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An INAR model with discrete Laplace marginal distributions

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Abstract. In this paper, we first introduce a new thinning operator and derive some of its properties. Then, by using the thinning operator we define a new stationary time series with discrete Laplace marginal distributions with either positive or negative lag-one autocorrelation. We show that this time series is distributed as the difference of two independent NGINAR(1) time series and, using this fact, we discuss some of its properties. The Yule–Walker estimators for the unknown parameters are derived and their asymptotic properties are discussed.

1 Introduction

In many real-life situations, there are time series which represent data obtained from significantly correlated systems and which may consist of integer values including both, positive and negative numbers. These data may be the differences of two non-negative integer-valued counting processes. For example, the researcher may be interested in comparing the counting results of the same criminal activities in two town districts, simultaneously. Regarding this, the interactions among the population elements, which may result in newly generated random events (crimes, in this case), could be handled by implementation of a geometrically distributed counting sequence, as it was done in Ristić and Nastić (2012). On the other hand, possible negative values of the observed process may be obtained by using, in certain way, the difference of two thinning operators and producing, as one of the results of this paper, a new thinning operator which is similarly constructed as in Freeland (2010). Besides this one, there may be many other possible applications of this kind of model. One of them may be the modeling a series of steps up and down, corresponding respectively to the positive and negative values of an arbitrary non-negative integer-valued counting process.

Integer-valued time series model with signed thinning operator have been considered by many authors. Kim and Park (2008) defined the signed binomial thinning operator $\alpha\odot$ as

$$\alpha \odot X = \operatorname{sgn}(\alpha)\operatorname{sgn}(X)\sum_{j=1}^{|X|} W_j(\alpha),$$

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where $\{W_j(\alpha)\}$ represents a sequence of independent and identically Bernoulli distributed random variables with parameter $|\alpha|$, $\alpha \in [-1, 1]$, and $\operatorname{sgn}(x)$ is the sign function. Then they used the introduced signed binomial thinning operator to introduce the signed integer-valued autoregressive model of the order p as

$$X_t = \sum_{i=1}^{p} \alpha_i \odot X_{t-i} + \varepsilon_t, \qquad t \in \mathbb{Z}, \tag{1.1}$$

where $\{\varepsilon_t\}$ is a sequence of independent and identically distributed random variables with mean μ_ε and variance σ_ε^2, $\alpha_i \in [-1, 1]$ for $i = 1, 2, \ldots, p$, the random variables ε_t are uncorrelated with random variable X_{t-i} for $i \geq 1$, and the counting series incorporated in $\alpha_i \odot$ are mutually independent. Zhang et al. (2010) generalized the signed binomial thinning operator of Kim and Park (2008) and introduced the generalized signed operator in the following way

$$\alpha \circledast X = \operatorname{sgn}(\alpha) \operatorname{sgn}(X) \sum_{j=1}^{|X|} W_j^{(i)}(\alpha),$$

where $\{W_j^{(i)}(\alpha)\}$ represents a sequence of independent and identically generalized power series distributed random variables with mean $|\alpha_i|$ and variance β_i, $\alpha_i \in [-1, 1]$, and $\operatorname{sgn}(x)$ is the sign function. The signed integer-valued autoregressive model with this generalized signed thinning operator was defined similarly as model given in (1.1). A random coefficient version of the signed integer-valued autoregressive model can be found in Wang and Zhang (2010). Some other results about the signed integer-valued autoregressive model can be found in Kachour and Truquet (2011), Chesneau and Kachour (2012) and Truquet and Yao (2012).

Anyway, using an integer-valued AR model based on a new thinning operator in the corresponding data analysis might be a considerably challenging task. One such model will be introduced in Section 2 of this article. In Section 3, we will present its properties. Estimation methods of the unknown model parameters and usage of the obtained procedures on simulated time series will be considered in Section 4.

2 Construction of the model

In this section, we will construct a stationary integer-valued autoregressive process with discrete Laplace distribution and with either positive or negative lag-one autocorrelation. Before introducing the process, we begin with the definition of the discrete Laplace distribution introduced by Inusah and Kozubowski (2006). We say that a random variable X has the discrete Laplace distribution with parameter $p \in (0, 1)$, if its probability mass function is given by

$$P(X = x) = \frac{1-p}{1+p} p^{|x|}, \qquad x \in \mathbb{Z}. \tag{2.1}$$

Let us denote this distribution as DL(p). Inusah and Kozubowski (2006) have shown that a random variable X with the DL(p) distribution can be represented as the difference of two independent geometric distributed random variables with parameter p, supported on the set $\{0, 1, 2, \dots\}$. In our paper, instead of considering the random variable X with the DL(p) distribution, we will consider the random variable X with the DL($\mu/(1+\mu)$) distribution, where $\mu > 0$ will be the mean of the geometric distribution. Then (2.1) can be represented as

$$P(X = x) = \frac{1}{1+2\mu}\left(\frac{\mu}{1+\mu}\right)^{|x|}, \qquad x \in \mathbb{Z}, \mu > 0. \tag{2.2}$$

Let us first introduce the result of Kozubowski and Inusah (2006) about the skew discrete Laplace distribution. This result will be used to obtain the distribution of the innovation sequence of our process.

Theorem 2.1. *Let X and Y be two independent random variables with* Geom($\mu/(1+\mu)$) *and* Geom($\nu/(1+\nu)$) *distributions*, $\mu > 0$, $\nu > 0$, *respectively. Then the random variable $Z = X - Y$ has probability mass function given by*

$$P(Z = z) = \begin{cases} \dfrac{1}{1+\nu+\mu}\left(\dfrac{\mu}{1+\mu}\right)^{z}, & z \geq 0, \\ \dfrac{1}{1+\nu+\mu}\left(\dfrac{\nu}{1+\nu}\right)^{-z}, & z < 0. \end{cases} \tag{2.3}$$

We will say that a random variable with probability mass function given by (2.3) has the skew discrete Laplace distribution with two parameters $\mu > 0$ and $\nu > 0$ and we will denote it as SDL($\mu/(1+\mu), \nu/(1+\nu)$).

The characteristic function of a random variable with SDL($\mu/(1+\mu), \nu/(1+\nu)$) distribution can be easily obtained and it is given by $\varphi_Z(t) \equiv E(e^{itZ}) = (1+\mu-\mu e^{it})^{-1}(1+\nu-\nu e^{-it})^{-1}$, $t \in \mathbb{R}$. Then, the expectation and variance of the random variable Z with the skew discrete Laplace distribution with the parameters $\mu > 0$ and $\nu > 0$ are $E(Z) = \mu - \nu$ and $\mathrm{Var}(Z) = \mu(1+\mu) + \nu(1+\nu)$, respectively.

