762
			0.806	0.998		0.898	0.987
	(R_9, S_9)		0.1989	0.0625		0.3168	0.0760
			0.809	0.997		0.903	0.991

management perspective, the amount of water discharged in rivers, amount of water storage in the reservoirs, precipitation accumulation, and water content in snow pack. The maximum and the minimum water levels of the reservoir are generally observed in May and November respectively. Kızılaslan and Nadar (2016) and Nadar and Kızılaslan (2016) have studied such data in different context. We aim to make inference in view to take precautions of extreme drought situations under progressive Type II censoring. In complete sample case, with $k=5$ and $s=1$, Y_1 denotes the capacity of November 1970 and $X_{11}, \ldots, X_{15}$ denote capacities of September during 1971-1975. Similarly let Y_2 denotes the capacity of November 1976 and $X_{21}, \ldots, X_{25}$ denote capacities of September during 1977-1981. This process is continued up to 2017. Thus, N $(=8)$ data are obtained for Y. For computational simplifications, we divided each data point by 4,552,000 which is the total capacity of Shasta reservoir. The transformed data sets are given below as:

$$X = \begin{bmatrix} 0.719442 & 0.717597 & 0.728603 & 0.803669 & 0.784161 \\ 0.138533 & 0.753054 & 0.690092 & 0.729504 & 0.544859 \\ 0.794552 & 0.711797 & 0.434490 & 0.705470 & 0.463141 \\ 0.460452 & 0.359703 & 0.294343 & 0.369772 & 0.681406 \\ 0.689022 & 0.678561 & 0.507104 & 0.755947 & 0.730997 \\ 0.483226 & 0.561995 & 0.694063 & 0.479537 & 0.666704 \\ 0.412817 & 0.304148 & 0.389707 & 0.729082 & 0.733984 \\ 0.418714 & 0.254192 & 0.352043 & 0.617617 & 0.742939 \end{bmatrix} \quad \text{and}$$

$$Y = \begin{bmatrix} 0.767728 \\ 0.343146 \\ 0.724319 \\ 0.400454 \\ 0.443879 \\ 0.633937 \\ 0.686972 \\ 0.563329 \end{bmatrix}.$$

Table 9 *Goodness of fit for the real data* 1

	Data X			Data Y		
PDF	MLEs	K-S	p-value	MLEs	K-S	p-value
Burr XII	$\hat{\alpha}=7.1409$, $\beta=4.1848$	0.2244	0.0299	$\hat{\alpha}=9.8633$, $\beta=4.7597$	0.1901	0.8863
Burr III	$\hat{\alpha}=0.0614$, $\beta=26.6416$	0.3009	0.0011	$\hat{\alpha}=0.0721$, $\beta=23.2187$	0.3576	0.2017
IER	$\hat{\alpha}=1.6056$, $\beta=0.2620$	0.2521	0.0099	$\hat{\alpha}=4.3115$, $\beta=0.5583$	0.2094	0.8082

Table 10 *Estimates of $R_{s,k}$ for the complete data* 1

MLEs	$\hat{R}^{\mathrm{ML}}_{s,k}$	$\tilde{R}^{\mathrm{LA}}_{s,k}$	$\tilde{R}^{\mathrm{TK}}_{s,k}$	$\tilde{R}^{\mathrm{MH}}_{s,k}$	AS	HPD
$\hat{\alpha}_1=3.6209$ $\hat{\alpha}_2=3.7801$ $\hat{\beta}=2.2269$	0.8436	0.8420	0.8614	0.8877	(0.4259, 1.2612)	(0.8842, 0.8909)

We also verify whether Burr XII distributions can be used to analyze these two data sets. We fit the given data sets using Burr XII distributions and for comparison purposes, we also consider Burr III distribution and inverted Rayleigh (IER) distribution. We have computed MLEs of unknown parameters of all the competing models in Table 9 and have also reported values of Kolmogorov–Smirnov (K-S) statistic along with associated p-values. Based on these estimates, we observe that Burr XII distributions fit the data reasonably good. In Table 10, we have presented MLEs and Bayes estimates of the multicomponent reliability along with 95% asymptotic and HPD intervals for the complete data case. Bayes estimates are computed with respect to the noninformative prior distribution. It is seen that different estimates of the multicomponent reliability marginally remain close to each other. Next we compute different estimates of the reliability under progressive Type II censoring by arbitrarily considering two different schemes given below as:

Scheme 1: $R=(1,0,0,0,0)$, $S=(2,0,0,0,0,0,0)$ $(N=8,\ K=5,\ n=6,\ k=4,\ s=1)$,

Scheme 2: $R=(0,0,0,2)$, $S=(0,0,0,4)$ $(N=8,\ K=5,\ n=4,\ k=3,\ s=1)$.

