Brazilian Journal of Probability and Statistics
2010, Vol. 24, No. 3, 434–456
DOI: 10.1214/09-BJPS023

Some corrections of the score test statistic for Gaussian ARMA models

Bernardo M. Lagos[a], Pedro A. Morettin[b] and Lúcia P. Barroso[b]

[a] *University of Concepción*
[b] *University of São Paulo*

Abstract. In this article we compute three corrected score statistic versions: the Bartlett-type correction and the monotone corrected score statistics proposed by Kakizawa [*Biometrika* **83** (1996) 923–927] and Cordeiro, Ferrari and Cysneiros [*J. Stat. Comput. Simul.* **62** (1998) 123–136]. These corrected statistics are used to test the null hypothesis concerning some parameter of interest of an ARMA model, assumed to be Gaussian, stationary and invertible. We also consider the situations where nuisance parameters are present. The formulas are written in matrix form, appropriate for the use of symbolic or numerical languages. Some simulation results are also presented for the AR(1), MA(1) and ARMA(1, 1) models.

1 Introduction

After the advances obtained in correcting the likelihood ratio statistic (LR) using the Bartlett correction factor, Harris (1985) showed that a multiplicative transformation that corrects Rao's score test statistic (which we will denote by S) does not exist, but he obtained a transformation for the critical value from the reference distribution for a fixed nominal value that works [see also Lawley (1956)]. The same conclusion was shown in Harris (1987), Cox (1988) and Barndorff-Nielsen (1990). Proposals to solve the problem were suggested at about the same time by Chandra and Mukerjee (1991) and Cordeiro and Ferrari (1991) for the case of independent identically distributed (i.i.d.) random variables, and Taniguchi (1991) for the case where the observation are not independent. Cordeiro and Ferrari (1991) have shown the existence of a correction factor (called Bartlett-type correction factor) such that the corrected statistic, S_B, obtained through the multiplication of S by a second degree polynomial in S, has a χ_s^2 distribution, up to order n^{-1} under the null hypothesis.

Some important references about the corrections of the statistic S and its properties are given in Cribari-Neto and Cordeiro (1996). An extension of the correction obtained by Cordeiro and Ferrari (1991) was given by Kakizawa (1997). The analytical disadvantage of this type of correction is that it is not monotone, that is, if S

Key words and phrases. ARMA models, Bartlett-type correction, chi-square distribution, score statistics, monotone correction, time series.

Received June 2008; accepted March 2009.

increases this does not imply that S_B increases, and this has motivated works that try to eliminate this unpleasant characteristic. Kakizawa (1996), assuming (under the null hypothesis) that the distribution function of S has an asymptotic expansion in terms of the chi-square distribution, found a monotone transformation that has a chi-square distribution up to order n^{-k}, k being the number of terms in the expansion. He applied this technique to a special case of the Hotelling T^2 statistic and through a simulation study concluded that the transformation works well. In the same direction, Cordeiro, Ferrari and Cysneiros (1998) proposed another monotone transformation. The three statistics are asymptotically equivalent to order n^{-1}, where n is the sample size.

In the same direction Fujikoshi (2000) proposed another monotone transformation such that the first two moments of the transformed statistic coincide with those of a chi-square distribution with s degrees of freedom up to order $O(n^{-1})$, different in principle from the Bartlett correction, which requires only that the first moment coincides with that of a chi-square distribution with s degrees of freedom up to order $O(n^{-1})$. He has shown through a simulation study that the transformation performed better than the usual Bartlet correction.

Concerning the power of S, there are many studies since Rao (1973) that dealt with the alternative LR and Wald (W) statistics. In general terms, the power of the statistic depends on the proposed model and the problem considered [Peers (1971); Harris and Peers (1980)]. Numerical results via simulation were obtained by Sutradhar and Bartlett (1993) and for a non-i.i.d. case the same results were obtained by Taniguchi (1991).

On the other hand the different corrected score statistic entertained by Chandra and Mukerjee (1991), Cordeiro and Ferrari (1991) and Taniguchi (1991) have a null asymptotic distribution equal to a central chi-square distribution up to second order, but not under the alternative hypothesis, specifically under a sequence of Pitman alternatives H_n which converge to the null hypothesis at the rate $n^{-1/2}$. Rao and Mukerjee (1997), dealing with the case not necessarily i.i.d., defined a larger class of statistics than that considered by Taniguchi (1991), and compared the several Bartlett-type corrected score statistics with different power criteria. An study of second-order power for the LR, W and S statistics considering the first-order autorregressive and moving average models was developed in Lagos and Morettin (2004).

In this paper we compute three corrected score statistics, proposed by Cordeiro and Ferrari (1991), Kakizawa (1996) and Cordeiro, Ferrari and Cysneiros (1998), including nuisance parameters. In Section 2 we give a matrix formula for the calculations of the coefficients of the second-degree polynomial which defines the correction factor for S and introduce the monotone corrected statistics suggested by Kakizawa (1996) and Cordeiro, Ferrari and Cysneiros (1998). In Section 3 we derive the results for the models AR(1), MA(1) and ARMA(1, 1), assumed to be Gaussian, with and without nuisance parameters. In Section 4 we present some simulation results and conclude the work in Section 5.

2 Bartlett-type factor in matrix form

Consider a time series $\{Y_t, t = 0, \pm 1, \dots\}$ following a model of the ARMA (autoregressive moving average) family, namely,

$$(1 - \phi_1 B - \cdots - \phi_p B^p) Y_t = (1 - \theta_1 B - \cdots - \theta_q B^q) a_t, \tag{2.1}$$

where $B^k Y_t = Y_{t-k}$ and the a_t's are elements of a sequence of Gaussian noncorrelated random variables, with zero mean and variance $\sigma^2 > 0$. Moreover, the polynomials in $z \in \mathbb{C}$

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p,$$
$$\theta(z) = 1 - \theta_1 z - \cdots - \theta_q z^q$$

have their (noncommon) roots outside the unit circle ($|z| > 1$). This means that we have a stationary and invertible model [see Box, Jenkins and Reinsel (1994), for details]. The general problem is to test hypotheses about the parameters of interest, for example

$$H : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^{(0)} \quad \text{vs} \quad A : \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_1^{(0)}, \tag{2.2}$$

where $\boldsymbol{\beta}_1 = (\beta_1, \dots, \beta_s)^\top$ is an s-dimensional vector of parameters of interest and the remaining parameters $\boldsymbol{\beta}_2 = (\beta_{s+1}, \dots, \beta_r)^\top$ are considered fixed or nuisance and $\boldsymbol{\beta}_1^{(0)}$ is a specified vector. Here the vector $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_r)^\top$ include all the parameters ϕ_i's, θ_i's and σ^2.

The total log-likelihood $l(\beta)$ is given by

$$l(\boldsymbol{\beta}) = -\frac{1}{2}\{n \log(2\pi) + \log(|\boldsymbol{\Sigma}|) + \mathbf{Y}^\top \boldsymbol{\Sigma}^{-1} \mathbf{Y}\}, \tag{2.3}$$

where $\mathbf{Y} = (Y_1, \dots, Y_n)^\top$ and $\boldsymbol{\Sigma}$ depends on the vector $\boldsymbol{\beta} = (\beta_1, \dots, \beta_r)^\top$. The partition of the vector of parameters induces the same partition in the score vector $U(\beta)$, the Fisher information matrix, $\mathbf{K} = K(\beta)$, its inverse $\mathbf{K}^{-1}$, and the matrices $\mathbf{A}$ and $\mathbf{M}$ defined below. So we have $U(\boldsymbol{\beta}) = (U_1^\top(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2), U_2^\top(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2))^\top$,

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{pmatrix}, \qquad \mathbf{K}^{-1} = \begin{pmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & K_{22}^{-1} \end{pmatrix} \tag{2.4}$$

with $\mathbf{M} = \mathbf{K}^{-1} - \mathbf{A}$, and K_{22}^{-1} is the asymptotic covariance matrix of $\tilde{\boldsymbol{\beta}}_2$ and we suppose that the matrix $\mathbf{K}$ is positive definite. It follows that the score statistics S is defined by

$$S = \tilde{U}_1^\top \tilde{\mathbf{K}}^{11} \tilde{U}_1, \tag{2.5}$$

where $\mathbf{K}^{11}$ is the asymptotic covariance matrix of $\hat{\boldsymbol{\beta}}_1$, obtained from (2.4), and the notation $\sim$ is used to indicate functions evaluated at the maximum likelihood estimates, under the null hypothesis, in this case, $\tilde{\boldsymbol{\beta}} = (\boldsymbol{\beta}_1^{(0)\top}, \tilde{\boldsymbol{\beta}}_2^\top)^\top$. For large samples and regularity conditions being satisfied, S has asymptotically a χ_s^2 distribution under H. We use the following notation for the derivatives of the log-likelihood with respect to the components of $\boldsymbol{\beta}$,

$$U_{i_1 \cdots i_d} = \frac{\partial^d l(\boldsymbol{\beta})}{\partial \beta_{i_1} \cdots \partial \beta_{i_d}}, \qquad i_j \in \{1, \ldots, r\} \text{ and } d \in \{1, \ldots, 4\},$$

for the mixed cumulants of the log-likelihood $\kappa_{ij} = E(U_{ij})$, $\kappa_{ijk} = E(U_{ijk})$, $\kappa_{ij,k} = E(U_{ij}U_k)$, $\kappa_{i,j,k} = E(U_i U_j U_k)$, $\kappa_{i,j,kr} = E(U_i U_j U_{kr})$ and $\kappa_{i,j,k,r} = E(U_i U_j U_k U_r)$.

