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Marshall–Olkin Esscher transformed Laplace distribution and processes

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Abstract. In this article we consider a class of asymmetric distributions which belongs to one parameter regular exponential family. The Marshall–Olkin version of this family is also considered. Various properties are examined. Applications of these models in time series analysis are discussed. We also consider an application of Marshall–Olkin Esscher transformed Laplace distribution in financial modeling. A comparative study shows that Marshall–Olkin Esscher transformed Laplace distribution is a better fit to our data compared to asymmetric Laplace and Esscher transformed Laplace distributions.

1 Introduction

Among the important symmetric distributions such as uniform, triangular, cosine, logistic, Laplace, normal etc., the Laplace distribution has a special position because of its towering peak and heavy tails. For many years, this distribution was a popular topic in probability theory due to the simplicity of its characteristic function, density function and the distribution function, and thus enjoys numerous attractive probabilistic features. Under geometric summation, the Laplace distribution plays a role analogous to that of Gaussian distribution under ordinary summation, so that the Laplace distribution is applicable in stochastic modeling, economics and health sciences.

There has been some growing interest in the literature, in engineering and recently in finance, in using Laplace and related distributions in data modeling contexts that involve time. McGill (1962) showed that the Laplace distribution provides a characterization of the error in a timing device that is under periodic excitation. Hsu (1979) found that navigation errors for aircraft position are best fitted by a mixture of two Laplace distributions. Damsleth and El-Shaarawi (1989) employed an ARMA model with Laplace noise to fit weekly data on sulphate concentration in a Canadian watershed. Anderson and Arnold (1993) observed that IBM daily stock prize returns are adequately modeled by Linnik processes. The Linnik distribution is also a symmetric distribution with tail index $\alpha \in (0, 2]$. For

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$\alpha = 2$, the Linnik distribution coincides with the Laplace distribution. Mathew and Jayakumar (2003) discussed a generalized Linnik distribution and process.

In the last few decades, applications from environmental, financial, telecommunications, signal processing, image processing, climatology and the biomedical sciences have shown that data sets following a normal law are more often the exception rather than the rule. Therefore, probability distributions which can account for skewness and kurtosis and are more flexible than the normal, are often needed in statistical modeling of data from these contexts. A well-known approach introduced to model departure from normality is by modifying a symmetric probability density function of a random variable/vector in a multiplicative fashion, thereby introducing skewness. Azzallini (1985, 1986) thoroughly implemented this idea for the univariate normal distribution, thereby yielded the skew-normal distribution. An extension to the multivariate case was then introduced by Azzallini and Dalla Valle (1996) and it became a fruitful starting point for further developments. Since then various univariate and multivariate skew-symmetric distributions have been constructed and intensively studied in the last decade. Later this idea was successfully used for defining skew-elliptical families and their generalizations (see Genton (2004a)). In the following years, significant progress has been made towards the construction of so-called multivariate skew-symmetric and skew-elliptical distributions and their successful application to problems in areas such as engineering, environmetrics, economics and biomedical sciences. For more details see Genton (20004b) and Azzallini (2005).

In this work, we consider a transformation procedure known as Esscher transformation (exponential tilting), introduced for density approximations by Esscher (1932) and further developed as a general probabilistic method by Daniels (1954) and Barndorff-Nielsen (1979) for introducing asymmetry and skewness in a symmetric distribution. The Esscher transform of a single random variable is a well established concept in the risk theory literature. This method provides a means for creating a regular exponential family from a distribution whose cumulant generating function converges in the regular sense. It is a time-honored tool in actuarial science. Though initially the Esscher transform was developed to approximate the aggregate claim amount distribution around a point of interest, say, x_0, by applying with an analytic approximation (the Edgeworth series) to the transformed distribution with the parameter θ chosen such that the new mean is equal to x_0, nowadays Esscher transformation is used as an efficient technique for valuing derivative securities, if the logarithms of the prices of the primitive securities are governed by certain stochastic processes with stationary and independent increments. An Esscher transform of such a stock price process induces an equivalent probability measure on the process. The Esscher parameter or parameter vector is determined so that the discounted price of each primitive security is a martingale under the new probability measure. Straight forward consequences of the Esscher transformations include formulae for pricing options on the maximum and minimum of multiple risky assets.

Since Laplace distribution is a member of an elliptical class of distributions (see Fang, Kotz and Ng (1990)), and has many applications in stochastic modeling and risk analysis, we consider the Esscher transform of a standard symmetric Laplace distribution. We introduce Esscher transformed Laplace distribution in Section 2. In Section 3, we present the Marshall–Olkin form of the Esscher transformed Laplace distribution, which is obtained by the general method of adding a parameter to a family of distributions (introduced by Marshall and Olkin (1997)), and explore its properties. Estimation of the parameters is considered in Section 4. Autoregressive processes of order 1 and order k with Marshall–Olkin Esscher transformed Laplace marginals are developed in Section 5. In Section 6, we model financial data, using Marshall–Olkin Esscher transformed Laplace distribution and show that it is a better fit compared to asymmetric Laplace and Esscher transformed Laplace distributions. We conclude the paper by Section 7.

2 Esscher transformed Laplace distribution

The Esscher transform of a density $f(x)$ is defined as

$$f(x;\theta) = \frac{e^{\theta x} f(x)}{M(\theta)},$$

provided the moment generating function, $M(\theta) = \int e^{\theta x} f(x)\,dx$, exists in an interval containing zero. Here θ is known as an Esscher parameter.

For the symmetric standard Laplace distribution with p.d.f.,

$$f(x) = \frac{1}{2}\exp(-|x|), \qquad -\infty < x < \infty,$$

the moment generating function is given by

$$M(\theta) = \frac{1}{1-\theta^2}, \qquad |\theta| < 1.$$

Thus, the Esscher transformed Laplace density, denoted by ETL(θ), is

$$f(x;\theta) = \begin{cases} \dfrac{(1-\theta^2)}{2}\exp[x(1+\theta)], & x < 0, \\ \dfrac{(1-\theta^2)}{2}\exp[-x(1-\theta)], & x \geq 0 \end{cases} \tag{2.1}$$

for $\theta \in (-1, 1)$.

The cumulative distribution function (c.d.f.) corresponding to ETL(θ) is given by

$$F(x) = \begin{cases} \dfrac{(1-\theta)}{2}\exp[x(1+\theta)], & x < 0, \\ \dfrac{1-\theta}{2} + \dfrac{1+\theta}{2}(1-\exp[-x(1-\theta)]), & x \geq 0 \end{cases} \tag{2.2}$$

for $\theta \in (-1, 1)$.

This distribution satisfies many important statistical properties like infinite divisibility, geometric infinite divisibility, stability with respect to geometric summation, maximum entropy, finiteness of moments, etc. Moreover, this model provides more flexibility, allowing for more asymmetry, peakedness and tail heaviness than the normal model, which are common features of many financial data sets (see Boothe and Glassman (1987)).

For many years stable Paretian laws were considered for modeling such data sets, because they also account for asymmetry and heavier tails. Though the theory of stable Paretian distributions are well developed, because of the lack of analytical form of the densities and infinite second moment, their applications in practical modeling are still rather limited; for more details we refer to Samorodnitsky and Taqqu (1994). Again, the stable Paretian model does not account for peakedness around the origin which is seen in most financial data. Considering all these facts, the Esscher transformed Laplace model can be considered as a good alternative to the stable Paretian model.