Corresponding to the first censoring scheme the observed data are obtained as:

$$X=\begin{bmatrix}0.717597 & 0.728603 & 0.803669 & 0.784161\\ 0.359703 & 0.294343 & 0.369772 & 0.681406\\ 0.678561 & 0.507104 & 0.755947 & 0.730997\\ 0.561995 & 0.694063 & 0.479537 & 0.666704\\ 0.304148 & 0.389707 & 0.729082 & 0.733984\\ 0.254192 & 0.352043 & 0.617617 & 0.742939\end{bmatrix} \quad \text{and} \quad Y=\begin{bmatrix}0.767728\\ 0.400454\\ 0.443879\\ 0.633937\\ 0.686972\\ 0.563329\end{bmatrix}.$$

The estimated values of $R_{s,k}$ using maximum likelihood and Bayesian approaches are listed in Table 11. From this table, we observe that these estimates relatively remain close to each other. The 95% asymptotic confidence interval and HPD interval of system reliability are also reported in this table. We see that length of noninformative HPD interval is shorter than the corresponding asymptotic interval. We also note that results are very much consistent with the simulation study.

Similarly for the second censoring scheme the observed data are:

$$X=\begin{bmatrix}0.719442 & 0.717597 & 0.728603\\ 0.138533 & 0.753054 & 0.690092\\ 0.794552 & 0.711797 & 0.434490\\ 0.460452 & 0.359703 & 0.294343\end{bmatrix} \quad \text{and} \quad Y=\begin{bmatrix}0.767728\\ 0.343146\\ 0.724319\\ 0.400454\end{bmatrix}$$

Table 11 *Estimates of $R_{s,k}$ under censoring scheme* 1 *for the real data* 1

MLEs	$\hat{R}_{s,k}^{\mathrm{ML}}$	$\tilde{R}_{s,k}^{\mathrm{LA}}$	$\tilde{R}_{s,k}^{\mathrm{TK}}$	$\tilde{R}_{s,k}^{\mathrm{MH}}$	AS	HPD
$\hat{\alpha}_1 = 3.1876$ $\hat{\alpha}_2 = 2.4934$ $\hat{\beta} = 2.3547$	0.7324	0.7269	0.7248	0.7327	(0.1858, 1.2790)	(0.7179, 0.7476)

Table 12 *Estimates of $R_{s,k}$ under censoring scheme* 2 *for the real data* 1

MLEs	$\hat{R}_{s,k}^{\mathrm{ML}}$	$R_{s,k}^{\mathrm{LA}}$	$R_{s,k}^{\mathrm{TK}}$	$R_{s,k}^{\mathrm{MH}}$	AS	HPD
$\hat{\alpha}_1 = 2.4323$ $\hat{\alpha}_2 = 2.8435$ $\hat{\beta} = 2.3988$	0.7906	0.7907	0.7909	0.7910	(0.1855, 1.3958)	(0.7784, 0.8035)

Estimated values of the system reliability along with confidence intervals are listed in Table 12. We draw similar conclusions from this table as well.

6.2 Real data 2

In this example, we analyze another data which is initially discussed in Lyu et al. (1996) and one may also refer to "www.cse.cuhk.edu.hk/~lyu/book/reliability/DATA/CH4/SS3.DAT" for some more information on the data which represent failure times of certain software model. Here $X_{11}, \ldots, X_{15}$ denote failure times when 120–124 indexing unit fails and Y_1 is the failure time when 125th unit fails. Next $X_{21}, \ldots, X_{25}$ denote failure times for 126-130 units and Y_2 is failure time for 131st units. This process is continued up to the failure of 167th unit. We consider 8 systems each with 5 components. The complete data set is observed as:

