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On estimating the location parameter of the selected exponential population under the LINEX loss function

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Abstract. Suppose that $\pi_1, \pi_2, \dots, \pi_k$ be $k(\geq 2)$ independent exponential populations having unknown location parameters $\mu_1, \mu_2, \dots, \mu_k$ and known scale parameters $\sigma_1, \dots, \sigma_k$. Let $\mu_{[k]} = \max\{\mu_1, \dots, \mu_k\}$. For selecting the population associated with $\mu_{[k]}$, a class of selection rules (proposed by Arshad and Misra [*Statistical Papers* **57** (2016) 605–621]) is considered. We consider the problem of estimating the location parameter μ_S of the selected population under the criterion of the LINEX loss function. We consider three natural estimators $\delta_{N,1}$, $\delta_{N,2}$ and $\delta_{N,3}$ of μ_S, based on the maximum likelihood estimators, uniformly minimum variance unbiased estimator (UMVUE) and minimum risk equivariant estimator (MREE) of μ_i's, respectively. The uniformly minimum risk unbiased estimator (UMRUE) and the generalized Bayes estimator of μ_S are derived. Under the LINEX loss function, a general result for improving a location-equivariant estimator of μ_S is derived. Using this result, estimator better than the natural estimator $\delta_{N,1}$ is obtained. We also shown that the estimator $\delta_{N,1}$ is dominated by the natural estimator $\delta_{N,3}$. Finally, we perform a simulation study to evaluate and compare risk functions among various competing estimators of μ_S.

1 Introduction

In many researches, it is often the case that $k(\geq 2)$ populations are available for evaluation of their quality. The quality of a population is defined in terms of an unknown parameter associated with it. Usually the parameter of interest is the population mean, so that, the population correspond to the largest (or smallest) mean is called the best (or worst) population. The basic goal is to select the best (or worst) population. All other populations are dropped-out and the selected population is remain for further investigations. For example, in clinical research, one is interested in selecting the most effective treatment from a choice of k available treatments. These problems have received considerable attention in the literature and are known as ranking and selection problems. After one of the population is selected, a practical problem is to estimate the mean (or unknown parameter) of the selected population. For example, in industrial applications, a company not only wishes to select an electric generator from a choice of $k(\geq 2)$ available generators that produces maximum power out-put, but also wants an estimate of the average power out-put of the selected generator. In the literature, the problem described above is termed as estimation after selection problem. The main focused of estimation after selection problem is to obtain the various competing estimators of the parameter associated with selected population and derive the decision theoretic results under various loss functions. For more details, see the references: Vellaisamy, Kumar and Sharma (1988), Vellaisamy (1996, 2009), Misra and Singh (1993), Misra, Anand and Singh (1998), Parsian and Farsipour (1999), Misra and van der Meulen (2001), Vellaisamy and Punnen (2002), Gangopadhyay and Kumar (2005), Kumar, Mahapatra and Vellaisamy (2009), Nematollahi and Motamed-Shariati (2012), Nematollahi and Jozani (2016) and Nematollahi and Pagheh

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(2017). A plethora on these problems exist under the scenarios of equal nuisance parameters and/or sample sizes associated with the available populations, less work was considered for the scenarios of unequal nuisance parameters and/or sample sizes. Some of the contribution in this direction is due to Risko (1985), Abughalous and Miescke (1989), Abughalous and Bansal (1994), Misra and Dhariyal (1994), Misra and Arshad (2014), Arshad, Misra and Vellaisamy (2015), Arshad and Misra (2015a, 2015b, 2016, 2017) and Meena, Arshad and Gangopadhyay (2018).

In some situations, where the overestimation (positive error) may be more serious than the underestimation (negative error) or vice versa, the use of symmetric loss function is inappropriate. For such cases, Varian (1975) proposed the following asymmetric loss function

$$L(\mu,\delta)=e^{a(\delta-\mu)}-a(\delta-\mu)-1, \qquad \mu\in\mathbb{R},\delta\in\mathbb{D}, \tag{1.1}$$

where $a\neq 0$ is location parameter of the loss function, μ denote the unknown parameter and δ is an estimator of μ. The loss function (1.1) is known as linear-exponential (LINEX) loss function. In this paper, we study the problem of estimation after selection from exponential populations with unequal scale parameters and unequal sample sizes, under the LINEX loss function (1.1), we also provide some generalization of the results of Nematollahi and Jozani (2016).

Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$ be independent random sample of size n_i from population π_i, which are exponentially distributed and having the pdf

$$f_i(x)=\begin{cases}\dfrac{1}{\sigma_i}e^{-(\frac{x-\mu_i}{\sigma_i})}, & \text{if } x\geq\mu_i,\\ 0, & \text{otherwise,}\end{cases}$$

where $\mu_i\in\mathbb{R}$ denotes the unknown location parameter and σ_i denotes the known scale parameter, $i=1,\ldots,k$. Let $\mu_{[1]}\leq\mu_{[2]}\leq\cdots\leq\mu_{[k]}$ denote the ordered values of $\mu_1,\ldots,\mu_k$, and the population associated with $\mu_{[k]}$ is called the best population. Let $X_i=\min\{X_{i1},X_{i2},\ldots,X_{in_i}\}$, $i=1,2,\ldots,k$. Then, X_i has exponential distribution with unknown location parameter μ_i and known scale parameter $\frac{\sigma_i}{n_i}$. Without loss of generality, we assume that $n_1=n_2=\cdots=n_k=1$, so that $X_i\sim \text{Exp}(\mu_i,\sigma_i)$, $i=1,2,\ldots,k$. Note that $\boldsymbol{X}=(X_1,X_2,\ldots,X_k)$ is a complete and sufficient statistic for $\boldsymbol{\mu}=(\mu_1,\mu_2,\ldots,\mu_k)\in\Omega$, where Ω denotes the k parametric space. We consider those inference procedures which depend on observations only through the complete and sufficient statistic $\boldsymbol{X}=(X_1,X_2,\ldots,X_k)$.

For the goal of selecting the best population, Arshad and Misra (2016) considered the class $\mathbb{C}=\{\psi^{\boldsymbol{c}}:\psi^{\boldsymbol{c}}=(\psi_1^{\boldsymbol{c}},\ldots,\psi_k^{\boldsymbol{c}}),\boldsymbol{c}\in\mathbb{R}^k\}$ of selection rules, where

$$\psi_i^{\boldsymbol{c}}(\boldsymbol{x})=\begin{cases}1, & \text{if } x_i+c_i>\max\limits_{j\neq i}(x_j+c_j),\\ 0, & \text{otherwise,}\end{cases} \tag{1.2}$$

where $\boldsymbol{c}=(c_1,\ldots,c_k)$.

It follows from Arshad and Misra (2015a) that, for $k=2$, the selection rule $\boldsymbol{\psi}^{c^*}=(\psi_1^{c^*},\psi_2^{c^*})$, where

$$\begin{aligned}\psi_1^{c^*}(\boldsymbol{x})&=\begin{cases}1, & \text{if } x_1>x_2+c^*,\\ 0, & \text{if } x_1\leq x_2+c^*;\end{cases}\\ \psi_2^{c^*}(\boldsymbol{x})&=\begin{cases}0, & \text{if } x_1>x_2+c^*,\\ 1, & \text{if } x_1\leq x_2+c^*,\end{cases}\end{aligned} \tag{1.3}$$

and

$$c^* = \begin{cases} \sigma_2 \ln\left(\dfrac{\sigma_1+\sigma_2}{2\sigma_2}\right), & \text{if } \sigma_1 < \sigma_2, \\ \sigma_1 \ln\left(\dfrac{2\sigma_1}{\sigma_1+\sigma_2}\right), & \text{if } \sigma_1 \geq \sigma_2, \end{cases}$$

is generalized Bayes with respect to non-informative prior and is minimax under the 0–1 loss function.