2.1 Construction of the new thinning operator $\odot$

Let us first define a new thinning operator denoted by $\odot$. We will use a similar approach to Freeland (2010) with exception that we consider the negative binomial thinning operator $\alpha *$. The negative binomial thinning operator $\alpha *$ was introduced by Ristić et al. (2009) as $\alpha * X = \sum_{i=1}^{X} W_i$, where $\{W_i\}$ is a sequence of independent and identically distributed random variables with Geom($\alpha/(1+\alpha)$), $\alpha \in [0, 1)$, and W_i and X are independent for all $i \geq 1$. Let Z_{n-1} be a random variable with discrete Laplace distribution DL($\mu/(1+\mu)$), $\mu > 0$, given by (2.2). Let X_{n-1} and Y_{n-1} be two independent random variables with Geom($\mu/(1+\mu)$)

distributions. Then $Z_{n-1} \stackrel{d}{=} X_{n-1} - Y_{n-1}$ according to Theorem 2.1. We define a new thinning operator $\odot$ as

$$(\alpha \odot Z_{n-1})|Z_{n-1} \stackrel{d}{=} (\alpha * X_{n-1} - \alpha * Y_{n-1})|(X_{n-1} - Y_{n-1}), \tag{2.4}$$

where the counting series in $\alpha * X_{n-1}$ and $\alpha * Y_{n-1}$ are mutually independent random variables with $\mathrm{Geom}(\alpha/(1+\alpha))$ distributions. Another thinning operator with the skew discrete Laplace distribution is introduced in Barreto-Souza and Bourguignon (2013). In this manuscript, the authors introduced the thinning operator $\odot$ as $\alpha \odot Z_{n-1} \stackrel{d}{=} \alpha * X_{n-1} - \alpha * Y_{n-1}$, where $\{X_n, n \geq 0\}$ and $\{Y_n, n \geq 0\}$ are two independent NGINAR(1) processes with geometric marginal distributions and with means $\mu_1 > 0$ and $\mu_2 > 0$, respectively.

As the first result, we derive the conditional probability of the random variable $\alpha \odot Z_{n-1}$ for given Z_{n-1}, that is, we derive the probabilities $g_\alpha(j,k) \equiv P(\alpha \odot Z_{n-1} = j | Z_{n-1} = k)$ for $j, k \in \mathbb{Z}$. Let us consider the case $j \geq 0$ and $k \geq 0$. To simplify the expression, let $p_\alpha(j,k,l) \equiv P(\alpha * X_{n-1} - \alpha * Y_{n-1} = j | X_{n-1} = l + k, Y_{n-1} = l)$. Then from the definition of the new thinning operator (2.4) and since the random variables X_{n-1}, Y_{n-1} and Z_{n-1} have $\mathrm{Geom}(\mu/(1+\mu))$, $\mathrm{Geom}(\mu/(1+\mu))$ and $\mathrm{DL}(\mu/(1+\mu))$ distributions, respectively, we have that

$$g_\alpha(j,k) = \frac{1+2\mu}{(1+\mu)^2} \sum_{l=0}^{\infty} p_\alpha(j,k,l) \left(\frac{\mu}{1+\mu}\right)^{2l}. \tag{2.5}$$

By using the fact that the random variables $\alpha * X_{n-1}|\{X_{n-1} = k + l\}$ and $\alpha * Y_{n-1}|\{Y_{n-1} = l\}$ have $\mathrm{NB}(k+l, \alpha/(1+\alpha))$ and $\mathrm{NB}(l, \alpha/(1+\alpha))$ distributions, respectively, we obtain that

$$p_\alpha(j,k,l) = \frac{\alpha^j}{(1+\alpha)^{k+2l+j}} \binom{j+k+l-1}{j} \times {}_2F_1\left(l, j+k+l; j+1; \left(\frac{\alpha}{1+\alpha}\right)^2\right), \tag{2.6}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function defined as

$${}_2F_1(a,b;c;z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}, \qquad |z| < 1,$$

and $(q)_m$ is the Pochhammer symbol. Now, replacing (2.6) in (2.5), we obtain that

$$g_\alpha(j,k) = \frac{(1+2\mu)\alpha^j}{(1+\mu)^2(1+\alpha)^{k+j}} \sum_{l=0}^{\infty} \left(\frac{\mu}{(1+\alpha)(1+\mu)}\right)^{2l} \binom{j+k+l-1}{j} \times {}_2F_1\left(l, j+k+l; j+1; \left(\frac{\alpha}{1+\alpha}\right)^2\right).$$

Similarly, we can show that if $j \geq 0$ and $k < 0$, then

$$g_\alpha(j,k) = \frac{(1+2\mu)\alpha^j}{(1+\mu)^2(1+\alpha)^{-k+j}} \sum_{l=0}^{\infty} \left(\frac{\mu}{(1+\alpha)(1+\mu)}\right)^{2l} \binom{j+l-1}{j} \times {}_2F_1\left(l+j, l-k; j+1; \left(\frac{\alpha}{1+\alpha}\right)^2\right).$$

Finally, by using the above routines we obtain that in the case $j < 0$ we have that $g_\alpha(j,k) = g_\alpha(-j,-k)$, for $k \in \mathbb{Z}$.

Let us now consider the characteristic function of the random variable $\alpha \odot Z_{n-1}$. This is given by the following theorem.

Theorem 2.2. *The characteristic function of a random variable $\alpha \odot Z_{n-1}$ is given by*

$$\varphi(t) = \frac{(1+\alpha-\alpha e^{it})(1+\alpha-\alpha e^{-it})}{[1+\alpha(1+\mu)-\alpha(1+\mu)e^{it}][1+\alpha(1+\mu)-\alpha(1+\mu)e^{-it}]}. \tag{2.7}$$

The equation (2.7) also represents the characteristic function of the random variable $\alpha * X_{n-1} - \alpha * Y_{n-1}$, where X_{n-1} and Y_{n-1} are two independent and identically distributed random variables with Geom$(\mu/(1+\mu))$ distributions and $\alpha*$ is the negative binomial thinning operator. Thus, we have the following result.

Corollary 2.1.

(a) $\alpha \odot Z_{n-1} \stackrel{d}{=} \alpha * X_{n-1} - \alpha * Y_{n-1}$;
(b) $E(\alpha \odot Z_{n-1}) = 0$;
(c) $\mathrm{Var}(\alpha \odot Z_{n-1}) = 2\alpha\mu(1+2\alpha+\alpha\mu)$;
(d) $0 \odot Z_{n-1} \stackrel{d}{=} 0$;
(e) $1 \odot Z_{n-1} \stackrel{d}{\neq} Z_{n-1}$.

Conditional properties of the random variable $\alpha \odot Z_{n-1}$ for given Z_{n-1} follows from the following theorem.

Theorem 2.3. *The conditional expectation and conditional variance of the random variable $\alpha \odot Z_{n-1}$ for given Z_{n-1} are respectively, given as*

$$\begin{aligned} E(\alpha \odot Z_{n-1}|Z_{n-1}) &= \alpha Z_{n-1}, \\ \mathrm{Var}(\alpha \odot Z_{n-1}|Z_{n-1}) &= \alpha(1+\alpha)|Z_{n-1}| + \frac{2\alpha(1+\alpha)\mu^2}{1+2\mu}. \end{aligned} \tag{2.8}$$

The following theorem gives another representation of the random variable $\alpha \odot Z_{n-1}$ by the negative binomial thinning operator $\alpha*$.

