A primer on the characterization of the exchangeable Marshall–Olkin copula via monotone sequences

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Abstract. While derivations of the characterization of the d-variate exchangeable Marshall–Olkin copula via d-monotone sequences relying on basic knowledge in probability theory exist in the literature, they contain a myriad of unnecessary relatively complicated computations. We revisit this issue and provide proofs where all undesired artefacts are removed, thereby exposing the simplicity of the characterization. In particular, we give an insightful analytical derivation of the monotonicity conditions based on the monotonicity properties of the survival probabilities.

1 Introduction

We start by recalling the definition of the exchangeable Marshall–Olkin distribution and the definition of a d-monotone sequence.

A random vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d)$ in $(0, \infty)^d$ is said to follow a d-variate Marshall–Olkin ($\mathcal{MO}_d$) distribution if its survival function $S : [0, \infty)^d \to [0, 1]$ satisfies the lack-of-memory (LM) property

$$S(\mathbf{t} + \mathbf{v}) = S(\mathbf{t})S(\mathbf{v}) \tag{1.1}$$

for all $\mathbf{t}, \mathbf{v} \in [0, \infty)^d$ such that $v_1 = \cdots = v_d$. A random vector $\boldsymbol{\tau}$ is called exchangeable if its survival (or, equivalently, its distribution) function S is symmetric, that is,

$$S(t_1, \ldots, t_d) = S(t_{\pi(1)}, \ldots, t_{\pi(d)})$$

for all permutations $\pi : \mathscr{S}_d \to \mathscr{S}_d$ and all $\mathbf{t}$ in the domain of definition of S, where we denote $\mathscr{S}_d = \{1, \ldots, d\}$ with the convention that $\mathscr{S}_0 = \varnothing$. For identically distributed continuous random variables, exchangeability is equivalent to the symmetry of the (survival) copula.

Let $\{a_n\}_{n \in \mathbb{N}_0}$ be a sequence of real numbers. For each $j \in \mathbb{N}$, let ∇^j denote the jth difference operator

$$\nabla^j a_n = \nabla^{j-1} a_n - \nabla^{j-1} a_{n+1}$$

and set $\nabla^0 a_n = a_n$. By induction, it readily follows that

$$\nabla^j a_n = \sum_{i=0}^{j} (-1)^i \binom{j}{i} a_{n+i}; \tag{1.2}$$

see, for example, Gnedin and Pitman (2008). Originally, d-monotone sequences (see Definition 1.1) were referred to as "finite completely monotonic"; see Schoenberg (1932). In this paper, we carry forward the recent terminology of Mai and Scherer (2009).

Definition 1.1 (d-monotone sequence). A sequence $\{a_n\}_{n=0}^{d-1}$ of real numbers is called d-monotone if

$$\nabla^j a_n \geq 0 \qquad \forall j, n = 0, \ldots, d-1 \text{ with } j + n \leq d - 1. \tag{1.3}$$

Remark 1.2. In view of Hausdorff's moment problem, if X is a random variable with values in $[0, 1]$, then the sequence defined by $a_n = \mathbb{E}[X^n]$, $n = 0, \ldots, d - 1$, is d-monotone; see Hausdorff (1921). If X is chosen to be a constant $b \in [0, 1]$, then we have $a_n = b^n$.

By looking at the distribution of a first-passage-time model, in the spirit of Section 5 in Esary and Marshall (1973), Mai and Scherer derived the characterization of the extendible Marshall–Olkin copula with completely monotone sequences; see Theorem 3.3 in Mai and Scherer (2009). A natural extension is Theorem 1.3 below.

In what follows, we denote by $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(d)}$ the ordered list of $t_1, \ldots, t_d \in [0, \infty)$ and take the convention $t_{(0)} = 0$.