$$X = \begin{bmatrix} 0.0096 & 0.0036 & 0.01200 & 1.0752 & 1.6488 \\ 5.4600 & 1.1976 & 0.01680 & 0.1428 & 1.5264 \\ 4.7568 & 0.0912 & 0.03840 & 0.3372 & 1.5408 \\ 14.7636 & 2.9400 & 1.53360 & 1.4160 & 1.3236 \\ 1.1004 & 0.0024 & 5.93592 & 1.8036 & 14.4912 \\ 2.1240 & 0.0096 & 2.34720 & 0.1296 & 0.0012 \\ 7.9716 & 7.8276 & 0.00360 & 0.0036 & 0.0048 \\ 6.7380 & 5.2896 & 16.94160 & 0.5292 & 6.9864 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0.0037 \\ 2.9736 \\ 1.4952 \\ 0.5508 \\ 3.3420 \\ 1.9572 \\ 0.0108 \\ 0.0816 \end{bmatrix}.$$

We divided each data point by 5000 for computational simplifications. We fit the given data set using the Burr XII distribution along with Lomax model and inverted exponentiated Pareto distribution (IEPD). We have computed MLEs of unknown parameters of proposed distributions in Table 13 and have also reported values of K-S statistic with corresponding p-values. From this table, we observe that the Burr XII distributions fits the data reasonably good. We mention that censoring schemes defined for the previous data are used for the real data 2 as well. The observed data under the first censoring scheme are given as:

$$X = \begin{bmatrix} 0.0036 & 0.01200 & 1.0752 & 1.6488 \\ 2.9400 & 1.53360 & 1.4160 & 1.3236 \\ 0.0024 & 5.93592 & 1.8036 & 14.4912 \\ 0.0096 & 2.34720 & 0.1296 & 0.0012 \\ 7.8276 & 0.00360 & 0.0036 & 0.0048 \\ 5.2896 & 16.94160 & 0.5292 & 6.9864 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0.0037 \\ 0.5508 \\ 3.3420 \\ 1.9572 \\ 0.0108 \\ 0.0816 \end{bmatrix}.$$

Table 13 *Goodness of fit for the real data* 2

	Data X			Data Y		
PDF	MLEs	K-S	p-value	MLEs	K-S	p-value
Burr XII	$\alpha = 1.3772, \beta = 0.5518$	0.2256	0.03409	$\alpha = 1.6727, \beta = 0.6254$	0.2492	0.6172
Lomax	$\alpha = 0.3157, \beta = 33.2103$	0.2794,	0.0038	$\alpha = 0.5076, \beta = 10.7780$	0.2634	0.5503
IEPD	$\alpha = 0.6556, \beta = 0.4095$	0.2912	0.0023	$\alpha = 0.9175, \beta = 0.4951$	0.25662,	0.5995

Table 14 *Estimates of $R_{s,k}$ for the complete data* 2

MLEs	$\hat{R}_{s,k}^{\mathrm{ML}}$	$\tilde{R}_{s,k}^{\mathrm{LA}}$	$\tilde{R}_{s,k}^{\mathrm{TK}}$	$\tilde{R}_{s,k}^{\mathrm{MH}}$	AS	HPD
$\hat{\alpha}_1 = 1.239$ $\hat{\alpha}_2 = 1.6258$ $\hat{\beta} = 0.73436$	0.89163	0.8878	0.8817	0.8924	(0.7563, 1.0269)	(0.8724, 0.9102)

Table 15 *Estimates of $R_{s,k}$ under censoring scheme* 1 *for the real data* 2

MLEs	$\hat{R}_{s,k}^{\mathrm{ML}}$	$\tilde{R}_{s,k}^{\mathrm{LA}}$	$\tilde{R}_{s,k}^{\mathrm{TK}}$	$\tilde{R}_{s,k}^{\mathrm{MH}}$	AS	HPD
$\hat{\alpha}_1 = 1.1474$ $\hat{\alpha}_2 = 2.0395$ $\hat{\beta} = 0.57406$	0.91713	0.9117	0.9115	0.8905	(0.76803, 1.0662)	(0.8819, 0.9016)

Table 16 *Estimates of $R_{s,k}$ under censoring scheme* 2 *for the real data* 2

MLEs	$\hat{R}_{s,k}^{\mathrm{ML}}$	$R_{s,k}^{\mathrm{LA}}$	$R_{s,k}^{\mathrm{TK}}$	$R_{s,k}^{\mathrm{MH}}$	AS	HPD
$\hat{\alpha}_1 = 1.2305$ $\hat{\alpha}_2 = 0.8676$ $\hat{\beta} = 0.5396$	0.64889	0.6569	0.6518	0.6749	(0.3006, 0.9971)	(0.5999, 0.7675)