Theorem 2.4. *Let* Z, X *and* Y *be random variables with* $\mathrm{DL}(\mu/(1+\mu))$, $\mathrm{Geom}(\mu/(1+\mu))$ *and* $\mathrm{Geom}(\mu/(1+\mu))$ *distributions, respectively. Let* $\{D_j, j \geq 1\}$ *be a sequence of independent random variables with* $\mathrm{DL}(\alpha/(1+\alpha))$ *distributions and suppose that the random variables* Z, X, Y, D_j, *and the random variables involved in* $\alpha*$ *are independent. Then*

$$\alpha \odot Z \stackrel{d}{=} \mathrm{sgn}(Z)\big(\alpha * |Z|\big) + \sum_{j=1}^{\min(X,Y)} D_j, \tag{2.9}$$

where $\sum_{j=1}^{\min(X,Y)} D_j = 0$ *when* $\min(X, Y) = 0$ *and* $\mathrm{sgn}(\cdot)$ *is the sign function.*

It is interesting to note that the right-hand side of the equation (2.9) can be interpreted as sum of two thinning operators. The first term in this equation represents the signed thinning operator introduced in Latour and Truquet (2008) whose counting sequence has geometric distribution. The second term represents a thinning operator whose counting sequence has discrete Laplace distribution and it is applied on a random variable $\min(X, Y)$ which represents a random variable with geometric distribution with mean $(\mu/(1+\mu))^2$.

2.2 Construction of the model

Now, let us first introduce a stationary integer-valued autoregressive time series with the discrete Laplace $\mathrm{DL}(\mu/(1+\mu))$ marginal distributions and with positive autocorrelations. We define the stationary integer-valued autoregressive time series $\{Z_n, n \geq 0\}$ as

$$Z_n = \alpha \odot Z_{n-1} + e_n, \qquad n \geq 1, \tag{2.10}$$

where $\{e_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) integer valued random variables such that $\mathrm{Cov}(e_n, Z_{n-l}) = 0$ for all $l \geq 1$, the thinning operator $\alpha\odot$ is the defined as in (2.4), and the counting series in $\alpha \odot Z_{n-1}$ are independent random variables independent of Z_n and e_m for all n and m. We will denote this model as DLINAR(1) time series (Discrete Laplace INteger-valued AutoRegressive time series of the first order).

First, we discuss the distribution of the random variable e_n. Its distribution is given by the following theorem. Note that the condition $\alpha \in (0, \mu/(1+\mu)]$ from this theorem provides that the distribution of the random variable e_n is valid.

Theorem 2.5. *If* $\alpha \in (0, \mu/(1+\mu)]$, *then the distribution of the random variable* e_n *is given as*

$$e_n \stackrel{d}{=} \begin{cases} \mathrm{DL}\Big(\dfrac{\mu}{1+\mu}\Big), & w.p.\ \Big(\dfrac{\mu-\alpha-\alpha\mu}{\mu-\alpha}\Big)^2, \\ \mathrm{SDL}\Big(\dfrac{\mu}{1+\mu}, \dfrac{\alpha}{1+\alpha}\Big), & w.p.\ \dfrac{\alpha\mu(\mu-\alpha-\alpha\mu)}{(\mu-\alpha)^2}, \\ \mathrm{SDL}\Big(\dfrac{\alpha}{1+\alpha}, \dfrac{\mu}{1+\mu}\Big), & w.p.\ \dfrac{\alpha\mu(\mu-\alpha-\alpha\mu)}{(\mu-\alpha)^2}, \\ \mathrm{DL}\Big(\dfrac{\alpha}{1+\alpha}\Big), & w.p.\ \Big(\dfrac{\alpha\mu}{\mu-\alpha}\Big)^2. \end{cases}$$

From the above theorem follows an interesting conclusion which proof is omitted since it is trivial.

Corollary 2.2. *If* $\alpha \in (0, \mu/(1+\mu)]$, *then* $e_n \stackrel{d}{=} \varepsilon_n - \eta_n$, *where* ε_n *and* η_n *are two independent and identically distributed random variables with distribution given as*

$$\begin{cases} \mathrm{Geom}\Big(\dfrac{\mu}{1+\mu}\Big), & w.p.\ \dfrac{\mu-\alpha-\alpha\mu}{\mu-\alpha}, \\ \mathrm{Geom}\Big(\dfrac{\alpha}{1+\alpha}\Big), & w.p.\ \dfrac{\alpha\mu}{\mu-\alpha}. \end{cases}$$

Remark 2.1. From Theorem 2.4, it follows that it is possible to construct a stationary time series model with discrete Laplace marginal distributions which is generated by the negative binomial thinning operator. Namely, we can consider a stationary time series model $\{Z_n\}$ with discrete Laplace marginal distributions given by $Z_n = \mathrm{sgn}(Z_{n-1})(\alpha * |Z_{n-1}|) + \xi_n$, $n \geq 1$, where $\{\xi_n\}$ is given by $\xi_n = \sum_{j=1}^{\min(X,Y)} D_j + e_n$, the random variables X, Y and D_j are given in Theorem 2.4 and the random variable e_n is given in Theorem 2.5.

Now, following the technique used in Freeland (2010), we introduce a stationary integer-valued autoregressive time series with discrete Laplace $\mathrm{DL}(\mu/(1+\mu))$ marginal distributions and with negative lag-one autocorrelation. We define the stationary integer-valued autoregressive time series $\{Z_n, n \geq 0\}$ as

$$Z_n = \alpha \odot (-Z_{n-1}) + e_n, \qquad n \geq 1, \tag{2.11}$$

where $Z_{n-1} \stackrel{d}{=} X_{n-1} - Y_{n-1}$, $\alpha \in (0, \mu/(1+\mu)]$, $\{e_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) integer valued random variables such that $\mathrm{Cov}(e_n, Z_{n-l}) = 0$ for all $l \geq 1$ and $e_n \stackrel{d}{=} (-1)^n(\varepsilon_n - \eta_n)$, for all $n \geq 0$, the thinning operator $\alpha\odot$ is the defined as in (2.4), and the counting series in $\alpha \odot (-Z_{n-1})$ are independent random variables independent of Z_n and e_m for all n and m.

3 Properties of the model

In this section, we derive and discuss some properties of DLINAR(1) time series. It is interesting to note that many properties of the DLINAR(1) time series can be obtained by considering the difference of two independent NGINAR(1) time series $\{X_n\}$ and $\{Y_n\}$. Ristić et al. (2009) introduced NGINAR(1) time series model $\{X_n\}$ as $X_n = \alpha * X_{n-1} + \varepsilon_n$, $n \geq 1$, where $\{X_n\}$ is a stationary autoregressive model with Geom$(\mu/(1+\mu))$ marginals, $\mu > 0$, $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables with distribution given in Corollary 2.2, X_{n-l} and ε_n are independent for all $l \geq 1$, and $\alpha *$ is the negative binomial thinning operator. Note that the NGINAR(1) time series model is valid for $\alpha \in (0, \mu/(1+\mu)]$. Thus, let $\alpha \in (0, \mu/(1+\mu)]$ and let $\{X_n\}$ and $\{Y_n\}$ be two independent NGINAR(1) time series given as $X_n = \alpha * X_{n-1} + \varepsilon_n$ and $Y_n = \alpha * Y_{n-1} + \eta_n$, where $\{\varepsilon_n\}$ and $\{\eta_n\}$ are mutually independent sequences of independent and identically distributed random variables, ε_n and X_{n-l} are independent for all $l > 1$, and η_n and Y_{n-l} are independent for all $l > 1$. The marginal distributions of the two NGINAR(1) time series are Geom$(\mu/(1+\mu))$ distributions. Thus, from Corollaries 2.1 and 2.2, we obtain that

$$X_n - Y_n = (\alpha * X_{n-1} - \alpha * Y_{n-1}) + (\varepsilon_n - \eta_n) \stackrel{d}{=} \alpha \odot Z_{n-1} + e_n = Z_n, \quad (3.1)$$

that is, we obtain that $Z_n \stackrel{d}{=} X_n - Y_n$, for all $n \geq 0$.

