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Extendibility of Marshall–Olkin distributions and inverse Pascal triangles

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Abstract. Necessary and sufficient conditions are derived on the parameters of a d-dimensional random vector with Marshall–Olkin distribution to be extendible to an infinite exchangeable sequence. Interpreted differently, this result allows to decide if the respective multivariate exponential distribution can be constructed by means of a model with conditionally independent and identically distributed components. The proof makes use of the solution of the truncated Hausdorff moment problem and a reparameterization of the Marshall–Olkin distribution.

1 Introduction

Multivariate distributions are often constructed from latent one-factor representations in the sense of De Finetti's classical theorem; see De Finetti (1937). This means that a random vector $(\tau_1, \dots, \tau_d)'$ is defined by $\tau_k := \text{functional}(E_k, F)$, $k = 1, \dots, d$, where $E_1, \dots, E_d$ is an i.i.d. sequence of random variables and F is an independent stochastic object. Such a construction is independent of the dimension d in the sense that the random vector $(\tau_1, \dots, \tau_d)'$ can immediately be extended to an infinite sequence $\{\tau_k\}_{k\in\mathbb{N}}$. This is achieved by simply extending the finite i.i.d. sequence $E_1, \dots, E_d$ to an infinite one, using the canonical product space methodology. For this reason, the multivariate distribution of $(\tau_1, \dots, \tau_d)'$ is called *extendible*. In particular, every extendible distribution is always exchangeable. The converse, however, is not true in general.

What if the random vector $(\tau_1, \dots, \tau_d)'$ is not constructed by means of a latent one-factor representation as above? This does not necessarily imply that its distribution is not extendible. There could still be a latent one-factor representation, that is, a different stochastic model of the above extendible nature yielding exactly the same multivariate law. Generally speaking, to find effective criteria whether a given multivariate distribution is extendible or not is a difficult—and, in general, unsolved—task, as already noted in Aldous (1985), Problems (1.11) and (1.12), pages 9–10. In the literature, one can find solutions to this problem only for some specific families of distributions. For instance, it is well known that a spherically symmetric distribution is extendible if and only if it is a mixture of zero-mean

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normal distributions with randomly drawn variance; see Schoenberg (1938) and Fang, Kotz and Ng (1990), Theorem 2.21. For the family of l^1-norm symmetric distributions, McNeil and Nešlehová (2009) show that extendibility is equivalent to the generating Williamson d-transform actually being a Laplace transform. This corresponds to checking whether a given generator function is completely monotone or not, which might be a difficult task. In the present reference, we provide a very effective criterion to decide whether a given d-dimensional Marshall–Olkin distribution is extendible or not.

Motivated by a multivariate extension of the lack-of-memory property, the Marshall–Olkin distribution is introduced by Marshall and Olkin (1967); see also Barlow and Proschan (1975), Galambos and Kotz (1978). Formally, a random vector $(\tau_1, \dots, \tau_d)'$ follows the Marshall–Olkin distribution if there are parameters $\lambda_I \geq 0, \varnothing \neq I \subset \{1, \dots, d\}$, with $\Lambda_i := \sum_{I: i \in I} \lambda_I > 0$ for all $i = 1, \dots, d$, such that for $t_1, \dots, t_d \geq 0$ its survival function has the form

$$\begin{aligned} \bar{F}(t_1, \dots, t_d) &:= \mathbb{P}(\tau_1 > t_1, \dots, \tau_d > t_d) \\ &= \exp\Bigg(- \sum_{I: \varnothing \neq I \subset \{1, \dots, d\}} \lambda_I \max_{i \in I}\{t_i\}\Bigg). \end{aligned} \tag{1.1}$$

Multivariate distributions of this kind are motivated by an exogenous shock model. For instance, they can be constructed as follows. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $E_I, \varnothing \neq I \subset \{1, \dots, d\}$, be a collection of $2^d - 1$ independent exponential random variables with $\mathbb{E}[E_I] = 1/\lambda_I > 0$, where $\lambda_I = 0$ is conveniently interpreted as $E_I \equiv \infty$ almost surely. The random vector $(\tau_1, \dots, \tau_d)'$, defined by

$$\tau_k := \min_{I: k \in I}\{E_I\}, \qquad k = 1, \dots, d, \tag{1.2}$$

is easily shown to have the survival function (1.1). Intuitively, $(\tau_1, \dots, \tau_d)'$ is interpreted as the vector of lifetimes of d components in a system. The random variable E_I corresponds to an exogenous shock destroying all components with indices in I. It follows that the lifetime τ_k of the kth component equals the minimum of all exogenous shocks affecting it, motivating the definition in (1.2). In this classical construction, dependence is caused by $2^d - d - 1$ different shocks, only d shocks are specific to a certain component.