The characteristic function of the ETL(θ) distribution is

$$\phi_X(t) = \left(1 + \frac{t^2}{1-\theta^2} - \frac{2it\theta}{1-\theta^2}\right)^{-1}. \tag{2.3}$$

Graphs of Laplce and ETL(θ) distributions for various values of θ are given in Figure 1(a), Figure 1(b) and Figure 1(c).

Note that after transformation on the Laplace model, we get an asymmetric unimodal distribution with high peakedness at zero. Here the Esscher parameter θ acts as the skewness parameter. ETL(θ) belongs to the one parameter regular exponential family. Families of this type are especially tractable for statistical inference.

Introducing a location parameter μ and a scale parameter σ, we have the location scale family, with p.d.f.

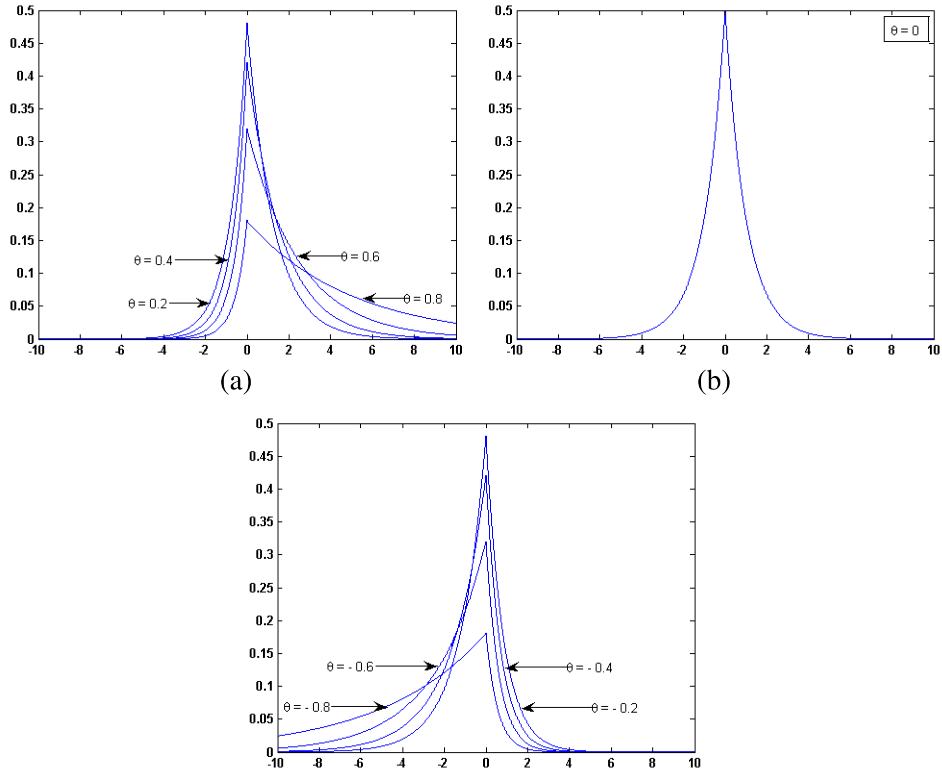
$$\begin{gathered} f(x;\theta,\mu,\sigma) = \left(\frac{1-\theta^2}{2\sigma}\right)\exp\left[\theta\left(\frac{x-\mu}{\sigma}\right) - \left|\frac{x-\mu}{\sigma}\right|\right], \\ -\infty < x < \infty, |\theta| < 1, \mu \in R, \sigma > 0, \end{gathered} \tag{2.4}$$

and c.d.f.

$$F(x) = \begin{cases} \dfrac{(1-\theta)}{2}\exp\left[\left(\dfrac{x-\mu}{\sigma}\right)(1+\theta)\right], & x < \mu, \\ 1 - \dfrac{1+\theta}{2}\exp\left[\left(\dfrac{\mu-x}{\sigma}\right)(1-\theta)\right], & x \geq \mu, \end{cases} \tag{2.5}$$

where $|\theta| < 1$ and $\sigma > 0$.

This family of distributions is quite useful for modeling asymmetric data such as financial data and file size data having high skewness and a heavy-tailed nature.



(a) (b) (c)

Figure 1 *Densities of Esscher transformed Laplace distribution for* (a) $\theta \in (-1, 0)$, (b) $\theta = 0$ (*standard symmetric Laplace*) *and* (c) $\theta \in (0, 1)$.

3 Marshall–Olkin Esscher transformed Laplace distribution

In this section we introduce the Marshall–Olkin Esscher transformed Laplace distribution using the method proposed by Marshall and Olkin (1997). They introduced a method of obtaining a new family of survival functions by adding a new parameter to the existing distribution given by

$$\bar{G}(x) = \frac{\bar{F}(x)}{\beta + (1-\beta)\bar{F}(x)}, \qquad x \in R,$$

where $\bar{F}$ is a survival function and $\beta > 0$. If $\beta = 1$, then we have $\bar{G} = \bar{F}$.

In the past decade, many distributions which belong to the Marshall–Olkin family of distributions have been investigated: Exponential (Marshall and Olkin (1997)), Weibull (Marshall and Olkin (1997)), Marshall–Olkin q-Weibull (Jose, Naik and Ristic (2010)) and Generalized Marshall–Olkin (Li and Pellerey (2011)).

Here we replace the survival function with the characteristic function $\phi(t)$ using the result that if $\phi(t)$ is a characteristic function of some arbitrary distribution, then

$$\psi_\beta(\phi(t)) = \frac{\beta\phi(t)}{1-(1-\beta)\phi(t)}, \qquad 0 < \beta \le 1, \tag{3.1}$$

forms another class of characteristic functions. Note that $\psi_\beta(\phi(t))$ is the characteristic function of the geometric sum of independently and identically distributed (i.i.d.) random variables with common characteristic function $\phi(t)$. That is, $\psi_\beta(\phi(t))$ is the characteristic function corresponding to $Y = \sum_{i=1}^{N} X_i$, where $X_1, X_2, \dots, X_N$ are i.i.d. with common characteristic function $\phi(t)$ and $N \sim \text{Geometric}(1-\beta)$, independent of X_i. These kinds of geometric sums are of special interest in financial modeling and time series analysis; see Kotz, Kozubowski and Podgórski (2001). When $\beta = 1$, $\psi_\beta(\phi(t)) = \phi(t)$.

Remark 3.1. Consider the transformation

$$\psi_\beta(\phi(t)) = \frac{\beta\phi(t)}{1-(1-\beta)\phi(t)}, \qquad 0 < \beta \le 1,$$

then

$$\begin{aligned}\psi_{\beta'}(\psi_\beta(\phi(t)) &= \frac{\beta'\psi_\beta(\phi(t))}{1-(1-\beta')\psi_\beta(\phi(t))}\\ &= \frac{\beta\beta'\phi(t)}{1-(1-\beta\beta')\phi(t)}\\ &= \psi_{\beta\beta'}(\phi(t)).\end{aligned}$$

If X is an ETL(θ) random variable with characteristic function (2.3), then, using (3.1), the characteristic function of the Marshall–Olkin Esscher transformed Laplace distribution is

$$\begin{aligned}\psi(t) &= \left[1 + \frac{1}{\beta}\left(\frac{t^2}{1-\theta^2} - \frac{2it\theta}{1-\theta^2}\right)\right]^{-1} \qquad \text{for } 0 < \beta \le 1, |\theta| < 1\\ &= \left(1 + \frac{1}{\lambda^2}[t^2 - 2it\theta]\right)^{-1}.\end{aligned} \tag{3.2}$$

We shall denote the Marshall–Olkin Esscher transformed Laplace distribution as MOETL(λ, κ), where

$$\lambda = \sqrt{\beta(1-\theta^2)} \tag{3.3}$$

and

$$\kappa = \frac{\lambda}{\theta + \sqrt{\lambda + \theta^2}}. \tag{3.4}$$

Here $|\theta| < 1$ and $\beta > 0$ so that $\lambda > 0$ and $\kappa > 0$. Obviously, when $\beta = 1$, the distribution in (3.2) reduces to the one parameter ETL distribution.