The following result is valid for DLINAR(1) time series model given either by (2.10) or (2.11).

Theorem 3.1. *The* DLINAR(1) *time series* $\{Z_n\}$ *is a strict stationary and ergodic Markov process.*

Theorem 3.2. *The k-step ahead conditional mean for* DLINAR(1) *time series model given by* (2.10) *is given as* $E(Z_n|Z_{n-k}) = \alpha^k Z_{n-k}$. *The k-step ahead conditional mean for* DLINAR(1) *time series model given by* (2.11) *is given as* $E(Z_n|Z_{n-k}) = (-\alpha)^k Z_{n-k}$.

Now, we have the following result.

Theorem 3.3. DLINAR(1) *time series* $\{Z_n\}$ *given by* (2.10) *is positively correlated time series with* $\mathrm{Corr}(Z_n, Z_{n-k}) = \alpha^k$, $k \geq 0$.

Let us consider now the DLINAR(1) time series $\{Z_n\}$ given by (2.11). Let $\alpha \in (0, \mu/(1+\mu)]$ and let $\{X_n\}$ and $\{Y_n\}$ be two independent NGINAR(1) time series given as $X_n = \alpha * X_{n-1} + \varepsilon_n$ and $Y_n = \alpha * Y_{n-1} + \eta_n$, where $\{\varepsilon_n\}$ and $\{\eta_n\}$ are mutually independent sequences of independent and identically distributed random variables, ε_n and X_{n-l} are independent for all $l > 1$, and η_n and Y_{n-l} are independent for all $l > 1$. The marginal distributions of the two NGINAR(1) time series are Geom($\mu/(1+\mu)$) distributions. Let $Z_{n-1} \stackrel{d}{=} X_{n-1} - Y_{n-1}$. Thus, from Corollaries 2.1 and 2.2, we obtain that

$$Y_n - X_n = (\alpha * Y_{n-1} - \alpha * X_{n-1}) + (\eta_n - \varepsilon_n) \stackrel{d}{=} \alpha \odot (-Z_{n-1}) + e_n = Z_n,$$

that is, we obtain that $Z_n \stackrel{d}{=} Y_n - X_n$ when $Z_{n-1} \stackrel{d}{=} X_{n-1} - Y_{n-1}$. Now, we have the following result.

Theorem 3.4. DLINAR(1) *time series* $\{Z_n\}$ *given by* (2.11) *is correlated time series with* $\mathrm{Corr}(Z_n, Z_{n-k}) = (-\alpha)^k$, $k \geq 0$.

In the next theorem, we give the MA(∞) representation of the DLINAR(1) time series model given by (2.10). Similar result can be obtain for the DLINAR(1) time series model given by (2.11).

Theorem 3.5. *The* DLINAR(1) *time series* $\{Z_n\}$ *can be represented in distribution as* MA(∞) *time series*

$$Z_n \stackrel{d}{=} \sum_{j=0}^{\infty} \alpha \odot^{(j)} e_{n-j}, \tag{3.2}$$

where $\alpha \odot^{(j)} e_{n-j} \stackrel{def}{=} \alpha *^{(j)} \varepsilon_{n-j} - \alpha *^{(j)} \eta_{n-j}$, $\alpha *^{(0)} \varepsilon_n = \varepsilon_n$, $\alpha *^{(0)} \eta_n = \eta_n$, $\alpha *^{(j)} \varepsilon_{n-j} \stackrel{def}{=} \alpha * (\alpha *^{(j-1)} \varepsilon_{n-j})$ *and* $\alpha *^{(j)} \eta_{n-j} \stackrel{def}{=} \alpha * (\alpha *^{(j-1)} \eta_{n-j})$, $j \geq 1$.

4 Estimation of the unknown parameters

In this section, we consider the estimation of the unknown parameters of the DLINAR(1) time series $\{Z_n\}$ given by (2.10). Similar results can be obtained for the DLINAR(1) time series $\{Z_n\}$ given by (2.11). Similarly, as in Freeland (2010) we have that $E(Z_n|Z_{n-1}) = \alpha Z_{n-1}$ which implies that by the conditional least squares method of estimation only the parameter α can be estimated. Because of that we will consider the estimation of the unknown parameters α and μ by the Yule–Walker method of estimation. Let us suppose that we have a random sample $(Z_1, Z_2, \dots, Z_N)$ of size N. From Theorem 3.3, we have that $\mathrm{Corr}(Z_n, Z_{n-1}) = \alpha$ and since $E(Z_n) = 0$, for $n \geq 0$, we obtain that the Yule–Walker estimator of the unknown parameter α is given by

$$\hat{\alpha}_{\mathrm{YW}} = \left(\sum_{n=2}^{N} Z_n Z_{n-1}\right)\left(\sum_{n=1}^{N} Z_n^2\right)^{-1}. \tag{4.1}$$

The second parameter μ can be estimated by using the fact that $E(Z_n^2) = 2\mu(1+\mu)$. Thus solving this equation, we obtain that the Yule–Walker estimator of the parameter μ is given by

$$\hat{\mu}_{\mathrm{YW}} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2}{N}\sum_{n=1}^{N} Z_n^2}. \tag{4.2}$$

Let us now discuss the asymptotic properties of the estimators (4.1) and (4.2). First, we start with the asymptotic properties of the estimator $\hat{\alpha}_{\mathrm{YW}}$. We have the following theorem.

Theorem 4.1. *The Yule–Walker estimator $\hat{\alpha}_{\mathrm{YW}}$ is a strongly consistent estimator of the parameter α and*

$$\sqrt{N-1}(\hat{\alpha}_{\mathrm{YW}} - \alpha) \xrightarrow{d} \mathcal{N}\left(0, \frac{(1+\mu)(2\mu(3+2\alpha) + 2\alpha^2\mu(1-2\mu)) + \alpha(1+\alpha)}{2\mu(1+\mu)(1+2\mu)}\right),$$

as $N \to \infty$.