The data sets corresponding to the second censoring scheme are given as:

$$X = \begin{bmatrix} 0.0096 & 0.0036 & 0.01200 \\ 5.4600 & 1.1976 & 0.01680 \\ 4.7568 & 0.0912 & 0.03840 \\ 14.7636 & 2.9400 & 1.53360 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0.0037 \\ 2.9736 \\ 1.4952 \\ 0.5508 \end{bmatrix}.$$

In Table 14, we have presented MLEs and Bayes estimates of the multicomponent reliability along with 95% asymptotic and HPD intervals for the complete data. Bayes estimates are computed with respect to the noninformative prior distribution. We observe that tabulated estimates of the multicomponent reliability remain marginally close to each other. We also compute different estimates of the multicomponent reliability under progressive Type II censoring with schemes as given for the real data 1. In Tables 15 and 16, we have tabulated various estimates of the reliability under censoring schemes 1 and 2, respectively. It is observed that tabulated estimates of the reliability remain arbitrarily close to each other. Also length of noninformative HPD intervals are relatively shorter than corresponding asymptotic intervals. We note that these inferences are very much consistent with the simulation study.

7 Conclusions

In this article, we have obtained various point and interval estimates of the multicomponent reliability $R_{s,k}$ assuming Burr XII distributions under progressive Type II censoring. In fact, we provided maximum likelihood and several Bayesian estimates of this parametric function under the assumptions that the common shape parameter may be known or unknown. We used Lindly method, Tierney and Kadane method and Metropolis Hasting algorithm to obtain desired Bayes estimates of $R_{s,k}$ under different sampling situations. Additionally UMVUE and exact Bayes estimates are also computed when the common parameter is known. We constructed asymptotic and HPD intervals of the reliability using progressively censored samples. By means of a simulation study, we observed that Bayes estimates in general perform better than the corresponding MLEs as far MSE and bias values are concerned. Also asymptotic confidence intervals in general are wider than the corresponding HPD intervals. We also observed that proposed asymptotic and HPD intervals show satisfactory coverage probabilities. Overall, better estimation results for the system reliability may be obtained with the known common shape parameter.

Appendix A

$$l_1 = \frac{nk}{\alpha_1} - \sum_{i=1}^{n}\sum_{j=1}^{n}(R_j+1)\log\big(1+x_{ij}^{\beta}\big), \qquad l_2 = \frac{n}{\alpha_2} - \sum_{i=1}^{n}(S_i+1)\log\big(1+y_i^{\beta}\big),$$

$$l_3 = \frac{n(k+1)}{\beta} + \sum_{i=1}^{n}\sum_{j=1}^{k}\log x_{ij} + \sum_{i=1}^{n}\log y_i - \sum_{i=1}^{n}\sum_{j=1}^{k}\frac{(\alpha_1(R_j+1)+1)x_{ij}^{\beta}\log x_{ij}}{(1+x_{ij}^{\beta})}$$

$$- \sum_{i=1}^{n}\frac{(\alpha_2(S_i+1)+1)y_i^{\beta}\log y_i}{(1+y_i^{\beta})}, \qquad l_{11} = \frac{-nk}{\alpha_1^2}, \qquad l_{12} = l_{21} = 0, \qquad l_{22} = -\frac{n}{\beta^2},$$

$$l_{13} = -\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{(R_j+1)x_{ij}^{\beta}\log x_{ij}}{(1+x_{ij}^{\beta})} = l_{31}, \qquad l_{23} = l_{32} = \sum_{i=1}^{n}\frac{(S_i+1)y_i^{\beta}\log y_i}{(1+y_i^{\beta})},$$

$$l_{33} = -\frac{n(k+1)}{\beta^2} - \sum_{i=1}^{n}\sum_{j=1}^{k}\frac{(\alpha_1(R_j+1)+1)x_{ij}^{\beta}(\log x_{ij})^2}{(1+x_{ij}^{\beta})^2}$$

$$- \sum_{i=1}^{n}\frac{(\alpha_2(S_i+1)+1)y_i^{\beta}(\log y_i)^2}{(1+y_i^{\beta})^2},$$