For some applications, the general probabilistic model (1.2) is too flexible, and, hence, inconvenient. For example, a simulation of the random vector $(\tau_1, \dots, \tau_d)'$ in dimensions $d \gg 2$ is quite expensive, due to the large number of involved shocks. Moreover, many practical applications rely on a much simpler kind of probabilistic model: they assume that the random variables $\tau_1, \dots, \tau_d$ are i.i.d. conditioned on a single latent factor. Such approaches are not only useful for efficient simulations, but also for the approximation of the distribution of functionals from $(\tau_1, \dots, \tau_d)'$. This is typically achieved by exploiting the latent factor construction and using stochastic limit theorems. A popular example in the context of credit-risk modeling is the derivation of an approximate loss distribution for large homogeneous credit portfolios. References are Frey and McNeil (2001) for a general treatment, Vasicek (1987), Schönbucher (2002), Kalemanova, Schmid and Werner (2005), Albrecher, Ladoucette and Schoutens (2007) for models with specific factor distributions.

For a given Marshall–Olkin distribution it is difficult to decide if a latent factor representation is available, since construction (1.2) is obviously not based on a single latent factor in the case $d \geq 3$. Only for $d = 2$ it is immediate that τ_1 and τ_2 are i.i.d. conditioned on $E_{\{1,2\}}$ if and only if $\lambda_{\{1\}} = \lambda_{\{2\}}$. This article provides necessary and sufficient conditions on the parameters λ_I of a given d-dimensional Marshall–Olkin distribution to be extendible. To this end, it is convenient to arrange the parameters in a geometric scheme of triangular form, called an inverse Pascal triangle. The extendibility is then equivalent to the extendibility of such triangles. A solution to the latter problem can be derived combining the results of Mai and Scherer (2011) and Karlin and Shapley (1953). Finally, in the extendible case a latent factor representation of Marshall–Olkin distributions based on Lévy subordinators is available; see Mai and Scherer (2011). The rest of the article is organized as follows. Section 2 presents the main result and Section 3 concludes.

2 Extendibility criteria

If the random vector $(\tau_1, \dots, \tau_d)'$ defined in (1.2) is extendible to an infinite exchangeable sequence, then necessarily the distribution of all subvectors have to depend only on their lengths—by definition of exchangeability. Making this more precise, the following lemma clarifies which Marshall–Olkin distributions are exchangeable.

Lemma 2.1 (Exchangeable Marshall–Olkin distribution). *On a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *let* $(\tau_1, \dots, \tau_d)'$ *be a random vector with Marshall–Olkin distribution, that is, with survival function* (1.1) *for parameters* $\lambda_I \geq 0$, $\emptyset \neq I \subset \{1, \dots, d\}$, *such that* $\sum_{I:k \in I} \lambda_I > 0$, $k = 1, \dots, d$. *Then* $(\tau_1, \dots, \tau_d)'$ *is exchangeable if and only if the parameters satisfy the following condition*:

$$|I| = |\tilde{I}| \quad \Rightarrow \quad \lambda_I = \lambda_{\tilde{I}}. \tag{2.1}$$

Proof. First suppose that (2.1) is valid. Without loss of generality, assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space from the original construction of Marshall and Olkin (1967), on which $(\tau_1, \dots, \tau_d)'$ is constructed by (1.2). Rewriting this definition, observe that

$$\tau_k := \min_{i=1,\dots,d} \big\{ \min\{E_I | I \subset \{1, \dots, d\}, k \in I, |I| = i\} \big\}, \qquad k = 1, \dots, d.$$

For all $i, k = 1, \dots, d$ there are precisely $d-1$ choose $i-1$ subsets I of $\{1, \dots, d\}$ with i elements containing k. By assumption, their associated parameters λ_I are identical and, in particular, independent of k. It follows for $\{i, k\} \subset \{1, \dots, d\}$ that the distribution of

$$\min\{E_I \,|\, I \subset \{1, \dots, d\}, k \in I, |I| = i\},$$

and, therefore, the distribution of τ_k is independent of k. This implies that $(\tau_1, \dots, \tau_d)'$ is exchangeable.

Conversely, assume that $(\tau_1, \dots, \tau_d)'$ is exchangeable. This means that the survival function (1.1) is invariant with respect to its arguments. In order to simplify notation, write $\bar{F}(\vec{t})$ instead of $\bar{F}(t_1, \dots, t_d)$, where $\vec{t} := (t_1, \dots, t_d)'$. Moreover, the ith unit vector in $\mathbb{R}^d$ is denoted by $\vec{e}_i$. Condition (2.1) is shown by induction over the cardinality of subsets of $\{1, \dots, d\}$. To begin with, verify $\lambda_{\{1\}} = \lambda_{\{2\}} = \dots = \lambda_{\{d\}}$: for each $k = 2, \dots, d$, exchangeability implies that

$$\sum_{\substack{\emptyset \neq I \subset \{1,\dots,d\} \\ I \neq \{1\}}} \lambda_I = -\log \bar{F}\left(\sum_{i=2}^{d} \vec{e}_i\right) = -\log \bar{F}\left(\sum_{\substack{i=1 \\ i \neq k}}^{d} \vec{e}_i\right) = \sum_{\substack{\emptyset \neq I \subset \{1,\dots,d\} \\ I \neq \{k\}}} \lambda_I.$$