Let μ_S be the location parameter of the population selected by the selection rule $\psi^{\boldsymbol{c}}$, given in (1.2). Let $C_i = \{\boldsymbol{x} \in \chi : x_i + c_i > \max_{j \neq i}(x_j + c_j)\} = \{\boldsymbol{x} \in \chi : x_i + c_i > x_j + c_j, \forall j \neq i, j = 1, \ldots, k\}, i = 1, \ldots, k$, be the partition of the sample space χ. Then

$$\mu_S = \sum_{i=1}^{k} \mu_i I_{C_i}(\boldsymbol{X}),$$

where $I_{C_i}(\cdot)$ denotes the indicator function of the set C. Note that μ_S is a random parameter which depends on $\boldsymbol{X} = (X_1, X_2, \ldots, X_k)$. Our goal is to estimate the location parameter μ_S under the LINEX loss function, given in (1.1). It is easy to verify that, in the component estimation problems, X_i and $X_i - \sigma_i$ are the maximum likelihood estimator (MLE) and the uniformly minimum variance unbiased estimator (UMVUE) of μ_i's, respectively. Also, under the LINEX loss function, the minimum risk unbiased estimator (UMRUE) and the minimum risk equivariant (MRE) estimator of μ_i is $X_i - \alpha_i$, where $\alpha_i = -\frac{1}{a}\ln(1 - a\sigma_i), i = 1, \ldots, k$. Note that, the UMRUE and the MRE are same under the LINEX loss function.

Now we defined some natural estimators of μ_S based on the MLE, UMVUE and MRE (or UMRUE). Natural estimators of μ_S are given by

$$\delta_{N,1}(\boldsymbol{X}) = \sum_{i=1}^{k} X_i I_{C_i}(\boldsymbol{X}), \qquad \text{(based on MLE)}, \tag{1.4}$$

$$\delta_{N,2}(\boldsymbol{X}) = \sum_{i=1}^{k} (X_i - \sigma_i) I_{C_i}(\boldsymbol{X}), \qquad \text{(based on UMVUE)}, \tag{1.5}$$

$$\delta_{N,3}(\boldsymbol{X}) = \sum_{i=1}^{k} (X_i - \alpha_i) I_{C_i}(\boldsymbol{X}), \qquad \text{(based on MRE)}, \tag{1.6}$$

where $\alpha_i = \alpha_i(\sigma_i, a) = -\frac{1}{a}\ln(1 - a\sigma_i)$, $a\sigma_i < 1$.

The rest of the paper is organized as follows. In Section 2, under the criterion of the LINEX loss fuction (1.1), we derive the UMRUE of μ_S and the generalized Bayes estimator with respect to a non-informative prior distribution. In Section 3, we derive a general result for improvement of certain location equivariant estimators by employing the idea of Brewster and Zidek (1974). A conclusion on this result, an estimator dominating over the natural estimator $\delta_{N,1}$ is obtained. We also provide an example to demonstrate how the various estimators of μ_S are computed. In Section 4, we perform a simulation study to evaluate and compare risk functions among various competing estimators of μ_S.

2 UMRUE and generalized Bayes estimator

In this section, we derive the UMRUE and the generalized Bayes estimator of μ_S under the LINEX loss function, given in (1.1).

Definition 1 (Nematollahi and Jozani (2016)). An estimator $\delta(\boldsymbol{X})$ of the location parameter μ_S is said to be risk unbiased estimator under the LINEX loss function (1.1) if

$$E_{\boldsymbol{\mu}}[e^{a\delta(\boldsymbol{X})}] = E_{\boldsymbol{\mu}}[e^{a\mu_S}], \qquad \forall \boldsymbol{\mu} \in \Omega.$$

The following lemma is a generalization of Theorem 1 of Nematollahi and Jozani (2016), which is useful for finding the UMRUE of μ_S.

Lemma 1. *Let $X_1, X_2, \ldots, X_k$ be k independent random variables from* $\mathrm{Exp}(\mu_i, \sigma_i)$, $i = 1, \ldots, k$. *Let $U_1(\boldsymbol{X}), U_2(\boldsymbol{X}), \ldots, U_k(\boldsymbol{X})$ be k real valued functions of $\boldsymbol{X}$ such that*

(a) $E_{\boldsymbol{\mu}}(|e^{aX_i}U_i(\boldsymbol{X})|) < \infty, \forall \boldsymbol{\mu} \in \Omega$.
(b) *The integral*

$$\int_{x_i}^{\infty} U_i(x_1, \ldots, x_{i-1}, y, x_{i+1} \ldots, x_k) e^{-(\frac{y}{\sigma_i})} \, dy < \infty, \qquad \forall \boldsymbol{x} \in \chi.$$

Then, the function

$$K_i(\boldsymbol{x}) = e^{ax_i}\left[U_i(\boldsymbol{x}) - a\int_{x_i}^{\infty} U_i(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots) e^{-(y-x_i)/\sigma_i} \, dy\right]$$

satisfies

$$E\big(K_i(\boldsymbol{X})\big) = E\big(e^{a\mu_i}U_i(\boldsymbol{X})\big), \qquad \forall \boldsymbol{\mu} \in \Omega.$$

Theorem 1. *The UMRUE of the location parameter μ_S of the selected population is given by*

$$\delta^U(\boldsymbol{X}) = \frac{1}{a}\ln\left[\sum_{i=1}^{k} e^{aX_i} I_{C_i}(\boldsymbol{X}) - a\sum_{i=1}^{k} \sigma_i e^{aX_i} e^{-(\frac{\max(0, Y_i - X_i)}{\sigma_i})}\right], \tag{2.1}$$

where $Y_i = \max_{j \neq i}(X_j + c_j) - c_i$, $i = 1, 2, \ldots, k$.

Proof. We have

$$E\big(e^{a\mu_S}\big) = E\left[\sum_{i=1}^{k} e^{a\mu_i} I_{C_i}(\boldsymbol{X})\right] = \sum_{i=1}^{k} E\big[e^{a\mu_i} I_{C_i}(\boldsymbol{X})\big]. \tag{2.2}$$

Let $K_i(\boldsymbol{X})$ be an estimator of $e^{a\mu_i} I_{C_i}(\boldsymbol{X})$ such that $E[K_i(\boldsymbol{X})] = E[e^{a\mu_i} I_{C_i}(\boldsymbol{X})]$, $i = 1, 2, \ldots, k$. Using Lemma 1, we get

$$\begin{aligned} K_i(\boldsymbol{X}) &= e^{aX_i} I_{C_i}(\boldsymbol{X}) - a e^{aX_i} \\ &\quad \times \int_{X_i}^{\infty} I_{C_i}(X_1, \ldots, X_{i-1}, y, X_{i+1}, \ldots, X_k) e^{-(y-X_i)/\sigma_i} \, dy \\ &= e^{aX_i} I_{C_i}(\boldsymbol{X}) - a e^{aX_i} \int_{\max\{X_i, Y_i\}}^{\infty} e^{-(y-X_i)/\sigma_i} \, dy \\ &= e^{aX_i} I_{C_i}(\boldsymbol{X}) - a\sigma_i e^{aX_i} e^{-(\frac{\max(0, Y_i - X_i)}{\sigma_i})}. \end{aligned}$$

