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Local limit theorems for shock models

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Abstract. In many physical systems, failure occurs when the stress after shock n first exceed a critical level x. We consider the number of shocks $\tau(x)$ to failure and obtain more detailed information that is usually obtained about asymptotic distribution by using local limit theorems. We consider extreme and cumulative shock models with both univariate and multivariate shock types. We derive the limiting distribution of $\tau(x)$ and the rate of convergence to that. For the extreme shock model, rate of convergence for regularly varying shock distributions is found using the weighted Kolmogorov probability metric. For the cumulative shock model, we examine the rate of convergence to Gaussian densities.

1 Introduction

In modeling of technical processes, one is often interested in the reliability of the system. A common cause of failure in mechanical, physical and electronic systems appears in situations when the applied load exceeds the strength. "Load" refers to mechanical stress, a voltage or internally generated stress such as temperature or absorbed energy. "Strength" refers to any resisting physical property such as hardness, melting point and so on. The failures of mechanical, physical or electronic systems, as a result of applied loads, occur primarily due to one or more of the following causes:

1. Overload, leading directly to failure;
2. Fatigue damage, that is caused when a system is under repeated stress. Fatigue damage is cumulative so that repeated or cyclical stress above a critical level will eventually result in failure of the system;
3. Material destruction, that is usually a very complex physical process that consists of many micro-defects. Each of these micro-defects is harmless, but the accumulation of many micro-defects may lead to the destruction of the material. See, for example, Landau and Lifshits (1976).

Two basic shock models are the extreme shock model where failure occurs the first time a shock is received that exceeds a critical level and the cumulative shock model when the sum of shocks received to date first exceeds a critical level.

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To model the above situations, let $\varepsilon(t)$ denote a measure for the functioning of a physical system. Such a measure can be, for example, the absorbed energy up to time t, the temperature at time t, the amount of contamination or pollution at time t, the accumulated damage up to time t and so on. We assume that $\varepsilon(t)$ is a nondecreasing, right-continuous function of $t \geq 0$ with $\varepsilon(0) = 0$. If $\varepsilon(t)$ reaches a certain critical level, E, the system fails or the system is fully contaminated and it should either be replaced by a new system or some control action should be undertaken. We are interested in the time $\tau(E)$ to reach a given level attainment E, defined as

$$\tau(E) = \inf\{t : \varepsilon(t) > E\}.$$

In general, the function $\varepsilon(t)$ is random and, therefore, also $\tau(E)$ is a random element.

This paper obtains the distribution of $\tau(E)$ and studies asymptotic properties of $\tau(E)$ for several models. The paper by Gut and Hüsler (1999) (and references therein) show that in the extreme shock model the time of shock, $\tau(E)$ is geometric with mean $p(E) = 1/\Pr(X > E)$, where X is the value of a generic shock. They also show that $p(E)\tau(E)$ converges in distribution to an exponential distribution with rate 1 as x approaches the upper limit of shocks $x_F = \sup\{x : \Pr(X < x) < 1\}$. Along similar lines, Gut (1990) applies the central limit theorem to obtain convergence in distribution to a normal distribution for a cumulative shock model.

In contrast to these results, one of the main contributions of this paper is to use different methods from what are normally used to obtain more detailed information about the asymptotic properties for the shock number of failure by means of local limit theorems. We are interested in the shock number where failure occurs and set $t = 0, 1, 2, \ldots$. Other papers also consider the time between shocks, for example, the two papers by Gut (1990) and Gut and Hüsler (1999) referred to above, but that is not the focus of this paper. In general, local limit theorems provide a uniform convergence result to a density function g over a given region. Convergence of densities to a density can imply total variation convergence (e.g., Shorack (2000), p. 380), which is a stronger form of convergence than convergence in distribution. More specifically, in this paper were are interested in results of the form

$$\lim_{n\to\infty} \sup \bigl|\Pr\bigl(\tau(E) = n\bigr) - g(E, n)\bigr| = 0, \tag{1}$$

where the limiting function is $g(E, n)$. We identify specific types of limiting functions that can apply to different models. As well, we examine rates of convergence to the limiting function by scaling time in an appropriate way.

2 Shock models

We consider two main classes of models: extreme shock models and cumulative shock models and, for each, consider univariate and multivariate versions.

2.1 Univariate extreme shock models

In the univariate extreme shock model, we assume that time is discrete and a system fails due to one extreme shock, cf. Shanthikumar and Sumita (1983), Gut and Hüsler (1999). In this case, we set

$$\varepsilon(n) = \max(X_1, X_2, \dots, X_n) = X_{n:n}, \qquad n \geq 1,$$

where X_i, $i = 1, 2, \dots$ are nonnegative, independent and identically distributed random variables and $X_{k:n}$, $k = i = 1, 2, \dots, n$ denote the order statistics of X_i, $i = 1, 2, \dots, n$.

We can interpret X_i as the size of the ith shock and $\varepsilon(n)$ as the strongest shock after n such shocks. The time of a given level attainment x is given by

$$\tau_e(x) = \min\{n : \max(X_i, i = 1, 2, \dots, n) > x\}$$

and it is of interest to study the (discrete) probability distribution

$$P(n, x) \equiv \Pr(\tau_e(x) = n), \qquad n \geq 1.$$

We can also consider a k-out-of-n shock model. In this type of model, the system fails if there are k out of n shocks above a critical level. In this case, one studies for each fixed k random variables of the type

$$\tau_{e,k}(x) = \min\{n : X_{n-k+1:n} > x\}.$$

A mixture of these two models can be studied as well where the system fails if there is one big shock or if there are several smaller shocks. In this case, the system fails at time $\delta_k(x, y)$, where

$$\delta_k(x, y) = \min\{n : X_{n:n} > x \text{ or } X_{n-k+1:n} > y\}.$$

2.2 Univariate cumulative shock model

In this model, we assume that the damage is cumulative. We set $\varepsilon(0) = 0$ and have

$$\varepsilon(n) = \sum_{i=1}^{n} X_i, \qquad n \geq 1,$$

where the X_i, $i = 1, 2, \dots$ are nonnegative, independent and identically distributed random variables. We can interpret X_i as the size of the ith shock and $\varepsilon(n)$ as the total (cumulative) damage after n such shocks. Cumulative shock models of this type have been studied before. We mention, for example, the paper of Sumita and Shanthikumar (1985), Gut (1990) and the papers of Virchenko (1998) and Virchenko et al. (1999). Now, the time of a given level attainment x is given by

$$\tau_c(x) = \min\left\{n : \sum_{i=1}^{n} X_i > x\right\}$$

and it is of interest to study the (discrete) probability distribution

$$P(n, x) \equiv \Pr(\tau_c(x) = n), \qquad n \geq 1.$$

2.3 Multivariate models

In reliability models, it is possible that a system is subject to different types of damage leading to failure. We consider here two different types of damage denoted by (X, Y) and i.i.d. copies (X_i, Y_i). For the multivariate extreme shock models, we consider, among others, system failure that occurs at $\tau_1(x, y)$ or $\tau_2(x, y)$, where

$$\tau_1(x, y) = \min\{n : X_{n:n} > x \text{ and } Y_{n:n} > y\},$$
$$\tau_2(x, y) = \min\{n : X_{n:n} > x \text{ or } Y_{n:n} > y\}.$$

In the first case, a failure occurs when both types of damage exceed a critical level. In the second case, the system fails when one of the types of damage exceeds a critical level.

Related models are $\tau_{1,k,l}(x, y)$ and $\tau_{2,k,l}(x, y)$, where

$$\tau_{1,k,l}(x, y) = \min\{n : X_{n-k+1:n} > x \text{ and } Y_{n-k+1:n} > y\},$$
$$\tau_{2,k,l}(x, y) = \min\{n : X_{n-k+1:n} > x \text{ or } Y_{n-l+1:n} > y\}.$$

We can also consider multivariate cumulative shock models. In this case, we study $\tau_3(x, y)$ or $\tau_4(x, y)$, where

$$\tau_3(x, y) = \min\left\{n : \sum_{i=1}^{n} X_i > x \text{ and } \sum_{i=1}^{n} Y_i > y\right\},$$
$$\tau_4(x, y) = \min\left\{n : \sum_{i=1}^{n} X_i > x \text{ or } \sum_{i=1}^{n} Y_i > y\right\}.$$

There are many feasible other alternative types of failure models.