Subtracting the sum of all parameters on both sides, this in turn verifies $\lambda_{\{1\}} = \lambda_{\{2\}} = \dots = \lambda_{\{d\}}$. By the induction hypothesis, assume that all parameters λ_I corresponding to subsets $I \subset \{1, \dots, d\}$ of cardinality $|I| \leq k$ are identical. Then, prove that all parameters λ_I corresponding to subsets $I \subset \{1, \dots, d\}$ of cardinality $|I| = k+1$ are identical. To this end, let I_0 be an arbitrary subset of $\{1, \dots, d\}$ of cardinality $|I_0| = k+1$. Then

$$\sum_{\substack{\emptyset \neq I \subset \{1,\dots,d\} \\ I \not\subseteq I_0}} \lambda_I = -\log \bar{F}\left(\sum_{\substack{i=1 \\ i \notin I_0}}^{d} \vec{e}_i\right) = -\log \bar{F}\left(\sum_{i=k+2}^{d} \vec{e}_i\right) = \sum_{\substack{\emptyset \neq I \subset \{1,\dots,d\} \\ I \not\subseteq \{1,\dots,k+1\}}} \lambda_I.$$

Subtracting the sum of all parameters on both sides, this implies

$$\lambda_{I_0} + \sum_{\substack{\emptyset \neq I \subset I_0 \\ |I| \leq k}} \lambda_I = \lambda_{\{1,\dots,k+1\}} + \sum_{\substack{\emptyset \neq I \subset \{1,\dots,k+1\} \\ |I| \leq k}} \lambda_I.$$

Using the induction hypothesis, this verifies that $\lambda_{\{1,\dots,k+1\}} = \lambda_{I_0}$. Since I_0 was arbitrary with cardinality $k+1$, one may conjecture that all parameters λ_I with $|I| = k+1$ are identical. The claim is established. □

Lemma 2.1 implies that for exchangeable Marshall–Olkin distributions, a parameter λ_I is only allowed to depend on the cardinality $|I|$ of I. In this case, denote by $\lambda_{|I|,d} := \lambda_I$, and a reduced set of only d parameters $(\lambda_{1,d}, \dots, \lambda_{d,d}) \in [0, \infty)^d \setminus$

$\{(0,\ldots,0)\}$ is obtained. It follows that the distribution of $(\tau_{\pi(1)},\ldots,\tau_{\pi(d)})'$ is invariant under permutations π on $\{1,\ldots,d\}$. Furthermore, it follows from Theorem 2.3 in Mai and Scherer (2011) that the survival function of such an exchangeable Marshall–Olkin distribution can be simplified to

$$\mathbb{P}(\tau_1 > t_1,\ldots,\tau_d > t_d) = \exp\left(-\sum_{k=1}^{d} t_{(d+1-k)} \sum_{i=0}^{d-k} \binom{d-k}{i} \lambda_{i+1,d}\right), \qquad (2.2)$$

where $0 \le t_{(1)} \le \cdots \le t_{(d)}$ denotes the ordered list of $t_1,\ldots,t_d \ge 0$.

Given a nonzero row vector $(\lambda_{1,d},\ldots,\lambda_{d,d})$ of d nonnegative numbers, it is possible to compute the following triangular scheme:

$$\begin{array}{ccccccccc}
 & & & & \lambda_{1,1} & & & & \\
 & & & \lambda_{1,2} & & \lambda_{2,2} & & & \\
 & & \iddots & & \cdots & & \ddots & & \\
 & \lambda_{1,d-1} & & & & & & \lambda_{d-1,d-1} & \\
\lambda_{1,d} & & \lambda_{2,d} & & \cdots & & \lambda_{d-1,d} & & \lambda_{d,d},
\end{array}$$

where the nth row is related to the row below via $\lambda_{k,n} = \lambda_{k,n+1} + \lambda_{k+1,n+1}$, $k = 1,\ldots,n$, $n = 1,\ldots,d-1$. Since this resembles the relation in Pascal's classical triangle, it is therefore called *Pascal's rule*. With the given row $(\lambda_{1,d},\ldots,\lambda_{d,d})$, one can only compute the tip of the triangle, since Pascal's rule is applied from the bottom to the top. Therefore, such a triangular array of nonnegative numbers is henceforth called an *inverse Pascal triangle*. Translated into the language of the Marshall–Olkin distribution, it is easy to verify that the parameters $(\lambda_{1,n},\ldots,\lambda_{n,n})$ in the nth row of the triangle are precisely the parameters of the Marshall–Olkin distribution of a subvector of $(\tau_1,\ldots,\tau_d)'$ of length n. For example,