It follows from Equation (2.2) and Definition 1 that the risk unbiased estimator of μ_S is

$$\begin{aligned} \delta_U(\boldsymbol{X}) &= \frac{1}{a}\ln\left[\sum_{i=1}^{k} K_i(\boldsymbol{X})\right] \\ &= \frac{1}{a}\ln\left[\sum_{i=1}^{k} e^{aX_i} I_{C_i}(\boldsymbol{X}) - a\sum_{i=1}^{k} \sigma_i e^{aX_i} e^{-(\frac{\max(0, Y_i - X_i)}{\sigma_i})}\right], \end{aligned}$$

where $Y_i = \max_{j \neq i}(X_j + c_j) - c_i$, $i = 1, 2, \ldots, k$. $\square$

Remark 1. Let $X_{[1]} \leq \cdots \leq X_{[k]}$ denote the ordered values of $X_1, \ldots, X_k$. Let $\sigma_1 = \cdots = \sigma_k = \sigma$ (say), and $c_1 = \cdots = c_k = 0$, it follows from Theorem 1 that the UMRUE of μ_S is

$$\delta^U(\boldsymbol{X}) = \frac{1}{a}\ln\left[e^{aX_{[k]}} - a\sigma e^{aX_{[k]}} - a\sigma\sum_{i=1}^{k-1} e^{aX_{[i]}}e^{-(X_{[k]}-X_{[i]})/\sigma}\right]$$
$$= X_{[k]} + \frac{1}{a}\ln\left[(1-a\sigma) - a\sigma\sum_{i=1}^{k-1} e^{(1+a\sigma)[X_{[i]}-X_{[k]}]}\right].$$

This result was derived by Nematollahi and Jozani (2016), and also reported in Nematollahi and Pagheh (2017).

Now we prove that, under the LINEX loss function (1.1), the natural estimator $\delta_{N,3}$ and the generalized Bayes estimator of μ_S are same.

Theorem 2. *Under the LINEX loss function* (1.1), *the natural estimator* $\delta_{N,3}$ *is the generalized Bayes estimator with respect to the non-informative prior distribution.*

Proof. Consider the non-informative prior distribution

$$\Pi_{\boldsymbol{\mu}}(\mu_1, \ldots, \mu_k) = 1, \qquad \forall(\mu_1, \ldots, \mu_k) \in \Omega.$$

The posterior probability density function of $\boldsymbol{\mu}$, given $\boldsymbol{X} = \boldsymbol{x}$, is given by

$$\Pi^*_{\boldsymbol{\mu}}(\mu_1, \ldots, \mu_k|\boldsymbol{x}) = \begin{cases} \prod_{i=1}^{k}\left\{\frac{1}{\sigma_i}e^{-(\frac{x_i-\mu_i}{\sigma_i})}\right\}, & \text{if } \mu_i \leq x_i, i = 1, \ldots, k, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

The posterior risk of an estimator δ under the LINEX loss is given by

$$r(\delta, \boldsymbol{x}) = E_{\Pi^*}\left[e^{a(\delta(\boldsymbol{X})-\mu_S)} - a\big(\delta(\boldsymbol{X}) - \mu_S\big) - 1|\boldsymbol{X} = \boldsymbol{x}\right]. \tag{2.4}$$

The generalized Bayes estimator $\delta^G(\boldsymbol{X})$, which minimizes the posterior risk (2.4), is given by

$$\delta^G(\boldsymbol{x}) = -\frac{1}{a}\ln\left[E_{\Pi^*}\left[e^{-a\mu_S}|\boldsymbol{X} = \boldsymbol{x}\right]\right]$$
$$= -\frac{1}{a}\ln\left[\sum_{i=1}^{k} E_{\Pi^*}\left(e^{-a\mu_i}\right)I_{C_i}(\boldsymbol{x})\right].$$

Using the posterior distribution (2.3), we get

$$\delta^G(\boldsymbol{X}) = \sum_{i=1}^{k}\left[X_i + \frac{1}{a}\ln(1 - a\sigma_i)\right]I_{C_i}(\boldsymbol{X}) = \delta_{N,3}(\boldsymbol{X}).$$

□

3 Improving a location equivariant estimator

In this section, we will show that the natural estimator $\delta_{N,1}(\boldsymbol{X})$ is inadmissible under the LINEX loss function (1.1). The following lemmas and definition are useful in deriving a sufficient condition for the inadmissibility of a location-equivariant estimator of μ_S.

Lemma 2. *Let $X_1, \dots, X_k$ be independent random variables such that X_i has the exponential distribution with location parameter μ_i and scale parameter $\sigma_i, i = 1, \dots, k$. Let $T_j = X_j - X_1, j = 2, \dots, k$. Then, for fixed $\boldsymbol{t} = (t_2, \dots, t_k) \in \mathbb{R}^{k-1}$, the conditional distribution of X_1, given that $\boldsymbol{T} = \boldsymbol{t}$, is exponential with location parameter $\mu_{\boldsymbol{t}} = \max\{\mu_1, \max_{j \neq 1}(\mu_j - t_j)\}$ and scale parameter $\sigma_0 = (\sum_{i=1}^{k} \frac{1}{\sigma_i})^{-1}$.*

The proof of the previous lemma is straightforward and hence omitted.

Let $\mathbb{V}_1 = \{(t_2, \dots, t_k) \in \mathbb{R}^{k-1} : t_j < c_1 - c_j, j = 2, \dots, k\}$ and

$$\mathbb{V}_l = \Big\{(t_2, \dots, t_k) \in \mathbb{R}^{k-1} : t_l > \max\Big(c_1 - c_l, \max_{\substack{2 \le j \le k \\ j \neq l}} (t_j + c_j - c_l)\Big)\Big\},$$

$$l = 2, \dots, k,$$

so that, $\{\mathbb{V}_1, \dots, \mathbb{V}_k\}$ forms a partition of $\mathbb{R}^{k-1}$. Define

$$\varphi(\boldsymbol{t}, \boldsymbol{\mu}) = -\frac{1}{a} \ln\Bigg[\sum_{i=1}^{k} E\big(e^{a(X_1 - \mu_i)} | \boldsymbol{T} = \boldsymbol{t}\big) I_{\mathbb{V}_i}(\boldsymbol{t})\Bigg], \qquad \boldsymbol{t} \in \mathbb{R}^{k-1}, \boldsymbol{\mu} \in \Omega. \tag{3.1}$$

Using Lemma 2, we have

$$\begin{aligned} &\varphi(\boldsymbol{t}, \underline{\mu}) \\ &\quad = -\frac{1}{a} \ln\Bigg[\frac{1}{1 - a\sigma_0} \sum_{i=1}^{k} e^{a(\max\{\mu_1, \max_{j \neq 1}(\mu_j - t_j)\} - \mu_i)} I_{\mathbb{V}_i}(\boldsymbol{t})\Bigg], \\ &\quad \boldsymbol{t} \in \mathbb{R}^{k-1}, a\sigma_0 < 1 \\ &\quad = \begin{cases} -\frac{1}{a} \ln\Big[\frac{1}{1 - a\sigma_0} e^{a(\max\{0, \max_{j \neq 1}(\mu_j - \mu_1 - t_j)\})}\Big], \\ \qquad \text{if } \boldsymbol{t} \in \mathbb{V}_1, \\ -\frac{1}{a} \ln\Big[\frac{1}{1 - a\sigma_0} e^{a(\max\{\mu_1 - \mu_l, \max_{\substack{j \neq 1 \\ j \neq l}} (\mu_j - \mu_l - t_j), -t_l\})}\Big], \\ \qquad \text{if } \underline{t} \in \mathbb{V}_l, l = \{2, 3, \dots, k\}. \end{cases} \end{aligned} \tag{3.2}$$