$$l_{333} = \frac{2n(k+1)}{\beta^3} + \sum_{i=1}^{n}\sum_{j=1}^{k}\frac{(\alpha_1(R_j+1)+1)(x_{ij}^{\beta})^2(\log x_{ij})^3(1-x_{ij}^{\beta})}{(1+x_{ij}^{\beta})^2}$$

$$- \sum_{i=1}^{n}\frac{(\alpha_2(S_i+1)+1)(y_i^{\beta})^2(\log y_i)^3(1-y_i^{\beta})}{(1+y_i^{\beta})^2}, \qquad l_{222} = \frac{2n}{\alpha_2^3}, \qquad l_{112} = l_{223} = 0,$$

$$l_{331} = -\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{(R_j+1)x_{ij}^{\beta}(\log x_{ij})^2}{(1+x_{ij}^{\beta})^2} - \sum_{i=1}^{n}\frac{(S_i+1)y_i^{\beta}(\log y_i)^2}{(1+y_i^{\beta})^2},$$

$$u = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^j\frac{\alpha_2}{(\alpha_1(i+j)+\alpha_2)},$$

$$u_1 = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^{j+1}\frac{\alpha_2(i+j)}{(\alpha_1(i+j)+\alpha_2)^2},$$

$$u_2 = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^{j}\frac{\alpha_1(i+j)}{(\alpha_1(i+j)+\alpha_2)^2},$$

$$u_{11} = 2\sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^{j}\frac{\alpha_1(i+j)^2}{(\alpha_1(i+j)+\alpha_2)^3},$$

$$u_{12} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^{j+1}\frac{(i+j)[(i+j)(\alpha_1-2\alpha_2)+\alpha_2]}{(\alpha_1(i+j)+\alpha_2)^3},$$

$$u_{22} = 2\sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^{j+1}\frac{\alpha_1(i+j)}{(\alpha_1(i+j)+\alpha_2)^3},$$

$$u_3 = u_{13} = u_{23} = u_{32} = u_{33} = 0.$$

Appendix B

To obtain $|\Sigma|$, we first note that

$$\begin{aligned}
&\delta(\alpha_1, \alpha_1, \beta) \\
&= \frac{1}{n}\Bigg[(nk + a_1 - 1)\log\alpha_1 + (n + a_2 - 1)\log\alpha_2 + \big(n(k+1) + a_3 - 1\big)\log\beta \\
&\quad - \alpha_1\left(b_1 + \sum_{i=1}^{n}\sum_{j=1}^{n}(R_j+1)\log\big(1+x_{ij}^{\beta}\big)\right) - \alpha_2\left(b_2 + \sum_{i=1}^{n}(S_i+1)\log\big(1+y_i^{\beta}\big)\right) \\
&\quad - \left(\sum_{i=1}^{n}\log y_i + \sum_{i=1}^{n}\sum_{j=1}^{k}\log x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n}\log(1+x_{ij}^{\beta}) + \sum_{i=1}^{n}(S_i+1)\log\big(1+y_i^{\beta}\big)\right) \\
&\quad + \beta\left(\sum_{i=1}^{n}\log y_i + \sum_{i=1}^{n}\sum_{j=1}^{k}\log x_{ij} - b_3\right)\Bigg].
\end{aligned}$$

Then we compute $(\hat{\alpha}_{1\delta}, \hat{\alpha}_{2\delta}, \hat{\beta}_{\delta})$ by solving following equations

$$\frac{\partial\delta}{\partial\alpha_1} = \frac{(nk + a_1 - 1)}{n\alpha_1} - \frac{1}{n}\left(b_1 + \sum_{i=1}^{n}\sum_{j=1}^{n}(R_j+1)\log\big(1+x_{ij}^{\beta}\big)\right) = 0,$$

$$\frac{\partial\delta}{\partial\alpha_2} = \frac{(n + a_2 - 1)}{n\alpha_2} - \frac{1}{n}\left(b_2 + \sum_{i=1}^{n}(S_i+1)\log\big(1+y_i^{\beta}\big)\right) = 0,$$

$$\begin{aligned}
\frac{\partial\delta}{\partial\beta} &= \frac{(n(k+1) + a_3 - 1)}{n\beta} + \frac{1}{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{k}\big(\alpha_1(R_j+1)+1\big)\frac{x_{ij}^{\beta}\log x_{ij}}{(1+x_{ij}^{\beta})}\right. \\
&\quad \left. + \sum_{i=1}^{n}\big(\alpha_2(S_i+1)+1\big)\frac{y_i^{\beta}\log y_i}{(1+y_i^{\beta})}\right) - \frac{1}{n}\left(b_3 - \sum_{i=1}^{n}\sum_{j=1}^{k}\log x_{ij} - \sum_{i=1}^{n} y_i\right) = 0.
\end{aligned}$$