The expressions that determine the coefficients involved in the Edgeworth expansion of the null distribution of S, given by Harris (1985), and that define the corrected statistic S_B proposed by Cordeiro and Ferrari (1991), are given by

$$\begin{aligned} A_1 = {} & 3 \sum{}' (\kappa_{ijk} + 2\kappa_{i,jk})(\kappa_{rst} + 2\kappa_{rs,t}) a_{ij} a_{st} m_{rk} \\ & - 6 \sum{}' (\kappa_{ijk} + 2\kappa_{i,jk}) \kappa_{r,s,t} a_{ij} a_{kr} m_{st} \\ & + 6 \sum{}' (\kappa_{i,jk} - \kappa_{i,j,k})(\kappa_{rst} + 2\kappa_{rs,t}) a_{js} a_{kt} m_{ir} \\ & - 6 \sum{}' (\kappa_{i,j,k,r} + \kappa_{i,j,kr}) a_{kr} m_{ij}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} A_2 = {} & -3 \sum{}' \kappa_{i,j,k} \kappa_{r,s,t} a_{kr} m_{ij} m_{st} + 6 \sum{}' (\kappa_{ijk} + 2\kappa_{i,jk}) \kappa_{r,s,t} a_{ij} m_{kr} m_{st} \\ & - 6 \sum{}' \kappa_{i,j,k} \kappa_{r,s,t} a_{kt} m_{ir} m_{js} + 3 \sum{}' \kappa_{i,j,k,r} m_{ij} m_{kr} \end{aligned} \tag{2.7}$$

and

$$A_3 = 3 \sum{}' \kappa_{i,j,k} \kappa_{r,s,t} m_{ij} m_{kr} m_{st} + 2 \sum{}' \kappa_{i,j,k} \kappa_{r,s,t} m_{ir} m_{js} m_{kt}, \tag{2.8}$$

where $\sum'$ denotes summation over suffices taking the range $1, \ldots, r$, a_{ij}'s and m_{ij}'s are the (i, j) elements of the matrices $\mathbf{A}$ and $\mathbf{M}$, respectively, given in (2.4). In this way the corrected score statistics is determined by

$$S_B = S\{1 - (c + bS + aS^2)\}, \tag{2.9}$$

where the factor that multiplies S is a Bartlett-type correction, as a function of the statistic itself and of the coefficients a, b and c, which are of order n^{-1} and given by

$$a = \frac{A_3}{12s(s+2)(s+4)}, \qquad b = \frac{A_2 - 2A_3}{12s(s+2)}, \qquad c = \frac{A_1 - A_2 + A_3}{12s}. \tag{2.10}$$

The corrected statistic S_B has the property that it is distributed according to a χ_s^2 when terms of order less than n^{-1} are neglected. In the case that the computation of the A's involve unknown parameters, estimates are given by $\tilde{A} = A(\tilde{\boldsymbol{\beta}})$, $\tilde{\boldsymbol{\beta}}$ being a consistent estimator for β, under H (e.g., the maximum likelihood estimator) and this does not affect the order of approximation, $O(n^{-3/2})$.

However, the improved statistic S_B is not always a monotone transformation. To overcome this, Kakizawa (1996) suggested a monotone transformation involving the score statistic and the coefficient a, b and c, given by

$$S_K = S_B + \frac{1}{4}\left\{c^2 S + 2bcS^2 + \left(2ac + \frac{4}{3}b^2\right)S^3 + 3abS^4 + \frac{9}{5}a^2 S^5\right\}. \tag{2.11}$$

Cordeiro, Ferrari and Cysneiros (1998) proposed an alternative expression for the improved score statistic which is a monotone transformation of S. It is expressed in terms of the normal distribution function $\Phi(\cdot)$, given by

$$S_C = \sqrt{\frac{\pi}{3a}}\exp\left(\frac{b^2}{3a} - c\right)\left\{\Phi\left(\sqrt{6a}S + \sqrt{\frac{2}{3a}}b\right) - \Phi\left(\sqrt{\frac{2}{3a}}b\right)\right\} \tag{2.12}$$

if $a > 0$ and by

$$S_C = \frac{1}{2b}\exp(-c)\{1 - \exp(-2bS)\} \tag{2.13}$$

if $a = 0$ and $b \neq 0$ (a is always nonnegative). If $a = b = 0$, S_B is a monotone transformation of S and $S_C = S_B$.

The model ARMA(p, q) considered in (2.1) has a spectral density $f_{\boldsymbol{\beta}}(\lambda)$ given by

$$f_{\boldsymbol{\beta}}(\lambda) = \frac{\sigma^2}{2\pi}\frac{|1 - \sum_{j=1}^{q}\theta_j e^{ij\lambda}|^2}{|1 - \sum_{j=1}^{p}\phi_j e^{ij\lambda}|^2}, \qquad -\pi < \lambda < \pi;$$

see Brockwell and Davis (1991). Consider the following functions, needed for computing the cumulants:

$$\begin{aligned}
f_{\boldsymbol{\beta}}^{i_1\cdots i_d}(\lambda) &= \frac{\partial^d f_{\boldsymbol{\beta}}(\lambda)}{\partial\beta_{i_1}\cdots\partial\beta_{i_d}}, \qquad h_{i_1\cdots i_d}(\lambda) = f_{\boldsymbol{\beta}}^{i_1\cdots i_d}(\lambda) f_{\boldsymbol{\beta}}^{-1}(\lambda),\\
\mathcal{U}_{i_1\cdots i_d}(\lambda) &= \frac{\partial^d \log f_{\boldsymbol{\beta}}(\lambda)}{\partial\beta_{i_1}\cdots\partial\beta_{i_d}}, \qquad i_j \in \{1, \ldots, r\} \text{ and } d \in \{1, \ldots, 4\},\\
h_{i,j}(\lambda) &= h_i(\lambda)h_j(\lambda), \qquad h_{i,jk}(\lambda) = h_i(\lambda)h_{jk}(\lambda), \qquad \text{etc.},\\
\mathcal{U}_{i,j}(\lambda) &= \mathcal{U}_i(\lambda)\mathcal{U}_j(\lambda), \qquad \mathcal{U}_{i,jk}(\lambda) = \mathcal{U}_i(\lambda)\mathcal{U}_{jk}(\lambda), \qquad \text{etc.}
\end{aligned}$$

Consider also the integrals of the functions $\mathcal{U}(\lambda)$'s and $h(\lambda)$'s given by

$$\mathbf{I}_\cdot = \int_{-\pi}^{\pi}\mathcal{U}_\cdot(\lambda)\,d\lambda = \mathbf{I}(\mathcal{U}_\cdot), \qquad \mathcal{I}_\cdot = \int_{-\pi}^{\pi} h_\cdot(\lambda)\,d\lambda = \mathcal{I}(h_\cdot).$$

To operate with the integrals $\mathbf{I}$ and $\mathcal{I}$ as functions of $\mathcal{U}$'s and h's, respectively, we shall use the notation

$$\begin{aligned}
&\mathbf{I}_{i,j} = \mathbf{I}(\mathcal{U}_{i,j}), \qquad \mathbf{I}_{ij} = \mathbf{I}(\mathcal{U}_{ij}), \qquad \mathbf{I}_{i,j,k} = \mathbf{I}(\mathcal{U}_{i,j,k}),\\
&\mathbf{I}_{i,jk} = \mathbf{I}(\mathcal{U}_{i,jk}), \qquad \mathbf{I}_{ijk} = \mathbf{I}(\mathcal{U}_{ijk}), \qquad \text{etc.,}
\end{aligned}$$

$$\mathcal{I}_{i,j} = \mathcal{I}(h_{i,j}), \qquad \mathcal{I}_{ij} = \mathcal{I}(h_{ij}), \qquad \mathcal{I}_{i,j,k} = \mathcal{I}(h_{i,j,k}),$$
$$\mathcal{I}_{i,jk} = \mathcal{I}(h_{i,jk}), \qquad \mathcal{I}_{i,j,k} = \mathcal{I}(h_{i,j,k}), \qquad \text{etc.}$$

The solution of the integrals $\mathbf{I}$ and $\mathcal{I}$ is carried out by tranforming the spectral density of an ARMA(p,q) model, $f_\beta(\lambda)$, as follows:

$$f_\tau(\lambda) = \frac{\sigma^2}{2\pi}\frac{\prod_{j=1}^{p}|1-\delta_j e^{i\lambda}|^2}{\prod_{j=1}^{q}|1-\rho_j e^{i\lambda}|^2}, \qquad -\pi < \lambda < \pi,$$

where $\tau = (\rho_1, \ldots, \rho_p, \delta_1, \ldots, \delta_q, \sigma^2)^\top$ and the ρ_i's, δ_i's denote the noncommon roots of the characteristic polynomials $\phi(\cdot)$ and $\theta(\cdot)$ respectively, which may be complex. We consider the case of real roots in what follows.