Lemma 3. *Let $\varphi(\boldsymbol{t}, \boldsymbol{\mu})$ be the function as defined in (3.2). Then, for $a < 0$,*

$$\varphi_S = \sup_{\boldsymbol{\mu} \in \Omega} \varphi(\boldsymbol{t}, \underline{\mu}) = \begin{cases} -\frac{1}{a} \ln\Big[\frac{1}{1 - a\sigma_0}\Big], & \textit{if } \boldsymbol{t} \in \mathbb{V}_1, \\ -\frac{1}{a} \ln\Big[\frac{e^{-at_l}}{1 - a\sigma_0}\Big], & \textit{if } \boldsymbol{t} \in \mathbb{V}_l, l = \{2, 3, \dots, k\}, \end{cases}$$

for $0 < a < \frac{1}{\sigma_0}$

$$\varphi_I = \inf_{\boldsymbol{t} \in \Omega} \varphi(\boldsymbol{t}, \underline{\mu}) = \begin{cases} -\frac{1}{a} \ln\Big[\frac{1}{1 - a\sigma_0}\Big], & \textit{if } \boldsymbol{t} \in \mathbb{V}_1, \\ -\frac{1}{a} \ln\Big[\frac{e^{-at_l}}{1 - a\sigma_0}\Big], & \textit{if } \boldsymbol{t} \in \mathbb{V}_l, l = \{2, 3, \dots, k\}. \end{cases}$$

Proof. When $a < 0$, for fixed $\boldsymbol{t} \in \mathbb{V}_1$ and $\boldsymbol{\mu} \in \Omega$,

$$\begin{aligned} \varphi(\boldsymbol{t}, \boldsymbol{\mu}) &= -\frac{1}{a} \ln\Big[\Big(\frac{1}{1 - a\sigma_0}\Big) e^{a(\max\{0, \max_{j \neq 1}(\mu_j - \mu_1 - t_j)\})}\Big], \qquad \boldsymbol{t} \in \mathbb{R}^{k-1}, \boldsymbol{\mu} \in \Omega \\ &\le -\frac{1}{a} \ln\Big[\Big(\frac{1}{1 - a\sigma_0}\Big)\Big], \end{aligned}$$

and the equality is attained when $\max_{j\neq 1}(\mu_j - \mu_1 - t_j) \leq 0$. Therefore, $\varphi_S = \sup_{\boldsymbol{\mu}\in\Omega} \varphi(\boldsymbol{t}, \boldsymbol{\mu}) = -\frac{1}{a}\ln[(\frac{1}{1-a\sigma_0})]$, $1 - a\sigma_0 > 0$.

For fixed $\boldsymbol{t} \in \mathbb{V}_l$, $l = \{2, 3, \ldots, k\}$

$$\varphi(\boldsymbol{t}, \boldsymbol{\mu}) = -\frac{1}{a}\ln\left[\left(\frac{1}{1-a\sigma_0}\right)e^{a(\max\{\mu_1-\mu_l, \max_{j\neq 1, j\neq l}(\mu_j-\mu_l-t_j), -t_l\})}\right],$$

$$\boldsymbol{t} \in \mathbb{R}^{k-1}, \boldsymbol{\mu} \in \Omega$$

$$\leq -\frac{1}{a}\ln\left[\left(\frac{e^{-at_l}}{1-a\sigma_0}\right)\right],$$

and the equality is attained when $\max_{\substack{j\neq 1\\ j\neq l}}(\mu_j - \mu_1 - t_j) \leq -t_l$ and $\mu_1 - \mu_l \leq -t_l$. Therefore, $\varphi_I = \sup_{\underline{\mu}\in\Omega} \varphi(\boldsymbol{t}, \boldsymbol{\mu}) = -\frac{1}{a}\ln[(\frac{e^{-at_l}}{1-a\sigma_0})]$.

When $0 < a < \frac{1}{\sigma_0}$, for fixed $\boldsymbol{t} \in \mathbb{V}_1$, $\inf_{\boldsymbol{\mu}\in\Omega} \varphi(\boldsymbol{t}, \boldsymbol{\mu}) = -\frac{1}{a}\ln[(\frac{1}{1-a\sigma_0})]$. Similarly, for fixed $\boldsymbol{t} \in \mathbb{V}_l$, $l = \{2, 3, \ldots, k\}$, $\inf_{\boldsymbol{\mu}\in\Omega} \varphi(\boldsymbol{t}, \boldsymbol{\mu}) = -\frac{1}{a}\ln[(\frac{e^{-at_l}}{1-a\sigma_0})]$. Hence the result follows. □

Definition 2. An estimator $\delta(X_1, \ldots, X_k)$ of μ_S is said to be location-equivariant if

$$\delta(X_1 + p, \ldots, X_k + p) = \delta(X_1, \ldots, X_k) + p, \qquad \forall p \in \mathbb{R}.$$

Clearly, any such estimator will be of the form

$$\delta(X_1, \ldots, X_k) = X_1 + \phi(T_2, \ldots, T_k),$$

where $T_i = X_i - X_1$, $i = 2, \ldots, k$, and $\delta : \mathbb{R}^{k-1} \to \mathbb{R}$ is a real-valued function.

Next, under the LINEX loss function (1.1), we demonstrate the approach of Brewster and Zidek (1974) to derive a general result providing a sufficient condition for the inadmissibility of a location-equivariant estimator of μ_S.

Theorem 3. *Suppose that $\delta(\boldsymbol{X}) = X_1 + \phi(\boldsymbol{T})$ be location-equivariant estimator of μ_S, where $\boldsymbol{T} = (T_2, \ldots, T_k) = (X_2 - X_1, \ldots, X_k - X_1)$, and $\phi(\cdot)$ is a real-valued function defined on $\mathbb{R}^{k-1}$.*

(i) *Suppose that for $a < 0$ and $P_{\boldsymbol{\mu}}(\{\phi(\boldsymbol{T}) > \varphi_S(\boldsymbol{T})\} > 0\ \forall \boldsymbol{T} \in \Omega$, where $\varphi_S(\boldsymbol{T})$ is as defined in Lemma 3. Then, the estimator δ is inadmissible and is dominated by $\delta_1(\boldsymbol{X}) = X_1 + \phi_1(\boldsymbol{T})$, where $\phi_1(\boldsymbol{T}) = \min\{\phi(\boldsymbol{T}), \varphi_S(\boldsymbol{T})\}$.*
(ii) *Suppose that for $0 < a < \frac{1}{\sigma_0}$ and $P_{\boldsymbol{\mu}}(\{\phi(\boldsymbol{T}) < \varphi_I(\boldsymbol{T})\} > 0\ \forall \boldsymbol{\mu} \in \Omega$, where $\varphi_I(\boldsymbol{T})$ is as defined in Lemma 3. Then, the estimator δ is inadmissible and is dominated by $\delta_2(\boldsymbol{X}) = X_1 + \phi_2(\boldsymbol{T})$, where $\phi_2(\boldsymbol{T}) = \max\{\phi(\boldsymbol{t}), \varphi_I(\boldsymbol{t})\}$.*

Proof.

(i) Consider

$$R(\boldsymbol{\mu}, \delta) - R(\boldsymbol{\mu}, \delta_1) = E\big[D_{\boldsymbol{\mu}}(\boldsymbol{T})\big],$$

where, for $\boldsymbol{t} \in \mathbb{R}^{k-1}$ and $\boldsymbol{\mu} \in \Omega$,

$$\begin{aligned} D_{\boldsymbol{\mu}}(\boldsymbol{t}) &= E\big[e^{a(\delta(\boldsymbol{X})-\mu_S)} - a\big(\delta(\boldsymbol{X}) - \mu_S\big) - 1|\boldsymbol{T} = \boldsymbol{t}\big] \\ &\quad - E\big[e^{a(\delta_1(\boldsymbol{X})-\mu_S)} - a\big(\delta_1(\boldsymbol{X}) - \mu_S\big) - 1|\boldsymbol{T} = \boldsymbol{t}\big] \\ &= E\big[e^{a(\delta(\boldsymbol{X})-\mu_S)} - e^{a(\delta_1(\boldsymbol{X})-\mu_S)}|\boldsymbol{T} = \boldsymbol{t}\big] - aE\big[\delta(\boldsymbol{X}) - \delta_1(\boldsymbol{X})|\boldsymbol{T} = \boldsymbol{t}\big] \end{aligned}$$

$$= E\big[e^{a(X_1+\phi(\boldsymbol{t})-\mu_S)} - e^{a(X_1+\phi_1(\boldsymbol{t})-\mu_S)}|\boldsymbol{T}=\boldsymbol{t}\big] - a\big[\phi(\boldsymbol{t})-\phi_1(\boldsymbol{t})\big]$$
$$= \big[e^{a\phi(\boldsymbol{t})} - e^{a\phi_1(\boldsymbol{t})}\big]\sum_{i=1}^{k} E\big[e^{a(X_i-\mu_i)}|\boldsymbol{T}=\boldsymbol{t}\big]I_{\mathbb{V}_i}(\boldsymbol{t}) - a\big(\phi(\boldsymbol{t})-\phi_1(\boldsymbol{t})\big)$$
$$= \big[e^{a\phi(\boldsymbol{t})} - e^{a\phi_1(\boldsymbol{t})}\big]e^{-a\varphi(\boldsymbol{t},\boldsymbol{\mu})} - a\big(\phi(\boldsymbol{t})-\phi_1(\boldsymbol{t})\big).$$

