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A note on the "L-logistic regression models: Prior sensitivity analysis, robustness to outliers and applications"

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Abstract. Da Paz, Balakrishnan and Bazan [Braz. J. Probab. Stat. **33** (2019), 455–479] introduced the L-logistic distribution, studied its properties including estimation issues and illustrated a data application. This note derives a closed form expression for moment properties of the distribution. Some computational issues are discussed.

1 Introduction

Da Paz, Balakrishnan and Bazan (2019) introduced the L-logistic distribution. Its probability density and cumulative distribution functions are specified by

$$f(x) = \frac{b(1-m)^b m^b x^{b-1}(1-x)^{b-1}}{[(1-m)^b x^b + m^b(1-x)^b]^2}$$

and

$$F(x) = \left[1 + \frac{m^b(1-x)^b}{x^b(1-m)^b}\right]^{-1},$$

respectively, for $0 < x < 1$, $0 < m < 1$ and $b > 0$.

Da Paz, Balakrishnan and Bazan (2019) derived properties of the L-logistic distribution, including quantile function, mode of its probability density function, moments, skewness and kurtosis. But no closed form was stated for the moments. It was expressed as the following integral:

$$E(X^t) = \int_0^1 \left[1 + \frac{1-m}{m}\left(\frac{1-x}{x}\right)^{1/b}\right]^{-t} dx. \tag{1}$$

Da Paz, Balakrishnan and Bazan (2019) stated that "The integral in (1) cannot be expressed in an analytical form".

Moments are perhaps the most important properties of any distribution. For example, the first four moments can be used to describe data fairly well. They are also useful in estimation and inference. Hence, it is important to have closed form expressions for the moments.

The aim of this note is to show that (1) can be expressed in closed form, involving the multivariate Wright generalized hypergeometric function, formally defined by

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p)\\(\beta_1, B_1), \ldots, (\beta_q, B_q)\end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j n)}{\prod_{k=1}^{q}\Gamma(\beta_k + B_k n)} \frac{z^n}{n!} \tag{2}$$

for $-\infty < z < \infty$, where $-\infty < \alpha_j < \infty$, $-\infty < \beta_k < \infty$, $A_j \neq 0$ and $B_k \neq 0$ for $j = 1, 2, \ldots, p$ and $k = 1, 2, \ldots, q$, see Wright (1935), Mathai and Saxena (1978), Srivastava,

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Gupta and Goyal (1982), Kilbas, Srivastava and Trujillo (2006, equation (1.9)) and Mathai, Saxena and Haubold (2010). The series in (2) converges for when

$$\Delta = 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. \tag{3}$$

In (2), $\Gamma(\cdot)$ denotes the gamma function defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t)\, dt.$$

A related function is the beta function defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt.$$

They are related by $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

The multivariate Wright generalized hypergeometric function has been applied in many areas: asymptotic analysis for the symbol error rate of single- and multiple-branch MRC and EGC receivers (El Ayadi and Ismail, 2014); analysis of effective capacity over generalized fading channels (Ji et al., 2014); expressions for the symbol error rate and capacity of the H-function fading channel (Alhennawi et al., 2016); effective rate analysis of wireless communication systems over MISO fading channels (You et al., 2017); to mention just a few. Various software for computing the multivariate Wright generalized hypergeometric function have been developed, see the cited papers and references therein.

Section 2 gives a closed form expression for (1). Some computational issues associated with this expression are discussed in Section 3.

2 Main result

Theorem 1 expresses the moments of the L-logistic distribution in terms of the multivariate Wright generalized hypergeometric function.

Theorem 1. *Let X denote a random variable having the L-logistic distribution. Then its tth moment can be expressed as*

$$E\left(X^t\right) = \frac{1}{\Gamma(t)}\, {}_3\Psi_0 \left[\left. \begin{array}{c} (t,1), \left(1, -\frac{1}{b}\right), \left(1, \frac{1}{b}\right) \\ \underline{\qquad\qquad} \end{array} \right| -c \right], \tag{4}$$

where $c = \frac{1-m}{m}$.