An approach exact for the geometry of AR(1) models were considered by van Garderen (1999), suggesting how to compute the cumulants geometrically, which in turn allows to compute the Bartlett correction factor exactly for AR(p) models.

To write the Bartlett-type correction factor in matrix form we proceed as in Cordeiro and Ferrari (1991). First we simplify some terms involved in the coefficients A_1 and A_2 given in (2.6) and (2.7), respectively, using the cumulants approximated in terms of the integrals $\mathbf{I}$'s provided in Lagos and Morettin (2004). The remaining elements are written using the "exact" cumulants κ's (although in the special models we use the approximations). For $\kappa_{ijk} + 2\kappa_{i,jk}$, $\kappa_{i,jk} - \kappa_{i,j,k}$, $\kappa_{i,j,k,l} - \kappa_{i,j,kl}$, we have

$$\begin{aligned}
\kappa_{ijk} + 2\kappa_{i,jk} = \delta_{ijk} &\approx \frac{1}{4\pi}(\mathbf{I}_{i,jk} - \mathbf{I}_{i,j,k} - \mathbf{I}_{j,ik} - \mathbf{I}_{k,ij}),\\
\kappa_{i,jk} - \kappa_{i,j,k} = \xi_{ijk} &\approx \frac{1}{4\pi}(\mathbf{I}_{i,jk} - 3\mathbf{I}_{i,j,k}), \qquad (2.14)\\
\kappa_{i,j,k,l} + \kappa_{i,j,kl} = \omega_{ijkl} &\approx \frac{1}{2\pi}(2\mathbf{I}_{i,j,k,l} + \mathbf{I}_{i,j,kl}).
\end{aligned}$$

With these coefficients we can write (2.6), (2.7) and (2.8) as follows: $A_1 = A_{11} + A_{12} + A_{13} + A_{14}$, $A_2 = A_{21} + A_{22} + A_{23} + A_{24}$ and $A_3 = A_{31} + A_{32}$, where

$$\begin{aligned}
A_{11} &= 3\sum{}'\delta_{ijk}\delta_{kst}a_{ij}a_{st}m_{kr}, & A_{12} &= -6\sum{}'\delta_{ijk}\kappa_{r,s,t}a_{ij}a_{kr}m_{st},\\
A_{13} &= 6\sum{}'\xi_{ijk}\delta_{rst}a_{js}a_{kt}m_{ir}, & A_{14} &= -6\sum{}'\omega_{ijkr}a_{kr}m_{ij},\\
A_{21} &= -3\sum{}'\kappa_{i,j,k}\kappa_{r,s,t}a_{kr}m_{ij}m_{st}, & A_{22} &= 6\sum{}'\delta_{ijk}\kappa_{r,s,t}a_{ij}m_{kr}m_{st}, \quad (2.15)\\
A_{23} &= -6\sum{}'\kappa_{i,j,k}\kappa_{r,s,t}a_{kt}m_{ir}m_{js}, & A_{24} &= 3\sum{}'\kappa_{i,j,k,r}m_{ij}m_{kr},\\
A_{31} &= 3\sum{}'\kappa_{i,j,k}\kappa_{r,s,t}m_{ij}m_{kr}m_{st}, & A_{32} &= 2\sum{}'\kappa_{i,j,k}\kappa_{r,s,t}m_{ir}m_{js}m_{kt},
\end{aligned}$$

with $\sum'$ is given as in the expression (2.8). Now writing the following matrices:

$$
\begin{aligned}
&\mathbf{N}^{(r)} = (\kappa_{r,s,t}), \qquad \mathbf{K}_1^{(ij)} = (\kappa_{i,j,k,r}), \qquad \boldsymbol{\Delta}^{(k)} = (\delta_{ijk}),\\
&\overline{\boldsymbol{\Delta}}^{(i)} = (\delta_{ijk}), \qquad \boldsymbol{\Xi}^{(k)} = (\xi_{ijk}), \qquad \boldsymbol{\Omega}^{(ij)} = (\omega_{ijkr}),\\
&\mathbf{Tr}^{11} = \big(\mathrm{tr}\big(\mathbf{A}\boldsymbol{\Delta}^{(k)}\big)\,\mathrm{tr}\big(\mathbf{A}\overline{\boldsymbol{\Delta}}^{(r)}\big)\big), \qquad \mathbf{Tr}^{12} = \big(\mathrm{tr}\big(\mathbf{A}\boldsymbol{\Delta}^{(k)}\big)\,\mathrm{tr}\big(\mathbf{M}\mathbf{N}^{(r)}\big)\big),\\
&\mathbf{Tr}^{13} = \big(\mathrm{tr}\big(\mathbf{A}\boldsymbol{\Xi}^{(k)}\mathbf{M}\boldsymbol{\Delta}^{(t)}\big)\big), \qquad \mathbf{Tr}^{14} = \big(\mathrm{tr}\big(\mathbf{A}\boldsymbol{\Omega}^{(ij)}\big)\big),\\
&\mathbf{Tr}^{21} = \big(\mathrm{tr}\big(\mathbf{M}\mathbf{N}^{(k)}\big)\,\mathrm{tr}\big(\mathbf{M}\mathbf{N}^{(r)}\big)\big), \qquad \mathbf{Tr}^{22} = \big(\mathrm{tr}\big(\mathbf{A}\boldsymbol{\Delta}^{(k)}\big)\,\mathrm{tr}\big(\mathbf{M}\mathbf{N}^{(r)}\big)\big),\\
&\mathbf{Tr}^{23} = \big(\mathrm{tr}\big(\mathbf{M}\mathbf{N}^{(k)}\mathbf{M}\mathbf{N}^{(t)}\big)\big), \qquad \mathbf{Tr}^{24} = \big(\mathrm{tr}\big(\mathbf{M}\mathbf{K}_1^{(ij)}\big)\big),
\end{aligned}
$$

where $\mathrm{tr}(\mathbf{A})$ denotes the trace of $\mathbf{A}$ and the matrices $\boldsymbol{\Delta}^{(k)}$ and $\overline{\boldsymbol{\Delta}}^{(i)}$ differ in the fixed index. Then we can write the coefficients A_{ij}'s as follows:

$$
\begin{aligned}
&A_{11} = 3\,\mathrm{tr}(\mathbf{M}\mathbf{Tr}^{11}), \qquad A_{12} = -6\,\mathrm{tr}(\mathbf{A}\mathbf{Tr}^{12}),\\
&A_{13} = 6\,\mathrm{tr}(\mathbf{A}\mathbf{Tr}^{13}), \qquad A_{14} = -6\,\mathrm{tr}(\mathbf{M}\mathbf{Tr}^{14}),\\
&A_{21} = -3\,\mathrm{tr}(\mathbf{A}\mathbf{Tr}^{21}), \qquad A_{22} = 6\,\mathrm{tr}(\mathbf{M}\mathbf{Tr}^{22}), \qquad A_{23} = -6\,\mathrm{tr}(\mathbf{A}\mathbf{Tr}^{23}),\\
&A_{24} = 3\,\mathrm{tr}(\mathbf{M}\mathbf{Tr}^{24}), \qquad A_{31} = 3\,\mathrm{tr}(\mathbf{M}\mathbf{Tr}^{21}), \qquad A_{32} = 2\,\mathrm{tr}(\mathbf{M}\mathbf{Tr}^{23}).
\end{aligned}
$$

Finally, for the coefficients A_1, A_2 and A_3, we have

$$
\begin{aligned}
A_1 &= 3\,\mathrm{tr}(\mathbf{M}[\mathbf{Tr}^{11} - 2\mathbf{Tr}^{14}]) - 6\,\mathrm{tr}(\mathbf{A}[\mathbf{Tr}^{12} - \mathbf{Tr}^{13}]),\\
A_2 &= -3\,\mathrm{tr}(\mathbf{A}\{\mathbf{Tr}^{21} + 2\mathbf{Tr}^{23}\} - \mathbf{M}\{2\mathbf{Tr}^{22} + \mathbf{Tr}^{24}\}),\\
A_3 &= \mathrm{tr}(\mathbf{M}\{3\mathbf{Tr}^{21} + 2\mathbf{Tr}^{23}\}).
\end{aligned}
\tag{2.16}
$$

These coefficients are evaluated at the maximum likelihood estimates restricted to the null hypothesis. Then, the coefficients given in (2.9) are computed in order to obtain the corrected statistic S_B given in (2.9).