Theorem 1.3 (Mai and Scherer (2009)). *Let $d \geq 2$ and $\{a_i\}_{i=0}^{d-1}$ be a sequence of real numbers. The function C defined on $[0, 1]^d$ by*

$$C(\mathbf{u}) = \prod_{i=1}^{d} u_{(i)}^{a_i - 1} \tag{1.4}$$

is a copula if and only if $a_0 = 1$ and $\{a_i\}_{i=0}^{d-1}$ is d-monotone.

Mai and Scherer (2011) proved a very closely related result which gives that copulas as in (1.4) are the survival copulas of exchangeable Marshall–Olkin distributions.

Theorem 1.4 (Mai and Scherer (2011)). *The parametric family of all exchangeable $\mathcal{MO}_d$ survival copulas $C : [0, 1]^d \to [0, 1]$ is given by the set*

$$\left\{ C(\mathbf{u}) = \prod_{i=1}^{d} u_{(i)}^{a_i - 1} \ \Big| \ a_0 = 1, \{a_i\}_{i=0}^{d-1} \text{ is } d\text{-monotone} \right\}. \tag{1.5}$$

The proof of Theorem 1.3 in Mai and Scherer (2009) relies on Pickand's representation of extreme value copulas, whereas the proof of Theorem 1.4 in Mai and Scherer (2011) uses Li's representation of the Marshall–Olkin copula and a reparametrization of the exchangeable Marshall–Olkin distribution. Although simplified in Mai (2010), both proofs are still long and involve a great deal of combinatorial computations. Ressel (2011, 2013) gave alternative non-combinatorial proofs based on the sophisticated theory of completely monotone functions on Abelian semigroups. However, while the author shows that Mai (2010) and Mai and Scherer (2009, 2011, 2012) do not take into account a key property of the copula which drastically simplifies the problem of the characterization, no remedy is provided for the plethora of combinatorial arguments found in their work. The purpose of the present paper is to address this issue, thereby presenting the characterization of the d-variate exchangeable Marshall–Olkin copula via monotone sequences in all its simplicity.

2 Prerequisites on the exchangeable Marshall–Olkin copula

Marshall and Olkin (1967) showed that the survival function $S : [0, \infty)^d \to [0, 1]$ of a $\mathcal{MO}_d$-distributed random vector $\boldsymbol{\tau}$ is given by

$$S(\mathbf{t}) = \mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) = \exp\left(- \sum_{\varnothing \neq I \subseteq \mathscr{S}_d} \widetilde{\lambda}_I \max_{i \in I} \{t_i\} \right) \tag{2.1}$$

for some admissible parameters $\{\widetilde{\lambda}_I : \varnothing \neq I \subseteq \mathscr{S}_d\}$ satisfying $\widetilde{\lambda}_I \geq 0$ for all $\varnothing \neq I \subseteq \mathscr{S}_d$ and $\sum_{I : i \in I} \widetilde{\lambda}_I > 0$ for all $i \in \mathscr{S}_d$. The vector $\boldsymbol{\tau}$ represents lifetimes of d components in a system

subject to independent shocks which cause the different groups of components to fail. For each $\varnothing \neq I \subseteq \mathscr{S}_d$, the parameter $\widetilde{\lambda}_I$ describes the failure rate governing the occurrence of the shock which kills the group of components with indices in I. Hence, $\boldsymbol{\tau}$ is exchangeable precisely when all the parameters $\widetilde{\lambda}_I$ depend only on the cardinalities $|I|$; see Lemma 2.1 below. This result is commonly used in the literature in dimension $d = 2$; see, for example, Giesecke (2003) and Nelsen (2006), p. 47. In arbitrary dimension, it is established in Mai (2010), p. 62 ff.; see also Mai and Scherer (2012), p. 123 ff. Both Mai and Scherer (2011) and Ressel (2013) use this result without providing a proof for it.