Proof. Using the Taylor expansion, we can express (1) as

$$\begin{aligned} E\left(Y^t\right) &= \int_0^1 \sum_{k=0}^{\infty} \binom{-t}{k} c^k \left(\frac{1-x}{x}\right)^{k/b} dx \\ &= \sum_{k=0}^{\infty} \binom{-t}{k} c^k \int_0^1 \left(\frac{1-x}{x}\right)^{k/b} dx \\ &= \sum_{k=0}^{\infty} \binom{-t}{k} c^k B\left(1 - \frac{k}{b}, 1 + \frac{k}{b}\right) \\ &= \sum_{k=0}^{\infty} \binom{-t}{k} c^k \Gamma\left(1 - \frac{k}{b}\right)\Gamma\left(1 + \frac{k}{b}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{(-t)(-t-1)\cdots(-t-k+1)}{k!} c^k \Gamma\left(1-\frac{k}{b}\right)\Gamma\left(1+\frac{k}{b}\right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k t(t+1)\cdots(t+k-1)}{k!} c^k \Gamma\left(1-\frac{k}{b}\right)\Gamma\left(1+\frac{k}{b}\right) \\
&= \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(t+k)}{\Gamma(t)k!} \Gamma\left(1-\frac{k}{b}\right)\Gamma\left(1+\frac{k}{b}\right).
\end{aligned}
$$

The result now follows from the definition in (2). The proof is complete. $\square$

The condition for convergence in (3) holds for all values of $t > 0$, $0 < m < 1$ and $b > 0$. Hence, (4) is a valid expression for all values of $t > 0$, $0 < m < 1$ and $b > 0$.

It follows that the mean, variance, skewness and kurtosis of X are

$$
E(X) = {}_3\Psi_0\left[\left. \begin{matrix} (1,1), \left(1, -\frac{1}{b}\right), \left(1, \frac{1}{b}\right) \\ \underline{\qquad\qquad} \end{matrix} \right| -c \right],
$$

$$
\begin{aligned}
\mathrm{Var}(X) = {}& {}_3\Psi_0\left[\left. \begin{matrix} (2,1), \left(1, -\frac{1}{b}\right), \left(1, \frac{1}{b}\right) \\ \underline{\qquad\qquad} \end{matrix} \right| -c \right] \\
& - \left\{ {}_3\Psi_0\left[\left. \begin{matrix} (1,1), \left(1, -\frac{1}{b}\right), \left(1, \frac{1}{b}\right) \\ \underline{\qquad\qquad} \end{matrix} \right| -c \right] \right\}^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Skewness}(X) = {}& \left[\frac{1}{2}\, {}_3\Psi_0\left[\left. \begin{matrix} (3,1), \left(1, -\frac{1}{b}\right), \left(1, \frac{1}{b}\right) \\ \underline{\qquad\qquad} \end{matrix} \right| -c \right] \right. \\
& - 3\, {}_3\Psi_0\left[\left. \begin{matrix} (1,1), \left(1, -\frac{1}{b}\right), \left(1, \frac{1}{b}\right) \\ \underline{\qquad\qquad} \end{matrix} \right| -c \right] \\
& \times {}_3\Psi_0\left[\left. \begin{matrix} (2,1), \left(1, -\frac{1}{b}\right), \left(1, \frac{1}{b}\right) \\ \underline{\qquad\qquad} \end{matrix} \right| -c \right] \\
& \left. + 2 \left\{ {}_3\Psi_0\left[\left. \begin{matrix} (1,1), \left(1, -\frac{1}{b}\right), \left(1, \frac{1}{b}\right) \\ \underline{\qquad\qquad} \end{matrix} \right| -c \right] \right\}^3 \right] \Big/ [\mathrm{Var}(X)]^{3/2}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Kurtosis}(X) = {}& \left[\frac{1}{6}\, {}_3\Psi_0\left[\left. \begin{matrix} (4,1), \left(1, -\frac{1}{b}\right), \left(1, \frac{1}{b}\right) \\ \underline{\qquad\qquad} \end{matrix} \right| -c \right] \right. \\
& - 2\, {}_3\Psi_0\left[\left. \begin{matrix} (1,1), \left(1, -\frac{1}{b}\right), \left(1, \frac{1}{b}\right) \\ \underline{\qquad\qquad} \end{matrix} \right| -c \right] \\
& \times {}_3\Psi_0\left[\left. \begin{matrix} (3,1), \left(1, -\frac{1}{b}\right), \left(1, \frac{1}{b}\right) \\ \underline{\qquad\qquad} \end{matrix} \right| -c \right]
\end{aligned}
$$

$$
+6\left\{{}_3\Psi_0\left[\begin{array}{c}(1,1),\left(1,-\frac{1}{b}\right),\left(1,\frac{1}{b}\right)\\ \hline\end{array}\middle| -c\right]\right\}^2
$$