Moreover, we use the rejection rate via simulation to compare the performances of the three corrected score statistics, S_B, S_K, S_C and its original form. We can also compare the power functions of the tests, under a sequence of contiguous alternatives.

3 Applications

In order to illustrate in detail the computation of the Bartlett-type correction factor in matrix form, we consider an AR(1) model, given in (2.1) with $p = 1$ and $q = 0$. The problem is to test $H_1 : \phi = \phi^{(0)}$ against $A : \phi \neq \phi^{(0)}$, when σ^2 is considered as a nuisance parameter. Note that the vector of parameters in question is $\beta = (\phi, \sigma^2)^\top$, with $|\phi| < 1$, $\beta_1 = \phi$ and $\beta_2 = \sigma^2$. To obtain the matrices in this special case the software MAPLE V [Abell and Braselton (1994)] was used. The matrices reduce to

$$\mathbf{K}=\begin{pmatrix}\dfrac{1}{1-\phi^2} & 0\\ 0 & \dfrac{1}{2\sigma^4}\end{pmatrix},\qquad \mathbf{K}^{-1}=\begin{pmatrix}1-\phi^2 & 0\\ 0 & 2\sigma^4\end{pmatrix},$$

$$\mathbf{M}=\begin{pmatrix}1-\phi^2 & 0\\ 0 & 0\end{pmatrix},\qquad \mathbf{A}=\begin{pmatrix}0 & 0\\ 0 & 2\sigma^4\end{pmatrix},$$

$$\boldsymbol{\Delta}^{(1)}=\begin{pmatrix}-\dfrac{4\phi}{(1-\phi^2)^2} & -\dfrac{1}{\sigma^2(1-\phi^2)}\\ -\dfrac{1}{\sigma^2(1-\phi^2)} & 0\end{pmatrix},$$

$$\boldsymbol{\Delta}^{(2)}=\begin{pmatrix}-\dfrac{1}{\sigma^2(1-\phi^2)} & 0\\ 0 & 0\end{pmatrix},$$

$$\boldsymbol{\Xi}^{(1)}=\begin{pmatrix}-\dfrac{8\phi}{(1-\phi^2)^2} & -\dfrac{3}{\sigma^2(1-\phi^2)}\\ -\dfrac{3}{\sigma^2(1-\phi^2)} & 0\end{pmatrix},$$

$$\boldsymbol{\Xi}^{(2)}=\begin{pmatrix}-\dfrac{3}{\sigma^2(1-\phi^2)} & 0\\ 0 & -\dfrac{2}{\sigma^6}\end{pmatrix},$$

$$\boldsymbol{\Omega}^{(11)}=\begin{pmatrix}\dfrac{2(7+17\phi^2)}{(1-\phi^2)^3} & \dfrac{12\phi}{\sigma^2(1-\phi^2)^2}\\ \dfrac{12\phi}{\sigma^2(1-\phi^2)^2} & \dfrac{2}{\sigma^4(1-\phi^2)}\end{pmatrix},$$

$$\boldsymbol{\Omega}^{(12)}=\begin{pmatrix}\dfrac{14\phi}{\sigma^2(1-\phi^2)^2} & \dfrac{4}{\sigma^2(1-\phi^2)}\\ \dfrac{4}{\sigma^2(1-\phi^2)} & 0\end{pmatrix}=\boldsymbol{\Omega}^{(21)},$$

$$\boldsymbol{\Omega}^{(22)}=\begin{pmatrix}\dfrac{4}{\sigma^2(1-\phi^2)} & 0\\ 0 & \dfrac{1}{\sigma^8}\end{pmatrix},\qquad \mathbf{N}^{(1)}=\begin{pmatrix}\dfrac{6\phi}{(1-\phi^2)^2} & \dfrac{2}{\sigma^2(1-\phi^2)}\\ \dfrac{2}{\sigma^2(1-\phi^2)} & 0\end{pmatrix},$$

$$\mathbf{N}^{(2)}=\begin{pmatrix}\dfrac{2}{\sigma^2(1-\phi^2)} & 0\\ 0 & \dfrac{1}{\sigma^6}\end{pmatrix}.$$

With these we get

$$\mathbf{Tr}^{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{Tr}^{12} = \mathbf{Tr}^{22},$$

$$\mathbf{Tr}^{13} = \begin{pmatrix} \dfrac{6}{1-\phi^2} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{Tr}^{14} = \begin{pmatrix} \dfrac{4}{1-\phi^2} & 0 \\ 0 & \dfrac{2}{\sigma^4} \end{pmatrix},$$

$$\mathbf{Tr}^{21} = \begin{pmatrix} \dfrac{36\phi^2}{(1-\phi^2)^2} & \dfrac{12\phi}{(1-\phi^2)\sigma^2} \\ \dfrac{12\phi}{(1-\phi^2)\sigma^2} & \dfrac{4}{\sigma^4} \end{pmatrix} = \mathbf{Tr}^{23} \quad \text{and}$$

$$\mathbf{Tr}^{24} = \begin{pmatrix} \dfrac{6(3+7\phi^2)}{(1-\phi^2)^2} & \dfrac{18\phi}{(1-\phi^2)\sigma^2} \\ \dfrac{18\phi}{(1-\phi^2)\sigma^2} & \dfrac{6}{\sigma^4} \end{pmatrix}.$$

From these computed matrices we obtain the coefficients A_1, A_2 and A_3 given in the expressions (2.16), and the coefficients a, b and c given by the expressions (2.10)

$$A_1 = -\frac{24}{n}, \qquad A_2 = \frac{18(11\phi^2 - 1)}{n(1-\phi^2)} \quad \text{and} \quad A_3 = \frac{180\phi^2}{n(1-\phi^2)}$$

and

$$a = \frac{\phi^2}{n(1-\phi^2)}, \qquad b = -\frac{1+9\phi^2}{2n(1-\phi^2)}, \qquad c = -\frac{1}{2n}.$$

Then, the corrected score statistic becomes

$$S_B = S\left\{1 - \frac{1}{n}\left(-\frac{1}{2} - \frac{1+9\phi^2}{2(1-\phi^2)}S + \frac{\phi^2}{1-\phi^2}S^2\right)\right\},$$

from which we notice that the correction does not depend of the nuisance parameter σ^2, evaluating S_B, S_K and S_C for $\phi = \phi^{(0)}$.

A summary of the cases considered, for which the correction factor was calculated is given in Table 1. The results for a moving average model of order 1, MA(1), are the same as for the AR(1) case. The elements of the matrices become complicated as the number of parameters increases. One example is the case of an ARMA(1, 1) model, where we want to test the hypothesis $H : \phi = \phi^{(0)}$ against

Table 1 *Bartlett-type correction factor*

	Parameter		A's			Coefficients of the polynomial in S		
Model	Interest	Nuisance	A_1	A_2	A_3	a	b	c
AR(1)	ϕ or θ	none	0	$\frac{12\phi}{n}$	$\frac{180\phi^2}{n(1-\phi^2)}$	$\frac{\phi^2}{n(1-\phi^2)}$	$-\frac{\phi(\phi^2+30\phi-1)}{3n(1-\phi^2)}$	$\frac{\phi(\phi^2+15\phi-1)}{n(1-\phi^2)}$
	σ^2	none	0	$-\frac{36\sigma^2}{n}$	$\frac{24\sigma^2}{n}$	$\frac{2\sigma^2}{15n}$	$-\frac{7\sigma^2}{3n}$	$\frac{5\sigma^2}{n}$
MA(1)	ϕ or θ	σ^2	$-\frac{24}{n}$	$\frac{18(11\phi^2-1)}{n(1-\phi^2)}$	$\frac{180\phi^2}{n(1-\phi^2)}$	$\frac{\phi^2}{n(1-\phi^2)}$	$-\frac{1+9\phi^2}{2n(1-\phi^2)}$	$-\frac{1}{2n}$
	σ^2	ϕ	$-\frac{6}{n}$	$\frac{12}{n}$	$\frac{40}{n}$	$\frac{2}{9n}$	$-\frac{17}{9n}$	$\frac{11}{6n}$

$A: \phi \neq \phi^{(0)}$, the vector of nuisance parameters being $(\theta, \sigma^2)^\top$. We get for the coefficients A's, a, b and c:

$$A_1 = \frac{24(\phi^2\theta^2 - 2 + 2\phi\theta - \phi^2)}{n(\phi\theta - 1)^2},$$

$$A_2 = -\frac{18(1 + \phi^2 - 2\phi\theta + 22\phi^3\theta - 11\phi^2\theta^2 + \phi^4\theta^2 - 12\phi^4)}{n(1 - \phi^2)(\phi\theta - 1)^2},$$

$$A_3 = \frac{180\phi^2(-2\phi\theta + 2\phi^2 + \theta^2)}{n(1 - \phi^2)(\phi\theta - 1)^2},$$

$$a = \frac{\phi^2(-2\phi\theta + \phi^2 + \theta^2)}{n(1 - \phi^2)(\phi\theta - 1)^2},$$

$$b = -\frac{1 + \phi^2 - 2\phi\theta - 18\phi^3\theta + 9\phi^2\theta^2 + \phi^4\theta^2 + 8\phi^4}{2n(1 - \phi^2)(\phi\theta - 1)^2},$$

$$c = \frac{\phi^2\theta^2 + 2\phi^2 + 2\phi\theta - 5}{2n(\phi\theta - 1)^2}.$$

We observe that this correction also does not depend of the nuisance parameter σ^2, a situation that was also noticed for the corrections for the likelihood ratio statistic [Lagos and Morettin (2004)].