Lemma 2.1. *Let $\boldsymbol{\tau}$ follow a $\mathcal{MO}_d$ distribution with survival function* (2.1). *Then $\boldsymbol{\tau}$ is exchangeable if and only if*

$$\widetilde{\lambda}_I = \widetilde{\lambda}_J \qquad \forall \text{ non-empty } I, J \subseteq \mathscr{S}_d \text{ with } |I| = |J|. \tag{2.2}$$

Proof. We first show that the survival function of $\boldsymbol{\tau}$ is symmetric if $\widetilde{\lambda}_I$'s satisfy the condition (2.2). To be able to refer to indices of an ordered list without abusing the notation, we define for each $\mathbf{t} \in [0, \infty)^d$, a permutation $\pi_{\mathbf{t}} : \mathscr{S}_d \to \mathscr{S}_d$ depending on $\mathbf{t}$ such that $t_{\pi_{\mathbf{t}}(1)} \leq t_{\pi_{\mathbf{t}}(2)} \leq \cdots \leq t_{\pi_{\mathbf{t}}(d)}$. Further, for all $i \in \mathscr{S}_d$, we denote $\lambda_i = \widetilde{\lambda}_{\{1,\ldots,i\}}$. Then for all $\mathbf{t} \in [0, \infty)^d$, the survival function of $\boldsymbol{\tau}$ simplifies to

$$\begin{aligned}
S(\mathbf{t}) &= \exp\left(- \sum_{\varnothing \neq I \subseteq \mathscr{S}_d} \lambda_{|I|} \max_{i \in I}\{t_i\}\right) \\
&= \exp\left(- \sum_{i=1}^{d} t_{(i)} \sum_{\substack{I \subseteq \{\pi_{\mathbf{t}}(1),\ldots,\pi_{\mathbf{t}}(i)\} \\ \pi_{\mathbf{t}}(i) \in I}} \lambda_{|I|}\right) \\
&= \exp\left(- \sum_{i=1}^{d} t_{(i)} \sum_{j=1}^{i} \binom{i-1}{j-1} \lambda_j\right),
\end{aligned} \tag{2.3}$$

which is clearly symmetric. For an inductive proof of the reverse implication, we refer the reader to Mai (2010), p. 62 ff., or Mai and Scherer (2012), p. 123 ff.

Alternatively, if $\widetilde{\lambda}_I$'s satisfy the condition (2.2), exchangeability of $\boldsymbol{\tau}$ follows immediately from the formulas (21) and (11) in Shenkman (2017). And, conversely, in view of the formulas (2), (8) and (20) in Shenkman (2017), one has

$$\widetilde{\lambda}_I = \sum_{J \subseteq I} (-1)^{|J|} \ln \mathbb{P}(\tau_i > 1, i \in \overline{I} \cup J), \qquad \varnothing \neq I \subseteq \mathscr{S}_d.$$

Hence, exchangeability of $\boldsymbol{\tau}$ implies (2.2). □

The first derivation of the exchangeable Marshall–Olkin copula in arbitrary dimension (see Lemma 2.2 below) which only requires elementary copula theory appeared in Mai (2010); see also Mai and Scherer (2012). In dimension $d = 2$, it can be found in, for example, Spizzichino (2009) or Cuadras (2009). Observe that admissible parameters $\{\lambda_i : i \in \mathscr{S}_d\}$ of an exchangeable $\mathcal{MO}_d$ distribution with survival function (2.3) satisfy $\lambda_i \geq 0$ for all $i \in \mathscr{S}_d$ and

$$\vartheta = \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda_{i+1} > 0. \tag{2.4}$$

Define the parameter sequence $\{a_i\}_{i=0}^{d-1}$ by

$$a_{i-1} = \frac{1}{\vartheta} \sum_{j=0}^{d-i} \binom{d-i}{j} \lambda_{j+1}, \qquad i \in \mathscr{S}_d, \tag{2.5}$$

while noting that $a_0 = 1$.