The probability density function and cumulative distribution function of the MOETL(λ, κ) distribution are respectively given by

$$f(x; \lambda, \kappa) = \frac{\lambda\kappa}{1+\kappa^2} \begin{cases} \exp\left[\frac{\lambda x}{\kappa}\right], & x < 0, \\ \exp[-\lambda\kappa x], & x \geq 0 \end{cases} \tag{3.5}$$

and

$$F(x) = \begin{cases} \frac{\kappa^2}{1+\kappa^2} \exp\left[\frac{\lambda x}{\kappa}\right], & x < 0, \\ 1 - \frac{1}{1+\kappa^2} \exp[-\lambda\kappa x], & x \geq 0, \end{cases} \tag{3.6}$$

where $\lambda > 0$ and $\kappa > 0$ are the parameters of the distribution.

The rth raw moment of the MOETL(λ, κ) distribution denoted by α_r is given by

$$\alpha_r = r!\left(\frac{1}{\lambda\kappa}\right)^r \left(\frac{1+(-1)^r\kappa^{2(r+1)}}{1+\kappa^2}\right),$$

so that

$$\text{Mean} = \frac{1-\kappa^2}{\lambda\kappa}$$

and

$$\text{Variance} = \frac{1+\kappa^4}{\lambda^2\kappa^2}.$$

Figure 2 represents the probability density plots of the MOETL distribution for $\theta = 0.2$, $\theta = -0.2$ and for various values of β.

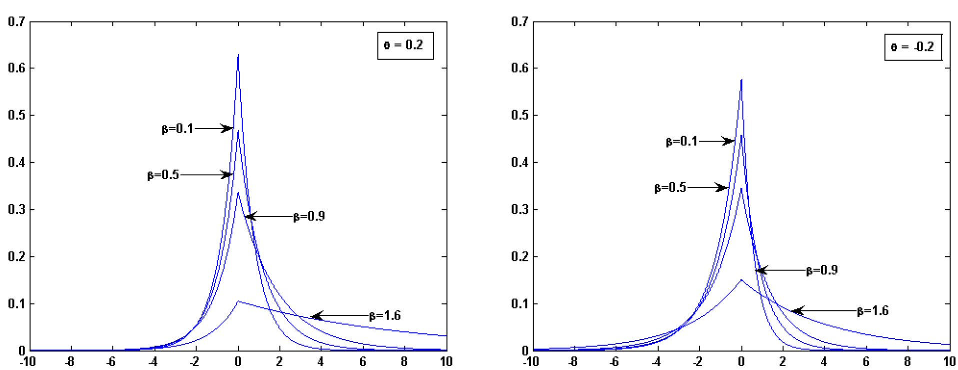


Figure 2 *Probability density plots of* MOETL(λ, κ).

It is clear that the distributions are unimodal with mode equal to zero. The coefficient of skewness and kurtosis of the distribution are given by

$$\gamma_1 = \frac{4\theta^2 + 3\lambda^2}{2\theta^2}$$

and

$$\gamma_2 = 6 + \frac{8\theta^2\lambda^2 + 3\lambda^4}{2\theta^4},$$

where λ is given in (3.3).

The distribution is positively skewed and lepto kurtic. When θ is positive, as β increases, the tail heaviness of the distribution increases at the right tail. When θ is negative, as β increases, the tail heaviness of the distribution increases at the left tail. So this distribution can have more flexible applications than the Esscher transformed Laplace distribution. Since we are mostly dealing with heavy-tailed distributions and Esscher transformed Laplace distribution is light heavy-tailed, this distribution serves as a competing alternative to it, in data modeling, in the areas of environmental, financial, telecommunications, signal processing, image processing, climatology and biomedical sciences.

We have the following theorems. The proofs of the theorems are given in Appendix sections A.1, A.2, A.3, A.4 and A.5, respectively.

Theorem 3.1. *If $X_1, X_2, \dots$ are i.i.d. as* $\mathrm{MOETL}(\lambda, \kappa)$ *distribution and N, a geometric random variable independent of $X_1, X_2, \dots$ with parameter p, $0 < p < 1$, then the geometric compound $S_N = X_1 + X_2 + X_3 + \cdots + X_N$ has the* $\mathrm{MOETL}(\sqrt{p}\lambda, \kappa)$ *distribution.*

Theorem 3.2. *If $X \sim$* $\mathrm{MOETL}(\lambda, \kappa)$ *distribution and if Z and W are independent random variables such that $Z \sim N(0, 1)$ and W an exponential random variable with mean $\frac{1}{\beta}$, then*

$$X \stackrel{d}{=} \frac{2\theta}{1-\theta^2} W + \frac{\sqrt{2}}{\sqrt{1-\theta^2}} \sqrt{W} Z, \tag{3.7}$$

where $\stackrel{d}{=}$ represents 'distributed as.'

Theorem 3.3. *If $X \sim$* $\mathrm{MOETL}(\lambda, \kappa)$ *distribution and I_k is a discrete random variable which take the values $-\kappa$ and $\frac{1}{\kappa}$ with probabilities $(\frac{\kappa^2}{1+\kappa^2})$ and $(\frac{1}{1+\kappa^2})$, respectively and W is a standard exponential random variable, then*

$$X \stackrel{d}{=} \left(\frac{1}{\lambda}\right) I_k W. \tag{3.8}$$

Theorem 3.4. *The* $\mathrm{MOETL}(\lambda, \kappa)$ *distribution is infinitely divisible.*

Proposition 3.1. *Let* $X \sim \text{MOETL}(\lambda, \kappa)$. *Then* X *is infinitely divisible, admitting the representation*

$$X \overset{d}{=} \sum_{i=1}^{n} X_{n_i},$$

where X_{n_i}*'s are i.i.d. variables which are the difference of two gamma variables with means* $\frac{1}{n\kappa\lambda}$ *and* $\frac{\kappa}{n\lambda}$, *respectively.*

Theorem 3.5. *If* $X \sim \text{MOETL}(\lambda, \kappa)$, *then* X *is geometrically infinitely divisible and for all* $p \in (0, 1)$,

$$X \overset{d}{=} \sum_{i=1}^{\gamma_p} X_p^{(i)}, \tag{3.9}$$

where γ_p *is a geometric random variable with mean* $\frac{1}{p}$, *the random variables* $X_p^{(i)}$*'s are i.i.d.* $\text{MOETL}(\lambda, \kappa)$ *for each* p *and* γ_p *and* $X_p^{(i)}$*'s are independent.*

4 Estimation of parameters

In this section, for estimating the parameters, we use the method of maximum likelihood and the method of moments.