Here, $\varphi(\boldsymbol{t},\boldsymbol{\mu})$ is as given in (3.1). Now suppose that if $\phi(\boldsymbol{t}) \le \varphi_S(\boldsymbol{t})$, i.e., $\phi_1(\boldsymbol{t}) = \phi(\boldsymbol{t})$, then $D_{\boldsymbol{\mu}}(\boldsymbol{t}) = 0$. If $\phi(\boldsymbol{t}) > \varphi_S(\boldsymbol{t})$ that is, $\phi_1(\boldsymbol{t}) = \varphi_S(\boldsymbol{t})$, then

$$\begin{aligned} D_{\boldsymbol{\mu}}(\boldsymbol{t}) &= \big[e^{a\phi(\boldsymbol{t})} - e^{a\varphi_S(\boldsymbol{t})}\big]e^{-a\varphi(\boldsymbol{t},\boldsymbol{\mu})} - a\big(\phi(\boldsymbol{t})-\varphi_S(\boldsymbol{t})\big) \\ &\ge \big[e^{a\phi(\boldsymbol{t})} - e^{a\varphi_S(\boldsymbol{t})}\big]e^{-a\varphi_S(\boldsymbol{t})} - a\big(\phi(\boldsymbol{t})-\varphi_S(\boldsymbol{t})\big) \\ &= \big[e^{a\{\phi(\boldsymbol{t})-\varphi_S(\boldsymbol{t})\}} - 1\big] - a\big[\phi(\boldsymbol{t})-\varphi_S(\boldsymbol{t})\big]. \end{aligned}$$

Using the inequality $e^x > 1+x$, $\forall x \neq 0$, that is, $D_{\boldsymbol{\mu}}(\boldsymbol{t}) \ge 0$. Since $P_{\boldsymbol{\mu}}\{\varphi(\boldsymbol{t}) > \varphi_S(\boldsymbol{t})\} > 0$, $\forall \boldsymbol{\mu} \in \Omega$, we get

$$R(\boldsymbol{\mu},\phi) - R(\boldsymbol{\mu},\phi_1) > 0, \qquad \forall \boldsymbol{\mu} \in \Omega.$$

Hence, the result follows.

(ii) Similar to the proof of (i), hence omitted. □

The natural estimator $\delta_{N,1}$ of μ_S, given in (1.4), can be written as $\delta_{N,1} = X_1 + \phi_{N,1}(\boldsymbol{T})$, where

$$\phi_{N,1}(\boldsymbol{T}) = \sum_{i=2}^{k} T_i I_{\mathbb{V}_i}(\boldsymbol{T}).$$

Using the results of Theorem 3, we get the following corollary.

Corollary 1. *Under the LINEX loss function* (1.1) *and* $a < 0$, *the natural estimator* $\delta_{N,1}$ *is inadmissible and is dominated by the estimator*

$$\delta^I_{N,1}(\boldsymbol{X}) = X_1 + \phi^I_{N,1}(\boldsymbol{T}), \tag{3.3}$$

where $\phi^I_{N,1}(\boldsymbol{T}) = \min\{\phi_{N,1}(\boldsymbol{t}), \varphi_S(\boldsymbol{t})\}$, *and* $\varphi_S(\boldsymbol{t})$ *is as defined in Lemma* 3.

Remark 2. It can be verify that, the UMVUE and the natural estimators $\delta_{N,2}$ and $\delta_{N,3}$ are not fulfill the sufficient condition for the inadmissibility of an estimator of μ_S. Therefore, the improvement of the UMVUE and the natural estimators $\delta_{N,2}$ and $\delta_{N,3}$ are not possible by using Theorem 3.

Now consider a general class $\mathcal{Q}_0 = \{\delta_{\boldsymbol{p}} : \boldsymbol{p} = (p_1,\ldots,p_k) \in \Omega\}$ of estimators of the form

$$\delta_{\boldsymbol{p}}(\boldsymbol{X}) = \sum_{i=1}^{k} (X_i + p_i) I_{C_i}(\boldsymbol{X}), \qquad \boldsymbol{p} = (p_1,\ldots,p_k) \in \Omega.$$

Clearly, the natural estimators $\delta_{N,1}, \delta_{N,2}$ and $\delta_{N,3}$ are the members of the class $\mathcal{Q}_0$ for the choices $\boldsymbol{p} = (0,0,\ldots,0)$, $\boldsymbol{p} = (-\sigma_1,-\sigma_2,\ldots,-\sigma_k)$ and $\boldsymbol{p} = (-\alpha_1,-\alpha_2,\ldots,-\alpha_k)$. Let $\mathcal{Q}_1 = \{\delta_{\boldsymbol{p}} \in \mathcal{Q}_0 : p_i > -\alpha_i \ \forall i = 1,2,\ldots,k\}$ be a subclass of $\mathcal{Q}_0$.

In the following theorem, we will derive estimators dominating over a general estimator $\delta_{\boldsymbol{p}} \in \mathcal{Q}_1$, under the LINEX loss function (1.1).

Theorem 4. *Let $\delta_{\boldsymbol{q}} \in \mathcal{Q}_1$ be a given estimators of μ_S, let $\boldsymbol{q} = (q_1, \ldots, q_k) \in \mathbb{R}^k$ such that $q_i \in (-\alpha_i, p_i)$, $i = 1, \ldots, k$. Then the estimator*

$$\delta_{\boldsymbol{q}} = \sum_{i=1}^{k} (X_i + q_i) I_{C_i}(\boldsymbol{X}), \qquad \boldsymbol{q} = (q_1, \ldots, q_k) \in \Omega,$$

dominate the given estimator $\delta_{\boldsymbol{p}}$ under the criterion of the LINEX loss function (1.1).