2.4 Other models

It is clear that one can also combine the extreme and cumulative shock models and study

$$\tau_m(x, y) = \min\bigl(\tau_e(x), \tau_c(y)\bigr).$$

Such models were studied by Gut (1988, 2001) and Mallor et al. (2006).

Another approach is to consider shock-run models, cf. Mallor and Omey (2001), Mallor et al. (2006). In such a model, the system fails if there is a run of consecutive shocks above a certain level. In this case, one can study, for each fixed $k = 1, 2, \dots,$ random variables of the type

$$\tau_r(x; k) = \min\{n : X_n \in R, X_{n+1} \in R, \dots, X_{n-k+1} \in R\},$$

where a critical shock is defined by a region R.

In this paper, we study into detail the asymptotic local behavior of the variables $\tau_e(x)$, $\tau_{e,k}(x)$ and $\tau_{1,k,l}(x, y)$. We plan to study other models in a forthcoming paper.

3 Regularly varying functions and extreme value theory

In this section, we review the regularly varying function theory and the extreme value theory that will be used in the paper.

3.1 Regularly varying functions

Univariate case. The tail distribution function $\overline{F}(x)$ is regularly varying at infinity with index a if it satisfies

$$\lim_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^a \qquad \text{for all } x > 0.$$

We denote this by $\overline{F}(x) \in RV(a)$. For properties and applications of this class and related classes of functions, we refer to de Haan (1970) and Bingham et al. (1987).

Multivariate case. Let $r(x) \in RV(\alpha)$, $s(x) \in RV(\beta)$ with $\alpha, \beta > 0$. We say that the tail $\overline{F} = 1 - F(x, y)$ is in the class $RV(r, s, \lambda)$ if it satisfies

$$\lim_{t\to\infty} t\overline{F}\big(r(t)x, s(t)y\big) = \lambda(x, y), \tag{2}$$

for all $x, y > 0$ with $\min(x, y) < \infty$.

Using local uniform convergence, from (2) we obtain that

$$\lim_{t\to\infty} t\overline{F}\big(r(tx), s(ty)\big) = \lambda\big(x^\alpha, y^\beta\big).$$

Using

$$t\overline{F}\big(r(tzx), s(tzy)\big) = \frac{1}{z} tz \lim_{t\to\infty} t\overline{F}\big(r(tzx), s(tzy)\big),$$

we find that

$$\lambda\big(z^\alpha x^\alpha, z^\beta y^\beta\big) = \frac{1}{z}\lambda\big(x^\alpha, y^\beta\big).$$

Since x and y were arbitrary, we obtain that

$$z\lambda\big(z^\alpha x, z^\beta y\big) = \lambda(x, y).$$

For more detail, refer to Mallor and Omey (2006).

3.2 Extreme value theory

Univariate case . The random variable X is in the max-domain of attraction of the non-degenerate random variable Z if there exist normalizing constants $a(n) > 0$ and $b(n) \in \Re$ such that as $n \to \infty$,

$$\frac{X_{n:n} - b(n)}{a(n)} \xrightarrow{d} Z,$$

or equivalently

$$F^n\big(a(n)x + b(n)\big) \to G(x),$$

where $G(x)$ is the d.f. of Z.

This is the classical setting of extreme value theory, cf. de Haan (1970). It can be shown that the d.f. of Z is one of three extreme value types: $\phi_a(x) = \exp(-x^{-a})$ $(x \geq 0, a > 0)$, $\psi_a(x) = \phi_a(-1/x)$ or $\Lambda(x) = \phi_1(\exp(x))$.

The domain of attraction of $\Lambda(x)$ is related to the gamma class. A d.f. $F(x)$ is in the class $\Gamma(g)$ if the tail satisfies

$$\lim_{t\to\infty} \frac{\overline{F}(t + g(t)x)}{\overline{F}(t)} = \exp(-x) \qquad \text{for all } x \in \Re.$$

The next proposition is well known. Recall that we assume that $F(x) < 1$ for all x.

Proposition 3.1.

(i) *Suppose that* $\overline{F}(x) \in RV(-\alpha)$, *with* $\alpha > 0$. *Then*

$$\lim_{n\to\infty} F^n\big(a(n)x\big) = \phi_\alpha(x),$$

where $a(n) > 0$ *is such that* $n\overline{F}(a(n)) \to 1$.

(ii) *Suppose that* $F \in \Gamma(g)$ *and let* $U(x)$ *denote the inverse of* $-\log(\overline{F}(x))$. *Then*

$$\lim_{n\to\infty} F^n\big(a(n)x + b(n)\big) = \Lambda(x),$$

where $b(n) = U(\log(n))$ *and* $a(n) = U(\log(ne)) - U(\log(n))$.

Multivariate case. The vector (X, Y) is in the max-domain of attraction of the nondegenerate vector (Z, V) if there exist normalizing constants $a(n) > 0, c(n) > 0$ and $b(n), d(n) \in \Re$, such that

$$\left(\frac{X_{n:n} - b(n)}{a(n)}, \frac{Y_{n:n} - d(n)}{c(n)}\right) \overset{d}{\Longrightarrow} (Z, V),$$

or, equivalently,

$$F^n\big(a(n)x + b(n), c(n)y + d(n)\big) \to G(x, y), \tag{3}$$

where G is the d.f. of (Z, V).

From the convergence of the marginals, the d.f. of Z and of V is one of the three types that were discussed in Section 3.2. Also, the marginal convergence determines the values of the normalizing constants.

A sufficient condition for (3) can be formulated in terms of multivariate regular variation. Suppose that $r(x) \in RV(\alpha)$ and $s(x) \in RV(\beta)$ where $\alpha, \beta > 0$. Without loss of generality, we assume that r and s are increasing functions.

If (2) holds, then (3) holds with $G(x, y) = \exp -\lambda(x, y)$ and with $a(n) = r(n)$, $c(n) = s(n)$ and $b(n) = d(n) = 0$.

From (3), we get that the marginals satisfy:

$$\lim_{t\to\infty} t\overline{F}_1\bigl(r(t)x\bigr) = \lambda(x, \infty),$$

$$\lim_{t\to\infty} t\overline{F}_2\bigl(s(t)y\bigr) = \lambda(\infty, y).$$

In order to have nondegenerate limits in (3), we require that $\lambda(x, \infty) > 0$ and $\lambda(\infty, y) > 0$.

In the relation for the first marginal, replace t by $r^i(t)$, where $r^i(t)$ is the inverse of $r(t)$ to give

$$\lim_{t\to\infty} r^i(t)\overline{F}_1(tx) = \lambda(x, \infty).$$

It follows that $\overline{F}_1(x)$ is regularly varying with index $-1/\alpha$. Moreover, $\overline{F}_1(t) \sim c_1 1/r^i(t)$. In a similar way, we have that $\overline{F}_2(t)$ is regularly varying with index $-1/\beta$ and that $\overline{F}_2(t) \sim c_2 1/s^i(t)$. In (3), we obtain that the marginals of the limiting distribution are given by $\phi_{1/\alpha}$ and $\phi_{1/\beta}$.

4 Univariate extreme shock model

4.1 Single shock model

Using the notation as in Section 1, the time of a given level attainment x is given by

$$\tau_e(x) = \min\bigl\{n : \max(X_1, X_2, \dots, X_n) > x\bigr\},$$

where $X, X_1, X_2, \dots$ are positive random variables having the same distribution as X. We assume that X has an infinite endpoint.

We are interested in the (discrete) probability distribution

$$P(n, x) \equiv \Pr\bigl(\tau_e(x) = n\bigr), \qquad n \ge 1.$$

It is easy to see that

$$\Pr\bigl(\tau_e(x) > n\bigr) = \Pr(X_{n:n} \le x) = F^n(x),$$

where $F(x)$ denotes the distribution function of X. From here, it follows that

$$\Pr\bigl(\tau_e(x) = n + 1\bigr) = F^n(x)\overline{F}(x),$$

where $\overline{F}(x) = 1 - F(x)$ is the tail of $F(x)$.