$$\lambda_{1,1} = \lambda_{1,2} + \lambda_{2,2} = \lambda_{1,3} + 2\lambda_{2,3} + \lambda_{3,3} = \cdots = \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda_{i+1,d}$$

is precisely the exponential rate of the random variables τ_k, $k = 1,\ldots,d$. Hence, the extendibility of Marshall–Olkin distributions is equivalent to the extendibility of such triangular schemes. The question whether a Marshall–Olkin distribution can be extended (by one dimension) translates to whether a new row can be appended below a given triangle without violating Pascal's rule. For example, the row vector $(\lambda_{1,3},\lambda_{2,3},\lambda_{3,3}) = (1,3,1)$ does not allow any extension, since $1+1<3$. In contrast, the random vector $(\lambda_{1,3},\lambda_{2,3},\lambda_{3,3}) = (1,1,1)$ even allows for an infinite extension via symmetric splitting:

$$
\begin{array}{ccccccccccc}
 & & & 1 & & 1 & & 1 & & & \\
 & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \\
 & \frac{1}{4} & & \frac{1}{4} & & \frac{1}{4} & & \frac{1}{4} & & \frac{1}{4} & \\
\frac{1}{8} & & \frac{1}{8} & & \frac{1}{8} & & \frac{1}{8} & & \frac{1}{8} & & \frac{1}{8} \\
\iddots & & & & & \vdots & & & & & \ddots
\end{array}
$$

The following lemma gives a necessary and sufficient condition whether it is possible to append one additional row below a given inverse Pascal's triangle without violating Pascal's rule.

Lemma 2.2 (Extending the dimension by one). *Given* $(\lambda_{1,d}, \ldots, \lambda_{d,d}) \in [0, \infty)^d$, *it is possible to add* (*at least*) *one row at the bottom of the associated inverse Pascal triangle without violating Pascal's rule*, *if and only if* $[m, M] \cap R_d \neq \varnothing$, *where*

$$
m := \max_{\substack{j=1,\ldots,d \\ j \text{ odd}}} \left\{ \sum_{i=j}^{d} (-1)^i \lambda_{i,d} \right\},
$$

$$
M := \min_{\substack{j=1,\ldots,d \\ j \text{ even}}} \left\{ \sum_{i=j}^{d} (-1)^i \lambda_{i,d} \right\},
$$

$$
R_d := \begin{cases} [0, \infty), & d \text{ even}, \\ (-\infty, 0], & d \text{ odd}. \end{cases}
$$

Proof. The existence of an extension is equivalent to the existence of a vector $(\lambda_{1,d+1}, \ldots, \lambda_{d+1,d+1}) \in [0, \infty)^{d+1}$ such that $\lambda_{k,d} = \lambda_{k,d+1} + \lambda_{k+1,d+1}$, $k = 1, \ldots, d$. Solving this linear equation system for the $d + 1$ unknown parameters $\lambda_{1,d+1}, \ldots, \lambda_{d+1,d+1}$, this is equivalent to the existence of a number $\kappa \in \mathbb{R}$ such that

$$
\begin{pmatrix}
(-1)^1 \sum_{i=1}^{d} (-1)^i \lambda_{i,d} \\
(-1)^2 \sum_{i=2}^{d} (-1)^i \lambda_{i,d} \\
(-1)^3 \sum_{i=3}^{d} (-1)^i \lambda_{i,d} \\
\vdots \\
(-1)^d \sum_{i=d}^{d} (-1)^i \lambda_{i,d} \\
0
\end{pmatrix}
+ \kappa
\begin{pmatrix}
1 \\ -1 \\ 1 \\ \vdots \\ (-1)^{d-1} \\ (-1)^d
\end{pmatrix}
\in [0, \infty)^{d+1}.
$$

Hence, such a parameter κ has to be in the set $[m, M] \cap R_d$ as claimed, establishing the claim. □

Unfortunately, Lemma 2.2 cannot easily be applied iteratively. It only provides a useful criterion for a random vector $(\tau_1, \ldots, \tau_d)'$ to be 1-extendible, that is, to find an exchangeable random vector $(\tilde{\tau}_1, \ldots, \tilde{\tau}_{d+1})'$ with Marshall–Olkin distribution such that all d-dimensional margins are identical in law with $(\tau_1, \ldots, \tau_d)'$. However, it is not clear whether there is an infinite extension $\{\tilde{\tau}_k\}_{k\in\mathbb{N}}$. A useful criterion for an infinite extension, and, hence, for an implicitly given conditional i.i.d. structure, is provided by the following theorem, which is the major result of this paper. The proof is based on a composition of the results in the authors' reference Mai and Scherer (2011) and the solution to the so-called truncated Hausdorff moment problem provided in Karlin and Shapley (1953).