4.1 Maximum likelihood estimation

For convenience, we reparametrize the distribution given in (3.5), by putting $\kappa^2 = \frac{\eta}{\delta}$ and $\lambda^2 = \eta\delta$ so that the reparametrized model is given by

$$f(x; \eta, \delta) = \frac{\eta\delta}{\eta + \delta} \begin{cases} \exp[\delta x], & x < 0, \\ \exp[-\eta x], & x \geq 0. \end{cases} \tag{4.1}$$

Here $\eta > 0$ and $\delta > 0$ are the model parameters. We will use the notation $\text{MOETL}(\eta, \delta)$ to refer this distribution. The skewness and kurtosis of the distribution depends on η and δ. A value of η greater than δ suggests that the right tail is thinner and there is less probability concentration to the right side of zero than the left side. Similarly, if δ is greater than η, the left tail will be thinner and there will be less probability concentration to the left of zero than the right side. When $\eta = \delta$, the distribution becomes symmetric.

For sample data $D = (X_1, X_2, \ldots, X_n)$ where X_i's are i.i.d. like X following $\text{MOETL}(\eta, \delta)$ given by (4.1), then the log likelihood function of η and δ given the observations is obtained as

$$\begin{aligned} \text{LL}(\eta, \delta | D) &= n \log\left(\frac{\eta\delta}{\eta + \delta}\right) + \sum_{(X_i \in D / X_i < 0)} (\delta X_i) + \sum_{(X_i \in D / X_i \geq 0)} (-\eta X_i) \\ &= n \log\left(\frac{\eta\delta}{\eta + \delta}\right) + \delta S_l - \eta S_r, \end{aligned} \tag{4.2}$$

where $S_l = \sum_{(X_i \in D/X_i<0)} X_i$ and $S_r = \sum_{(X_i \in D/X_i \geq 0)} X_i$.

Solving the equations obtained by taking the partial derivatives of the log likelihood function with respect to η and δ and equating to zero, we get the estimates of η and δ as

$$\hat{\eta} = \frac{n}{-S_l + \sqrt{S_r(-S_l)}} \tag{4.3}$$

and

$$\hat{\delta} = \frac{n}{S_r + \sqrt{S_r(-S_l)}}. \tag{4.4}$$

Using the estimates of η and δ, we can estimate the parameters λ and κ in (3.5).

4.2 Method of moments

In this section, we derive the moment estimators for the parameters λ and κ in the MOETL distribution. Let $(X_1, X_2, \dots, X_n)$ be a random sample from a MOETL(λ, κ) distribution given by the characteristic function (3.2). Here the first and second sample moments are $m_1' = \frac{1}{n}\sum_{i=1}^n X_i$ and $m_2' = \frac{1}{n}\sum_{i=1}^n X_i^2$. From the characteristic function (3.2), the first and second raw moments of the distribution are

$$\alpha_1 = E(X) = \frac{2\theta}{\lambda^2} \quad \text{and} \quad \alpha_2 = E(X^2) = \frac{8\theta^2 - 2\lambda^2}{\lambda^4}.$$

Equating the raw moments of the sample and population, we obtain the moment estimators as

$$\hat{\theta} = \frac{m_1'}{2(m_1')^2 - m_2'},$$

$$\hat{\lambda} = \sqrt{\frac{2}{2(m_1')^2 - m_2'}} \quad \text{and}$$

$$\hat{\beta} = \frac{2[2(m_1')^2 - m_2']}{[2(m_1')^2 - m_2']^2 - (m_1')^2}.$$

5 Autoregressive models

Recently there has been growing interest in the construction of time series models with non-Gaussian marginal distributions, because of the wide applications of such models in socioeconomic fields. The pioneering work in this area is by Gaver and Lewis (1980). Subsequently, Lawrence and Lewis (1981), Dewald and Lewis (1985), Anderson and Arnold (1993), Jayakumar and Pillai (1993), Lekshmi and Jose (2004), Jose, Tomy and Sreekumar (2008), Tomy and Jose (2009) and Trindade, Zhu and Andrews (2010) developed autoregressive models with different marginal distributions.

5.1 First order autoregressive processes with MOETL(λ, κ) marginal

This section characterizes two autoregressive models with MOETL(λ, κ) marginal distribution.

5.1.1 AR(1) *Model I.* The first order autoregressive process is defined by the structural relationship

$$X_n = aX_{n-1} + \boldsymbol{\varepsilon}_n, \qquad |a| < 1, \tag{5.1}$$

where $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \ldots$ are i.i.d. random variables and $\boldsymbol{\varepsilon}_n$ is independent of $X_1, X_2, \ldots, X_{n-1}$.

Theorem 5.1. *The process* $\{X_n\}$ *in* (5.1) *is strictly stationary with* MOETL(λ, κ) *if and only if* $\boldsymbol{\varepsilon}_n$ *has the characteristic function*

$$\phi_{\boldsymbol{\varepsilon}_n}(t) = \frac{\lambda^2 + a^2t^2 - 2iat\theta}{\lambda^2 + t^2 - 2it\theta} \tag{5.2}$$

and X_0 *is* MOETL(λ, κ).

Proof is given in Appendix section A.6.

Remark 5.1. If $X_0 \sim$ any arbitrary distribution, then

$$X_n = aX_{n-1} + \boldsymbol{\varepsilon}_n = a^n X_0 + \sum_{k=0}^{n-1} a^k \boldsymbol{\varepsilon}_{n-k}, \qquad |a| < 1.$$

$$\begin{aligned}
\phi_{X_n}(t) &= \phi_{X_0}(a^n t) \prod_{k=0}^{n-1} \phi_{\boldsymbol{\varepsilon}}(a^k t) \\
&= \phi_{X_0}(a^n t) \prod_{k=0}^{n-1} \frac{\lambda^2 + a^{k+2}t^2 + 2ia^{k+1}t\theta}{\lambda^2 + t^2 + 2it\theta} \\
&= [1 + \lambda^2(t^2 - 2it\theta)]^{-1}.
\end{aligned}$$

That is, the process (5.1) is stationary with MOETL(λ, κ) marginal distribution.

Again, for any positive integer k, the sum

$$\begin{aligned}
T_k &= X_n + X_{n+1} + \cdots + X_{n+k-1} \\
&= \sum_{j=0}^{k-1} a^j X_n + a^{j-1}\boldsymbol{\varepsilon}_{n+1} + a^{j-2}\boldsymbol{\varepsilon}_{n+2} + \cdots + \boldsymbol{\varepsilon}_{n+j} \\
&= \frac{1-a^k}{1-a} X_n + \sum_{j=1}^{k-1} \frac{1-a^{k-j}}{1-a} \boldsymbol{\varepsilon}_{n+j},
\end{aligned}$$

has the characteristic function

$$\begin{aligned}
\phi_{T_k}(t) &= \phi_{X_n}\left(\frac{1-a^k}{1-a}t\right)\prod_{j=1}^{k-1}\phi_{\boldsymbol{\varepsilon}}\left(\frac{1-a^{k-j}}{1-a}t\right) \\
&= 1\Big/\left(1+\frac{1}{\lambda^2}\left[\left(\frac{1-a^k}{1-a}t\right)^2 - 2i\left(\frac{1-a^k}{1-a}t\right)\theta\right]\right) \\
&\quad \times \prod_{j=1}^{k-1}\left[\left(\lambda^2+\left(\frac{1-a^{k-j}}{1-a}\right)^2 a^2t^2 - 2ai\left(\frac{1-a^{k-j}}{1-a}t\right)\theta\right)\right. \\
&\quad \left.\Big/\left(\lambda^2+\left(\frac{1-a^{k-j}}{1-a}\right)^2 t^2 - 2i\left(\frac{1-a^{k-j}}{1-a}t\right)\theta\right)\right].
\end{aligned}$$