4.2 *k*-out-of-*n* shocks model

For $\tau_{e,k}(x)$, we have

$$\tau_{e,k}(x) = \min\{n : X_{n-k+1:n} > x\}$$

and

$$\Pr\big(\tau_{e,k}(x) > n\big) = \Pr(X_{n-k+1:n} \le x) = \sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}(x)\overline{F}^{i}(x).$$

For $k \ge 2$, we find that

$$\begin{aligned}\Pr\big(\tau_{e,k}(x) = n+1\big) &= \Pr\big(\tau_{e,k}(x) > n\big) - \Pr\big(\tau_{e,k}(x) > n+1\big)\\ &= \sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}(x)\overline{F}^{i}(x) - \sum_{i=0}^{k-1} \binom{n+1}{i} F^{n+1-i}(x)\overline{F}^{i}(x).\end{aligned}$$

Entering $F^{n+1}(x)$ in the formula leads to

$$\begin{aligned}\Pr\big(\tau_{e,k}(x) = n+1\big) = &\sum_{i=0}^{k-1} \binom{n}{i} \big(F^{n-i}(x) - F^{n+1}(x)\big)\overline{F}^{i}(x)\\ &- \sum_{i=0}^{k-1} \binom{n+1}{i} \big(F^{n+1-i}(x) - F^{n+1}(x)\big)\overline{F}^{i}(x)\\ &+ F^{n+1}(x)\sum_{i=0}^{k-1}\left(\binom{n}{i} - \binom{n+1}{i}\right)\overline{F}^{i}(x)\end{aligned}$$

and we find that

$$\begin{aligned}\Pr\big(\tau_{e,k}(x) = n+1\big) = &\sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}(x)\big(1 - F^{i+1}(x)\big)\overline{F}^{i}(x)\\ &- \sum_{i=0}^{k-1} \binom{n+1}{i} F^{n+1-i}(x)\big(1 - F^{i}(x)\big)\overline{F}^{i}(x)\\ &+ F^{n+1}(x)\sum_{i=0}^{k-1} \frac{-i}{n+1-i}\overline{F}^{i}(x).\end{aligned}$$

4.3 Limiting distributions

For fixed x, we see that $\tau_e(x)$ has a geometric distribution. For fixed x, we also see that as $n \to \infty$, we have $\Pr(\tau_e(x) = n+1) \to 0$. To obtain a nondegenerate limit, we use extreme value theory. The form of the limiting distributions is given by the following proposition.

Proposition 4.1. *Suppose that* $\overline{F}(x) \in RV(-\alpha)$, *with* $\alpha > 0$. *Then*

(i)

$$\lim_{n\to\infty} n \Pr\bigl(\tau_e\bigl(a(n)x\bigr) = n+1\bigr) = x^{-\alpha}\phi_\alpha(x)$$

and

(ii)

$$\lim_{n\to\infty} n \Pr\bigl(\tau_{e,k}\bigl(a(n)x\bigr) = n+1\bigr) = \phi_\alpha(x) H_{k;\alpha}(x),$$

where $H_{k,\alpha}(x) = x^{-\alpha k}/(k-1)!$.

Proof. To prove (i), we have

$$\begin{aligned} n \Pr\bigl(\tau_e\bigl(a(n)x\bigr) = n+1\bigr) &= F^n\bigl(a(n)x\bigr) n\overline{F}\bigl(a(n)x\bigr) \\ &= F^n\bigl(a(n)x\bigr)\frac{\overline{F}(a(n)x)}{\overline{F}(a(n))} n\overline{F}\bigl(a(n)\bigr). \end{aligned}$$

With our choice of $a(n)$, we obtain that

$$n \Pr\bigl(\tau_e\bigl(a(n)x\bigr) = n+1\bigr) \to x^{-\alpha}\phi_a(x).$$

For the result (ii), we have

$$n \Pr\bigl(\tau_{e,k}\bigl(a(n)x\bigr) = n+1\bigr) = I - II + III,$$

where

$$I = n\sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}\bigl(a(n)x\bigr)\bigl(1 - F^{i+1}\bigl(a(n)x\bigr)\bigr)\overline{F}^i\bigl(a(n)x\bigr),$$

$$II = n\sum_{i=1}^{k-1} \binom{n+1}{i} F^{n+1-i}\bigl(a(n)x\bigr)\bigl(1 - F^{i}\bigl(a(n)x\bigr)\bigr)\overline{F}^i\bigl(a(n)x\bigr),$$

$$III = nF^{n+1}\bigl(a(n)x\bigr)\sum_{i=1}^{k-1} \frac{-i}{n+1-i}\overline{F}^i\bigl(a(n)x\bigr).$$

First consider I. We have $F^{n-i}(a(n)x) \to \phi_\alpha(x)$. Note that $1 - F^{i+1}(a(n)x) \sim (i+1)\overline{F}(a(n)x)$. Using

$$\binom{n}{i} \sim \frac{n^i}{i!}$$

and $n\overline{F}(a(n)x) \to x^{-\alpha}$, we obtain that

$$I \to \phi_\alpha(x)\sum_{i=0}^{k-1} \frac{i+1}{i!} x^{-\alpha(i+1)}.$$

In a similar way, we obtain that

$$II = \phi_\alpha(x) \sum_{i=1}^{k-1} \frac{i}{i!} x^{-\alpha(i+1)},$$

$$III \to \phi_\alpha(x) \sum_{i=1}^{k-1} \frac{-i}{i!} x^{-\alpha i}.$$

It follows that $n\,\mathrm{Pr}(\tau_{e,k}(a(n)x) = n+1) \to \phi_\alpha(x) H_k(x)$, where

$$\begin{aligned} H_{k,\alpha}(x) &= \sum_{i=0}^{k-1} \frac{i+1}{i!} x^{-\alpha(i+1)} - \sum_{i=1}^{k-1} \frac{i}{i!} x^{-\alpha(i+1)} - \sum_{i=1}^{k-1} \frac{i}{i!} x^{-\alpha i} \\ &= \sum_{i=0}^{k-1} \frac{1}{i!} x^{-\alpha(i+1)} - \sum_{i=1}^{k-1} \frac{i}{i!} x^{-\alpha i} \\ &= \frac{1}{(k-1)!} x^{-\alpha k}. \end{aligned}$$

This proves the result. □

Remark. Note that the limit in Proposition 4.1(i) can be written as $x^{-\alpha}\phi_\alpha(x) = \alpha^{-1}\phi'_\alpha(x)$. This shows that we obtained a real local limit result.

For a d.f. in the domain of attraction of $\Lambda(x)$, we have a similar result.

Proposition 4.2. *Suppose that $F \in \Gamma(g)$ and let $a(n)$ and $b(n)$ be as in Proposition 3.1. Then*

$$\lim_{n\to\infty} n\,\mathrm{Pr}\big(\tau_e\big(a(n)x + b(n)\big) = n+1\big) = e^{-x}\Lambda(x)$$

and

$$\lim_{n\to\infty} n\,\mathrm{Pr}\big(\tau_{e,k}\big(a(n)x + b(n)\big) = n+1\big) = \Lambda(x) G_k(x),$$

where $G_k(x) = H_{k,1}(e^{-x})$.

Proof. To prove the first statement, we use

$$n\,\mathrm{Pr}\big(\tau_e\big(a(n)x + b(n)\big) = n+1\big) = F^n\big(a(n)x + b(n)\big)\overline{F}\big(a(n)x + b(n)\big).$$

Using

$$\lim_{n\to\infty} F^n\big(a(n)x + b(n)\big) = \Lambda(x),$$

and

$$n\overline{F}\big(a(n)x + b(n)\big) \to e^{-x},$$

we obtain the first statement.

For the second statement, we can proceed as in the proof of Proposition 4.1. □

Remark. If $F(x) = \phi_\alpha(x)$, we have $a(n) = n^{1/\alpha}$ and F is max-stable, that is, $F^n(n^{1/\alpha}x) = F(x)$. In this case, we use the notation $\tau_e^{(\alpha)}$ for τ_e and we have

$$\Pr\bigl(\tau_e^{(\alpha)}(n^{1/\alpha}x) = n+1\bigr) = \phi_\alpha(x)\bigl(1-\phi_\alpha(n^{1/\alpha}x)\bigr).$$

This shows that we cannot hope that $\tau_e(x)$ has some stability property.