4 A simulation study

We first set down some formulas which are useful for the developments of the simulations, having a completely specified model. In the simple AR(1) case the formula for the S statistics is specified. For the other cases this is too complicated, due to the difficulty in obtaining the score function and the Fisher information matrix of the vector of parameters β in closed form.

4.1 Autoregressive model of order one

Let us remember that the first-order autoregressive model, AR(1), is given by $Y_t - \phi Y_{t-1} = a_t$, where $|\phi| < 1$ and $\{a_t\}$ is a sequence of normal random variables with mean zero and variance σ^2 (>0). The log-likelihood is given by

$$\begin{aligned} l(\phi, \sigma^2|\mathbf{Y}) = &-\frac{n}{2}\log 2\pi\sigma^2 + \frac{1}{2}\log(1-\phi^2) \\ &- \frac{1}{2\sigma^2}\left[(1-\phi^2)Y_1^2 + \sum_{j=2}^{n}(Y_j - \phi Y_{j-1})^2\right]. \end{aligned} \tag{4.1}$$

Notice that the vector of parameters is $\beta = (\phi, \sigma^2)^\top$. We consider two tests, $H : \phi = \phi^{(0)}$ against $A : \phi \neq \phi^{(0)}$ with σ^2 as a nuisance parameter and $H : \sigma^2 = \sigma^{(0)2}$ against $A : \sigma^2 \neq \sigma^{(0)2}$ with ϕ as a nuisance parameter.

Considering the total score function $U(\beta)$, partitioned in the same form as β, that is, $U(\phi, \sigma^2) = (U_1(\phi, \sigma^2), U_2(\phi, \sigma^2))^\top$, taking the expected value of the Hessian matrix of the $l(\phi, \sigma^2|\mathbf{Y})$, we derive the total Fisher information matrix for β and its inverse,

$$\mathbf{K} = \begin{pmatrix} \frac{1}{1-\phi^2}\left(\frac{1+\phi^2}{1-\phi^2} + n - 2\right) & \frac{\phi}{\sigma^2(1-\phi^2)} \\ \frac{\phi}{\sigma^2(1-\phi^2)} & \frac{n}{2\sigma^4} \end{pmatrix},$$

$$\mathbf{K}^{-1} = \frac{1}{|\mathbf{K}|}\begin{pmatrix} \frac{n}{2\sigma^4} & -\frac{\phi}{\sigma^2(1-\phi^2)} \\ -\frac{\phi}{\sigma^2(1-\phi^2)} & \frac{1}{1-\phi^2}\left(\frac{1+\phi^2}{1-\phi^2} + n - 2\right) \end{pmatrix},$$

where the determinant of $\mathbf{K}$ is

$$|\mathbf{K}| = \frac{n(1+\phi^2)}{2\sigma^4(1-\phi^2)^2} + \frac{n(n-2)}{2\sigma^4(1-\phi^2)} - \frac{\phi^2}{\sigma^4(1-\phi^2)^2}.$$

For the computation of the score statistic S in (2.5) we need the matrices $\mathbf{K}^{11}$, if the interest is the parameter ϕ, evaluated at $\tilde{\beta} = (\phi^{(0)}, \tilde{\sigma}^2)$ and $\mathbf{K}^{22}$, if the interest is the parameter σ^2, evaluated at $\tilde{\beta} = (\tilde{\phi}, \sigma^{(0)2})$, resulting in

$$\mathbf{K}^{11} = -\frac{n(1-\phi^2)^2}{n - n^2 + 2\phi^2 - 3n\phi^2 + n^2\phi^2} \quad \text{and}$$

$$\mathbf{K}^{22} = \frac{2\sigma^4(1 - 3\phi^2 - n + n\phi^2)}{n - n^2 + 2\phi^2 - 3n\phi^2 + n^2\phi^2}.$$

We also observe that if the sample size is small or moderate the term $\frac{1}{2\sigma^2}(1-\phi^2)y_1^2$ in the log-likelihood (4.1) cannot be neglected, as is the case when the sample size is large, a fact that is assumed often in the literature.

The score functions are

$$U_1 = U_1(\phi, \sigma^2) = -\frac{\phi}{1-\phi^2} + \frac{\phi}{\sigma^2} y_1^2 + \frac{1}{\sigma^2} \sum_{j=2}^{n} y_{j-1}(y_j - \phi y_{j-1}),$$

$$U_2 = U_2(\phi, \sigma^2) = -\frac{n}{2\sigma^2} + \frac{1-\phi^2}{2\sigma^4} y_1^2 + \frac{1}{2\sigma^4} \sum_{j=2}^{n} (y_j - \phi y_{j-1})^2.$$

Therefore, if the interest is to test $H_1 : \phi = \phi^{(0)}$, the score statistic S given by (2.5) results

$$S = \frac{(\phi^{(0)} \tilde{\sigma}^2 - \phi^{(0)} y_1^2 + \phi^{(0)3} y_1^2 - K1 + K1 \phi^{(0)2})^2 n}{(-n^2 - 3n\phi^{(0)2} + n + 2\phi^{(0)2} + n^2 \phi^{(0)2}) \tilde{\sigma}^4},$$

$$\text{with } K1 = \sum_{j=2}^{n} y_{j-1}(y_j - \phi^{(0)} y_{j-1}).$$

If the interest is to test $H_2 : \sigma^2 = \sigma^{(0)2}$,

$$S = \frac{(n\sigma^{(0)2} - y_1^2 + y_1^2 \tilde{\phi}^2 - K2)^2 (1 - 3\tilde{\phi}^2 - n + n\tilde{\phi}^2)}{2\sigma^{(0)4} (n - 3n\tilde{\phi}^2 - n^2 + n^2 \tilde{\phi}^2 + 2\tilde{\phi}^2)},$$

$$\text{with } K2 = \sum_{j=2}^{n} (y_j - \tilde{\phi} y_{j-1})^2.$$

The numerical results are shown in the Tables 2, 3 and 4 for some fixed parameters.

4.2 Moving average model of order one

For the model MA(1), $Y_t = a_t - \theta a_{t-1}$, where $|\theta| < 1$ and $\{a_t\}$ is a sequence of independent normal random variables, with mean zero and variance $\sigma^2 > 0$, the log-likelihood function is given in (2.3), where $\mathbf{V} = V(\beta)$ is a matrix such that $\boldsymbol{\Sigma} = \sigma^2 \mathbf{V}$. The procedure used to write $l(\boldsymbol{\beta})$ in a convenient way for the maximization procedure is the prediction error decomposition form, given in Brockwell and Davis (1991). The matrix of autocovariances $\boldsymbol{\Sigma}$ of the vector $\mathbf{Y}$ following a MA(1) model is a definite positive matrix (Toeplitz type), given by $\boldsymbol{\Sigma} = \sigma^2 \mathbf{V}$ where $\mathbf{V} = (v_{ij})_{i,j=1}^{n}$ with $v_{ii} = 1 + \theta^2$, $v_{ij} = -\theta$, for $|i - j| = 1$ and $v_{ij} = 0$, for $|i - j| > 1$, for $i, j = 1, \ldots, n$, and $\boldsymbol{\Sigma}^{-1} = \sigma^{-2} \mathbf{V}^{-1}$, where $\mathbf{V}^{-1}$ has a typical element computed from Galbraith and Galbraith (1974).

We consider two tests, $H : \theta = \theta^{(0)}$ against $A : \theta \neq \theta^{(0)}$ with σ^2 as a nuisance parameter and $H : \sigma^2 = \sigma^{(0)2}$ against $A : \sigma^2 \neq \sigma^{(0)2}$ with θ as a nuisance parameter.