Lemma 2.2. *Let $\boldsymbol{\tau}$ have an exchangeable $\mathcal{MO}_d$ distribution with survival function* (2.3) *and* $\{a_i\}_{i=0}^{d-1}$ *be as in* (2.5). *Then the survival copula of $\boldsymbol{\tau}$ is given by*

$$C(\mathbf{u}) = \prod_{i=1}^{d} u_{(i)}^{a_{i-1}}, \qquad \mathbf{u} \in [0,1]^d.$$

Proof. Notice that for all $i \in \mathcal{S}_d$, the marginal survival function S_i of τ_i is given by $S_i(t) = \exp(-t \sum_{i=1}^{d} \binom{d-1}{i-1} \lambda_i) = \exp(-t\vartheta)$ for all $t \in [0, \infty)$, where ϑ is as in (2.4), and $S_i^{-1}(u) = -\ln u / \vartheta$. It follows from Sklar's theorem for survival functions (see, for example, Georges et al. (2001)) that the survival copula of $\boldsymbol{\tau}$ is given by

$$C(\mathbf{u}) = S(S_1^{-1}(u_1), \dots, S_d^{-1}(u_d))$$

$$= \exp\left(\sum_{i=1}^{d} \frac{\ln u_{(d-i+1)}}{\vartheta} \sum_{j=1}^{i} \binom{i-1}{j-1} \lambda_j\right)$$

$$= \prod_{i=1}^{d} u_{(d-i+1)}^{\frac{1}{\vartheta} \sum_{j=1}^{i} \binom{i-1}{j-1} \lambda_j}$$

$$= \prod_{i=1}^{d} u_{(i)}^{\frac{1}{\vartheta} \sum_{j=0}^{d-i} \binom{d-i}{j} \lambda_{j+1}}$$

$$= \prod_{i=1}^{d} u_{(i)}^{a_{i-1}}. \qquad \square$$

We conclude this section with a very simple, yet crucial observation, which is missing in the work of Mai (2010) and Mai and Scherer (2009, 2011, 2012); cf. Theorem 4 and the discussion around Corollary 2 in Ressel (2013).

Remark 2.3. It is readily seen that if the function C in Theorem 1.3 is a copula, then it is an exchangeable $\mathcal{MO}_d$ survival copula. Indeed, by Sklar's theorem for survival functions, the function S defined on $[0, \infty)^d$ by

$$S(\mathbf{t}) = C(e^{-t_1}, \dots, e^{-t_d}) = \prod_{i=1}^{d} e^{-a_{i-1} t_{(d-i+1)}} \tag{2.6}$$

is a survival function of a d-variate distribution whose univariate marginals are identically exponentially distributed with rate parameter a_0. Since S is symmetric and satisfies the LM property (1.1), it is the survival function of an exchangeable $\mathcal{MO}_d$ distribution. Hence, C is an exchangeable $\mathcal{MO}_d$ survival copula.

In passing, we mention that a similar observation makes the relatively long proof of Theorem 3.7 in Mai, Scherer and Shenkman (2013) superfluous.

3 Characterization by d-monotone sequences

From the triangular system given by the set of equations (2.5), we obtain, on the one hand, that the parameter sequence $\{a_i\}_{i=0}^{d-1}$ is non-negative and non-increasing. On the other hand, it is easy to see that the sole knowledge of its unique solution combined with the non-negativity

of the λ_i/ϑ's will yield additional explicit conditions. Lemma 3.1 below establishes, amongst other things, that

$$\lambda_i/\vartheta = \nabla^{i-1} a_{d-i}, \qquad i \in \mathscr{S}_d, \tag{3.1}$$

is the solution of the triangular system (set $n = i - 1$ and $k = d - i$ in (3.2) and compare to (2.5)). Alternatively, the system can be solved directly using the Möbius inversion formula; see Remark 3.3.