4.4 Combined single shock and k-out-of-n shocks model

The next model can be considered as a combination of τ_e and $\tau_{e,k}$. In this model, we have a system failure at time $\delta_k(x, y)$, where

$$\delta_k(x, y) = \min\{n : X_{n:n} > x \text{ or } X_{n-k+1:n} > y\}.$$

This means that we have a failure if we have either one "extra" large shock or a number of large shocks. In this model, we have

$$\Pr\bigl(\delta_k(x, y) > n\bigr) = \Pr(X_{n:n} \le x, X_{n-k+1:n} \le y).$$

If $x \le y$, we have

$$\Pr\bigl(\delta_k(x, y) > n\bigr) = \Pr(X_{n:n} \le x) = F^n(x).$$

If $y \le x$, we have

$$\Pr\bigl(\delta_k(x, y) > n\bigr) = \sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}(y)\bigl(F(x) - F(y)\bigr)^i.$$

We have the following result.

Proposition 4.3. *Suppose that $\overline{F}(x) \in RV(-\alpha)$, with $\alpha > 0$. For $y \le x$, we have*

$$\lim_{n\to\infty} n \Pr\bigl(\delta_k\bigl(a(n)x, a(n)y\bigr) = n+1\bigr) = \phi_\alpha(y) H_k(x, y),$$

where

$$H_k(x, y) = \sum_{i=0}^{k-1} \frac{1}{i!} y^{-\alpha}\bigl(y^{-\alpha} - x^{-\alpha}\bigr)^i - \sum_{i=1}^{k-1} \frac{i}{i!}\bigl(y^{-\alpha} - x^{-\alpha}\bigr)^i.$$

Proof. See Appendix A. □

4.5 Rates of convergence

We obtain a rate of convergence result for Proposition 4.1 in this section. Probability metrics have been used by Smith (1982) and Omey and Rachev (1988) to obtain rates of convergence results in extreme value theory. Among others, they considered the distance ρ_r between distribution functions or random variables. Let

X, Y denote random variables with d.f. $F(x)$ and $G(x)$ respectively. For $r \geq 0$, we define

$$\rho_r(X, Y) = \rho_r(F, G) = \sup_x |x|^r \big|F(x) - G(x)\big|.$$

For $r = 0$, we find the uniform metric.

To formulate our results, we apply a monotone transformation and replace X by X^α allowing us to assume without loss of generality that $\alpha = 1$. The main result of this section is the following theorem.

Theorem 4.1. *Suppose that $r > 1$ and that $\rho_r(F, \phi_1) < \infty$. Then, as $n \to \infty$,*

$$\sup_{x\geq 0}\big|\Pr\big(\tau_e(x) = n+1\big) - \phi_1(x/n)\big(1 - \phi_1(x)\big)\big| = O(1)n^{-r}$$

and

$$\sup_{x\geq 0}\big|n\Pr\big(\tau_e(nx) = n+1\big) - n\phi_1(x)\big(1 - \phi_1(nx)\big)\big| = O(1)n^{1-r}.$$

Proof. For convenience, we write $\phi(x)$ in the place of $\phi_1(x)$. To prove the theorem, first note that

$$\Pr\big(\tau_e(x) = n+1\big) - \phi^n(x)\overline{\phi}(x) = A + B,$$

where

$$A(x) = \big(F^n(x) - \phi^n(x)\big)\overline{F}(x),$$
$$B(x) = \phi^n(x)\big(\overline{F}(x) - \overline{\phi}(x)\big).$$

First, consider $B(x)$. Since $\phi^n(x) = \phi(x/n)$, our assumptions imply that

$$\big|B(x)\big| \leq \phi^n(x)\big|F(x) - \phi(x)\big| \leq \rho_r(F, \phi)x^{-r}\phi(x/n).$$

Since for $r > 0$, we have $\phi(x)/x^r \leq C(r) = (r/e)^r$, we obtain that

$$\sup_{x\geq 0}\big|B(x)\big| \leq \rho_r(F, G)C(r)n^{-r}.$$

Next, we consider $A(x)$. Using the inequality

$$\big|a^n - b^n\big| \leq n\max\big(a^{n-1}, b^{n-1}\big)|a - b|,$$

we see that

$$\big|A(x)\big| \leq n\overline{F}(x)\max\big(F^{n-1}(x), \phi^{n-1}(x)\big)\big|F(x) - \phi(x)\big|.$$

Using $\rho_r(F, \phi) < \infty$, we have $x|F(x) - \phi(x)| = O(x^{-r+1}) = o(1)$ and it follows that $x\overline{F}(x) \to 1$. As a consequence, $x\overline{F}(x)$ is bounded by, say, K. We find that

$$\big|A(x)\big| \leq nK\rho_r(F, \phi)x^{-r-1}\max\big(F^{n-1}(x), \phi^{n-1}(x)\big).$$

Consider the case where $F(x) \le \phi(x)$. In this case, we have that

$$\begin{aligned}|A(x)| &\le nK\rho_r(F,\phi)x^{-r-1}\phi^{n-1}(x)\\ &= nK\rho_r(F,\phi)x^{-r-1}\phi\big(x/(n-1)\big)\\ &\le K\rho_r(F,G)C(r+1)n(n-1)^{-1-r}.\end{aligned}$$

Next, consider the case where $\phi(x) \le F(x)$. In this case, we use $\log(z) \le z-1$ ($z \ge 1$) to find that

$$0 \le \log\left(\frac{F(x)}{\phi(x)}\right) \le \frac{F(x)-\phi(x)}{\phi(x)}.$$

Now, fix $x^\circ > 0$ such that $\phi(x^\circ) > 0$. For $x \ge x^\circ$, we have

$$0 \le (n-1)\log\left(\frac{F(x)}{\phi(x)}\right) \le K(n-1)x^{-r},$$

where $K = \rho_r(F,\phi)/\phi(x_0)$. Now, choose x_n in such a way that $(n-1)x_n^{-r} = 1$. If n is sufficiently large, we find that $x_n \ge x^\circ$. For $x \ge x_n$ and $n \ge N^\circ$, we have

$$0 \le (n-1)\log\left(\frac{F(x)}{\phi(x)}\right) \le K,$$

or equivalently that

$$F^{n-1}(x) \le e^K\phi^{n-1}(x).$$

Now we can proceed as in the first case to obtain

$$\sup_{x\ge x_n}|A(x)| = O(1)n^{-r}.$$

If $x^\circ \le x \le x_n$, we can use $F(x) \le F(x_n)$ to see that

$$F^{n-1}(x) \le F^{n-1}(x_n) \le e^K\phi^{n-1}(x_n)$$

and then we find that

$$\begin{aligned}|A(x)| &\le n\overline{F}(x)\phi^{n-1}(x_n)e^K|F(x)-\phi(x)|\\ &\le e^K n\phi\big(x_n/(n-1)\big),\end{aligned}$$

for some positive constant K. Since $r > 1$ and by our choice of x_n, it follows that $n\phi(x_n/(n-1)) \to 0$ exponentially fast. We conclude that

$$\sup_{x\ge x^\circ}|A(x)| = O(1)n^{-r}.$$

Finally, we consider the case where $x \le x^\circ$. In this case, we have

$$|A(x)| \le F^n\big(x^\circ\big) + \phi^n\big(x^\circ\big).$$

Both terms of the right-hand side of this inequality tend to zero exponentially fast. This proves the result. □

Remark. Note that Theorem 4.1 is not a statement about convergence in distribution, but rather a statement about convergence of density functions. This limits the optimal use of probability metrics. For nonnegative functions $\psi(\cdot)$, one can prove the following result following the same lines as in the proof of Theorem 4.1.