Consider the finite inverse Pascal triangle associated with a given exchangeable Marshall–Olkin distribution in dimension d. Instead of parameterizing the Marshall–Olkin distribution by the numbers $\lambda_{1,d}, \ldots, \lambda_{d,d}$ in the bottom line of the triangle, the former reference suggests to parameterize it by the numbers $\lambda_{1,1}, \lambda_{1,2}, \ldots, \lambda_{1,d}$—that is, by the left column of the triangle. It is shown in Mai and Scherer (2011) that this sequence of real numbers satisfies a certain monotonicity property. The extension of the triangle is equivalent to the extension of this sequence to an infinite sequence $\{\lambda_{1,n}\}_{n\in\mathbb{N}}$, without violating this monotonicity property. More precisely, one has to show that the finite sequence $\lambda_{1,1}, \ldots, \lambda_{1,d}$ can be extended to an infinite completely monotone sequence $\{\lambda_{1,n}\}$. Hausdorff (1921) showed that this is equivalent to the fact that there is a random variable X on $[0, 1]$ such that $\lambda_{1,n}/\lambda_{1,1} = \mathbb{E}[X^{n-1}]$, $n \in \mathbb{N}$. Deciding whether a given finite sequence of numbers is the initial part of a completely monotone sequence is known as the *truncated Hausdorff moment problem*. Its solution in terms of convenient, necessary and sufficient conditions on the given finite sequence is provided in Karlin and Shapley (1953). Summing up, one obtains the following Theorem 2.3.

Its formal proof uses the language of copula theory. Standard textbooks on copulas are Joe (1997), Nelsen (1999), McNeil, Frey and Embrechts (2005). In particular, recall that the survival function of a random vector $(\tau_1, \ldots, \tau_d)'$ with continuous margins can always be written as

$$\mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) = \hat{C}\big(\mathbb{P}(\tau_1 > t_1), \ldots, \mathbb{P}(\tau_d > t_d)\big)$$

for a unique distribution function $\hat{C}$ on $[0, 1]^d$ with uniform marginals, called the *survival copula* of $(\tau_1, \ldots, \tau_d)'$. Furthermore, recall that a Lévy subordinator is a nondecreasing stochastic process with stationary and independent increments, starting from zero; see Bertoin (1999) for further details on such processes.

Theorem 2.3 (Infinite extendibility). *Fix $d \geq 2$. Consider a row vector $(\lambda_{1,d}, \ldots, \lambda_{d,d}) \in [0, \infty)^d \setminus \{(0, \ldots, 0)\}$ and a random vector $(\tau_1, \ldots, \tau_d)'$ following the associated exchangeable Marshall–Olkin distribution, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The following statements are equivalent*:

(a) *There exists an infinite inverse Pascal triangle whose dth row equals* $(\lambda_{1,d}, \ldots, \lambda_{d,d})$.
(b) $(\tau_1, \ldots, \tau_d)'$ *is extendible to an infinite exchangeable sequence such that every finite subvector has a Marshall–Olkin distribution.*
(c) *Defining* $b_k := \sum_{i=0}^{d-k-1} \binom{d-k-1}{i} \lambda_{i+1,d}$ *for* $k = 0, \ldots, d-1$, *the Hankel determinants* $\hat{H}_1, \check{H}_1, \hat{H}_2, \check{H}_2, \ldots, \hat{H}_{d-1}, \check{H}_{d-1}$ *are all nonnegative, which for* $l \in \mathbb{N}$ *with* $2l \leq d-1$ *and for* $k \in \mathbb{N}_0$ *with* $2k+1 \leq d-1$ *are defined by*

$$\hat{H}_{2l} := \begin{vmatrix} b_0 & \cdots & b_l \\ \vdots & & \vdots \\ b_l & \cdots & b_{2l} \end{vmatrix}, \qquad \check{H}_{2l} := \begin{vmatrix} b_1 - b_2 & \cdots & b_l - b_{l+1} \\ \vdots & & \vdots \\ b_l - b_{l+1} & \cdots & b_{2l-1} - b_{2l} \end{vmatrix},$$

$$\hat{H}_{2k+1} := \begin{vmatrix} b_1 & \cdots & b_{k+1} \\ \vdots & & \vdots \\ b_{k+1} & \cdots & b_{2k+1} \end{vmatrix},$$

$$\check{H}_{2k+1} := \begin{vmatrix} b_0 - b_1 & \cdots & b_k - b_{k+1} \\ \vdots & & \vdots \\ b_k - b_{k+1} & \cdots & b_{2k} - b_{2k+1} \end{vmatrix}.$$

(d) *There exists a probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *supporting i.i.d. standard exponential random variables* $\tilde{E}_1, \ldots, \tilde{E}_d$ *and an independent Lévy subordinator* $\tilde{S} = \{\tilde{S}_t\}_{t \geq 0}$, *such that* $(\tau_1, \ldots, \tau_d)' \stackrel{d}{=} (\tilde{\tau}_1, \ldots, \tilde{\tau}_d)'$, *where*

$$\tilde{\tau}_k := \inf\{t > 0 : \tilde{S}_t > \tilde{E}_k\}, \qquad k = 1, \ldots, d.$$

Proof. The subsequent proof strategy is (d) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (c) $\Rightarrow$ (d).