The joint characteristic function of (X_n, X_{n+1}) is

$$\begin{aligned}
\phi_{X_n,X_{n+1}}(t_1,t_2) &= E\left[\exp\left(it_1X_n + it_2(aX_n+\boldsymbol{\varepsilon}_{n+1})\right)\right] \\
&= \left[\frac{\lambda^2}{\lambda^2+(t_1+at_2)^2-2ia(t_1+at_2)}\right]\left[\frac{\lambda^2+a^2t_2^2-2iat_2\theta}{\lambda^2+t_2^2-2it_2\theta}\right].
\end{aligned}$$

The autocorrelation function

$$\begin{aligned}
\rho(k) &= \operatorname{Corr}(X_n, X_{n-k}) \\
&= a^{|k|}, \qquad k = 0, \pm 1, \pm 2, \ldots,
\end{aligned}$$

which can be assumed either positive or negative in practical situations. Of course,

$$\phi_{X_n,X_{n+1}}(t_1,t_2) \neq \phi_{X_n,X_{n+1}}(t_2,t_1)$$

reveals that the AR process (5.1) with MOETL(λ, κ) marginal distribution is not time reversible.

In the exponential autoregressive process of Gaver and Lewis (1980) the structure (5.1) takes the form

$$X_n = \begin{cases} aX_{n-1} & \text{with probability } p, \\ aX_{n-1}+\boldsymbol{\varepsilon}_n & \text{with probability } 1-p, \end{cases} \tag{5.3}$$

where $0 \leq p \leq 1$. This model suffers from the 'zero defect' which caused successive values of the process to be fixed multiples of previous values. In order to overcome this defect, Lawrence and Lewis (1981) developed a new exponential auto regressive model which we refer to as AR(1) Model II.

5.1.2 AR(1) *Model II.* This is the autoregressive model with general structure

$$X_n = \begin{cases} \boldsymbol{\varepsilon}_n & \text{with probability } p, \\ \delta X_{n-1}+\boldsymbol{\varepsilon}_n & \text{with probability } 1-p, \end{cases} \tag{5.4}$$

where $0 \leq p \leq 1$ and $0 \leq \delta \leq 1$.

In terms of characteristic functions, (5.4) can be written as

$$\phi_{X_n}(t) = \phi_{\boldsymbol{\varepsilon}_n}(t)[p + (1-p)\phi_{X_{n-1}}(\delta t)].$$

Assuming stationarity, it becomes

$$\begin{aligned} \phi_X(t) &= \phi_{\boldsymbol{\varepsilon}}(t)[p + (1-p)\phi_X(\delta t)] \quad \text{and} \\ \phi_{\boldsymbol{\varepsilon}}(t) &= \frac{\phi_X(t)}{p + (1-p)\phi_X(\delta t)}. \end{aligned} \tag{5.5}$$

Consider the AR(1) model defined by the structure,

$$X_n = \begin{cases} \boldsymbol{\varepsilon}_n & \text{with probability } \dfrac{1}{\beta}, \\ X_{n-1} + \boldsymbol{\varepsilon}_n & \text{with probability } 1 - \dfrac{1}{\beta}, \end{cases} \tag{5.6}$$

where $n \geq 1$, $\beta > 1$ and $\{\boldsymbol{\varepsilon}_n\}$ is a sequence of independently and identically distributed random variables. This model is equivalent to the model (5.4).

Theorem 5.2. *A necessary and sufficient condition that* $\{X_n\}$ *in* (5.6) *is strictly stationary with* ETL(θ) *marginal is that* $\{\boldsymbol{\varepsilon}_n\}$ *is distributed as* MOETL(λ, κ).

Proof is given in Appendix subsection A.7.

5.2 Bivariate process of (X_{n-1}, X_n)

The joint characteristic function of (X_{n-1}, X_n) in the autoregressive model (5.6) is

$$\begin{aligned} \phi_{X_{n-1},X_n}(t_1, t_2) &= \phi_{\boldsymbol{\varepsilon}_n}(t_2)\left[\frac{1}{\beta}\phi_{X_{n-1}}(t_1) + \left(1 - \frac{1}{\beta}\right)\phi_{X_{n-1}}(t_1 + t_2)\right] \\ &= \left(1 + \lambda^2(t_2^2 - 2it_2)\right)^{-1} \\ &\quad \times \left[\frac{1}{\beta}\left(1 + \frac{t_1^2}{1-\theta^2} - \frac{2it_1\theta}{1-\theta^2}\right)^{-1}\right. \\ &\qquad \left. + \left(1 - \frac{1}{\beta}\right)\left(1 + \frac{(t_1+t_2)^2}{1-\theta^2} - \frac{2i(t_1+t_2)\theta}{1-\theta^2}\right)^{-1}\right]. \end{aligned}$$

This process is not time reversible, since

$$\phi_{X_n,X_{n+1}}(t_1, t_2) \neq \phi_{X_n,X_{n+1}}(t_2, t_1).$$

Again, since $\{X_n\}$ is stationary with $X_n \stackrel{d}{=} \text{ETL}(\theta)$ and $\boldsymbol{\varepsilon}_n \stackrel{d}{=} \text{MOETL}(\lambda, \kappa)$, we have $E(X_n) = \frac{2\theta}{1-\theta^2}$, $\text{Var}(X_n) = \frac{2(1+\theta^2)}{(1-\theta^2)^2}$ and $E(\boldsymbol{\varepsilon}_n) = \frac{2\theta}{\beta(1-\theta^2)}$. Then

$$E[X_{n+1}/X_n = x] = \left(1 - \frac{1}{\beta}\right)x + \frac{2\theta}{\beta(1-\theta^2)}.$$

Hence, the regression of X_{n+1} on $X_n = x$ is linear in x.

By some algebraic evaluations in (A.4) given in Appendix section A.7, we have

$$\operatorname{Cov}(X_n, X_{n-k}) = \left(1 - \frac{1}{\beta}\right)^k \operatorname{Cov}(X_{n-k}, X_{n-k}).$$

So the autocorrelation function

$$\rho(k) = \left(1 - \frac{1}{\beta}\right)^k,$$

which is always positive and, hence, the variables are positively correlated.

5.3 Sample path behavior

The simulated sample path using 100 observations generated from the MOETLAR(1) process for different values of θ and β is given in Figure 3. The sample path behavior seems to be distinctive and is adjustable through the parameters θ and β. This makes the model very rich. The model identification can be done as described in Sim (1994).

For an AR(1) process, the first order autocorrelation function decays exponentially, while the partial autocorrelation function cuts off after the first lag. If these two functions of the sample are consistent with that of the AR(1) model, we can identify the model as AR(1) and the parameter β of the model can be estimated using the sample autocorrelation.

Here the estimate of β is $\hat{\beta} = \frac{1}{1-r_1}$ (see Sim (1994)), where r_1 is the sample autocorrelation of first order.