Proposition 4.4. *Suppose that $r > 1$ and that $\int_0^\infty x^r \psi(x)|F(x) - \phi_1(x)|\,dx < \infty$. Then we have*

$$\int_0^\infty \psi(x)\big|nP\big(\tau_e(x) = n+1\big) - \phi_1(x/n)\overline{\phi}_1(x)\big| = O(1)n^{-r}.$$

Remark. Under the conditions of the theorem, we have $\overline{F}(x)/\overline{\phi}_1(x) \to 1$, and we see that $x\overline{F}(x) \to 1$. From Proposition 4.1, we find that

$$\lim_{n\to\infty} n\Pr\big(\tau_e(nx) = n+1\big) = \phi_1(x)\big(-\log\phi_1(x)\big). \tag{4}$$

Our theorem gives that

$$\sup_{x\geq 0}\big|n\Pr\big(\tau_e(nx) = n+1\big) - n\phi_1(x)\big(1-\phi_1(nx)\big)\big| = O(1)n^{1-r}.$$

It follows that

$$\begin{aligned}
&\big|n\Pr\big(\tau_e(nx) = n+1\big) - \phi_1(x)\big(-\log\phi_1(x)\big)\big| \\
&\quad\leq \big|n\Pr\big(\tau_e(nx) = n+1\big) - n\phi_1(x)\big(1-\phi_1(nx)\big)\big| \\
&\qquad + \phi_1(x)\big|-\log\phi_1(x) - n\big(1-\phi_1(nx)\big)\big|.
\end{aligned}$$

Using $|1 - x - \exp(-x)| \leq \frac{1}{2}x^2$, we see that

$$\phi_1(x)\big|-\log\phi_1(x) - n\big(1-\phi_1(nx)\big)\big| \leq \frac{1}{n}\frac{\phi_1(x)}{x^2} \leq \frac{K}{n},$$

for some positive constant K. We conclude that

$$\sup_{x\geq 0}\big|n\Pr\big(\tau_e(nx) = n+1\big) - \phi_1(x)\big(-\log\phi_1(x)\big)\big| = O(1)n^{1-r} + O(1)n^{-1}.$$

This bound is sharp and shows that the rate of convergence in (4) is given by $O(1)n^{-1}$if $r \geq 2$. If $r > 2$, it is better to approximate $n\Pr(\tau_e(nx) = n+1)$ by $n\phi_1(x)(1-\phi_1(nx))$ than by its limit.

5 Multivariate extreme shock model

In this section, we study local limit theorems for multivariate extreme shock models.

Consider nonnegative and i.i.d. random vectors $(X, Y), (X_1, Y_1), (X_2, Y_2), \ldots$. The d.f. of (X, Y) will be denoted by $F(x, y)$. The marginal distributions will be denoted by F_1 and F_2.

We first consider systems in which failure occurs at discrete time $\tau_1(x, y)$ or $\tau_2(x, y)$ where

$$\tau_1(x, y) = \min\{n : X_{n:n} > x \text{ and } Y_{n:n} > y\},$$
$$\tau_2(x, y) = \min\{n : X_{n:n} > x \text{ or } Y_{n:n} > y\}.$$

It is clear that

$$\Pr\big(\tau_2(x, y) > n\big) = \Pr(X_{n:n} \le x, Y_{n:n} \le y) = F^n(x, y)$$

and

$$\begin{aligned}\Pr\big(\tau_1(x, y) > n\big) &= \Pr(X_{n:n} \le x \text{ or } Y_{n:n} \le y)\\ &= F_1^n(x) + F_2^n(y) - F^n(x, y).\end{aligned}$$

It follows that

$$\Pr\big(\tau_2(x, y) = n + 1\big) = F^n(x, y)\overline{F}(x, y),$$
$$\Pr\big(\tau_1(x, y) = n + 1\big) = F_1^n(x)\overline{F}_1(x) + F_2^n(y)\overline{F}_2(y) - F^n(x, y)\overline{F}(x, y).$$

5.1 Limiting distribution

Assuming that (X, Y) is in the class $RV(r, s, \lambda)$ it is straightforward to obtain the following result.

Proposition 5.1. *Suppose that* (2) *holds. Then, as $n \to \infty$*

$$\begin{aligned}&n \Pr\big(\tau_2\big(a(n)x, c(n)y\big) = n + 1\big)\\ &\qquad \to G(x, y)\lambda(x, y),\\ &n \Pr\big(\tau_1\big(a(n)x, c(n)y\big) = n + 1\big)\\ &\qquad \to G_1(x)\lambda(x, \infty) + G_2(y)\lambda(\infty, y) - G(x, y)\lambda(x, y).\end{aligned}$$

5.2 A rate of convergence result

To derive a rate of converge result, we introduce the following weighted Kolmogorov metric. Using $M(x, y) = \min(|x|, |y|)$ and d.f. F and G, we define

$$\rho_r(F, G) = \sup_{x,y\ge 0, \min(x,y)<\infty} M^r(x, y)\big|F(x, y) - G(x, y)\big|.$$

(See Rachev (1991).) Note that this definition implies that the marginals have a finite ρ_r-distance.

For simplicity, we assume that (3) and (2) hold with $G_1(x) = G_2(x) = \phi_1(x)$. In this case, we have $\alpha = \beta = 1$ and

$$z\lambda(zx, zy) = \lambda(x, y).$$

Theorem 5.1. *Suppose that $r > 1$ and that $\rho_r(F, G) < \infty$. Then*

$$\sup_{x,y\geq 0}\left|\Pr\bigl(\tau_2(x, y) = n + 1\bigr) - G\left(\frac{x}{n}, \frac{y}{n}\right)\overline{G}(x, y)\right| = O(1)n^{-r}.$$

Proof. The proof follows similar lines to that of Theorem 4.1 and is given in Appendix B. □

6 Cumulative shock model

6.1 Univariate shock model

Using the notation as in Section 2, in the univariate cumulative shock model the time of a given level attainment x is given by

$$\tau_c(x) = \min\left\{n : \varepsilon(n) = \sum_{i=1}^{n} X_i > x\right\},$$

where $X_1, X_2, \ldots$ are nonnegative random variables having the same distribution as X. We are interested in the (discrete) probability distribution

$$P(n, x) \equiv \Pr\bigl(\tau_c(x) = n\bigr), \qquad n \geq 1.$$

We prove a local limit theorem for $\tau_c(x)$.

Clearly, this setting is that of classical renewal theory (cf. Feller (1971)) and many properties are known. If $\mathrm{E}(X) = \mu < \infty$, we have

$$\frac{1}{n}\varepsilon(n) \overset{\text{a.s.}}{\to} \mu \qquad \text{as } n \to \infty$$

and, as a consequence, $\tau_c(x)$ is finite for each level $x > 0$. Moreover, we have

$$\frac{1}{x}\tau_c(x) \overset{\text{a.s.}}{\to} \frac{1}{\mu} \qquad \text{as } x \to \infty.$$

If $\operatorname{var}(\xi) = \sigma^2 < \infty$, then $\tau_c(x)$ obeys a central limit theorem (cf. Feller (1971), Chapter XI.5) and as $x \to \infty$, we have

$$\frac{\tau_c(x) - x/\mu}{\sqrt{x}} \overset{d}{\Longrightarrow} Z,$$

where $Z \sim N(0, \sigma^2/\mu^3)$. Feller (1971), Chapter XI. 5, also gives results if $\sigma^2 = \infty$ assuming that X is in the domain of attraction of a stable law with parameter $\alpha < 2$. For an overview, we refer to Mallor and Omey (2006).

In the present paper, we assume that $\sigma^2 < \infty$. Let $g(x)$ denote the standard Gaussian density function, that is, $g(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ and let $\phi(z)$ denote the common characteristic function of the X_i. The main result of this section is the following.

Theorem 6.1. *Suppose that* $|\phi(z)|^m$ *is integrable for some* $m \geq 1$*. Then as* $n \to \infty$,

$$\sup_x \left| \frac{\sigma\sqrt{n}}{\mu} \Pr(n+1, x) - g\left(\frac{x - \mu n}{\sigma\sqrt{n}}\right) \right| = o(1).$$

Moreover, if also $EX_1^3 < \infty$, *then*

$$\sqrt{n} \sup_x \left| \frac{\sigma\sqrt{n}}{\mu} \Pr(n+1, x) - g\left(\frac{x - \mu n}{\sigma\sqrt{n}}\right) \right| = O(1).$$

Before proceeding to the proof of the theorem, we present some preliminary results.

By assumption, we have $\sigma^2 < \infty$, and hence $\varepsilon(n)$ satisfies the central limit theorem. From Feller (1971), we have the following results.

Lemma 6.1 (Feller (1971), Chapter XV.5). *Suppose that* $|\phi(z)|^m$ *is integrable for some integer* $m \geq 1$*. Then for all* $n \geq m$ *the random variable* $(\varepsilon(n) - n\mu)/\sigma\sqrt{n}$ *has a density* $g_n(x)$ *and* $g_n(x)$ *converges uniformly to the density* $g(x)$.