- (d) $\Rightarrow$ (b): On $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, extend the sequence $\tilde{E}_1, \ldots, \tilde{E}_d$ to an infinite i.i.d. sequence $\{\tilde{E}_k\}_{k \in \mathbb{N}}$ and define the associated exchangeable sequence $\{\tilde{\tau}_k\}_{k \in \mathbb{N}}$ of first passage times. Since $(\tau_1, \ldots, \tau_d)' \stackrel{d}{=} (\tilde{\tau}_1, \ldots, \tilde{\tau}_d)'$, the extendibility of $(\tau_1, \ldots, \tau_d)'$ is established. Moreover, it is shown in Mai and Scherer (2011) that all finite subvectors of $\{\tilde{\tau}_k\}_{k \in \mathbb{N}}$, in particular, the n-margins for $n > d$, follow a certain Marshall–Olkin distribution.
- (b) $\Rightarrow$ (a): Given the exchangeable sequence $\{\tilde{\tau}_k\}_{k \in \mathbb{N}}$, one has for each $n \in \mathbb{N}$ the associated parameters $(\lambda_{1,n}, \ldots, \lambda_{n,n})$ of the exchangeable Marshall–Olkin distribution of $(\tilde{\tau}_1, \ldots, \tilde{\tau}_n)'$. Since the Marshall–Olkin distribution is automatically consistent with Pascal's rule, the claim is immediate.
- (a) $\Rightarrow$ (c): By virtue of Pascal's rule, the value $b_0 := \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda_{i+1,d}$ is independent of $d \in \mathbb{N}$. This means that

$$\begin{aligned} b_0 &= \lambda_{1,1} = \lambda_{1,2} + \lambda_{2,2} \\ &= \lambda_{1,3} + 2\lambda_{2,3} + \lambda_{3,3} = \cdots = \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda_{i+1,d}. \end{aligned}$$

Moreover, again by Pascal's rule, one observes that $b_k = \lambda_{1,k+1}$ for $k = 0, \ldots, d-1$, since

$$\begin{aligned}\lambda_{1,k+1} &= \lambda_{1,k+2} + \lambda_{2,k+2} \\ &= \lambda_{1,k+3} + 2\lambda_{2,k+3} + \lambda_{3,k+3} = \cdots = \sum_{i=0}^{d-k-1} \binom{d-k-1}{i} \lambda_{i+1,d}.\end{aligned}$$

Hence, extending $(b_0, \ldots, b_{d-1})$, one may define $b_k := \lambda_{1,k+1}$, $k \in \mathbb{N}_0$, to obtain an infinite sequence $\{b_k\}_{k\in\mathbb{N}_0}$. Note, in particular, that for all $n \in \mathbb{N}$ with $n \geq k+1$ Pascal's rule implies that

$$b_{k-1} = \lambda_{1,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} \lambda_{i+1,n}, \qquad k = 1, \ldots, n.$$

Using (2.2), it follows that the function

$$(t_1, \ldots, t_n) \mapsto \exp\left(-\sum_{k=1}^{n} t_{(n+1-k)} b_{k-1}\right), \qquad t_1, \ldots, t_n \geq 0,$$

is the survival function of a Marshall–Olkin distribution for all $n \geq 2$. Since the margins of this distribution are exponential with parameter b_0, the corresponding survival copula $\hat{C}$ is given by

$$\hat{C}(u_1, \ldots, u_n) = u_{(1)}^{a_0} u_{(2)}^{a_1} u_{(3)}^{a_2} \cdots u_{(n)}^{a_{n-1}}, \qquad u_1, \ldots, u_n \in [0, 1],$$

where $a_k := b_k/b_0$, $k \in \mathbb{N}_0$. Corollary 3.4 in Mai and Scherer (2009) implies that there exists a random variable X on the unit interval $[0, 1]$ such that $a_k := \mathbb{E}[X^k]$, $k \in \mathbb{N}_0$. Hence, the finite sequence $(a_0, \ldots, a_{n-1})$ is the initial sequence of moments of a random variable. Then Theorem 1.4.3 in Dette and Studden (1997) implies that the associated Hankel determinants $\hat{H}_1$, $\check{H}_1$, $\hat{H}_2$, $\check{H}_2$, …, $\hat{H}_{d-1}$, $\check{H}_{d-1}$ are all nonnegative, when the b's are replaced by the a's. This is the so-called *truncated Hausdorff moment problem* and was originally solved by Karlin and Shapley (1953). Since the a's and b's differ only via the constant multiple factor b_0, which does not affect the sign of the determinants, the claim follows immediately.