Theorem 4.2. *The Yule–Walker estimator $\hat{\mu}_{\mathrm{YW}}$ of the parameter μ has asymptotic normal distribution and is strongly consistent.*

4.1 Simulation

To check the performance of the obtained Yule–Walker estimates, we simulate 10,000 series each with 500 elements for different true values of the parameters α and μ. The series $\{Z_n\}$ are simulated by using the fact that our time series model is distributed as the difference of two independent NGINAR(1) time series $\{X_n\}$ and $\{Y_n\}$. Thus, we first simulate X_1 and independently of it, we simulate Y_1, each from geometric distribution with parameter $\mu/(1+\mu)$ and derive $Z_1 = X_1 - Y_1$. Then for $n = 2, 3, \ldots, 500$ we derive X_n from $X_n = \alpha * X_{n-1} + \varepsilon_n$ and derive Y_n from $Y_n = \alpha * Y_{n-1} + \eta_n$, where ε_n and η_n are simulated from the distribution given in Corollary 2.2. Finally, for each $n = 2, 3, \ldots, 500$ we derive $Z_n = X_n - Y_n$. For the parameter α, we use the true values 0.1, 0.3, 0.6 and 0.8, and for the parameter μ we use the true values 0.5, 1, 2, 5 and 10. Due to the parameters restriction $\alpha \in (0, \mu/(1+\mu)]$, some pairs of true values of the parameters α and μ are not considered, that is, $\alpha = 0.8$ and $\mu = 1$. For each pair of the considered true values of the parameters α and μ, we concentrate on three different sample sizes 100, 200 and 500. By using the Yule–Walker method, we obtain the estimates for each subsample. Since for small true values of the parameter α it is possible to obtain negative estimates for it, we set, in this case $\hat{\alpha} = 0$ since this estimate is near to possible true value of parameter α. Also, for large true values of the parameter α

it is possible to obtain estimates greater than $\mu/(1+\mu)$. In these cases, we set $\hat{\alpha} = \hat{\mu}/(1+\hat{\mu})$. So, the estimated value for α will be:

$$\hat{\alpha} = \begin{cases} 0, & \left(\sum_{n=2}^{N} z_n z_{n-1}\right)\left(\sum_{n=1}^{N} z_n^2\right)^{-1} \in (-\infty, 0], \\ \left(\sum_{n=2}^{N} z_n z_{n-1}\right)\left(\sum_{n=1}^{N} z_n^2\right)^{-1}, & \\ & \left(\sum_{n=2}^{N} z_n z_{n-1}\right)\left(\sum_{n=1}^{N} z_n^2\right)^{-1} \in (0, \hat{\mu}/(1+\hat{\mu})], \\ \hat{\mu}/(1+\hat{\mu}), & \\ & \left(\sum_{n=2}^{N} z_n z_{n-1}\right)\left(\sum_{n=1}^{N} z_n^2\right)^{-1} \in (\hat{\mu}/(1+\hat{\mu}), \infty). \end{cases}$$

In Table 1, the sample means and standard deviations of the estimates are given for each pair of the true values. Also, we provide numbers of how often $\hat{\alpha}$ is negative (L) and how often the restriction $\alpha \in (0, \mu/(1+\mu)]$ is not valid (U). We can conclude that we obtain estimates that converge to the true values of the parameters with decreasing standard deviations. Also, we can conclude that the number (L) of the negative estimates $\hat{\alpha}$ and the number (U) of the estimates $\hat{\alpha}$ greater than $\mu/(1+\mu)$ decrease as the sample size increases.

5 Conclusion

In this paper, we introduced an integer-valued autoregressive process based on a new thinning operator $\odot$ and with the discrete Laplace marginal distribution (DLINAR(1)). The thinning operator of the model was based on the negative binomial thinning of Ristić et al. (2009), which has been used in deriving some of its properties. Since the construction of DLINAR(1) was inspired by the work of Freeland (2010), some of its features have been obtained by similar approach. However, while in Freeland (2010) the marginal Skellam distribution is defined by means of Bessel functions, in our case the discrete Laplace distribution of the process was obtained more easily by direct calculation. Besides the features of the thinning operator, we presented the full characterization of the newly introduced process including its existence, stationarity, ergodicity, correlation and regression properties. Estimation of model parameters was carried out using the method of moments which asymptotic characterization has been thoroughly done and also numerically presented using subsamples of different sizes of 10,000 simulated data series each of length 500.

Further research in this field might be performed in some new directions. First, two NGINAR(1) models $\{X_n\}$ and $\{Y_n\}$ with different parameters in marginal distributions can be considered, for example, X_n with Geom$(\mu/(1+\mu))$ distribution

Table 1 *Some numerical results of the estimators for some true values of the parameters α and μ*

N	$\alpha = 0.1, \mu = 0.5$		$\alpha = 0.1, \mu = 1$		$\alpha = 0.1, \mu = 2$	
100	0.1058(0.0861)	0.4971(0.0917)	0.1071(0.0861)	0.9941(0.1556)	0.1057(0.0842)	1.9864(0.2722)
	$L = 1660, U = 124$		$L = 1599, U = 1$		$L = 1574, U = 1$	
200	0.1010(0.0674)	0.4980(0.0649)	0.1015(0.0666)	0.9962(0.1099)	0.1010(0.0653)	1.9935(0.1938)
	$L = 886, U = 14$		$L = 838, U = 0$		$L = 809, U = 0$	
500	0.0996(0.0464)	0.4989(0.0410)	0.0997(0.0452)	0.9980(0.0705)	0.0995(0.0444)	1.9961(0.1231)
	$L = 164, U = 0$		$L = 145, U = 0$		$L = 155, U = 0$	
N	$\alpha = 0.1, \mu = 5$		$\alpha = 0.1, \mu = 10$		$\alpha = 0.3, \mu = 0.5$	
100	0.1054(0.0835)	4.9724(0.6153)	0.1067(0.0829)	9.9113(1.1664)	0.2628(0.0885)	0.4953(0.1064)
	$L = 1575, U = 0$		$L = 1523, U = 0$		$L = 55, U = 3386$	
200	0.1014(0.0646)	4.9902(0.4360)	0.1002(0.0639)	9.9446(0.8250)	0.2787(0.0662)	0.4978(0.0765)
	$L = 790, U = 0$		$L = 789, U = 0$		$L = 2, U = 3003$	
500	0.0994(0.0439)	4.9994(0.2764)	0.0989(0.0433)	9.9767(0.5255)	0.2908(0.0452)	0.4992(0.0491)
	$L = 136, U = 0$		$L = 141, U = 0$		$L = 0, U = 2284$	
N	$\alpha = 0.3, \mu = 1$		$\alpha = 0.3, \mu = 2$		$\alpha = 0.3, \mu = 5$	
100	0.2888(0.1039)	0.9908(0.1755)	0.2903(0.1004)	1.9805(0.3063)	0.2914(0.0967)	4.9572(0.6688)
	$L = 33, U = 160$		$L = 40, U = 0$		$L = 25, U = 0$	
200	0.2938(0.0755)	0.9957(0.1252)	0.2944(0.0721)	1.992(0.2161)	0.2961(0.0695)	4.9763(0.4764)
	$L = 0, U = 17$		$L = 0, U = 0$		$L = 3, U = 0$	
500	0.2977(0.0485)	0.9985(0.0801)	0.2979(0.0460)	1.9958(0.1359)	0.2983(0.0444)	4.9906(0.3045)
	$L = 0, U = 0$		$L = 0, U = 0$		$L = 0, U = 0$	