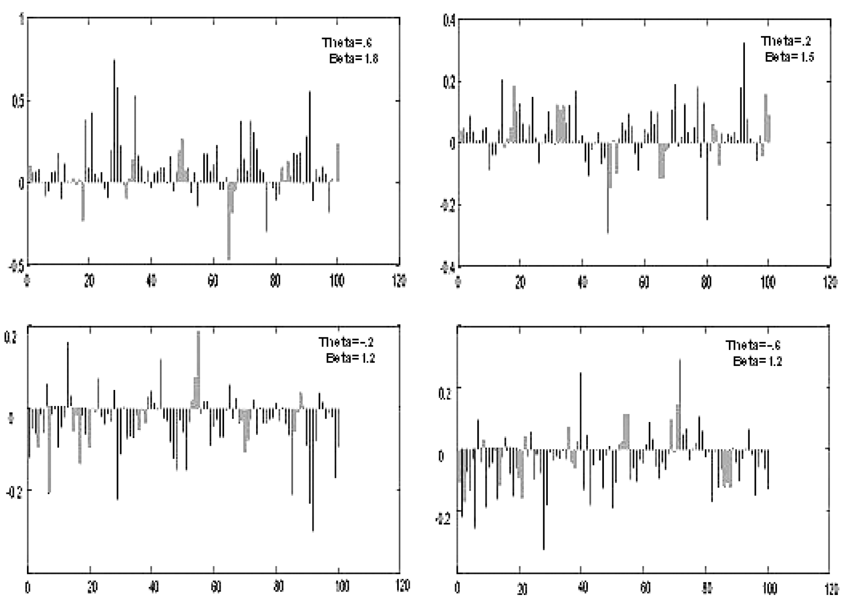


Figure 3 *Sample path of MOETLAR process.*

5.4 Extension to higher order processes

In this section, we consider the kth order autoregressive model constructed by Lawrence and Lewis (1981) with structure

$$X_n = \begin{cases} \varepsilon_n & \text{with probability } p_0, \\ X_{n-1} + \varepsilon_n & \text{with probability } p_1, \\ X_{n-2} + \varepsilon_n & \text{with probability } p_2, \\ \dots & \\ X_{n-k} + \varepsilon_n & \text{with probability } p_k, \end{cases} \tag{5.7}$$

where $\sum_{i=0}^{k} p_i = 1, 0 < p_i < 1, i = 0, 1, 2, \dots, k$ and $\{\varepsilon_n\}$ is a sequence of i.i.d. MOETL(λ, κ), independent of $\{X_n, X_{n-1}, X_{n-2}, \dots\}$.

In terms of characteristic function, (5.7) can be written as

$$\begin{aligned} \phi_{X_n}(t) &= p_0\phi_{\varepsilon_n}(t) + p_1\phi_{X_{n-1}}(t)\phi_{\varepsilon_n}(t) \\ &\quad + p_2\phi_{X_{n-2}}(t)\phi_{\varepsilon_n}(t) + \cdots + p_k\phi_{X_{n-k}}(t)\phi_{\varepsilon_n}(t). \end{aligned}$$

Assuming stationarity, we get

$$\phi_{\varepsilon}(t) = \frac{\phi_X(t)}{p + (1-p)\phi_X(t)}.$$

This shows that the previous results in this section can be applied in higher order cases also

6 A case study

This section deals with an application of Marshall–Olkin Esscher transformed Laplace distribution in modeling exchange rates. We consider a data on daily observations of US Dollar–Indian Rupee foreign exchange rates. The data consists of 2516 observations starting from 01/09/1998 to 29/08/2008. The data can be downloaded from the NSE website. A time series plot of the data is provided in Figure 4. From the graph it can be seen that the exchange rate shows an increase initially and then decrease and again a slow increase. The first order autocorrelation of the series $\{X_n\}$ is obtained as $\rho_0 = 0.9987$. To make the series stationary, we take the first order autocorrelated difference of the $\{X_n\}$. The new series obtained is $\{Y_n\}$ where $Y_n = X_n - \rho_0 X_{n-1}$. This series is standardized by subtracting its mean and dividing by its standard deviation. The autocorrelation of the resulting series is insignificant. Each observation in the series is multiplied by 10. The observations are classified into classes of width one unit. A histogram is constructed and is given in Figure 5. The graph resembles the shape of the Marshall–Olkin Esscher transformed Laplace distribution presented in Figure 2. We estimate the values of λ and κ, say, $\hat{\lambda}$ and $\hat{\kappa}$, respectively, from the observed data using the MLEs. We get $\hat{\lambda}$ = 0.9816 and $\hat{\kappa}$=0.8682. We construct frequency curve of the Marshall–Olkin Esscher transformed Laplace density with these λ and κ and super impose

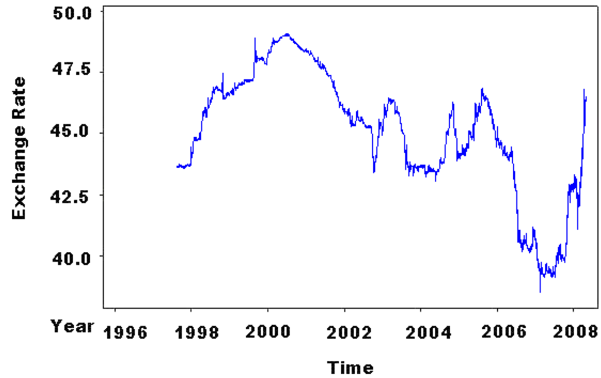


Figure 4 *Time series of daily exchange rate of US Dollar–Indian Rupee.*

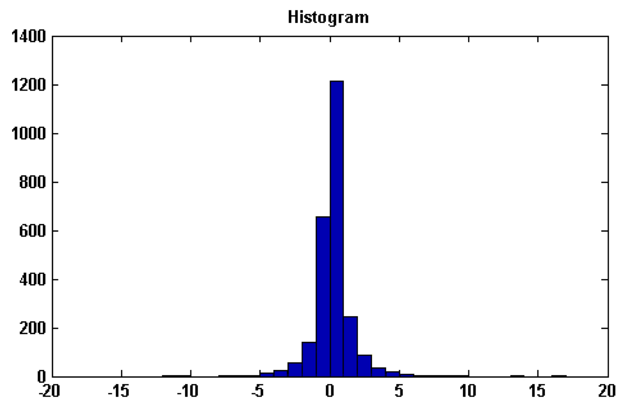


Figure 5 *Histogram of the observed series.*

this curve on the histogram of the observed data. Figure 6(c) represents the histogram of the observed data and embedded Marshall–Olkin Esscher transformed Laplace frequency curve. Since the Marshall–Olkin Esscher transformed Laplace distribution is considered as a more flexible model than the Esscher transformed Laplace distribution and since the Esscher transformed Laplace distribution is a special case of the asymmetric Laplace distribution, a comparison with both Esscher transformed Laplace distribution and asymmetric Laplace distribution is also done by fitting these probability distributions to the same observed data. The estimated value of the parameter θ of the Esscher transformed Laplace distribution is $\hat{\theta} = 0.12432$ and the estimated values of the parameters of the asymmetric Laplace distribution are $\hat{\kappa} = 1.101$ and $\hat{\sigma} = 0.7812$. Figures 6(a) and 6(b), respectively, represent the histogram of the observed data and the embedded asymmetric Laplace frequency curve and the histogram of the observed data and the embedded Esscher transformed Laplace frequency curve. Figures 7(a), 7(b) and 7(c) are, re-