Lemma 3.1. *Let $\{a_n\}_{n=0}^{d-1}$ be a sequence of real numbers. Then we have for all $n = 0, \ldots, d - 1$ and all $k = 0, \ldots, d - 1 - n$ that*

$$a_n = \sum_{i=0}^{k} \binom{k}{i} \nabla^i a_{n+k-i}. \tag{3.2}$$

Proof. For $d = 1$, the claim holds trivially. Let $d \geq 2$. We use a nested induction with an outer backward induction on n and an inner forward induction on k. It is easily seen that (3.2) is true for $n = d - 1$ and $k = 0$. Assume that (3.2) is true for all $n = j, \ldots, d - 1$ for some $j \in \{1, \ldots, d - 1\}$ and all $k = 0, \ldots, d - 1 - n$. For the induction step to succeed, it suffices to show that (3.2) holds for $n = j - 1$ and all $k = 0, \ldots, d - 1 - n$. For $n = j - 1$ and $k = 0$, (3.2) holds trivially. Assume it holds for $n = j - 1$ and all $k = 0, \ldots, l$ for some $l \in \{0, \ldots, d - 2 - n\}$. Then, to conclude the proof, it remains to show that (3.2) holds for $n = j - 1$ and $k = l + 1$:

$$a_{j-1} = \sum_{i=0}^{l} \binom{l}{i} \nabla^i a_{j-1+l-i}$$

$$= \sum_{i=0}^{l} \binom{l}{i} \nabla^i (\nabla a_{j-1+l-i} + a_{j+l-i})$$

$$= \sum_{i=1}^{l+1} \binom{l}{i-1} \nabla^i a_{j+l-i} + \sum_{i=0}^{l} \binom{l}{i} \nabla^i a_{j+l-i}$$

$$= \sum_{i=1}^{l} \binom{l+1}{i} \nabla^i a_{j+l-i} + \nabla^{l+1} a_{j-1} + a_{j+l}$$

$$= \sum_{i=0}^{l+1} \binom{l+1}{i} \nabla^i a_{j+l-i},$$

where the first equality is due to the induction hypothesis and the penultimate equality to Pascal's rule. Hence the lemma. $\square$

Lemma 3.1 is a special case of the formula (A.1) in Shenkman (2017) and a generalized version of a result found in the proof of Lemma A.1 in Mai and Scherer (2011). It can be used to reduce the d-monotonicity conditions in (1.3) to only d conditions.

Remark 3.2 (Reduced conditions). Schoenberg (1932) noticed that the set of conditions in (1.3) is equivalent to

$$\nabla^{i-1} a_{d-i} \geq 0, \qquad i \in \mathscr{S}_d. \tag{3.3}$$

While a direct proof can be given, see Schoenberg (1932) or Mai and Scherer (2009), the claim simply follows from Lemma 3.1. More precisely, assume (3.3) and set $k = d - 1 - (j + n)$ in (3.2). Then observe that for all $j, n = 0, \ldots, d - 1$ with $j + n \leq d - 1$,

$$\nabla^j a_n = \sum_{i=0}^{d-1-(j+n)} \binom{d - 1 - (j + n)}{i} \nabla^{j+i} a_{d-1-(j+i)} \geq 0,$$

where $j \leq j + i \leq d - 1 - n$, that is, $0 \leq j + i \leq d - 1$.

We are now in a position to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. If C is a copula, then, by Remark 2.3, it is the survival copula of an exchangeable $\mathcal{MO}_d$ distribution. Hence, in view of Lemma 2.2, there are some $\lambda_i \geq 0$, $i \in \mathscr{S}_d$, satisfying (2.4), such that the sequence $\{a_i\}_{i=0}^{d-1}$ is given by (2.5). In particular, one has $a_0 = 1$. Moreover, it follows from (3.1) together with Remark 3.2 that $\{a_i\}_{i=0}^{d-1}$ is d-monotone. Conversely, given a d-monotone sequence $\{a_i\}_{i=0}^{d-1}$ with $a_0 = 1$, define $\lambda_i = \nabla^{i-1} a_{d-i} \geq 0$, $i \in \mathscr{S}_d$. It follows from Lemma 3.1 with $k = d - 1 - n$ that $\{a_i\}_{i=0}^{d-1}$ are given by

$$a_{i-1} = \sum_{j=0}^{d-i} \binom{d - i}{j} \lambda_{j+1}, \qquad i \in \mathscr{S}_d.$$