Table 1—Continued

N	$\alpha = 0.3$, $\mu = 10$		$\alpha = 0.6$, $\mu = 2$		$\alpha = 0.6$, $\mu = 5$	
100	0.2922(0.0947)	9.944(1.2792)	0.5691(0.0921)	1.9623(0.4289)	0.5794(0.0884)	4.9287(0.916)
	$L = 18$, $U = 0$		$L = 0$, $U = 1261$		$L = 0$, $U = 0$	
200	0.2954(0.0678)	9.968(0.9178)	0.5851(0.0690)	1.9837(0.3094)	0.5881(0.0635)	4.9535(0.6613)
	$L = 0$, $U = 0$		$L = 0$, $U = 767$		$L = 0$, $U = 0$	
500	0.2980(0.0435)	9.9846(0.5813)	0.5941(0.0453)	1.9921(0.1991)	0.5951(0.0399)	4.9776(0.4197)
	$L = 0$, $U = 0$		$L = 0$, $U = 154$		$L = 0$, $U = 0$	
N	$\alpha = 0.6$, $\mu = 10$		$\alpha = 0.8$, $\mu = 5$		$\alpha = 0.8$, $\mu = 10$	
100	0.5812(0.0845)	9.8588(1.7045)	0.7612(0.0775)	4.8427(1.3668)	0.7706(0.0714)	9.716(2.5077)
	$L = 0$, $U = 0$		$L = 0$, $U = 1397$		$L = 0$, $U = 4$	
200	0.5902(0.0608)	9.9214(1.2326)	0.7793(0.0545)	4.9121(1.0115)	0.7844(0.0502)	9.8291(1.8366)
	$L = 0$, $U = 0$		$L = 0$, $U = 984$		$L = 0$, $U = 0$	
500	0.5963(0.0383)	9.9737(0.7798)	0.7914(0.0346)	4.9589(0.6541)	0.7930(0.0317)	9.9216(1.1843)
	$L = 0$, $U = 0$		$L = 0$, $U = 350$		$L = 0$, $U = 0$	

and Y_n with $\text{Geom}(\nu/(1+\nu))$ distribution. This assumption would make the model much more realistic. Second, two NGINAR(1) models $\{X_n\}$ and $\{Y_n\}$ can be generated by different equations, for example, X_n given by $X_n = \alpha * X_{n-1} + \varepsilon_n$ and $\{Y_n\}$ given by $Y_n = \beta * Y_{n-1} + \eta_n$. Third, our derivation implies two independent NGINAR(1) time series. A new model $\{Z_n\}$ can be considered by including not independent NGINAR(1) time series models, for example, by taking the difference $Z_n = X_n - Y_n$ of the bivariate process $\{(X_n, Y_n)\}$ defined by Ristić et al. (2012). Finally, DLINAR(1) model can be generalized to higher order than 1 following the technique used in Nastić et al. (2012).

Appendix: Proofs

Proof of Theorem 2.2. It is much easier to derive it by using the definition of the operator (2.4). We have that

$$\begin{aligned}
\varphi_{\alpha\odot Z_{n-1}}(t) &= \sum_{z=-\infty}^{\infty} E\big(e^{it(\alpha\odot Z_{n-1})}|Z_{n-1}=z\big)P(Z_{n-1}=z) \\
&= \sum_{z=-\infty}^{-1}\sum_{y=0}^{\infty} E\big(e^{it(\alpha*X_{n-1}-\alpha*Y_{n-1})}|X_{n-1}=y, Y_{n-1}=y-z\big) \\
&\qquad \times P(X_{n-1}=y, Y_{n-1}=y-z) \qquad\qquad \text{(A.1)} \\
&\quad + \sum_{z=0}^{\infty}\sum_{y=0}^{\infty} E\big(e^{it(\alpha*X_{n-1}-\alpha*Y_{n-1})}|X_{n-1}=y+z, Y_{n-1}=y\big) \\
&\qquad \times P(X_{n-1}=y+z, Y_{n-1}=y).
\end{aligned}$$

For given $X_{n-1} = y$ and $Y_{n-1} = y - z$, the random variables $\alpha * X_{n-1}$ and $\alpha * Y_{n-1}$ are conditionally independent random variables with $\text{NB}(y, \alpha/(1+\alpha))$ and $\text{NB}(y-z, \alpha/(1+\alpha))$ distributions, respectively. This implies that

$$\begin{aligned}
&E\big(e^{it(\alpha*X_{n-1}-\alpha*Y_{n-1})}|X_{n-1}=y, Y_{n-1}=y-z\big) \\
&\quad = \big(1+\alpha-\alpha e^{it}\big)^{-y}\big(1+\alpha-\alpha e^{-it}\big)^{-y+z}.
\end{aligned} \tag{A.2}$$

In a similar way, we obtain that

$$\begin{aligned}
&E\big(e^{it(\alpha*X_{n-1}-\alpha*Y_{n-1})}|X_{n-1}=y+z, Y_{n-1}=y\big) \\
&\quad = \big(1+\alpha-\alpha e^{it}\big)^{-y-z}\big(1+\alpha-\alpha e^{-it}\big)^{-y}.
\end{aligned} \tag{A.3}$$

Finally, replacing (A.2) and (A.3) in (A.1) and after some calculations, we obtain that the characteristic function of the random variable $\alpha \odot Z_{n-1}$ is given by (2.7). □

Proof of Theorem 2.3. From Corollary 2.1, we obtain for $k \geq 0$ is

$$\begin{aligned}
&E\big((\alpha \odot Z_{n-1})^k | Z_{n-1} = z\big)\\
&\quad = E\big((\alpha * X_{n-1} - \alpha * Y_{n-1})^k | X_{n-1} - Y_{n-1} = z\big)\\
&\quad = \begin{cases} \dfrac{1+2\mu}{(1+\mu)^2} \displaystyle\sum_{y=0}^{\infty} \left(\frac{\mu}{1+\mu}\right)^{2y} E\big((\alpha * X_{n-1} - \alpha * Y_{n-1})^k | \\ \qquad X_{n-1} = y+z, Y_{n-1} = y\big), \qquad z \geq 0, \\ \dfrac{1+2\mu}{(1+\mu)^2} \displaystyle\sum_{x=0}^{\infty} \left(\frac{\mu}{1+\mu}\right)^{2x} E\big((\alpha * X_{n-1} - \alpha * Y_{n-1})^k | \\ \qquad X_{n-1} = x, Y_{n-1} = x - z\big), \qquad z < 0. \end{cases}
\end{aligned}$$

From the properties of the NGINAR(1) time series it follows that

$$\begin{aligned}
E(\alpha * X - \alpha * Y | X = x, Y = y) &= \alpha x - \alpha y,\\
E\big((\alpha * X - \alpha * Y)^2 | X = x, Y = y\big) &= \alpha(1+\alpha)(x+y) + \alpha^2 (x-y)^2.
\end{aligned}$$