Proof. Consider the risk difference.

$$\begin{aligned} R(\boldsymbol{\mu}, \delta_p) - R(\boldsymbol{\mu}, \delta_q) &= E\big[e^{a(\delta_p(\boldsymbol{X}) - \mu_S)} - a\big(\delta_p(\boldsymbol{X}) - \mu_S\big)\big] \\ &\quad - E\big[e^{a(\delta_q(\boldsymbol{X}) - \mu_S)} - a\big(\delta_q(\boldsymbol{X}) - \mu_S\big)\big] \\ &= E\left[\sum_{i=1}^{k} \big\{e^{a(X_i + p_i - \mu_i)} - e^{a(X_i + q_i - \mu_i)}\big\} I_{C_i}(\boldsymbol{X})\right] \\ &\quad - aE\left[\sum_{i=1}^{k} (p_i - q_i) I_{C_i}(\boldsymbol{X})\right] \\ &= \sum_{i=1}^{k} \big(e^{ap_i} - e^{aq_i}\big) E\big[e^{a(X_i - \mu_i)} I_{C_i}(\boldsymbol{X})\big] \\ &\quad - a\sum_{i=1}^{k} (p_i - q_i) E\big[I_{C_i}(\boldsymbol{X})\big]. \end{aligned} \tag{3.4}$$

For each $i = \{1, \ldots, k\}$, and $\boldsymbol{\mu} \in \Omega$, define

$$\begin{aligned} B_i(\boldsymbol{\mu}) &= E\big(e^{a(X_i - \mu_i)} I_{C_i}(\boldsymbol{X})\big) \\ &= E\big[e^{-a\mu_i} e^{aX_i} I_{C_i}(\boldsymbol{X})\big]. \end{aligned}$$

Let $M_i(\boldsymbol{X})$ be an estimator of $B_i(\boldsymbol{\mu})$ such that $E[M_i(\boldsymbol{\mu})] = B_i(\boldsymbol{\mu})$. Now using Lemma 1, we get

$$\begin{aligned} M_i(\boldsymbol{X}) &= e^{-aX_i}\bigg[e^{aX_i} I_{C_i}(\boldsymbol{X}) \\ &\qquad + a\int_{X_i}^{\infty} e^{ay} I_{C_i}(X_1, \ldots, X_i, y, X_{i+1}, \ldots, X_k) e^{-(y - X_i)/\sigma_i}\, dy\bigg] \\ &= I_{C_i}(\boldsymbol{X}) + e^{(\frac{1 - a\sigma_i}{\sigma_i})X_i} \int_{\max\{X_i, Y_i\}}^{\infty} e^{-(\frac{1 - a\sigma_i}{\sigma_i})y}\, dy \\ &= I_{C_i}(\boldsymbol{X}) + \left(\frac{a\sigma_i}{1 - a\sigma_i}\right) e^{(\frac{1 - a\sigma_i}{\sigma_i})X_i} e^{-(\frac{1 - a\sigma_i}{\sigma_i})(\max\{X_i, Y_i\})} \\ &= I_{C_i}(\boldsymbol{X}) + \left(\frac{a\sigma_i}{1 - a\sigma_i}\right) e^{-(\frac{1 - a\sigma_i}{\sigma_i})(\max\{0, Y_i - X_i\})}. \end{aligned} \tag{3.5}$$

Using Equations (3.4) and (3.5), we get

$$\begin{aligned}
&R(\boldsymbol{\mu}, \delta_p) - R(\boldsymbol{\mu}, \delta_q) \\
&\quad = \sum_{i=1}^{k} (e^{ap_i} - e^{aq_i}) E\big[M_i(\boldsymbol{X})\big] - a \sum_{i=1}^{k} (p_i - q_i) E\big[I_{C_i}(\boldsymbol{X})\big] \\
&\quad = \sum_{i=1}^{k} E\Bigg((e^{ap_i} - e^{aq_i}) \bigg[I_{C_i}(\boldsymbol{X}) + \left(\frac{a\sigma_i}{1 - a\sigma_i}\right) e^{-(\frac{1-a\sigma_i}{\sigma_i})(\max\{0, Y_i - X_i\})} \bigg] \\
&\qquad - a(p_i - q_i) I_{C_i}(\boldsymbol{X}) \Bigg) \\
&\quad = \sum_{i=1}^{k} E\big[\psi_i(\boldsymbol{X})\big] \qquad \text{(say).}
\end{aligned} \tag{3.6}$$

For each fixed $i \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned}
\psi_i(\boldsymbol{X}) &= (e^{ap_i} - e^{aq_i}) I_{C_i}(\boldsymbol{X}) - a(p_i - q_i) I_{C_i}(\boldsymbol{X}) \\
&\qquad + \frac{a\sigma_i (e^{ap_i} - e^{aq_i})}{(1 - a\sigma_i)} e^{-(\frac{1-a\sigma_i}{\sigma_i})(\max\{0, Y_i - X_i\})} \\
&= (e^{ap_i} - e^{aq_i}) I_{C_i}(\boldsymbol{X}) - a(p_i - q_i) I_{C_i}(\boldsymbol{X}) + \frac{a\sigma_i (e^{ap_i} - e^{aq_i})}{(1 - a\sigma_i)} I_{C_i}(\boldsymbol{X}) \\
&\qquad + \frac{a\sigma_i (e^{ap_i} - e^{aq_i})}{(1 - a\sigma_i)} e^{-(\frac{1-a\sigma_i}{\sigma_i})(Y_i - X_i)} I_{C_i^c}(\boldsymbol{X}) \\
&= \left[\frac{(e^{ap_i} - e^{aq_i})}{(1 - a\sigma_i)} - a(p_i - q_i) \right] I_{C_i}(\boldsymbol{X}) \\
&\qquad + \frac{a\sigma_i (e^{ap_i} - e^{aq_i})}{(1 - a\sigma_i)} e^{-(\frac{1-a\sigma_i}{\sigma_i})(Y_i - X_i)} I_{C_i^c}(\boldsymbol{X}) \\
&= \left[\frac{e^{aq_i}}{(1 - a\sigma_i)} (e^{ap_i - aq_i} - 1) - a(p_i - q_i) \right] I_{C_i}(\boldsymbol{X}) \\
&\qquad + \frac{a\sigma_i (e^{ap_i} - e^{aq_i})}{(1 - a\sigma_i)} e^{-(\frac{1-a\sigma_i}{\sigma_i})(Y_i - X_i)} I_{C_i^c}(\boldsymbol{X}),
\end{aligned}$$

where C_i^c denotes the complement of the set C_i. Since $(1 - a\sigma_i) > 0$, and $e^{ap_i - aq_i} - 1 > (ap_i - aq_i)$, it follows that

$$\begin{aligned}
\psi_i(\boldsymbol{X}) &> \left(\frac{e^{aq_i}}{1 - a\sigma_i} - 1 \right) a(p_i - q_i) I_{C_i}(\boldsymbol{X}) \\
&\qquad + \frac{a\sigma_i (e^{ap_i} - e^{aq_i})}{(1 - a\sigma_i)} e^{-(\frac{1-a\sigma_i}{\sigma_i})(Y_i - X_i)} I_{C_i^c}(\boldsymbol{X}) \\
&= \psi_i^*(\boldsymbol{X}) \qquad \text{(say).}
\end{aligned}$$

Here

$$\psi_i^*(\boldsymbol{X}) = \begin{cases} \left(\dfrac{e^{aq_i}}{1 - a\sigma_i} - 1 \right) a(p_i - q_i), & \text{if } \boldsymbol{X} \in C_i, \\ \dfrac{a\sigma_i (e^{ap_i} - e^{aq_i})}{1 - a\sigma_i} \, e^{-(\frac{1-a\sigma_i}{\sigma_i})(Y_i - X_i)}, & \text{if } \boldsymbol{X} \in C_i^c. \end{cases}$$

Since the sign of ψ_i^* depends on the sign of a, the following two cases arise:

(i) When $a < 0$.

We know that

$$q_i > -\alpha_i = \frac{1}{a}\ln(1 - a\sigma_i) \implies \left(\frac{e^{aq_i}}{1 - a\sigma_i} - 1\right) < 0$$

(since $a < 0$ and $p_i > q_i$)

$$\implies a(p_i - q_i)\left(\frac{e^{aq_i}}{1 - a\sigma_i} - 1\right) > 0$$

$$\implies \psi_i^*(\boldsymbol{X}) > 0 \qquad \forall \boldsymbol{X} \in C_i.$$

Further, we have,

$$\begin{aligned} p_i > q_i &\implies e^{ap_i} < e^{aq_i} \\ &\implies \frac{(e^{ap_i} - e^{aq_i})}{(1 - a\sigma_i)} < 0 \\ &\implies \frac{a\sigma_i(e^{ap_i} - e^{aq_i})}{1 - a\sigma_i} e^{-(\frac{1-a\sigma_i}{\sigma_i})(Y_i - X_i)} > 0 \\ &\implies \psi_i^*(\boldsymbol{X}) > 0, \qquad \boldsymbol{X} \in C_i^c. \end{aligned}$$

Now, we can conclude that, for fixed $i \in \{1, 2, \ldots, k\}$, $\psi_i(\boldsymbol{X}) > \psi_i^*(\boldsymbol{X}) > 0$, $\forall \boldsymbol{X} \in \chi$. It follows from Equation (3.6) and $\psi_i(\boldsymbol{X}) > 0$, $\forall \boldsymbol{X} \in \chi$, $i = 1, 2, \ldots, k$, that $R(\boldsymbol{\mu}, \delta_{\boldsymbol{p}}) > R(\boldsymbol{\mu}, \delta_{\boldsymbol{q}})$.