In particular, one has $\sum_{i=0}^{d-1} \binom{d-1}{i} \lambda_{i+1} = a_0 > 0$. By Lemma 2.2, the survival copula of the $\mathcal{MO}_d$ distribution with parameters λ_i, $i \in \mathscr{S}_d$, is given by the function C in (1.4). Hence, C is a copula. $\qquad \square$

We remark that while proving Theorem 1.3, we have, in the process, shown Lemma A.1 in Mai and Scherer (2011). Additionally, note that because the univariate marginals of copulas are standardized to uniform distributions, the copula parameters $\{a_i\}_{i=0}^{d-1}$ determine the λ's only up to the parameter of the univariate marginals. Namely, for any d-monotone sequence $\{a_i\}_{i=0}^{d-1}$ with $a_0 = 1$ and any $\vartheta > 0$, we have that $\lambda_i = \vartheta \nabla^{i-1} a_{d-i}$, $i \in \mathscr{S}_d$, define admissible parameters of an exchangeable $\mathcal{MO}_d$ distribution whose univariate marginals follow an exponential distribution with rate parameter ϑ.

Proof of Theorem 1.4. The fact that (1.5) is a subset of exchangeable $\mathcal{MO}_d$ survival copulas follows directly from Theorem 1.3 and Remark 2.3. The converse inclusion is an immediate consequence of Lemma 2.2 (or for that matter Lemma 2.1 in Mai and Scherer (2011)) and Theorem 1.3. $\qquad \square$

Remark 3.3. To solve (2.5) for λ_i/ϑ, $i \in \mathscr{S}_d$, we set $\lambda_i/\vartheta = \nabla^{i-1} a_{d-i}$ for all $i \in \mathscr{S}_d$ and then applied Lemma 3.1 to infer that it is the solution. Lest the reader be unsatisfied by such an ansatz, we note that

$$a_{i-1} = \sum_{j=0}^{d-i} \binom{d - i}{j} \lambda_{j+1}, \qquad i \in \mathscr{S}_d, \tag{3.4}$$

can also be inverted directly with the help of the Möbius inversion formula. For completeness, we recall that if f and F are two functions defined on the subsets of $\mathscr{S}_d$ such that $F(I) = \sum_{J \subseteq I} f(J)$ for all $I \subseteq \mathscr{S}_d$, then $f(I) = \sum_{J \subseteq I} (-1)^{|I \setminus J|} F(J)$; see Lemma 3.1 in Genest, Quessy and Rémillard (2007) or Theorem 2 in Weiss (2009). To invert (3.4), define f on the

subsets of $\mathscr{S}_{d-1}$ by $f(I) = \lambda_{|I|+1}$ and for any $I \subseteq \mathscr{S}_{d-1}$, set

$$F(I) = \sum_{J \subseteq I} f(J)$$

$$= \sum_{j=0}^{|I|} \binom{|I|}{j} \lambda_{j+1}$$

$$= a_{d-|I|-1}.$$

Then, by application of the Möbius inversion formula, it follows for all $I \subseteq \mathscr{S}_{d-1}$ that

$$\lambda_{|I|+1} = \sum_{J \subseteq I} (-1)^{|I \setminus J|} F(J)$$

$$= \sum_{J \subseteq I} (-1)^{|I \setminus J|} a_{d-|J|-1}$$

$$= \sum_{j=0}^{|I|} (-1)^{|I|-j} \binom{|I|}{j} a_{d-j-1}$$

$$= \sum_{i=0}^{|I|} (-1)^{i} \binom{|I|}{i} a_{d-|I|-1+i}$$

$$= \nabla^{|I|} a_{d-|I|-1},$$

where the penultimate equality uses the substitution $i = |I| - j$ and the symmetry of the binomial coefficients, and the last equality is due to (1.2).

4 An alternative proof of necessity in Theorem 1.3

Mai and Scherer (2009) showed the necessity in Theorem 1.3 analytically thanks to some relatively complicated combinatorial argumentation. In the spirit of Shenkman (2017), we exploit the monotonicity properties of the survival probabilities to provide a much simpler proof.