Lemma 6.2 (Feller (1971), Chapter XVI). *Suppose that* $|\phi(z)|^m$ *is integrable for some integer* $m \geq 1$ *and suppose that* $\mathrm{E}(X_1^3) = a < \infty$*. Then as* $n \to \infty$, *we have*

$$g_n(x) - g(x) - \frac{a}{6\sigma^3\sqrt{n}}(x^3 - 3x)g(x) = o(1)\frac{1}{\sqrt{n}}$$

uniformly in x.

Under the condition of Lemma 6.1, we obtain that for $n \geq m$, the random variable $\varepsilon(n)$ also has a density. We denote it by $f_n(x)$. The relation between $f_n(x)$ and $g_n(x)$ is given by

$$g_n(x) = \sigma\sqrt{n} f_n(x\sigma\sqrt{n} + \mu n). \tag{5}$$

From Lemmas 6.1 and 6.2 and using the same notation, we obtain the following result.

Lemma 6.3. *Suppose that* $|\phi(x)|^m$ *is integrable for some* $m \geq 1$*. Then as* $n \to \infty$,

$$\sup_x \left| g_n(n\mu + \sqrt{n}\sigma x) - g(x) \right| \to 0.$$

Moreover, if also $\mathrm{E}(X^3) < \infty$, *then as* $n \to \infty$,

$$\sqrt{n} \sup_x \left| g_n(n\mu + \sqrt{n}\sigma x) - g(x) \right| = O(1).$$

Now we are ready to prove the main theorem.

Proof. Let $F(x) = \Pr(X \le x)$ denote the common distribution function of the X_i and let $F^{*n}(x) = \Pr(\varepsilon(n) \le x)$ denote the distribution function of $\varepsilon(n)$. The following relation is well known in renewal theory:

$$\begin{aligned}\Pr(n+1, x) &= \Pr\big(\varepsilon(n) \le x < \varepsilon(n+1)\big)\\ &= \int_0^x \big(1 - F(x-y)\big)\, dF^{*n}(y).\end{aligned}$$

If we introduce the notation

$$h(x) = \frac{1}{\mu}\big(1 - F(x)\big), \qquad x \ge 0$$

we find that

$$\frac{1}{\mu}\Pr(n+1, x) = \int_0^x h(x-y)\, dF^{*n}(y). \tag{6}$$

Note, that since $h(x)$ is a bounded probability density function, also $\frac{1}{\mu}\Pr(n+1, x)$ is a bounded probability density function.

For $n \ge m$, we can write

$$\frac{1}{\mu}\Pr(n+1, x) = \int_0^x h(x-y) f_n(y)\, dy$$

so that, using (5), we have

$$\frac{\sigma\sqrt{n}}{\mu}\Pr(n+1, x) = \int_0^x h(y) g_n\Big(\frac{x-y-\mu n}{\sigma\sqrt{n}}\Big)\, dy.$$

Now, we consider the three terms in the following relation:

$$\frac{\sigma\sqrt{n}}{\mu}\Pr(n+1, x) - g\Big(\frac{x-\mu n}{\sigma\sqrt{n}}\Big) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}I_1 &= \int_0^x h(y)\Big(g_n\Big(\frac{x-y-\mu n}{\sigma\sqrt{n}}\Big) - g\Big(\frac{x-y-\mu n}{\sigma\sqrt{n}}\Big)\Big)\, dy;\\ I_2 &= \int_0^x h(y)\Big(g\Big(\frac{x-y-\mu n}{\sigma\sqrt{n}}\Big) - g\Big(\frac{x-\mu n}{\sigma\sqrt{n}}\Big)\Big)\, dy;\\ I_3 &= g\Big(\frac{x-\mu n}{\sigma\sqrt{n}}\Big)\int_x^\infty h(y)\, dy.\end{aligned}$$

First, consider I_1. We clearly have

$$|I_1| \le \sup_x \big|g_n(x) - g(x)\big| \int_0^x h(y)\, dy.$$

Using Lemma 6.3, we obtain that

$$\sup_{x\geq 0} |I_1| \to 0 \qquad \text{as } n \to \infty.$$

Also, if $\mathrm{E}(\xi_1^3) < \infty$, we find that

$$\sqrt{n} \sup_{x\geq 0} |I_1| = O(1) \qquad \text{as } n \to \infty.$$

In the second term, I_2, we use the mean value theorem to see that

$$g\left(\frac{x-y-\mu n}{\sigma\sqrt{n}}\right) - g\left(\frac{x-\mu n}{\sigma\sqrt{n}}\right) = -\frac{y}{\sigma\sqrt{n}} g'(\theta),$$

where

$$\frac{x-y-\mu n}{\sigma\sqrt{n}} \leq \theta \leq \frac{x-\mu n}{\sigma\sqrt{n}}.$$

Since $|g'(x)|$ is bounded, we find that

$$|I_2| \leq \frac{K}{\sqrt{n}} \int_0^x yh(y)\,dy,$$

where K is a positive constant. Finally, since $\sigma^2 < \infty$, we obtain that

$$\int_0^x yh(y)\,dy \leq \int_0^\infty yh(y)\,dy = \frac{2}{\mu} \int_0^\infty y^2\,dF(y) < \infty.$$

We conclude that

$$\sup_{x\geq 0} |I_2| = O(1)\frac{1}{\sqrt{n}} \qquad \text{as } n \to \infty.$$

Finally, we consider I_3. In the case where $x > n\mu/2$, we have

$$|I_3| \leq g(0) \int_x^\infty h(y)\,dy \leq \frac{g(0)}{x} \int_x^\infty yh(y)\,dy \leq K\frac{1}{n},$$

for some positive constant K. On the other hand, if $0 \leq x \leq n\mu/2$, we find

$$|I_3| \leq g\left(\frac{-\mu n}{2\sigma\sqrt{n}}\right) \int_x^\infty h(y)\,dy \leq g\left(\frac{\mu\sqrt{n}}{2\sigma}\right)$$

and using $|xg(x)| \leq K$, we obtain that

$$|I_3| \leq K\frac{2\sigma}{\mu\sqrt{n}}.$$

Combining the two cases, we find that

$$\sup_{x\geq 0} |I_3| \leq \frac{K}{\sqrt{n}},$$

for some positive constant K. To complete the proof, we consider the case where $x < 0$. In this case, we trivially have

$$\left|\frac{\sigma\sqrt{n}}{\mu}\Pr(n+1,x) - g\left(\frac{x-\mu n}{\sigma\sqrt{n}}\right)\right| = g\left(\frac{x-\mu n}{\sigma\sqrt{n}}\right) \leq g\left(\frac{-\mu n}{\sigma\sqrt{n}}\right)$$

and consequently

$$\sup_{x<0}\left|\frac{\sigma\sqrt{n}}{\mu}\Pr(n+1,x) - g\left(\frac{x-\mu n}{\sigma\sqrt{n}}\right)\right| \leq g\left(\frac{\mu n}{\sigma\sqrt{n}}\right) \leq \frac{K}{\sqrt{n}}$$

for some constant K. This proves the result. □

6.2 Bivariate cumulative shock model

In the bivariate cumulative shock model, one studies processes of the following type:

$$\tau_c(x,y) = \min\left\{n : \sum_{i=1}^{n} X_i > x \text{ or } \sum_{i=1}^{n} Y_i > y\right\}.$$

Now we find that

$$\Pr\big(\tau_c(x,y) > n\big) = \Pr\left(\sum_{i=1}^{n} X_i \leq x, \sum_{i=1}^{n} Y_i \leq y\right) = F^{*n}(x,y),$$

where $F^{*n}(x,y)$ denotes the n-fold convolution of the d.f. F of the vector (X,Y). To study these types of processes, one needs multivariate central limit theory, local limit theorems and multivariate renewal theory. For an overview of univariate and multivariate renewal functions and generalized renewal functions, we refer to Mallor and Omey (2006).

Let $\tau_{c,X}(x)$ and $\tau_{c,Y}(y)$ be defined as (cf. Section 6)

$$\tau_{c,X}(x) = \min\left\{n : \sum_{i=1}^{n} X_i > x\right\}, \qquad \tau_{c,Y}(y) = \min\left\{n : \sum_{i=1}^{n} Y_i > y\right\}.$$

In this case, we see that $\tau_c(x,y) = \min(\tau_{c,X}(x), \tau_{c,Y}(y))$.