(ii) When $0 < a < \frac{1}{\sigma_0}$. The proof is similar to case (i), hence omitted. □

Remark 3. Since the natural estimator $\delta_{N,1}$ is a member of the class $\mathcal{Q}_1$ ($\delta_{\boldsymbol{p}}$ with $p_i = 0$, $i = 1, \ldots, k$), it follows from Theorem 4 that, for all $a < \frac{1}{\sigma_0}$, the natural estimator $\delta_{N,1}$ is inadmissible and is dominated by the natural estimator $\delta_{N,3}$ ($\delta_{\boldsymbol{p}}$ with $p_i = -\alpha_i, i = 1, 2, \ldots, k$).

Remark 4. For $0 < a < \frac{1}{\sigma_0}$, it is verify from Theorem 4 that the natural estimator $\delta_{N,2}$ is dominated by the natural estimator $\delta_{N,3}$. But, for $a < 0$, Theorem 4 fails to find an estimator dominating the natural estimator $\delta_{N,2}$. In Section 4, we also provide the risks values of these estimators, which suggest that the natural estimator $\delta_{N,3}$ performs better than other estimators except few cases.

In the following example, we demonstrate how the various estimators of the location parameter μ_S of the selected population are computed.

Example 1. Table 1 represent the mean of the daily power output (kw) for daily average wind speed greater than 3 (m/s) for two different regions in the United States under the same conditions. The datasets are compatible with hypothesis that the two populations are exponentially distributed. Furthermore, these values are important because they provide information about the value of the power output of wind turbines as well as to maintain the same power density. Detailed information can be found in the application (Wind Energy Resource Atlas of the United States) at http://www.osti.gov/bridge.

Let π_1 represent the population under Region A and π_2 represent the population under Region B. We assume that an equal number of samples are randomly taken from both the regions. We have fitted the available datasets and found the following: For the population

Table 1 *Data for Example 1*

Population	Observations						
Region A (π_1)	15.00	14.93	14.80	14.60	14.40	14.49	14.42
	14.15	14.24	14.13	14.08	13.91	14.00	14.11
	13.74						
Region B (π_2)	11.63	11.58	11.48	11.33	11.17	11.23	11.18
	10.97	11.04	10.96	10.92	10.79	10.86	10.94
	10.65						

π_1, the Kolmogorov–Smirnov (K–S) distance between the actual data and the fitted exponential (13.74, 0.5933) distribution is 0.33 and the corresponding p-value is 0.3855. Similarly, for the population π_2, the K–S distance between the actual data and the fitted exponential (10.65, 0.4653) distribution is 0.4 and the corresponding p-value is 0.1844. Therefore, the available datasets have provided sufficient evident to indicate that the two populations are exponentially distributed. The quality of the population is measured in terms of average power out-put, that is, the population $\pi_1 \equiv \text{Exp}(\mu_1, \sigma_1)$ is better than the population $\pi_2 \equiv \text{Exp}(\mu_2, \sigma_2)$ if $\mu_1 > \mu_2$, and the population π_2 is better than the population π_1 if $\mu_1 \le \mu_2$. For the goal of selecting the better population, we use the minimax selection rule $\boldsymbol{\psi}^{C^*}$, given in (1.3). From the data, we have $X_1 = 13.74$, $X_2 = 10.65$, $\sigma_1 = 0.5933$, $\sigma_2 = 0.4653$, and $c^* = 0.6772$.

We will construct the computation of the estimators for different values of a. The Region A has been selected as the best region by using the minimax selection rule defined in (1.3). Therefore, the estimators of μ_S defined in (1.4, 1.5, 1.6, 2.1, 3.3) are $\delta_{N,1} = 13.74$, $\delta^I_{N,1} = 12.6210$ ($a = 0.1$), 12.7337 ($a = -0.1$), $\delta_{N,2} = 13.1467$, $\delta_{N,3} = 13.1283$ ($a = 0.1$), 13.1636 ($a = -0.1$), $\delta_U = 13.1278$ ($a = 0.1$), 13.1627 ($a = -0.1$).

4 Risk comparisons of estimators

In this section, we perform a simulation study to assess the performances of various estimators of μ_S under the LINEX loss function (1.1). For numerical comparison of various competing estimators, we take $k = 2$ and use the minimax selection rule $\boldsymbol{\psi}^{c^*}$, given in (1.3), for selecting the best exponential population. Clearly $\boldsymbol{\psi}^{c^*}$ is a function of (σ_1, σ_2), that is, $\boldsymbol{\psi}^{c^*}$ varies for different configurations of (σ_1, σ_2). It can be seen that the risks of the competing estimators of μ_S, i.e., the UMRU estimator δ_U, the natural estimators $\delta_{N,1}, \delta_{N,2}$ and $\delta_{N,3}$, and the estimator $\delta^I_{N,1}$ (which improves on the natural estimator $\delta_{N,1}$) depend on unknown parameter (μ_1, μ_2) only through the difference $\mu = \mu_2 - \mu_1$. We have compared the risk functions of the five competing estimators of μ_S for various values of μ and for different configurations of a and (σ_1, σ_2). For notational convenience, let $R_1(\boldsymbol{\mu}) = R(\boldsymbol{\mu}, \delta_{N,1})$, $R_2(\boldsymbol{\mu}) = R(\boldsymbol{\mu}, \delta_{N,2})$, $R_3(\boldsymbol{\mu}) = R(\boldsymbol{\mu}, \delta_{N,3})$, $R_4(\boldsymbol{\mu}) = R(\boldsymbol{\mu}, \delta_U)$ and $R_5(\boldsymbol{\mu}) = R(\boldsymbol{\mu}, \delta^I_{N,1})$ denote the risk functions of the various estimators. Risk values are provided in Tables 2, 3, 4 and 5 for $\mu \in \{-4, -3.6, -3.2, -2.8, -2.4, -2, -1.6, -1.2, -0.8, -0.4, 0, 0.4, 0.8, 1.2, 1.6, 2, 2.4, 2.8, 3.2, 3.6, 4\}$, $(\sigma_1, \sigma_2) \in \{(2, 1), (1, 2)\}$ and $a \in \{0.1, -0.1\}$. The following observations are deduced from Tables 2, 3, 4 and 5.

1. The natural estimator $\delta_{N,1}$ is dominated by all other competing estimators for all configurations of a and (σ_1, σ_2).
2. The UMRUE δ_U is dominated by the natural estimator $\delta_{N,3}$ for all configurations of a and (σ_1, σ_2).