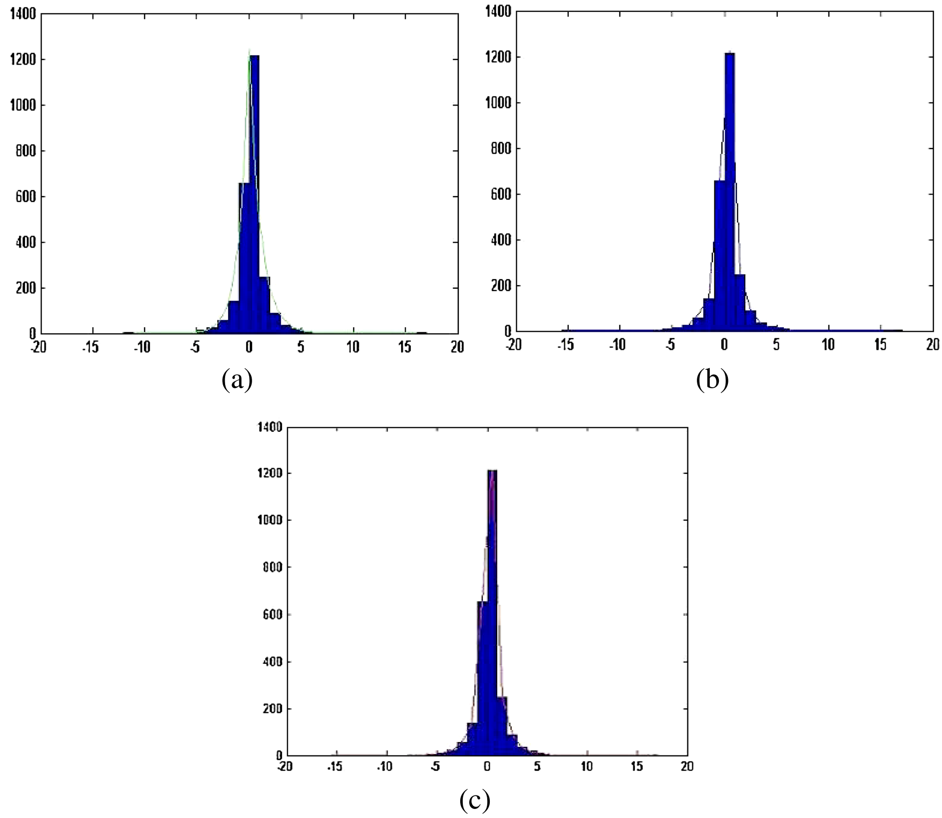


Figure 6 *The histogram and embedded frequency polygon of* (a) *asymmetric Laplace distribution* (b) *Esscher transformed Laplace distribution and* (c) *Marshall–Olkin Esscher transformed Laplace distribution*.

spectively, the P–P plots of asymmetric Laplace distribution, Esscher transformed Laplace distribution and Marshall–Olkin Esscher transformed Laplace distribution. All these graphs reveal that the Marshall–Olkin Esscher transformed Laplace distribution is a better model than both the Esscher transformed Laplace distribution and the asymmetric Laplace distribution to the currency exchange data. We check the goodness of fit, using the Kolmogrov distance measure also. For the asymmetric Laplace model, the value of the distance measure is 0.0987, for the Esscher transformation Laplace model, it is 0.07932 and for the Marshall–Olkin Esscher transformed Laplace model, the distance measure is 0.0684. So we conclude that the Marshall–Olkin Esscher transformed Laplace distribution is a better model compared to both the asymmetric Laplace model and the Esscher transformed Laplace model for the currency exchange data.

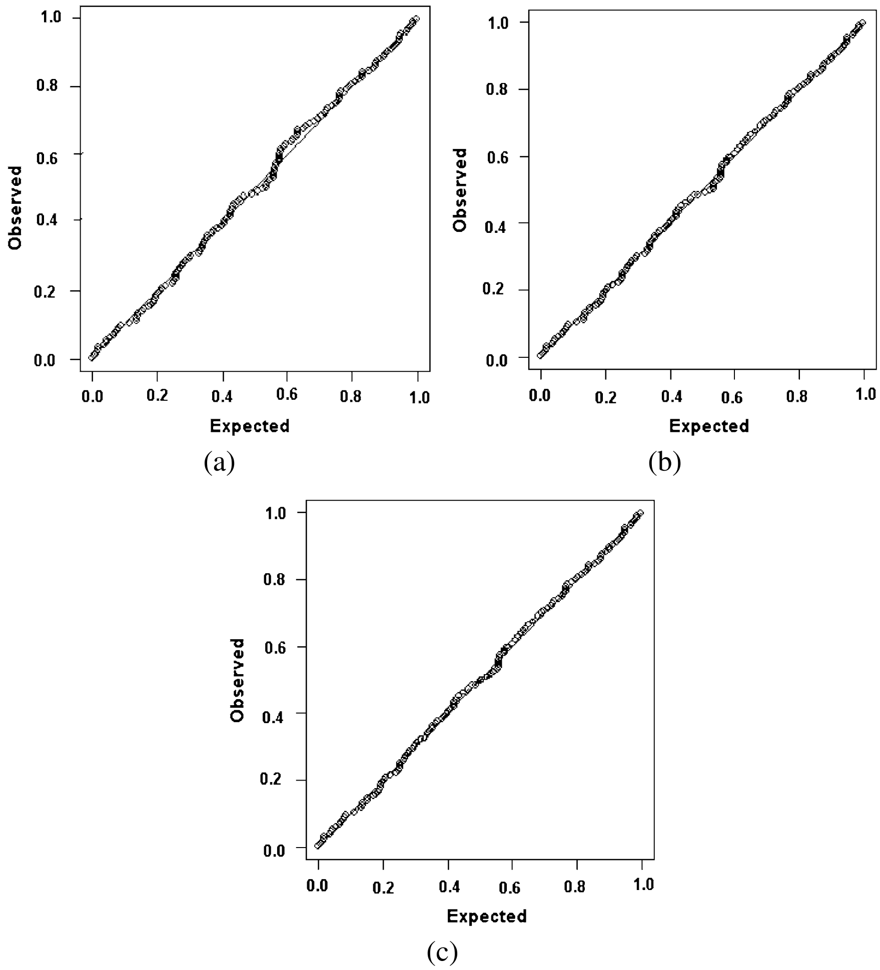


Figure 7 *P–P plot of* (a) *asymmetric Laplace distribution* (b) *Esscher transformed Laplace distribution and* (c) *Marshall–Olkin Esscher transformed Laplace distribution.*

7 Conclusion

Esscher transformation of a distribution provides a means for creating a regular exponential family from a distribution whose cumulant generating function converges in the regular sense. It can be seen that the class of p.d.f.s obtained by the Escher transformation on the the standard Laplace distribution is a subclass of the family of asymmetric Laplace distributions. Further, this family of distributions belongs to a regular one parameter exponential family of distributions and, therefore, the most efficient estimator of the parameter involved in the transformation can be obtained. This family exhibits asymmetry, sharp peakedness at zero and has heavier tails than normal distribution, and is suitable for modeling financial data related to currency exchange rate, stock price changes, interest rate, etc. It is illustrated that the Marshall–Olkin generalized family of distributions introduced in this paper are appropriate to model much heavier-tailed data such as currency exchange data.