Finally, $|\Sigma|$ can be computed using the following expressions,

$$\frac{\partial^2 \delta}{\partial \alpha_1^2} = -\frac{(nk + a_1 - 1)}{n\alpha_1^2},$$

$$\frac{\partial^2 \delta}{\partial \alpha_2^2} = -\frac{(n + a_2 - 1)}{n\alpha_2^2} - \frac{1}{n}\sum_{i=1}^{n} \frac{(S_i + 1)y_i^{\beta} \log y_i}{(1 + y_i^{\beta})},$$

$$\frac{\partial^2 \delta}{\partial \beta^2} = -\frac{1}{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{k}(\alpha_1(R_j + 1) + 1)\frac{x_{ij}^{\beta}(\log x_{ij})^2}{(1 + x_{ij}^{\beta})^2} + \sum_{i=1}^{n}(\alpha_2(S_i + 1) + 1)\frac{y_i^{\beta}(\log y_i)^2}{(1 + y_i^{\beta})^2}\right) - \frac{(n(k + 1) + a_3 - 1)}{n\beta^2},$$

$$\frac{\partial^2 \delta}{\partial \alpha_1\, \partial \alpha_2} = \frac{\partial^2 \delta}{\partial \alpha_2\, \partial \alpha_1} = 0,$$

$$\frac{\partial^2 \delta}{\partial \alpha_1\, \partial \beta} = \frac{\partial^2 \delta}{\partial \beta\, \partial \alpha_1} = -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} \frac{(R_j + 1)x_{ij}^{\beta} \log x_{ij}}{(1 + x_{ij}^{\beta})},$$

$$\frac{\partial^2 \delta}{\partial \alpha_2\, \partial \beta} = \frac{\partial^2 \delta}{\partial \beta\, \partial \alpha_2} = -\frac{1}{n}\sum_{i=1}^{n} \frac{(S_i + 1)y_i^{\beta} \log y_i}{(1 + y_i^{\beta})}.$$

Next, we observe that $|\Sigma_u^\star|$ depends on $u(\alpha_1, \alpha_1, \beta)$ and we take $R_{s,k} = u(\alpha_1, \alpha_2, \beta)$ for computing the desired Bayes estimate. Thus we have,

$$\delta_u^\star(\alpha_1, \alpha_2, \beta) = \delta(\alpha_1, \alpha_2, \beta) + \frac{1}{n}\log u(\alpha_1, \alpha_2, \beta).$$

Then we compute $(\hat{\alpha}_{1\delta^\star}, \hat{\alpha}_{2\delta^\star}, \hat{\beta}_{\delta^\star})$ by solving following system of equations

$$\frac{\partial \delta_u^\star}{\partial \alpha_1} = \frac{\partial \delta}{\partial \alpha_1} + \frac{1}{n}\frac{\partial}{\partial \alpha_1}\log u(\alpha_1, \alpha_2, \beta) = 0,$$

$$\frac{\partial \delta_u^\star}{\partial \alpha_2} = \frac{\partial \delta}{\partial \alpha_2} + \frac{1}{n}\frac{\partial}{\partial \alpha_2}\log u(\alpha_1, \alpha_2, \beta) = 0,$$

$$\frac{\partial \delta_u^\star}{\partial \beta} = \frac{\partial \delta}{\partial \beta} + \frac{1}{n}\frac{\partial}{\partial \beta}\log u(\alpha_1, \alpha_2, \beta) = 0.$$

Proceeding similarly, we are able to compute $|\Sigma_u^*|$ from the second order partial derivatives. The Bayes estimate of $R_{s,k}$ is now obtained as

$$\tilde{R}_{s,k}^{\mathrm{TK}} = \sqrt{\frac{|\Sigma_u^\star|}{|\Sigma|}}\exp\left[n\left\{\delta_u^\star(\hat{\alpha}_{1\delta^\star}, \hat{\alpha}_{2\delta^\star}, \hat{\beta}_{\delta^\star}) - \delta(\hat{\alpha}_{1\delta}, \hat{\alpha}_{2\delta}, \hat{\beta}_{\delta})\right\}\right].$$

Acknowledgments

The authors are grateful to a reviewer for encouraging comments and constructive suggestions that led to significant improvement in presentation and content of the manuscript. They also thank the Editor and an Associate Editor for helpful comments. Yogesh Mani Tripathi gratefully acknowledges the partial financial support for this research work under a grant EMR/2016/001401 Science and Engineering Research Board, India.