Proof of necessity in Theorem 1.3. If C is a copula, then, due to uniform marginals, $C(u, 1, \ldots, 1) = u$ for all $u \in [0, 1]$, and therefore $a_0 = 1$. By Sklar's theorem for survival functions, the symmetric function S defined on $[0, \infty)^d$ as in (2.6) is a survival function of an exchangeable random vector $\boldsymbol{\tau}$. Define the sequence $\{\beta_k\}_{k=0}^d$ by $\beta_0 = 1$ and

$$\beta_k = \prod_{i=1}^{k} e^{-a_{i-1}}, \qquad k \in \mathscr{S}_d.$$

Let $n \in \mathbb{N}$ arbitrary. Observe that for all $k = 0, \ldots, d$, $\sqrt[n]{\beta_k} = \mathbb{P}(\tau_i > 1/n, i \in \mathscr{S}_k)$, and, hence,

$$\nabla^{d-k} \sqrt[n]{\beta_k} = \sum_{j=0}^{d-k} (-1)^j \binom{d-k}{j} \mathbb{P}(\tau_i > 1/n, i \in \mathscr{S}_{k+j})$$

$$= \sum_{J \subseteq \{k+1,\ldots,d\}} (-1)^{|J|} \mathbb{P}(\tau_i > 1/n, i \in \mathscr{S}_k \cup J)$$

$$= \mathbb{P}(\tau_i > 1/n, i \in \mathscr{S}_k)$$

$$- \sum_{\varnothing \neq J \subseteq \{k+1,\ldots,d\}} (-1)^{|J|-1} \mathbb{P}(\tau_i > 1/n, i \in \mathscr{S}_k \cup J)$$

$$= \mathbb{P}(\tau_i > 1/n, i \in \mathscr{S}_k)$$

$$- \mathbb{P}\left(\bigcup_{j \in \{k+1,\ldots,d\}} = \{\tau_i > 1/n, i \in \mathscr{S}_k \cup \{j\}\} \right)$$

$$= \mathbb{P}\big(\{\tau_i > 1/n, i \in \mathscr{S}_k\} \cap \{\tau_i \leq 1/n, i \in \{k+1, \ldots, d\}\} \big) \geq 0,$$

where the first equality is due to (1.2), the second to exchangeability of $\boldsymbol{\tau}$ and the fourth to the inclusion-exclusion principle. In view of Remark 3.2, we conclude that $\{\sqrt[n]{\beta_k}\}_{k=0}^d$ is $(d+1)$-monotone for all $n \in \mathbb{N}$. Further, since $\{\beta_k\}_{k=0}^d$ is strictly positive, it follows for all $k = 0, \ldots, d-1$ that

$$\nabla^{d-k} \ln \beta_k = \lim_{n \to \infty} n \nabla^{d-k} \big(\sqrt[n]{\beta_k} - 1 \big) = \lim_{n \to \infty} n \nabla^{d-k} \sqrt[n]{\beta_k} \geq 0.$$

Finally, observe that for all $k = 0, \ldots, d-1$,

$$a_k = \sum_{i=1}^{k+1} a_{i-1} - \sum_{i=1}^{k} a_{i-1} = -\ln \beta_{k+1} + \ln \beta_k = \nabla \ln \beta_k$$

and, consequently, $\nabla^{d-1-k} a_k = \nabla^{d-k} \ln \beta_k \geq 0$ for all $k = 0, \ldots, d-1$. Hence, $\{a_i\}_{i=0}^{d-1}$ is d-monotone by Remark 3.2. $\square$

5 Conclusion

In this article, we have shown that the proofs of Theorems 1.3 and 1.4 provided by Mai and Scherer could be significantly simplified. Furthermore, we have given an alternative proof of necessity in Theorem 1.3 which is interesting in its own right since it illustrates well the analytical tractability of the exchangeable Marshall–Olkin distribution.

Acknowledgments

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