Table 2 *Estimated rejection rates for S, S_B, S_K, S_C, AR(1) model, ϕ is the interest, $\sigma^2 = 1$ is the nuisance*

		$n=20$				$n=30$				$n=40$				$n=50$			
$\phi^{(0)}$	$\alpha \rightarrow$	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%
−0.9	S	6.44	4.70	3.46	2.39	6.61	4.82	3.63	2.27	6.94	4.77	3.45	2.30	6.85	4.67	3.34	2.21
	S_B	8.79	6.25	4.62	2.67	8.44	5.48	3.43	0.00	8.41	5.17	3.07	0.00	8.24	4.66	2.52	0.00
	S_K	14.02	10.30	9.07	7.55	12.58	9.38	7.61	5.87	11.96	8.45	6.94	3.11	11.40	7.76	6.11	2.45
	S_C	18.45	13.05	10.72	9.58	15.19	10.60	8.95	7.21	13.57	9.34	7.71	5.51	12.51	8.39	6.62	0.00
−0.6	S	6.57	3.74	2.25	1.25	7.54	4.04	2.41	1.33	7.89	3.82	2.27	1.20	8.65	4.17	2.50	1.40
	S_B	9.25	4.73	2.42	0.00	9.56	4.54	2.50	0.00	9.65	4.46	2.15	0.00	10.01	5.00	2.41	0.00
	S_K	10.05	5.64	3.51	0.61	10.20	5.16	3.27	0.51	10.09	4.93	2.68	0.98	10.30	5.39	2.87	1.15
	S_C	10.27	5.81	3.66	1.21	10.31	5.24	3.31	1.16	10.15	5.06	2.70	1.06	10.36	5.46	2.87	1.23
−0.3	S	8.14	3.58	1.76	0.76	8.71	3.76	1.82	0.74	9.15	4.44	2.02	0.84	8.63	4.33	2.09	0.80
	S_B	9.90	5.10	2.48	1.19	9.97	4.69	2.41	1.00	10.09	5.23	2.58	1.06	9.40	4.78	2.41	0.95
	S_K	10.00	5.12	2.55	1.19	9.99	4.71	2.43	1.00	10.11	5.24	2.60	1.06	9.41	4.78	2.41	0.95
	S_C	10.08	5.17	2.61	1.21	10.05	4.76	2.47	1.01	10.11	5.24	2.62	1.06	9.42	4.81	2.41	0.95
0.0	S	8.22	3.56	1.39	0.39	8.93	4.19	2.07	0.81	9.13	4.34	1.90	0.70	9.28	4.39	2.02	0.74
	S_B	9.65	4.81	2.28	0.88	10.04	5.02	2.69	1.16	9.90	5.09	2.57	0.96	9.86	4.88	2.51	0.97
	S_K	9.70	4.84	2.31	0.92	10.04	5.04	2.70	1.18	9.91	5.10	2.58	0.97	9.87	4.89	2.51	0.97
	S_C	9.76	4.88	2.36	0.95	10.05	5.07	2.73	1.22	9.92	5.11	2.59	0.99	9.88	4.90	2.53	0.98
0.3	S	8.32	3.72	1.70	0.57	8.91	4.00	1.95	0.82	9.08	4.50	2.08	0.79	9.23	4.31	1.98	0.76
	S_B	10.29	5.14	2.54	1.04	10.29	5.20	2.56	1.14	10.17	5.28	2.54	1.01	9.92	4.99	2.34	0.85
	S_K	10.35	5.16	2.57	1.05	10.30	5.23	2.58	1.15	10.19	5.32	2.55	1.02	9.93	4.99	2.35	0.86
	S_C	10.46	5.25	2.59	1.08	10.36	5.25	2.60	1.18	10.21	5.33	2.55	1.03	9.93	4.99	2.35	0.86
0.6	S	7.01	3.65	2.29	1.28	7.77	3.99	2.28	1.27	7.93	3.84	2.11	1.11	8.62	4.24	2.45	1.26
	S_B	9.85	5.04	2.46	0.00	9.55	4.91	2.40	0.00	9.72	4.66	2.11	0.00	10.31	4.85	2.38	0.00
	S_K	10.70	5.95	3.40	0.61	10.01	5.44	3.09	0.41	10.08	5.03	2.61	0.81	10.60	5.16	2.77	1.04
	S_C	10.92	6.17	3.54	1.21	10.18	5.48	3.12	1.09	10.18	5.15	2.69	1.01	10.63	5.16	2.77	1.13
0.9	S	6.61	4.75	3.53	2.43	6.48	4.63	3.36	2.23	6.62	4.82	3.42	2.38	6.98	4.70	3.53	2.50
	S_B	9.14	6.52	4.82	2.81	8.38	5.51	3.56	0.00	8.11	4.73	2.80	0.00	8.44	4.81	2.30	0.00
	S_K	14.43	10.69	9.34	7.74	12.31	9.18	7.55	5.79	11.74	8.16	6.57	3.13	11.88	8.16	6.10	2.63
	S_C	18.43	13.33	11.07	9.79	14.41	10.36	8.75	7.20	12.99	9.02	7.35	5.45	13.10	8.89	6.69	0.00

To write an appropriate expression for the score statistic, the score vector and the Fisher information matrix are written in matrix form. For the first derivative of the autocovariance matrix with respect to θ, $\boldsymbol{\Sigma}_\theta$, we have $\boldsymbol{\Sigma}_\theta = \sigma^2 \mathbf{V}_\theta$ where $\mathbf{V}_\theta$ is the derivative of $\mathbf{V}$ with respect to θ and the first derivative with respect to σ^2 is $\boldsymbol{\Sigma}_{\sigma^2} = \mathbf{V}$. With the purpose of simplifying the calculations, we define $\overline{\boldsymbol{\Sigma}}_\theta = \boldsymbol{\Sigma}_\theta \boldsymbol{\Sigma}^{-1}$ and $\overline{\mathbf{V}}_\theta = \mathbf{V}_\theta \mathbf{V}^{-1}$. The score vector has elements

$$U_1(\theta, \sigma^2) = \frac{1}{2}\left[\frac{1}{\sigma^2}\mathbf{Y}^\top \mathbf{V}^{-1} \overline{\mathbf{V}}_\theta \mathbf{Y} - \operatorname{tr}(\overline{\mathbf{V}}_\theta)\right] \quad \text{and}$$

$$U_2(\theta, \sigma^2) = \frac{1}{2}\left[\frac{1}{\sigma^4}\mathbf{Y}^\top \mathbf{V}^{-1} \mathbf{Y} - \frac{n}{\sigma^2}\right].$$

Table 3 *Estimated rejection rates for S, S_B, S_K, S_C, AR(1) model, $\sigma^2 = 1$ is the interest, ϕ is the nuisance*