- (c) ⇒ (d): Assume (c) holds. With $b_0 := \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda_{i+1,d}$, define $a_k := b_k/b_0$ for $k = 0, \ldots, d-1$. By assumption (c), the Hankel determinants $\hat{H}_1$, $\check{H}_1$, $\hat{H}_2$, $\check{H}_2$, …, $\hat{H}_{d-1}$, $\check{H}_{d-1}$ are all nonnegative, when the b's are replaced by the a's. It follows from Theorem 1.4.3 in Dette and Studden (1997) that there exists a random variable $X \in [0, 1]$ with $(\mathbb{E}[X^0], \ldots, \mathbb{E}[X^{d-1}]) := (a_0, \ldots, a_{d-1})$. Due to Hausdorff's theorem [see Hausdorff (1921)], the sequence of moments of X is completely monotone. An application of Theorem 3.3 in Mai and Scherer (2009) implies the existence of a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ supporting i.i.d.

standard exponential random variables $\tilde{E}_1, \dots, \tilde{E}_d$ and an independent Lévy subordinator $\hat{S} = \{\hat{S}_t\}_{t \geq 0}$, such that $(b_0\tau_1, \dots, b_0\tau_d)' \stackrel{d}{=} (\hat{\tau}_1, \dots, \hat{\tau}_d)'$, where

$$\hat{\tau}_k := \inf\{t > 0 : \hat{S}_t > \tilde{E}_k\}, \qquad k = 1, \dots, d.$$

Defining $\tilde{S} := \{\hat{S}_{b_0 t}\}_{t \geq 0}$ and

$$\tilde{\tau}_k := \inf\{t > 0 : \tilde{S}_t > \tilde{E}_k\}, \qquad k = 1, \dots, d,$$

yields the claim. □

The crucial innovation of Theorem 2.3 is statement (c). Given an exchangeable Marshall–Olkin distribution, it gives an algorithm on how to decide whether the distribution is extendible or not—based only on its parameters $(\lambda_{1,d}, \dots, \lambda_{d,d})$. If (and only if) it is extendible, part (d) says that an alternative stochastic latent one-factor representation of this distribution via a Lévy subordinator is possible.

Example 2.4 (The trivariate case). Consider the simplest nontrivial case, that is, $d = 3$. In the original probabilistic model (1.2) the random variables τ_1, τ_2, τ_3 are not i.i.d. conditioned on the shock $E_{\{1,2,3\}}$, since the pairwise shocks $E_{\{1,2\}}, E_{\{1,3\}}, E_{\{2,3\}}$ induce additional dependence. Similarly, conditioned on the σ-algebra $\mathcal{G} := \sigma(E_{\{1,2,3\}}, E_{\{1,2\}}, E_{\{1,3\}}, E_{\{2,3\}})$, the random variables τ_1, τ_2, τ_3, although independent, are not identically distributed. Writing out the Hankel determinants, Theorem 2.3(c) implies that $(\tau_1, \tau_2, \tau_3)'$ is extendible if and only if its parameters $(\lambda_{1,3}, \lambda_{2,3}, \lambda_{3,3}) \in [0, \infty)^3 \setminus \{(0, 0, 0)\}$ satisfy $\lambda_{2,3}^2 \leq \lambda_{1,3}\lambda_{3,3}$. However, it is not obvious how (and even if) a σ-algebra $\mathcal{G}$ can be found such that τ_1, τ_2, τ_3 are i.i.d. conditioned on $\mathcal{G}$. Nevertheless, Theorem 2.3(d) explains how a conditionally i.i.d. random vector $(\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3)'$, which agrees in distribution with $(\tau_1, \tau_2, \tau_3)'$, can be constructed on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. On the latter probability space, $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3$ are i.i.d. conditioned on $\tilde{\mathcal{G}} := \sigma(\tilde{S}_t : t \geq 0)$, using the notation of Theorem 2.3(d).

Remark 2.5 (Link to regenerative composition structures). In the extendible case, the construction via Lévy subordinators is closely related to so-called *regenerative composition structures*, as introduced in Gnedin and Pitman (2005). More precisely, assume the notation of Theorem 2.3(d) above and apply the usual convention $\tilde{\tau}_{(1)} \leq \cdots \leq \tilde{\tau}_{(d)}$ for the ordered list of $\tilde{\tau}_1, \dots, \tilde{\tau}_d$. Whereas the present article studies the distribution of $(\tilde{\tau}_1, \dots, \tilde{\tau}_d)'$, the reference Gnedin and Pitman (2005) studies the distribution of $(P_1, \dots, P_{K_d})'$, where $K_d := |\{\tilde{\tau}_1, \dots, \tilde{\tau}_d\}|$ denotes the number of different first passage times, and

$$P_1 := \max\{j \in \mathbb{N} | \tilde{\tau}_{(j)} = \tilde{\tau}_{(1)}\},$$
$$P_2 := \max\{j \in \mathbb{N} | \tilde{\tau}_{(P_1+j)} = \tilde{\tau}_{(P_1+1)}\},$$
$$\vdots$$
$$P_{K_d} := \max\{j \in \mathbb{N} | \tilde{\tau}_{(P_{K_d-1}+j)} = \tilde{\tau}_{(P_{K_d-1}+1)}\}.$$