Appendix

A.1 Proof of Theorem 3.1

The characteristic function of S_N is

$$\psi_{S_N}(t) = \sum_{k=1}^{\infty} [\psi_X(t)]^k p(1-p)^{k-1}, \qquad \text{where } \psi_X(t) \text{ is given in (3.2)}$$
$$= \left[1 + \frac{1}{p\lambda^2}(t^2 - 2it\theta)\right]^{-1}.$$

Hence it follows from (3.2) that

$$S_N = X_1 + X_2 + X_3 + \cdots + X_N \sim \text{MOETL}(\sqrt{p}\lambda, \kappa) \text{ distribution.}$$

A.2 Proof of Theorem 3.2

Assume that W follows an an exponential distribution with p.d.f.

$$f(w) = \begin{cases} \beta \exp(-\beta w), & w > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Conditioning on W, we can express the characteristic function of the right-hand side of (3.7) as follows:

$$\phi_X(t) = E_W\left\{E\left(\exp\left[it\left(\frac{2\theta}{1-\theta^2}W + \frac{\sqrt{2}}{\sqrt{1-\theta^2}}\sqrt{W}Z\right)\right]\right)\Big|W\right\}$$
$$= \left(1 + \frac{1}{\lambda^2}[t^2 - 2it\theta]\right)^{-1},$$

which is the characteristic function of the MOETL(λ, κ) distribution given in (3.2).

A.3 Proof of Theorem 3.3

$$\phi_X(t) = \frac{\kappa^2}{1+\kappa^2}\left[\frac{1}{1+\kappa it/\lambda}\right] + \frac{1}{1+\kappa^2}\left[\frac{1}{1-it/(\kappa\lambda)}\right],$$

where κ and λ are given in (3.4) and (3.3), respectively. Hence,

$$\phi_X(t) = \left(1 + \frac{1}{\lambda^2}[t^2 - 2it\theta]\right)^{-1},$$

which is the characteristic function of the MOETL(λ, κ) given in (3.2).

A.4 Proof of Theorem 3.4

The characteristic function given by (3.2) can be written as

$$\psi(t) = \left[\left(1 + \frac{it\kappa}{\lambda}\right)^{-1/n}\left(1 - \frac{it}{\kappa\lambda}\right)^{-1/n}\right]^n,$$

where κ and λ are given in (3.4) and (3.3). That is, $\psi(t) = (\psi_n(t))^n$ for each integer $n \geq 1$, where $\psi_n(t)$ is the characteristic function of the difference of two gamma variables with means $\frac{1}{n\kappa\lambda}$ and $\frac{\kappa}{n\lambda}$ and variances $\frac{1}{n\kappa^2\lambda^2}$ and $\frac{\kappa^2}{n\lambda^2}$, respectively.

A.5 Proof of Theorem 3.5

Let g_p be the characteristic function of $X_p^{(i)}$. Conditioning on γ_p, the characteristic function of the right-hand side of (3.9) can be obtained as

$$\begin{aligned} E\left[e^{it\sum_{i=1}^{\gamma_p} X_p^{(i)}}\right] &= \sum_{i=1}^{\infty} E\left[e^{it\sum_{i=1}^{\gamma_p} X_p^{(i)}}\right](1-p)^{n-1}p \\ &= \frac{pg_p(t)}{1-(1-p)g_p(t)}. \end{aligned} \tag{A.1}$$

Now substitute

$$g_p(t) = \left(1 + \frac{1}{\lambda^2}(pt^2 - 2ipt\theta)\right)^{-1},$$

the characteristic function of MOETL(λ, κ) in (A.1), we get

$$E\left[e^{it\sum_{i=1}^{\gamma_p} X_p^{(i)}}\right] = \left(1 + \frac{1}{\lambda^2}(t^2 - 2it\theta)\right)^{-1},$$

which is the characteristics function of X in (3.2).

A.6 Proof of Theorem 5.1

Let the characteristic function of X_n and $\boldsymbol{\varepsilon}_n$ be $\phi_{X_n}(t)$, and $\phi_{\boldsymbol{\varepsilon}_n}(t)$, respectively.

By (5.1), for all t,

$$\phi_{X_n}(t) = \phi_{\boldsymbol{\varepsilon}_n}(t)\phi_{X_{n-1}}(at). \tag{A.2}$$

By the stationary property of (5.1), we have for all t,

$$\phi_{\boldsymbol{\varepsilon}}(t) = \frac{\phi_X(t)}{\phi_X(at)} = \frac{\lambda^2 + a^2t^2 - 2iat\theta}{\lambda^2 + t^2 - 2it\theta}.$$

If X_0 is of MOETL(λ, κ) and $\boldsymbol{\varepsilon}_n$ has the characteristic function (3.2), then, for $n = 1$, we have from (A.2)

$$\phi_{X_1}(t) = \left(1 + \frac{1}{\lambda^2}[t^2 - 2it\theta]\right)^{-1}. \tag{A.3}$$

The rest of the proof follows by mathematical induction, assuming that

$$X_{n-1} \stackrel{d}{=} \text{MOETL}(\lambda, \kappa).$$

A.7 Proof of Theorem 5.2

Let the characteristic function of $\{X_n\}$ be $\phi_{X_n}(t)$ and that of $\{\boldsymbol{\varepsilon}_n\}$ be $\phi_{\boldsymbol{\varepsilon}_n}(t)$.

From (5.6) for all t,

$$\phi_{X_n}(t) = \phi_{\boldsymbol{\varepsilon}_n}(t)\left[\frac{1}{\beta} + \left(1 - \frac{1}{\beta}\right)\phi_{X_{n-1}}(t)\right].$$

Assuming stationarity, it becomes

$$\begin{aligned} \phi_X(t) &= \phi_{\boldsymbol{\varepsilon}}(t)\left[\frac{1}{\beta} + \left(1 - \frac{1}{\beta}\right)\phi_X(t)\right] \quad \text{and} \\ \phi_{\boldsymbol{\varepsilon}}(t) &= \frac{\phi_X(t)}{1/\beta + (1 - 1/\beta)\phi_X(t)}. \end{aligned} \tag{A.4}$$

If

$$X \sim \mathrm{ETL}(\theta),$$

then

$$\phi_{\boldsymbol{\varepsilon}}(t) = [1 + \lambda^2(t^2 - 2it\theta)]^{-1},$$

which is the characteristic function of MOETL(λ, κ), so that

$$\boldsymbol{\varepsilon}_n \sim \mathrm{MOETL}(\lambda, \kappa).$$

Conversely, if $\{\boldsymbol{\varepsilon}_n\}$ is a sequence of i.i.d. MOETL(λ, κ) random variables and

$$X_0 \stackrel{d}{=} \mathrm{ETL}(\theta),$$

then from (A.4), when $n = 1$, we have

$$\begin{aligned} \phi_{X_1}(t) &= \frac{1}{\beta}\left[1 + \left(\frac{t^2}{1-\theta^2} - \frac{2it\theta}{1-\theta^2}\right)\right]^{-1} \\ &\quad + \left(1 - \frac{1}{\beta}\right)\left(1 + \frac{t^2}{1-\theta^2} - \frac{2it\theta}{1-\theta^2}\right)^{-1}[1 + \lambda^2(t^2 - 2it\theta)]^{-1} \\ &= \left(1 + \frac{t^2}{1-\theta^2} - \frac{2it\theta}{1-\theta^2}\right)^{-1}. \end{aligned}$$

That is,

$$X_1 \stackrel{d}{=} \mathrm{ETL}(\theta).$$

Assuming

$$X_{n-1} \stackrel{d}{=} \mathrm{ETL}(\theta),$$

it follows by mathematical induction that $\{X_n\}$ is a stationary process with Esscher transformed Laplace marginal distribution.

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