References

Aslam, M., Azam, M. and Jun, C.-H. (2016). Multiple dependent state repetitive group sampling plan for Burr XII distribution. *Quality Engineering* **28**, 231–237.

Awad, A. M. and Gharraf, M. K. (1986). Estimation of $P(Y < X)$ in the Burr case: A comparative study. *Communications in Statistics Simulation and Computation* **15**, 389–403. MR0848181 https://doi.org/10.1080/03610918608812514

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods, and Applications*. Springer. MR1763020 https://doi.org/10.1007/978-1-4612-1334-5

Balakrishnan, N. and Sandhu, R. A. (1995). A simple simulational algorithm for generating progressive Type-II censored samples. *American Statistician* **49**, 229–230.

Basirat, M., Baratpour, S. and Ahmadi, J. (2015). Statistical inferences for stress–strength in the proportional hazard models based on progressive Type-II censored samples. *Journal of Statistical Computation and Simulation* **85**, 431–449. MR3275457 https://doi.org/10.1080/00949655.2013.824449

Bhattacharyya, G. K. and Johnson, R. A. (1974). Estimation of reliability in a multicomponent stress-strength model. *Journal of the American Statistical Association* **69**, 966–970. MR0443269

Birnbaum, Z. W., et al (1956). On a use of the Mann–Whitney statistic. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*. MR0084909

Burr, I. W. (1942). Cumulative frequency functions. *The Annals of Mathematical Statistics* **13**, 215–232. MR0006644 https://doi.org/10.1214/aoms/1177731607

Chen, M.-H. and Shao, Q.-M. (1999). Monte Carlo estimation of Bayesian credible and HPD intervals. *Journal of Computational and Graphical Statistics* **8**, 69–92. MR1705909 https://doi.org/10.2307/1390921

Church, J. D. and Harris, B. (1970). The estimation of reliability from stress-strength relationships. *Technometrics* **12**, 49–54.

Dey, S., Mazucheli, J. and Anis, M. Z. (2017). Estimation of reliability of multicomponent stress–strength for a Kumaraswamy distribution. *Communications in Statistics Theory and Methods* **46**, 1560–1572. MR3574712 https://doi.org/10.1080/03610926.2015.1022457

Downtown, F. (1973). The estimation of $P(Y < X)$ in the normal case. *Journal of the American Statistical Association* **15**, 551–558. MR0334377

Draper, N. R. and Guttman, I. (1978). Bayesian analysis of reliability in multicomponent stress-strength models. *Communications in Statistics Theory and Methods* **7**, 441–451. MR0514232 https://doi.org/10.1080/03610927808827637

Enis, P. and Geisser, S. (1971). Estimation of the probability that $Y < X$. *Journal of the American Statistical Association* **66**, 162–168.

Gunasekera, S. (2015). Generalized inferences of $R = \Pr(X > Y)$ for Pareto distribution. *Statistical Papers* **56**, 333–351. MR3331866 https://doi.org/10.1007/s00362-014-0584-8

Hastings, W. K. (1970). Monte Carlo sampling methods using Markov chains and their applications. *Biometrika* **57**, 97–109. MR3363437 https://doi.org/10.1093/biomet/57.1.97

Kızılaslan, F. (2017). Classical and Bayesian estimation of reliability in a multicomponent stress–strength model based on the proportional reversed hazard rate mode. *Mathematics and Computers in Simulation* **136**, 36–62. MR3610080 https://doi.org/10.1016/j.matcom.2016.10.011

Kızılaslan, F. and Nadar, M. (2016). Estimation of reliability in a multicomponent stress–strength model based on a bivariate Kumaraswamy distribution. *Statistical Papers*, 1–34. MR3765947 https://doi.org/10.1007/s00362-016-0765-8

Kohansal, A. (2017). On estimation of reliability in a multicomponent stress-strength model for a Kumaraswamy distribution based on progressively censored sample. *Statistical Papers*, 1–40. MR4027641 https://doi.org/10.1007/s00362-017-0916-6

Lindley, D. V. (1980). Approximate Bayesian methods. *Trabajos de Estadística Y de Investigación Operativa* **31**, 223–245.