Finally, applying these results we obtain the proof of the theorem. □

Proof of Theorem 2.4. We will derive the characteristic function of the random variable $\operatorname{sgn}(Z)(\alpha * |Z|) + \sum_{j=1}^{\min(X,Y)} D_j$. Since the random variables $\operatorname{sgn}(Z)(\alpha * |Z|)$ and $\sum_{j=1}^{\min(X,Y)} D_j$ are independent, first we will derive the characteristic functions of the random variables $\operatorname{sgn}(Z)(\alpha * |Z|)$ and $\sum_{j=1}^{\min(X,Y)} D_j$ and then multiply them. From the definition of the random variable Z and the negative binomial thinning operator $\alpha *$, we obtain that

$$\begin{aligned}
&E\big(e^{it \operatorname{sgn}(Z)(\alpha * |Z|)}\big)\\
&\quad = \frac{1}{1+2\mu}\Bigg[\sum_{z=0}^{\infty}\Bigg(\left(\frac{1}{1+\alpha-\alpha e^{it}}\right)^z \\
&\qquad\qquad + \left(\frac{1}{1+\alpha-\alpha e^{-it}}\right)^z\Bigg)\left(\frac{\mu}{1+\mu}\right)^z - 1\Bigg] \qquad \text{(A.4)}\\
&\quad = \frac{(1+\mu)^2(1+\alpha-\alpha e^{it})(1+\alpha-\alpha e^{-it}) - \mu^2}{(1+2\mu)(1+\alpha(1+\mu)-\alpha(1+\mu)e^{it})(1+\alpha(1+\mu)-\alpha(1+\mu)e^{-it})}.
\end{aligned}$$

Let us now derive the characteristic function of the random variable $\sum_{j=1}^{\min(X,Y)} D_j$. We have that

$$\begin{aligned}
&E\big(e^{it \sum_{j=1}^{\min(X,Y)} D_j}\big)\\
&\quad = \sum_{0 \leq y \leq x < \infty} \left(\frac{1}{1+\alpha-\alpha e^{it}}\right)^y \left(\frac{1}{1+\alpha-\alpha e^{-it}}\right)^y \frac{\mu^{x+y}}{(1+\mu)^{x+y+2}} \qquad \text{(A.5)}
\end{aligned}$$

$$+ \sum_{0\le x<y<\infty} \left(\frac{1}{1+\alpha-\alpha e^{it}}\right)^x \left(\frac{1}{1+\alpha-\alpha e^{-it}}\right)^x \frac{\mu^{x+y}}{(1+\mu)^{x+y+2}}$$
$$= \frac{(1+2\mu)(1+\alpha-\alpha e^{it})(1+\alpha-\alpha e^{-it})}{(1+\mu)^2(1+\alpha-\alpha e^{it})(1+\alpha-\alpha e^{-it})-\mu^2}.$$

Finally, multiplying (A.4) and (A.5), we obtain (2.7), which represents the characteristic function of the random variable $\alpha \odot Z$. □

Proof of Theorem 2.5. Let $\varphi_e(t)$ represents the characteristic function of the random variable e_n. From (2.10), we obtain that

$$\varphi_e(t) = \frac{\varphi_{Z_n}(t)}{\varphi_{\alpha\odot Z_{n-1}}(t)}$$
$$= \frac{(1+\alpha(1+\mu)-\alpha(1+\mu)e^{it})(1+\alpha(1+\mu)-\alpha(1+\mu)e^{-it})}{(1+\mu-\mu e^{it})(1+\mu-\mu e^{-it})(1+\alpha-\alpha e^{it})(1+\alpha-\alpha e^{-it})}$$
$$= \left(\frac{a}{1+\mu-\mu e^{it}}+\frac{1-a}{1+\alpha-\alpha e^{it}}\right)\left(\frac{a}{1+\mu-\mu e^{-it}}+\frac{1-a}{1+\alpha-\alpha e^{-it}}\right),$$

where $a=(\mu-\alpha-\alpha\mu)/(\mu-\alpha)$. Next we have that

$$\varphi_e(t) = \frac{a^2}{(1+\mu-\mu e^{it})(1+\mu-\mu e^{-it})}+\frac{a(1-a)}{(1+\mu-\mu e^{it})(1+\alpha-\alpha e^{-it})}$$
$$+\frac{a(1-a)}{(1+\alpha-\alpha e^{it})(1+\mu-\mu e^{-it})}+\frac{(1-a)^2}{(1+\alpha-\alpha e^{it})(1+\alpha-\alpha e^{-it})}.$$

Finally, by using the facts that $(1+\mu-\mu e^{it})^{-1}(1+\mu-\mu e^{-it})^{-1}$, $(1+\mu-\mu e^{it})^{-1}(1+\alpha-\alpha e^{-it})^{-1}$, $(1+\alpha-\alpha e^{it})^{-1}(1+\mu-\mu e^{-it})^{-1}$, and $(1+\alpha-\alpha e^{it})^{-1}(1+\alpha-\alpha e^{-it})^{-1}$ are the characteristic functions of the random variables $\mathrm{DL}(\frac{\mu}{1+\mu})$, $\mathrm{SDL}(\frac{\mu}{1+\mu},\frac{\alpha}{1+\alpha})$, $\mathrm{SDL}(\frac{\alpha}{1+\alpha},\frac{\mu}{1+\mu})$, and $\mathrm{DL}(\frac{\alpha}{1+\alpha})$, respectively, we finish the proof of the theorem. □

Proof of Theorem 3.1. Let us first prove that the DLINAR(1) time series is a Markov process. From (2.9), we have that $\alpha \odot Z_{n-1} \stackrel{d}{=} \mathrm{sgn}(Z_{n-1})(\alpha * |Z_{n-1}|) + \xi_n$, where $\{\xi_n\}$ is a sequence of independent random variables distributed as the random variable $\sum_{j=1}^{\min(X,Y)} D_j$ and independent of Z_{n-1} and the counting series in $\alpha * |Z_{n-1}|$. Then

$$P(Z_n=z_n|Z_{n-1}=z_{n-1},\dots,Z_0=z_0) = \sum_{j=0}^{\infty}\binom{|z_{n-1}|+j-1}{j}\frac{\alpha^j}{(1+\alpha)^{|z_{n-1}|+j}}$$
$$\times P\big(\xi_n+e_n=z_n-j\cdot\mathrm{sgn}(z_{n-1})\big).$$

Since the last equation depend only on z_{n-1}, we obtain that the DLINAR(1) time series is a Markov process. Now, since the time series is a Markov process, it follows for $m \geq 1$ that

$$P(Z_{n+k}=z_{n+k}, 1 \leq k \leq m) = P(Z_{n+1}=z_{n+1}) \times \prod_{k=2}^{m} P(Z_{n+k}=z_{n+k}|Z_{n+k-1}=z_{n+k-1}).$$

Thus, to prove the strict stationarity it is enough to show that $P(Z_{n+k} = z_{n+k}|Z_{n+k-1} = z_{n+k-1}) = P(Z_n = z_{n+k}|Z_{n-1} = z_{n+k-1})$, for any $k \geq 1$. This easily follows from the fact that the random variables $\{\xi_n + e_n\}$ are identically distributed random variables. Finally, the ergodicity can be proved following the proof for the ergodicity from Zheng et al. (2006) and the fact that the σ-algebra generated by the random variables $Z_n, Z_{n-1}, \ldots$ is a subset of the σ-algebra generated by the independent random variables $\xi_n, e_n, W^{(n)}, \xi_{n-1}, e_{n-1}, W^{(n-1)}, \ldots,$ where $W^{(n)}$ contains all the counting series included in $\alpha * |Z_{n-1}|$. □