For convenience, let $\mu = \mathrm{E}(X)$ and $\nu = \mathrm{E}(Y)$. Using $\tau_{c,X}(x)/x \overset{\text{a.s.}}{\to} 1/\mu$ and $\tau_{c,Y}(y)/y \overset{\text{a.s.}}{\to} 1/\nu$, we get that as $\min(x,y) \to \infty$,

$$\frac{\tau_c(x\mu, y\nu)}{\min(x,y)} \overset{\text{a.s.}}{\to} 1.$$

Now, suppose that $\sigma_1^2 = \mathrm{var}(X) < \infty$ and $\sigma_2^2 = \mathrm{var}(Y) < \infty$. In this case, we have

$$\begin{aligned}
&\Pr\big(\tau_c(n\mu + \sqrt{n}x, n\nu + \sqrt{n}y) > n\big) \\
&\quad = \Pr\left(\sum_{i=1}^{n} X_i \leq n\mu + \sqrt{n}x, \sum_{i=1}^{n} Y_i \leq \nu n + \sqrt{n}y\right) \\
&\quad \to \Pr(\sigma_1 Z_1 \leq x, \sigma_2 Z_2 \leq y),
\end{aligned}$$

where (Z_1, Z_2) has a bivariate normal distribution with standard normal marginals. We obtain

$$\begin{aligned}\Pr\bigl(\tau_c(x, y) = n + 1\bigr) &= F^{*n}(x, y) - F^{*n+1}(x, y) \\ &= \int_0^x \int_0^y h(x - u, y - v)\, dF^{*n}(u, v),\end{aligned}$$

where

$$h(x, y) = 1 - F(x, y).$$

Now, assume that F has a density f and that the density satisfies the central limit theorem, that is,

$$g_n(x, y) \to \phi(x, y),$$

where

$$g_n(x, y) = \sigma_1\sigma_2 n f^{\otimes n}(\mu n + \sigma_1\sqrt{n}x, \nu n + \sigma_2\sqrt{n}y)$$

and $f^{\otimes n}$ is the n-fold convolution of f. The function ϕ is a bivariate normal density with standard normal marginals:

$$\phi(x, y) = C \exp \frac{-1}{2(1 - \rho^2)}(x^2 + y^2 - 2\rho xy).$$

Using this notation, we find that

$$\begin{aligned}&\Pr\bigl(\tau_c(x, y) = n + 1\bigr) \\ &\quad = \int_0^x \int_0^y h(x - u, y - v) f^{\otimes n}(u, v)\, du\, dv \\ &\quad = \int_0^x \int_0^y h(u, v) f^{\otimes n}(x - u, y - v)\, du\, dv \\ &\quad = \frac{1}{\sigma_1\sigma_2 n} \int_0^x \int_0^y h(u, v) g_n\biggl(\frac{x - u - \mu n}{\sigma_1\sqrt{n}}, \frac{y - v - \nu n}{\sigma_2\sqrt{n}}\biggr) du\, dv.\end{aligned}$$

It follows that

$$\begin{aligned}&\sigma_1\sigma_2 n \Pr\bigl(\tau_c(x, y) = n + 1\bigr) - \phi\biggl(\frac{x - \mu n}{\sigma_1\sqrt{n}}, \frac{y - \nu n}{\sigma_2\sqrt{n}}\biggr) \int_0^x \int_0^y h(u, v)\, du\, dv \\ &\quad = \int_0^x \int_0^y h(u, v)\biggl(g_n\biggl(\frac{x - u - \mu n}{\sigma_1\sqrt{n}}, \frac{y - v - \nu n}{\sigma_2\sqrt{n}}\biggr) \\ &\qquad\qquad - \phi\biggl(\frac{x - \mu n}{\sigma_1\sqrt{n}}, \frac{y - \nu n}{\sigma_2\sqrt{n}}\biggr)\biggr) du\, dv \\ &\quad = I + II,\end{aligned}$$

where

$$I = \int_0^x \int_0^y h(u,v)\Big(g_n\Big(\frac{x-u-\mu n}{\sigma_1\sqrt{n}}, \frac{y-v-\nu n}{\sigma_2\sqrt{n}}\Big) - \phi\Big(\frac{x-u-\mu n}{\sigma_1\sqrt{n}}, \frac{y-v-\nu n}{\sigma_2\sqrt{n}}\Big)\Big)\,du\,dv,$$

$$II = \int_0^x \int_0^y h(u,v)\Big(\phi\Big(\frac{x-u-\mu n}{\sigma_1\sqrt{n}}, \frac{y-v-\nu n}{\sigma_2\sqrt{n}}\Big) - \phi\Big(\frac{x-\mu n}{\sigma_1\sqrt{n}}, \frac{y-\nu n}{\sigma_2\sqrt{n}}\Big)\Big)\,du\,dv.$$

If we have uniform convergence for the densities, then we find that

$$|I| \le \sup_{x,y\ge 0}\big|g_n(x,y) - \phi(x,y)\big| \int_0^x \int_0^y h(u,v)\,du\,dv.$$

For the second term, we use the mean value theorem to obtain that

$$\phi\Big(\frac{x-u-\mu n}{\sigma_1\sqrt{n}}, \frac{y-v-\nu n}{\sigma_2\sqrt{n}}\Big) - \phi\Big(\frac{x-\mu n}{\sigma_1\sqrt{n}}, \frac{y-\nu n}{\sigma_2\sqrt{n}}\Big) = \frac{\vartheta\phi}{\vartheta x}\Big(\theta_1, \frac{y-v-\nu n}{\sigma_2\sqrt{n}}\Big)\frac{u}{\sigma_1\sqrt{n}} + \frac{\vartheta\phi}{\vartheta y}\Big(\frac{x-\mu n}{\sigma_1\sqrt{n}}, \theta_2\Big)\frac{v}{\sigma_2\sqrt{n}}.$$

Since the partial derivatives of ϕ are bounded, for some constant K we find that

$$|II| \le \frac{K}{\sqrt{n}} \int_0^x \int_0^y (u+v)h(u,v)\,du\,dv.$$

Our results at least show that

$$\sigma_1\sigma_2 n \Pr\big(\tau_c(x,y) = n+1\big) - \phi\Big(\frac{x-\mu n}{\sigma_1\sqrt{n}}, \frac{y-\nu n}{\sigma_2\sqrt{n}}\Big)\int_0^x \int_0^y h(u,v)\,du\,dv \to 0,$$

for fixed values of $x, y > 0$.

Appendix A: Proof of Proposition 4.3

Proof. Proceeding as in the proof of Proposition 4.1, we have

$$\Pr\big(\delta_k(x,y) = n+1\big) = I - II + III,$$

where

$$I = \sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}(y)\big(1 - F^{i+1}(y)\big)\big(F(x) - F(y)\big)^i,$$

$$II = \sum_{i=1}^{k-1} \binom{n+1}{i} F^{n+1-i}(y)\big(1 - F^i(y)\big)\big(F(x) - F(y)\big)^i,$$

$$III = F^{n+1}(y) \sum_{i=1}^{k-1} \binom{n}{i} \frac{-i}{n+1-i}\big(F(x) - F(y)\big)^i.$$

Now, we replace x and y by $a(n)x$ and $a(n)y$. For the first term, we get

$$nI = n\sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}\big(a(n)y\big)\big(1 - F^{i+1}\big(a(n)y\big)\big)\big(F\big(a(n)x\big) - F\big(a(n)y\big)\big)^i.$$

As in the proof of Proposition 4.1, we get that

$$nI \sim n\sum_{i=0}^{k-1} \frac{n^i}{i!}\phi_\alpha(y)(i+1)\big(1 - F\big(a(n)y\big)\big)\big(F\big(a(n)x\big) - F\big(a(n)y\big)\big)^i.$$

Using $n(1 - F(a(n)x)) \to x^{-\alpha}$ and $n(F(a(n)x) - F(a(n)y)) \to y^{-\alpha} - x^{-\alpha}$, we obtain that

$$nI \to \phi_\alpha(y) \sum_{i=0}^{k-1} \frac{i+1}{i!} y^{-\alpha}\big(y^{-\alpha} - x^{-\alpha}\big)^i.$$