Lyu, M. R., et al (1996). *Handbook of Software Reliability Engineering, Vol. 222*. IEEE Computer Society Press CA.

Maurya, R. K., Tripathi, Y. M., Rastogi, M. K. and Asgharzadeh, A. (2017). Parameter estimation for a Burr XII distribution under progressive censoring. *American Journal of Mathematical and Management Sciences* **36**, 259–276.

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H. and Teller, E. (1953). Equation of state calculations by fast computing machines. *Journal of Chemical Physics* **21**, 1087–1092.

Moore, D. and Papadopoulos, A. S. (2000). The Burr type XII distribution as a failure model under various loss functions. *Microelectronics and Reliability* **40**, 2117–2122.

Nadar, M. and Kızılaslan, F. (2016). Estimation of reliability in a multicomponent stress-strength model based on a Marshall–Olkin bivariate Weibull distribution. *IEEE Transactions on Reliability* **65**, 370–380. MR3374192 https://doi.org/10.15446/rce.v38n2.51674

Nadar, M., Kızılaslan, F. and Papadopoulos, A. (2014). Classical and Bayesian estimation of $P(Y < X)$ for Kumaraswamy's distribution. *Journal of Statistical Computation and Simulation* **84**, 1505–1529. MR3196018 https://doi.org/10.1080/00949655.2012.750658

Najarzadegan, H., Babaii, S., Rezaei, S. and Nadarajah, S. (2016). Estimation of $p(y < x)$ for the Levy distribution. *Hacettepe Bulletin of Natural Sciences and Engineering* **45**, 957–972. MR3588235 https://doi.org/10.15672/hjms.2015599877

Papadopoulos, A. S. (1978). The Burr distribution as a failure model from a Bayesian approach. *IEEE Transactions on Reliability* **27**, 369–371.

Rao, G. S., Aslam, M. and Kundu, D. (2015). Burr-XII distribution parametric estimation and estimation of reliability of multicomponent stress-strength. *Communications in Statistics Theory and Methods* **44**, 4953–4961. MR3429812 https://doi.org/10.1080/03610926.2013.821490

Raqab, M. Z. and Kundu, D. (2005). Comparison of different estimators of $P(Y < X)$ for a scaled Burr type X distribution. *Communications in Statistics Simulation and Computation* **34**, 465–483. MR2157322 https://doi.org/10.1081/SAC-200055741

Rastogi, M. K. and Tripathi, Y. M. (2012). Estimating the parameters of a Burr distribution under progressive type II censoring. *Statistical Methodology* **9**, 381–391. MR2871439 https://doi.org/10.1016/j.stamet.2011.10.002

Sinha, S. K. (1986). Bayesian estimation of the reliability function of the Inverse Gaussian distribution. *Statistics & Probability Letters* **4**, 319–323. MR0858325 https://doi.org/10.1016/0167-7152(86)90052-0

Soliman, A. A. (2005). Estimation of parameters of life from progressively censored data using Burr-XII model. *IEEE Transactions on Reliability* **54**, 34–42.

Surles, J. G. and Padgett, W. J. (1998). Inference for $P(Y < X)$ in the Burr type X model. *Journal of Applied Statistical Science* **7**, 225–238.

Tierney, L. and Kadane, J. B. (1986). Accurate approximations for posterior moments and marginal densities. *Journal of the American Statistical Association* **81**, 82–86. MR0830567 https://doi.org/10.1007/s00362-017-0916-6

Wingo, D. R. (1993). Maximum likelihood methods for fitting the Burr type XII distribution to multiply (progressively) censored life test data. *Metrika* **40**, 203–210. MR1235082 https://doi.org/10.1007/BF02613681

Zimmer, W. J., Keats, J. B. and Wang, F. K. (1998). The Burr XII distribution in reliability analysis. *Journal of Quality Technology* **30**, 386.

Department of Mathematics
Indian Institute of Technology Patna
Bihta-801106
India
E-mail: yogesh@iitp.ac.in