Table 2 *Comparison of risk functions for $a = 0.1$*

	$(\sigma_1, \sigma_2) = (2, 1); c^* \equiv c^*(\sigma_1, \sigma_2) = 0.5753$				
μ	$R_1(\mu)$	$R_2(\mu)$	$R_3(\mu)$	$R_4(\mu)$	$R_5(\mu)$
−4	0.0510	0.0238	0.0235	0.0239	0.0363
−3.6	0.0517	0.0242	0.0238	0.0243	0.0368
−3.2	0.0525	0.0247	0.0243	0.0249	0.0374
−2.8	0.0541	0.0255	0.0250	0.0260	0.0387
−2.4	0.0533	0.0247	0.0242	0.0255	0.0377
−2	0.0563	0.0262	0.0255	0.0272	0.0402
−1.6	0.0555	0.0251	0.0244	0.0267	0.0392
−1.2	0.0555	0.0242	0.0232	0.0265	0.0388
−0.8	0.0579	0.0248	0.0235	0.0280	0.0407
−0.4	0.0562	0.0229	0.0213	0.0276	0.0392
0	0.0558	0.0224	0.0205	0.0281	0.0391
0.4	0.0541	0.0218	0.0199	0.0285	0.0381
0.8	0.0544	0.0228	0.0208	0.0298	0.0388
1.2	0.0521	0.0222	0.0203	0.0295	0.0374
1.6	0.0497	0.0219	0.0200	0.0296	0.0359
2	0.0496	0.0229	0.0211	0.0307	0.0363
2.4	0.0464	0.0219	0.0202	0.0298	0.0339
2.8	0.0436	0.0210	0.0195	0.0290	0.0319
3.2	0.0436	0.0219	0.0204	0.0298	0.0323
3.6	0.0399	0.0203	0.0190	0.0284	0.0295
4	0.0381	0.0197	0.0185	0.0277	0.0282

Table 3 *Comparison of risk functions for $a = -0.1$*

	$(\sigma_1, \sigma_2) = (2, 1); c^* \equiv c^*(\sigma_1, \sigma_2) = 0.5753$				
μ	$R_1(\mu)$	$R_2(\mu)$	$R_3(\mu)$	$R_4(\mu)$	$R_5(\mu)$
−4	0.0350	0.0189	0.0188	0.0188	0.0257
−3.6	0.0349	0.0187	0.0186	0.0186	0.0256
−3.2	0.0353	0.0187	0.0186	0.0188	0.0258
−2.8	0.0357	0.0189	0.0189	0.0192	0.0261
−2.4	0.0360	0.0188	0.0188	0.0193	0.0268
−2	0.0368	0.0191	0.0192	0.0200	0.0268
−1.6	0.0371	0.0187	0.0188	0.0201	0.0269
−1.2	0.0378	0.0181	0.0185	0.0203	0.0272
−0.8	0.0381	0.0175	0.0180	0.0207	0.0274
−0.4	0.0391	0.0168	0.0177	0.0208	0.0281
0	0.0370	0.0154	0.0164	0.0196	0.0265
0.4	0.0368	0.0155	0.0166	0.0194	0.0267
0.8	0.0370	0.0163	0.0174	0.0197	0.0273
1.2	0.0353	0.0160	0.0170	0.0188	0.0261
1.6	0.0340	0.0160	0.0170	0.0182	0.0253
2	0.0320	0.0156	0.0165	0.0173	0.0240
2.4	0.0298	0.0150	0.0158	0.0164	0.0224
2.8	0.0286	0.0149	0.0156	0.0159	0.0215
3.2	0.0268	0.0143	0.0150	0.0151	0.0201
3.6	0.0255	0.0140	0.0145	0.0146	0.0192
4	0.0238	0.0133	0.0138	0.0138	0.0179

Table 4 *Comparison of risk functions for $a = 0.1$*

μ	$(\sigma_1, \sigma_2) = (1, 2); c^* \equiv c^*(\sigma_1, \sigma_2) = -0.5753$				
	$R_1(\mu)$	$R_2(\mu)$	$R_3(\mu)$	$R_4(\mu)$	$R_5(\mu)$
−4	0.0362	0.0182	0.0174	0.0193	0.0266
−3.6	0.0398	0.0201	0.0189	0.0211	0.0293
−3.2	0.0405	0.0199	0.0186	0.0211	0.0296
−2.8	0.0451	0.0218	0.0202	0.0233	0.0332
−2.4	0.0457	0.0214	0.0198	0.0235	0.0334
−2	0.0481	0.0219	0.0202	0.0245	0.0350
−1.6	0.0503	0.0223	0.0205	0.0255	0.0364
−1.2	0.0513	0.0218	0.0199	0.0257	0.0367
−0.8	0.0541	0.0225	0.0205	0.0270	0.0385
−0.4	0.0562	0.0231	0.0212	0.0282	0.0399
0	0.0559	0.0221	0.0207	0.0282	0.0392
0.4	0.0562	0.0227	0.0212	0.0283	0.0391
0.8	0.0568	0.0241	0.0229	0.0290	0.0398
1.2	0.0545	0.0237	0.0228	0.0278	0.0380
1.6	0.0551	0.0248	0.0240	0.0281	0.0388
2	0.0573	0.0267	0.0259	0.0291	0.0409
2.4	0.0538	0.0248	0.0243	0.0269	0.0382
2.8	0.0529	0.0246	0.0241	0.0268	0.0375
3.2	0.0528	0.0246	0.0242	0.0263	0.0375
3.6	0.0527	0.0248	0.0244	0.0261	0.0376
4	0.0517	0.0243	0.0240	0.0251	0.0369

Table 5 *Comparison of risk functions for $a = -0.1$*

μ	$(\sigma_1, \sigma_2) = (1, 2); c^* \equiv c^*(\sigma_1, \sigma_2) = -0.5753$				
	$R_1(\mu)$	$R_2(\mu)$	$R_3(\mu)$	$R_4(\mu)$	$R_5(\mu)$
−4	0.0230	0.0129	0.0133	0.0141	0.0172
−3.6	0.0252	0.0138	0.0143	0.0152	0.0189
−3.2	0.0260	0.0139	0.0145	0.0185	0.0195
−2.8	0.0289	0.0152	0.0159	0.0175	0.0210
−2.4	0.0304	0.0154	0.0162	0.0182	0.0229
−2	0.0316	0.0153	0.0162	0.0184	0.0236
−1.6	0.0339	0.0162	0.0171	0.0197	0.0253
−1.2	0.0352	0.0160	0.0170	0.0200	0.0260
−0.8	0.0369	0.0163	0.0174	0.0206	0.0272
−0.4	0.0369	0.0156	0.0167	0.0200	0.0268
0	0.0376	0.0157	0.0167	0.0200	0.0271
0.4	0.0389	0.0168	0.0177	0.0204	0.0280
0.8	0.0385	0.0175	0.0181	0.0199	0.0276
1.2	0.0376	0.0176	0.0182	0.0192	0.0270
1.6	0.0376	0.0189	0.0191	0.0196	0.0272
2	0.0366	0.0189	0.0190	0.0192	0.0266
2.4	0.0360	0.0189	0.0190	0.0190	0.0263
2.8	0.0362	0.0192	0.0192	0.0192	0.0266
3.2	0.0351	0.0186	0.0185	0.0184	0.0256
3.6	0.0354	0.0190	0.0189	0.0188	0.0260
4	0.0347	0.0188	0.0186	0.0186	0.0255

3. Improved estimator $\delta^I_{N,1}$ performs better than the natural estimator $\delta_{N,1}$ and worst than the natural estimator $\delta_{N,3}$ for all configurations of a and (σ_1, σ_2).
4. The natural estimator $\delta_{N,2}$ is dominated by the natural estimator $\delta_{N,3}$ for $0 < a < \frac{1}{\sigma_0}$ and all configurations (σ_1, σ_2).
5. For $a < 0$ and all configurations (σ_1, σ_2), the natural estimator $\delta_{N,2}$ performs better than the natural estimator $\delta_{N,3}$ except for few values of μ.

The above observations suggest that, in practical applications, the use of natural estimator $\delta_{N,3}$ is recommended for $0 < a < \frac{1}{\sigma_0}$ and the natural estimator $\delta_{N,2}$ is recommended for $a < 0$.

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