		$n=20$				$n=30$				$n=40$				$n=50$			
ϕ	$\alpha \rightarrow$	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%
−0.9	S	9.25	4.36	2.39	1.16	9.94	4.72	2.40	1.20	9.43	4.67	2.34	1.13	10.06	4.74	2.46	1.14
	S_B	10.43	5.12	2.72	1.02	10.78	5.47	2.83	1.27	10.15	5.16	2.67	1.12	10.58	5.20	2.69	1.21
	S_K	10.61	5.32	2.96	1.36	10.84	5.56	2.89	1.34	10.19	5.20	2.74	1.18	10.60	5.21	2.70	1.22
	S_C	10.65	5.39	2.99	1.36	10.84	5.59	2.89	1.35	10.20	5.20	2.74	1.18	10.60	5.21	2.70	1.23
−0.6	S	8.20	3.51	1.63	0.85	8.80	4.19	2.06	0.99	9.01	4.54	2.35	1.24	9.63	4.68	2.46	1.14
	S_B	9.26	4.47	2.05	0.78	9.62	4.82	2.38	0.99	9.55	4.97	2.64	1.25	10.14	5.13	2.76	1.17
	S_K	9.38	4.64	2.21	0.96	9.72	4.89	2.48	1.11	9.61	5.02	2.69	1.30	10.15	5.14	2.77	1.20
	S_C	9.50	4.68	2.23	0.97	9.73	4.91	2.49	1.11	9.62	5.03	2.71	1.30	10.16	5.14	2.77	1.20
−0.3	S	8.05	3.57	1.76	0.85	8.46	4.02	2.05	0.97	9.17	4.25	1.93	0.88	9.25	4.43	2.23	0.99
	S_B	9.21	4.41	2.29	0.86	9.25	4.61	2.31	0.97	9.80	4.75	2.37	0.92	9.88	4.88	2.53	1.05
	S_K	9.34	4.55	2.42	1.07	9.30	4.66	2.36	1.07	9.81	4.77	2.40	0.97	9.89	4.91	2.55	1.08
	S_C	9.39	4.58	2.46	1.09	9.33	4.66	2.36	1.07	9.85	4.79	2.41	0.97	9.93	4.93	2.55	1.08
0.0	S	8.84	3.98	1.91	1.01	8.72	4.06	2.06	1.04	9.20	4.43	2.49	1.19	9.78	4.72	2.38	1.02
	S_B	9.95	4.92	2.40	0.95	9.47	4.62	2.36	1.07	9.88	5.01	2.69	1.27	10.29	5.27	2.66	1.08
	S_K	10.07	5.07	2.54	1.13	9.53	4.69	2.43	1.18	9.88	5.04	2.72	1.33	10.30	5.30	2.68	1.10
	S_C	10.12	5.08	2.56	1.15	9.55	4.73	2.44	1.18	9.89	5.06	2.72	1.33	10.30	5.30	2.68	1.10
0.3	S	8.51	4.18	2.13	1.09	8.72	4.09	1.93	0.99	9.60	4.48	2.03	0.98	9.51	4.57	2.15	0.95
	S_B	9.75	4.94	2.44	1.00	9.55	4.64	2.26	0.92	10.29	5.00	2.36	0.98	9.89	5.13	2.43	1.01
	S_K	9.94	5.17	2.69	1.32	9.68	4.77	2.37	1.08	10.33	5.06	2.41	1.05	9.89	5.14	2.44	1.03
	S_C	9.99	5.21	2.71	1.33	9.68	4.79	2.37	1.09	10.35	5.08	2.41	1.05	9.89	5.14	2.44	1.03
0.6	S	8.43	3.85	1.91	0.89	8.27	3.81	1.86	0.78	9.24	4.38	2.21	1.01	9.52	4.66	2.17	0.91
	S_B	9.76	4.87	2.28	0.83	9.02	4.55	2.17	0.83	9.73	4.93	2.51	1.02	10.12	5.16	2.50	0.97
	S_K	9.91	5.03	2.50	1.10	9.10	4.60	2.23	0.93	9.76	5.00	2.57	1.08	10.13	5.17	2.50	0.98
	S_C	9.94	5.07	2.53	1.13	9.12	4.62	2.23	0.94	9.77	5.01	2.58	1.09	10.15	5.18	2.51	0.98
0.9	S	9.11	4.30	2.18	1.22	9.27	4.12	2.02	0.91	9.72	4.87	2.25	1.03	9.76	4.63	2.46	1.11
	S_B	10.16	5.10	2.46	1.04	10.23	4.93	2.39	0.87	10.34	5.43	2.73	1.05	10.17	5.07	2.71	1.11
	S_K	10.40	5.36	2.74	1.45	10.23	5.03	2.50	0.99	10.40	5.48	2.80	1.11	10.23	5.09	2.74	1.15
	S_C	10.44	5.38	2.76	1.47	10.24	5.04	2.51	0.99	10.40	5.51	2.80	1.11	10.23	5.09	2.75	1.15

The Fisher information matrix is given by

$$\mathbf{K} = \frac{1}{2}\begin{pmatrix} \operatorname{tr}(\overline{\boldsymbol{\Sigma}}_\theta \overline{\boldsymbol{\Sigma}}_\theta) & \dfrac{\operatorname{tr}(\overline{\mathbf{V}}_\theta)}{\sigma^2} \\ \dfrac{\operatorname{tr}(\overline{\mathbf{V}}_\theta)}{\sigma^2} & \dfrac{n}{\sigma^4} \end{pmatrix},$$

and its inverse is

$$\mathbf{K}^{-1} = \frac{1}{|\mathbf{K}|}\begin{pmatrix} \dfrac{n}{2\sigma^4} & -\dfrac{\operatorname{tr}(\overline{\mathbf{V}}_\theta)}{2\sigma^2} \\ -\dfrac{\operatorname{tr}(\overline{\mathbf{V}}_\theta)}{2\sigma^2} & \operatorname{tr}(\overline{\boldsymbol{\Sigma}}_\theta \overline{\boldsymbol{\Sigma}}_\theta) \end{pmatrix}, \qquad \text{with } |\mathbf{K}| = \frac{n \operatorname{tr}(\overline{\boldsymbol{\Sigma}}_\theta \overline{\boldsymbol{\Sigma}}_\theta) - (\operatorname{tr}(\overline{\mathbf{V}}_\theta))^2}{4\sigma^4}.$$

Table 4 *Estimated power (%) for S, S_B, S_K and S_C, AR(1) model, $\phi^{(0)} = -0.3$ is the interest, $\sigma^2 = 1$ fixed*

		$\phi^{(1)}$						$\phi^{(1)}$					
n		−0.9	−0.6	0.0	0.3	0.6	0.9	−0.9	−0.6	0.0	0.3	0.6	0.9
20				$\alpha = 10\%$						$\alpha = 5\%$			
	S	93.19	45.76	24.48	70.44	95.43	99.73	90.49	35.53	10.64	50.60	89.26	99.26
	S_B	57.16	48.12	29.26	74.69	96.26	99.79	53.05	38.84	15.59	59.20	92.26	99.52
	S_K	93.94	48.48	29.38	74.80	96.40	99.79	91.68	39.28	15.73	59.45	92.38	99.52
	S_C	93.96	48.52	29.50	74.96	96.43	99.80	91.71	39.44	15.92	59.71	92.51	99.52
				$\alpha = 2.5\%$						$\alpha = 1\%$			
	S	87.31	27.17	3.56	28.98	77.54	97.73	83.01	18.37	0.45	8.66	50.95	91.90
	S_B	48.29	31.08	6.62	40.50	84.97	98.74	41.34	22.04	1.54	17.81	66.46	95.81
	S_K	89.08	31.54	6.76	40.86	85.14	98.77	85.45	22.78	1.61	18.14	67.01	95.81
	S_C	89.14	31.71	6.94	41.37	85.33	98.80	85.56	22.92	1.64	18.60	67.61	95.98
30				$\alpha = 10\%$						$\alpha = 5\%$			
	S	98.19	59.14	37.79	89.19	99.61	99.98	97.11	49.49	22.34	78.07	98.66	99.96
	S_B	36.61	60.64	41.06	90.44	99.66	99.99	34.25	51.51	26.44	81.77	99.06	99.96
	S_K	98.41	61.21	41.15	90.46	99.66	99.99	97.44	52.24	26.51	81.85	99.06	99.96
	S_C	98.41	61.24	41.19	90.49	99.66	99.99	97.46	52.37	26.58	81.88	99.06	99.96
				$\alpha = 2.5\%$						$\alpha = 1\%$			
	S	95.94	40.09	10.82	62.45	96.63	99.94	94.11	30.43	2.99	38.88	90.68	99.77
	S_B	31.80	42.76	14.81	69.15	97.49	99.96	27.93	32.82	4.98	47.40	93.48	99.84
	S_K	96.38	43.66	14.89	69.29	97.51	99.96	94.90	33.89	5.03	47.66	93.54	99.84
	S_C	96.40	43.73	14.94	69.44	97.52	99.96	94.90	34.04	5.10	47.94	93.62	99.84
40				$\alpha = 10\%$						$\alpha = 5\%$			
	S	99.47	69.78	50.25	96.23	99.90	100.00	99.21	60.87	33.02	91.71	99.84	100.00
	S_B	22.47	69.82	52.69	96.66	99.95	100.00	21.13	61.43	36.46	92.93	99.86	100.00
	S_K	99.53	71.01	52.70	96.68	99.95	100.00	99.25	62.81	36.51	92.94	99.86	100.00
	S_C	99.53	71.01	52.75	96.69	99.95	100.00	99.25	62.81	36.60	92.95	99.86	100.00
				$\alpha = 2.5\%$						$\alpha = 1\%$			
	S	98.83	52.47	19.05	83.48	99.63	100.00	98.19	41.71	7.60	66.98	98.74	99.99
	S_B	19.78	53.21	22.95	86.39	99.68	100.00	17.51	42.83	9.88	72.26	99.03	99.99
	S_K	98.98	54.81	23.05	86.47	99.68	100.00	98.43	44.67	9.95	72.33	99.05	99.99
	S_C	98.98	54.84	23.09	86.52	99.68	100.00	98.43	44.74	10.00	72.39	99.05	99.99

Considering the parameter θ of interest and the nuisance parameter σ^2, the block of matrix $\mathbf{K}^{-1}$ to be used is $\mathbf{K}^{11}$ which does not depend of σ^2, which facilitates the calculations and if the parameter of interest is σ^2 the block is $\mathbf{K}^{22}$. These are given by

$$\mathbf{K}^{11} = \frac{2n}{n\,\mathrm{tr}(\overline{\boldsymbol{\Sigma}}_\theta \overline{\boldsymbol{\Sigma}}_\theta) - (\mathrm{tr}(\overline{\mathbf{V}}_\theta))^2} \quad \text{and} \quad \mathbf{K}^{22} = \frac{2\sigma^4\,\mathrm{tr}(\overline{\boldsymbol{\Sigma}}_\theta \overline{\boldsymbol{\Sigma}}_\theta)}{n\,\mathrm{tr}(\overline{\boldsymbol{\Sigma}}_\theta \overline{\boldsymbol{\Sigma}}_\theta) - (\mathrm{tr}(\overline{\mathbf{V}}_\theta))^2},$$

respectively.