Notice that $P_1, \dots, P_{K_d} \in \mathbb{N}$ and $P_1 + \cdots + P_{K_d} = d$, hence, $(P_1, \dots, P_{K_d})'$ defines a so-called *random partition* of d, and vast literature can be found on the study of such objects; see the references in Gnedin and Pitman (2005).

3 Conclusion

Necessary and sufficient conditions were derived on the parameters of a given d-dimensional Marshall–Olkin distribution to be extendible to an infinite exchangeable sequence. This constitutes an important and tractable subclass of multivariate exponential distributions.

References

Albrecher, H., Ladoucette, S. A. and Schoutens, W. (2007). A generic one-factor Lévy model for pricing synthetic CDOs. In *Advances in Mathematical Finance (Applied and Numerical Harmonic Analysis)* 259–278. Boston: Birkhäuser. MR2359372

Aldous, D. J. (1985). Exchangeability and related topis. In *École d'Été de Probabilités de Saint-Flour XIII-1983. Lecture Notes in Mathematics* **1117** 1–198. Berlin: Springer. MR0883646

Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*. New York: Holt, Rinehart and Winston. MR0438625

Bertoin, J. (1999). Subordinators: Examples and Applications. In *École d'Été de Probabilités de Saint-Flour XXVII-1997. Lecture Notes in Mathematics* **1717** 1–91. Berlin: Springer. MR1746300

De Finetti, B. (1937). La prévision: ses lois logiques, ses sources subjectives. *Annales de l'Institut Henri Poincaré* **7** 1–68. MR1508036

Dette, H. and Studden, W. J. (1997). *The Theory of Canonical Moments With Applications in Statistics, Probability, and Analysis. Wiley Series in Probability and Statistics*. New York: Wiley. MR1468473

Fang, K.-T., Kotz, S. and Ng, K.-W. (1990). *Symmetric Multivariate and Related Distributions*. London: Chapman & Hall. MR1071174

Frey, R. and McNeil, A. J. (2001). Modelling dependent defaults. Working paper, ETHZ, Swiss. Available at http://e-collection.ethbib.ethz.ch/show?type=bericht&nr=273.

Galambos, J. and Kotz, S. (1978). *Characterizations of Probability Distributions. Lecture Notes in Mathematics* **675**. Berlin: Springer. MR0513423

Gnedin, A. and Pitman, J. (2005). Regenerative composition structures. *The Annals of Probability* **33**(2), 445–479. MR2122798

Hausdorff, F. (1921). Summationsmethoden und Momentfolgen I. *Mathematische Zeitschrift* **9**(3–4), 74–109. MR1544453

Joe, H. (1997). *Multivariate Models and Dependence Concepts*. London: Chapman & Hall/CRC. MR1462613

Kalemanova, A., Schmid, B. and Werner, R. (2007). The normal inverse Gaussian distribution for synthetic CDO pricing. *The Journal of Derivatives* **14**(3), 80–93.

Karlin, S. and Shapley, L. S. (1953). Geometry of moment spaces. *Memoirs of the American Mathematical Society* **12**, 93. MR0059329

Mai, J.-F. and Scherer, M. (2009). Lévy-frailty copulas. *Journal of Multivariate Analysis* **100**(7), 1567–1585. MR2514148

Mai, J.-F. and Scherer, M. (2011). Reparameterizing Marshall–Olkin copulas with applications to sampling. *Journal of Statistical Computation and Simulation* **81**(1), 59–78. MR2747378

Marshall, A. W. and Olkin, I. (1967). A multivariate exponential distribution. *Journal of the American Statistical Association* **62**(317), 30–44. MR0215400

McNeil, A. J., Frey, R. and Embrechts, P. (2005). *Quantitative Risk Management*. Princeton, NJ: Princeton Univ. Press. MR2175089

McNeil, A. J. and Nešlehová, J. (2009). Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions. *The Annals of Statistics* **37**(5B), 3059–3097. MR2541455

Nelsen, R. B. (1999). *An Introduction to Copulas*. New York: Springer. MR1653203

Schoenberg, I. J. (1938). Metric spaces and positive definite functions. *Trans. Amer. Mat. Soc.* **44**, 522–536. MR1501980

Schönbucher, P. (2002). Taken to the limit: Simple and not-so-simple loan loss distributions. Working paper, Bonn Univ., Germany.

Vasicek, O. (1987). Probability of loss on a loan portfolio. Working paper, KMV Corporation.

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