Proof of Theorem 3.2. From (2.8) and the Markovian property of the model we obtain that $E(Z_n|Z_{n-k}) = \alpha^k Z_{n-k}$. Proof of the second part of this theorem is similar. □

Proof of Theorem 3.3. Since the series $\{Z_n\}$ is a strict stationary process with the $k-$step ahead conditional mean is $E(Z_n|Z_{n-k}) = \alpha^k Z_{n-k}$ and the variance of the process is finite, $\mathrm{Var}(Z_n) = 2\mu(1+\mu)$, we can easily obtain $\mathrm{Cov}(Z_n, Z_{n-k})$ in the following way

$$\begin{aligned}\mathrm{Cov}(Z_n, Z_{n-k}) &= \mathrm{Cov}(Z_{n-k}, Z_n) = \mathrm{Cov}\big(Z_{n-k}, E(Z_n|Z_{n-k})\big)\\ &= \mathrm{Cov}\big(Z_{n-k}, \alpha^k Z_{n-k}\big) = \alpha^k \,\mathrm{Var}(Z_{n-k}) = 2\alpha^k \mu(1+\mu).\end{aligned}$$

So, the autocorrelation function is $\mathrm{Corr}(Z_n, Z_{n-k}) = \alpha^k$. □

Proof of Theorem 3.5. From (3.1) and the definitions of $\alpha\odot^{(k)}$ and $\alpha*^{(k)}$, we have that

$$Z_n \stackrel{d}{=} \alpha * X_{n-1} + \varepsilon_n - \alpha * Y_{n-1} - \eta_n = \alpha *^{(k)} X_{n-k} - \alpha *^{(k)} Y_{n-k} + \sum_{j=0}^{k-1} \alpha \odot^{(j)} e_{n-j}.$$

Let us derive the distribution of the random variable $\alpha *^{(k)} X_{n-k}$. It is easy to show that its probability generating function is given by

$$\Phi_{\alpha*^{(k)}X_{n-k}}(s) = \left(1 + \alpha \sum_{j=0}^{k-1} \alpha^j (1-s)\right)\left(1 + \alpha\left(\sum_{j=0}^{k-1} \alpha^j + \mu\alpha^{k-1}\right)(1-s)\right)^{-1},$$

which implies that the random variable $\alpha *^{(k)} X_{n-k}$ has the following distribution

$$\alpha *^{(k)} X_{n-k} \stackrel{d}{=} \begin{cases} 0, & \text{w.p. } \dfrac{\sum_{j=0}^{k-1} \alpha^j}{\sum_{j=0}^{k-1} \alpha^j + \mu\alpha^{k-1}}, \\ \text{Geom}\left(\dfrac{\sum_{j=1}^{k} \alpha^j + \mu\alpha^k}{1 + \sum_{j=1}^{k} \alpha^j + \mu\alpha^k}\right), & \text{w.p. } \dfrac{\mu\alpha^{k-1}}{\sum_{j=0}^{k-1} \alpha^j + \mu\alpha^{k-1}}. \end{cases}$$

Thus, we have that $E(\alpha *^{(k)} X_{n-k}) = \alpha^k \mu \to 0$, and $\operatorname{Var}(\alpha *^{(k)} X_{n-k}) = \mu\alpha^k(\frac{1-\alpha^{k+1}}{1-\alpha} + \mu\alpha^k) \to 0$, as $k \to \infty$. The same conclusion we obtain for the random variable $\alpha *^{(k)} Y_{n-k}$, which completes the proof. □

Proof of Theorem 4.1. Since the DLINAR(1) time series $\{Z_n\}$ is a strict stationary and ergodic time series, it follows, according to the Theorem 4.1 (Du and Li, 1991), that the estimator $\hat{\alpha}_{\text{YW}}$ is a strongly consistent estimator of the parameter α. To derive the asymptotic distribution, we consider the statistic $\sqrt{N-1}(\hat{\alpha}_{\text{YW}} - \alpha)$. We have that

$$\begin{aligned} &\sqrt{N-1}(\hat{\alpha}_{\text{YW}} - \alpha) \\ &\quad = \left(\frac{N-1}{N} \cdot \frac{1}{\sqrt{N-1}} \sum_{n=2}^{N} Z_{n-1}(Z_n - \alpha Z_{n-1}) - \frac{\sqrt{N-1}}{N} \alpha Z_N^2\right) \\ &\qquad \times \left(\frac{1}{N} \sum_{n=1}^{N} Z_n^2\right)^{-1}. \end{aligned}$$

From the ergodicity of the DLINAR(1) time series follows that $\frac{1}{N}\sum_{n=1}^{N} Z_n^2 \xrightarrow{\text{w.p. } 1} 2\mu(1+\mu)$ as $N \to \infty$. Also, we have that $\frac{\sqrt{N-1}}{N}\alpha Z_N^2 = o(1)$, with probability 1, when $N \longrightarrow \infty$.

Now, let us consider the time series $\{Z_{n-1}(Z_n - \alpha Z_{n-1})\}$. This is a stationary ergodic martingale difference time series with

$$\begin{aligned} \sigma^2 &\equiv E\big(Z_{n-1}(Z_n - \alpha Z_{n-1})\big)^2 \\ &= \frac{2\mu(1+\mu)((1+\mu)(2\mu(3+2\alpha) + 2\alpha^2\mu(1-2\mu)) + \alpha(1+\alpha))}{1+2\mu} < \infty, \end{aligned}$$

which implies that

$$(N-1)^{-1/2} \sum_{n=2}^{N} Z_{n-1}(Z_n - \alpha Z_{n-1}) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \qquad N \to \infty.$$

Finally, by applying the Slutsky theorem we obtain the proof of the theorem. □

Proof of Theorem 4.2. Consider the estimator $\hat{\gamma}_Z(0) = \frac{1}{N}\sum_{k=1}^{N} Z_k^2$ as an estimate of the variance $\gamma_Z(0)$. According to the Theorem 1 (Silva and Silva, 2006), estimator $\hat{\gamma}_Z(0)$ is strongly consistent estimator of $\gamma_Z(0)$ and has asymptotic normal distribution. Yule–Walker estimator of the parameter μ is the function of the statistics $\hat{\gamma}_Z(0)$, $\hat{\mu}_{\mathrm{YW}} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 2\hat{\gamma}_Z(0)}$. According to the Proposition 6.4.3 (Brockwell and Davis, 1987), we conclude that $\hat{\mu}_{\mathrm{YW}}$ also has asymptotic normal distribution. Strong consistency of $\hat{\mu}_{\mathrm{YW}}$ follows from the strong consistency of $\hat{\gamma}_Z(0)$ and the continuity of the function $f(x) = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 2x}$, $x \geq 0$, since $\hat{\mu}_{\mathrm{YW}} = f(\hat{\gamma}_Z(0))$. □

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