$$
\times\, {}_3\Psi_0\left[\begin{array}{c}(2,1),\left(1,-\frac{1}{b}\right),\left(1,\frac{1}{b}\right)\\ \hline\end{array}\middle| -c\right]
$$

$$
\left.-3\left\{{}_3\Psi_0\left[\begin{array}{c}(1,1),\left(1,-\frac{1}{b}\right),\left(1,\frac{1}{b}\right)\\ \hline\end{array}\middle| -c\right]\right\}^4\right] \Big/ \left[\mathrm{Var}(X)\right]^2.
$$

3 Computational issues

An in-built routine for computing the Fox H function can be found in Mathematica. The routine is named FoxH and its source code is in http://community.wolfram.com/groups/-/m/t/57378 Using the relationship between Wright generalized hypergeometric and Fox H functions, we note that the following line can be used to compute (4):

$$
\left(1/\,\mathrm{Gamma}[t]\right) * \mathrm{FoxH}\left[a, b, (1-m)/m\right],
$$

where a is a list containing $\{1-t, 1\}$, $\{0, -1/b\}$, $\{0, 1/b\}$ and b is a list containing $\{0, 1\}$.

This one line command is an efficient way to compute (4). It avoids the need for numerical integration to compute the moments. Numerical integration needs coding, it wastes time and effort. The one line command can be employed to compute the moments directly without any effort. For example, FoxH[{{0, 1}, {0, $-1/b$}, {0, $1/b$}}, {{0, 1}}, $(1-m)/m$] will compute the mean of the L-logistic distribution. FoxH[{{-1, 1}, {0, $-1/b$}, {0, $1/b$}}, {{0, 1}}, $(1-m)/m$] will compute the second moment of the L-logistic distribution. FoxH[{{-2, 1}, {0, $-1/b$}, {0, $1/b$}}, {{0, 1}}, $(1-m)/m$] will compute the third moment of the L-logistic distribution, and so on.

Another advantage is that Mathematica like other algebraic manipulation packages allows for arbitrary precision. That is, the moments can be computed accurately to any desired level of accuracy. Numerical integration in the R software for example can be prone to roundoff errors.

The values of $E(X)$, $E(X^2)$, $\mathrm{Var}(X)$, $\mathrm{Skewness}(X)$ and $\mathrm{Kurtosis}(X)$ computed using the one line command are given in Table 1. The values of m and b selected are those in Table 1 of Da Paz, Balakrishnan and Bazan (2019). We see that the reported values of $E(X)$, $E(X^2)$ and $\mathrm{Var}(X)$ are the same as those reported in Table 1 in Da Paz, Balakrishnan and Bazan (2019) up to three decimal places.

The closed form expression in Section 2 can also be used for estimation purposes. Suppose $x_1, x_2, \ldots, x_n$ is a random sample assumed to follow the L-logistic distribution. Then

Table 1 *Values of mean, variance, skewness and kurtosis*

m	b	$E(X)$	$E(X^2)$	$\mathrm{Var}(X)$	$\mathrm{Skewness}(X)$	$\mathrm{Kurtosis}(X)$
0.2	1	0.2827975	0.1448017	0.06482725	1.008756	3.06419
0.2	3	0.2157058	0.05631151	0.009782524	1.192609	5.60486
0.5	1	0.5	0.3333333	0.08333333	0	1.8
0.5	3	0.5	0.268711	0.01871102	0	2.99388
0.8	1	0.7172025	0.5792067	0.06482725	−1.008756	3.06419
0.8	3	0.7842942	0.6248999	0.009782524	−1.192609	5.604859

estimates of m and b can be obtained by solving the equations

$$
{}_3\Psi_0\left[\begin{array}{c} (1,1), \underline{\left(1,-\frac{1}{b}\right)}, \left(1,\frac{1}{b}\right) \\ \end{array}\middle| -c\right] = \frac{1}{n}\sum_{j=1}^{n} x_j
$$

and

$$
{}_3\Psi_0\left[\begin{array}{c} (2,1), \underline{\left(1,-\frac{1}{b}\right)}, \left(1,\frac{1}{b}\right) \\ \end{array}\middle| -c\right] = \frac{1}{n}\sum_{j=1}^{n} x_j^2.
$$

Once again the in-built routine FoxH can be used to solve the equations.

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