In a similar way, we obtain that

$$nII \to \phi_\alpha(y) \sum_{i=1}^{k-1} \frac{i}{i!} y^{-\alpha}\big(y^{-\alpha} - x^{-\alpha}\big)^i$$

and

$$nIII \to -\phi_\alpha(y) \sum_{i=1}^{k-1} \frac{i}{i!}\big(y^{-\alpha} - x^{-\alpha}\big)^i.$$

The result follows. □

Appendix B: Proof of Theorem 5.1

Proof. To prove the result, first note that

$$\begin{aligned}\Pr\big(\tau_2(x,y) = n+1\big) - G\Big(\frac{x}{n}, \frac{y}{n}\Big)\overline{G}(x,y) &= F^n(x,y)\overline{F}(x,y) - G^n(x,y)\overline{G}(x,y) \\ &= A(x,y) + B(x,y),\end{aligned}$$

where

$$A(x,y) = \big(F^n(x,y) - G^n(x,y)\big)\overline{F}(x,y),$$
$$B(x,y) = G^n(x,y)\big(\overline{F}(x,y) - \overline{G}(x,y)\big).$$

Choose $x^\circ, y^\circ > 0$ such that $F_1(x^\circ), F_2(y^\circ), G(x^\circ, y^\circ) > 0$. If $x \le x^\circ$ (and similarly if $y \le y^\circ$), we have

$$\begin{aligned}\left|A(x,y)\right| + \left|B(x,y)\right| &\le F^n(x,y) + 2G^n(x,y)\\ &\le F_1^n(x^\circ) + 2G_1^n(x^\circ),\end{aligned}$$

which converges to 0 exponentially fast.

From now on, assume that $x \ge x^\circ$ and $y \ge y^\circ > 0$. First, consider $B(x,y)$. Our assumptions imply that

$$\begin{aligned}\left|B(x,y)\right| &\le G^n(x,y)\left|F(x,y) - G(x,y)\right|\\ &\le \rho_r(F,G)\bigl(\min(x,y)\bigr)^{-r} G\left(\frac{x}{n}, \frac{y}{n}\right).\end{aligned}$$

Suppose that $x \le y$. Since $G(x/n, y/n) \le G_1(x/n) = \phi_1(x/n)$, we obtain that

$$\bigl(\min(x,y)\bigr)^{-r} G\left(\frac{x}{n}, \frac{y}{n}\right) \le x^{-r}\phi_1(x/n) \le n^{-r}B(r).$$

If $y \le x$, in a similar way we obtain that

$$\bigl(\min(x,y)\bigr)^{-r} G\left(\frac{x}{n}, \frac{y}{n}\right) \le y^{-r}\phi_1(y/n) \le n^{-r}B(r).$$

We conclude that

$$\left|B(x,y)\right| \le \rho_r(F,G)B(r)n^{-r}.$$

Next, consider the term $A(x,y)$. Using the inequality

$$\left|a^n - b^n\right| \le n\max\bigl(a^{n-1}, b^{n-1}\bigr)|a-b|,$$

we see that

$$\left|A(x,y)\right| \le n\overline{F}(x,y)\max\bigl(F^{n-1}(x,y), G^{n-1}(x,y)\bigr)\left|F(x,y) - G(x,y)\right|,$$

and so,

$$\left|A(x,y)\right| \le n\overline{F}(x,y)\bigl(\min(x,y)\bigr)^{-r}\max\bigl(F^{n-1}(x,y), G^{n-1}(x,y)\bigr)\rho_r(F,G). \tag{7}$$

We show that $\min(x,y)\overline{F}(x,y)$ is bounded. Since $\rho_r(F,G) < \infty$, we have

$$\left|\min(x,y)\bigl(F(x,y) - G(x,y)\bigr)\right| \le \bigl(\min(x,y)\bigr)^{-r+1}\rho_r(F,G).$$

In the case $x \le y$, we obtain that

$$\left|x\overline{F}(x,y) - x\overline{G}(x,y)\right| \le x^{-r+1}\rho_r(F,G).$$

Now observe that

$$\begin{aligned}x\overline{G}(x,y) &\le x\overline{G}_1(x) + x\overline{G}_2(y)\\ &\le x\overline{\phi}_1(x) + y\overline{\phi}_1(y).\end{aligned}$$

Since $z\overline{\phi}_1(z) \le 1$ for $z \ge 0$, it follows that

$$x\overline{F}(x, y) \le x^{-r+1}\rho_r(F, G) + 2.$$

In a similar way, for $y \le x$ we get that

$$y\overline{F}(x, y) \le y^{-r+1}\rho_r(F, G) + 2.$$

It follows that $\min(x, y)\overline{F}(x, y)$ is bounded. Returning to (7), we just showed that

$$\big|A(x, y)\big| \le Kn\big(\min(x, y)\big)^{-r-1}\max\big(F^{n-1}(x, y), G^{n-1}(x, y)\big). \tag{8}$$

Consider the case where $F(x, y) \le G(x, y)$. In this case, we find that

$$\begin{aligned}\big|A(x, y)\big| &\le Kn\big(\min(x, y)\big)^{-r-1}G^{n-1}(x, y)\\ &\le Kn\big(\min(x, y)\big)^{-r-1}G\left(\frac{x}{n-1}, \frac{y}{n-1}\right).\end{aligned}$$

If $x \le y$, we have

$$\big|A(x, y)\big| \le Knx^{-r-1}G_1\left(\frac{x}{n-1}\right) \le KB(r+1)n^{-r}.$$

If $x \ge y$, we have a similar bound.

Next, consider the case where $G(x, y) \le F(x, y)$. Since $\log(z) \le z - 1$, $z \ge 1$, we have that

$$\begin{aligned}0 \le \log\left(\frac{F(x, y)}{G(x, y)}\right) &\le \frac{F(x, y) - G(x, y)}{G(x, y)}\\ &\le \frac{1}{G(x^\circ, y^\circ)}\big(\min(x, y)\big)^{-r}\rho_r(F, G)\\ &= C\big(\min(x, y)\big)^{-r},\end{aligned}$$

where $C > 0$ is a constant. It follows that

$$F^{n-1}(x, y) \le \exp\big(C(n-1)\big(\min(x, y)\big)^{-r}\big)G^{n-1}(x, y). \tag{9}$$

A similar inequality holds when we consider the marginals of F and G.

Now choose x_n in such a way that $(n-1)x_n^{-r} = 1$. Since $x_n \uparrow \infty$, we can choose n large enough to have $x_n > x^\circ$ and $x_n > y^\circ$.

For $x, y \ge x_n$ relation (9) shows that

$$\begin{aligned}F^{n-1}(x, y) &\le \exp\big(C(n-1)x_n^{-r}\big)G^{n-1}(x, y)\\ &= \exp(C)G\left(\frac{x}{n-1}, \frac{y}{n-1}\right).\end{aligned}$$

We can proceed as in the case $F \le G$ to obtain

$$\sup_{x, y \ge x_n}\big|A(x, y)\big| = O(1)n^{-r}.$$

For $x^\circ \le x \le x_n$ and $y^\circ \le y \le x_n$, we use $F(x,y) \le F(x_n,x_n)$ and (9) to show that

$$F^{n-1}(x,y) \le \exp(C) G^{n-1}(x_n,x_n).$$

From (8), we obtain that

$$\left|A(x,y)\right| \le K \exp(C) n \bigl(\min(x^\circ, y^\circ)\bigr)^{-r-1} G\left(\frac{x_n}{n-1}, \frac{x_n}{n-1}\right).$$

By our choice of x_n, we have that $nG(x_n/(n-1), x_n/(n-1)) \to 0$ exponentially fast.

Next, assume that $x^\circ < x \le x_n$ and that $y > x_n$. In this case, (8) implies that

$$\left|A(x,y)\right| \le Knx^{-r-1} F^{n-1}(x,y) \le K' n F_1^{n-1}(x_n),$$

for some constant K. It follows that

$$\left|A(x,y)\right| \le K'' n G_1^{n-1}(x_n).$$

Also, in this case we have $nG_1^{n-1}(x_n) \to 0$ exponentially fast. A similar result applies for the case $y^\circ < y < x_n < x$. Combining all the above cases, gives

$$\sup_{x,y\ge 0} \left|A(x,y)\right| = O(1) n^{-r}.$$

□

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