Table 5 *Estimated rejection rates for S, S_B, S_K, S_C,* MA(1) *model, θ is the interest, $\sigma^2 = 1$ is the nuisance*

		$n=20$				$n=30$				$n=40$				$n=50$			
$\theta^{(0)}$	$\alpha \rightarrow$	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%
−0.9	S	6.64	4.60	3.43	2.51	6.73	4.97	3.82	2.76	6.57	4.82	3.43	2.53	6.41	4.62	3.37	2.24
	S_B	8.09	5.99	4.73	2.79	6.83	4.95	3.37	0.00	7.12	4.78	2.70	0.00	6.61	4.14	2.13	0.00
	S_K	12.48	10.20	8.95	7.69	10.81	8.75	7.77	6.01	10.33	8.18	6.55	3.08	9.75	7.28	5.80	2.51
	S_C	15.24	11.84	10.59	9.32	12.01	9.68	8.51	7.39	11.21	8.84	7.35	5.36	10.63	7.79	6.16	0.00
−0.6	S	6.54	3.67	2.48	1.52	7.45	4.01	2.57	1.53	7.72	4.03	2.29	1.33	8.18	4.31	2.48	1.39
	S_B	8.80	4.51	2.21	0.00	9.31	4.56	2.33	0.00	9.35	4.62	2.26	0.00	9.28	4.87	2.42	0.00
	S_K	9.74	5.52	3.44	0.76	10.00	5.31	3.18	0.60	9.75	5.10	2.78	0.93	9.60	5.24	2.87	1.22
	S_C	10.01	5.72	3.52	1.46	10.17	5.43	3.22	1.33	9.87	5.16	2.83	1.13	9.67	5.29	2.89	1.31
−0.3	S	7.79	3.65	1.79	0.74	9.08	4.09	1.98	0.72	8.91	4.32	1.97	0.73	9.25	4.41	2.04	0.92
	S_B	9.62	5.00	2.58	1.18	10.51	5.19	2.49	0.97	9.92	5.08	2.51	0.99	9.97	5.00	2.47	1.09
	S_K	9.69	5.05	2.59	1.22	10.55	5.21	2.53	0.97	9.92	5.08	2.51	0.99	9.97	5.03	2.47	1.10
	S_C	9.78	5.11	2.60	1.24	10.59	5.21	2.53	0.98	9.96	5.09	2.55	0.99	9.97	5.03	2.47	1.10
0.0	S	8.57	3.65	1.60	0.36	8.95	3.99	1.74	0.52	8.88	4.36	1.98	0.77	9.71	4.76	2.37	0.80
	S_B	10.13	4.94	2.37	0.97	9.87	4.77	2.44	0.90	9.92	4.94	2.53	1.01	10.38	5.25	2.84	1.06
	S_K	10.16	4.95	2.40	0.99	9.88	4.79	2.45	0.94	9.95	4.94	2.54	1.02	10.38	5.27	2.85	1.07
	S_C	10.21	4.96	2.44	1.00	9.90	4.80	2.45	0.94	9.95	4.95	2.55	1.02	10.40	5.28	2.85	1.08
0.3	S	7.89	3.63	1.65	0.73	8.77	4.32	1.91	0.82	8.83	4.31	2.04	0.85	9.30	4.39	1.96	0.80
	S_B	9.95	5.05	2.60	1.09	10.07	5.34	2.70	1.06	9.94	4.91	2.52	1.07	10.06	4.97	2.43	0.97
	S_K	10.02	5.07	2.64	1.12	10.09	5.38	2.72	1.06	9.96	4.91	2.53	1.08	10.07	5.00	2.43	0.97
	S_C	10.09	5.13	2.68	1.14	10.11	5.40	2.73	1.07	9.97	4.95	2.55	1.08	10.09	5.00	2.43	0.97
0.6	S	6.47	3.85	2.50	1.39	7.60	4.09	2.62	1.53	8.03	3.89	2.30	1.14	8.22	4.12	2.43	1.17
	S_B	8.79	4.55	2.57	0.00	9.46	4.82	2.28	0.00	9.63	4.94	2.31	0.00	9.65	4.77	2.47	0.00
	S_K	9.66	5.49	3.65	0.71	10.14	5.57	3.24	0.59	10.05	5.29	2.73	0.80	9.87	5.04	2.78	0.90
	S_C	9.84	5.64	3.76	1.32	10.24	5.66	3.30	1.29	10.13	5.33	2.73	0.93	9.91	5.06	2.78	0.99
0.9	S	6.63	4.73	3.58	2.62	6.75	5.00	3.84	2.74	6.85	4.84	3.63	2.41	6.97	4.93	3.67	2.59
	S_B	8.00	6.05	4.50	2.71	7.31	5.13	3.29	0.00	6.95	4.63	2.79	0.00	6.98	4.54	2.16	0.00
	S_K	12.30	10.35	9.08	7.73	11.11	9.15	7.68	5.96	10.22	8.22	6.80	3.29	10.47	7.99	6.13	2.92
	S_C	15.63	11.75	10.64	9.55	12.44	10.07	8.80	7.32	11.09	8.77	7.41	5.47	11.28	8.27	6.66	0.00

For MA(1) model, the numerical results are shown in the Tables 5, 6 and 7 for some fixed parameters.

4.3 Autoregressive moving average model of order (1, 1)

For the ARMA(1, 1) model in (2.1), $\mathbf{V} = (1-\phi^2)^{-1}\mathbf{G}$, where

$$\mathbf{G} = \begin{pmatrix} 1-2\phi\theta+\theta^2 & (1-\phi\theta)(\phi-\theta) & (1-\phi\theta)(\phi-\theta)\phi & \dots & (1-\phi\theta)(\phi-\theta)\phi^{n-2} \\ (1-\phi\theta)(\phi-\theta) & 1-2\phi\theta+\theta^2 & (1-\phi\theta)(\phi-\theta) & \dots & (1-\phi\theta)(\phi-\theta)\phi^{n-3} \\ (1-\phi\theta)(\phi-\theta)\phi & (1-\phi\theta)(\phi-\theta) & 1-2\phi\theta+\theta^2 & \dots & (1-\phi\theta)(\phi-\theta)\phi^{n-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-\phi\theta)(\phi-\theta) \times\phi^{n-2} & (1-\phi\theta)(\phi-\theta) \times\phi^{n-3} & (1-\phi\theta)(\phi-\theta) \times\phi^{n-4} & \dots & 1-2\phi\theta+\theta^2 \end{pmatrix}. \quad (4.2)$$

Table 6 *Estimated rejection rates for* S, S_B, S_K, S_C, MA(1) *model*, $\sigma^2 = 1$ *is the interest*, θ *is the nuisance*

		$n=20$				$n=30$				$n=40$				$n=50$			
θ	$\alpha \rightarrow$	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%
−0.9	S	57.85	55.85	54.68	54.06	51.08	48.40	47.24	46.47	45.17	42.07	40.79	40.00	38.70	35.47	34.17	33.08
	S_B	5.35	3.12	1.97	0.96	5.87	3.11	1.77	0.72	6.40	3.30	1.82	0.82	6.82	3.41	2.02	0.81
	S_K	58.43	56.27	55.14	54.15	51.62	48.87	47.54	46.54	45.58	42.48	41.00	40.02	39.11	35.72	34.32	33.11
	S_C	58.47	56.30	55.14	54.16	51.63	48.87	47.55	46.55	45.58	42.50	41.00	40.02	39.11	35.72	34.32	33.11
−0.6	S	21.81	18.10	16.45	15.50	15.39	10.64	8.51	7.52	11.94	7.34	5.35	4.28	10.10	5.35	3.23	2.17
	S_B	8.33	4.29	2.20	0.72	9.42	4.57	2.16	0.86	9.26	4.54	2.26	0.97	9.35	4.30	2.14	0.82
	S_K	23.09	19.11	17.06	15.62	16.17	11.33	8.91	7.63	12.58	7.88	5.61	4.33	10.74	5.71	3.54	2.22
	S_C	23.12	19.17	17.12	15.63	16.19	11.35	8.91	7.64	12.58	7.91	5.61	4.33	10.75	5.72	3.54	2.22
−0.3	S	12.25	7.61	5.96	5.01	9.74	5.07	2.79	1.72	9.51	4.51	2.36	1.20	9.37	4.51	2.45	1.20
	S_B	9.57	4.63	2.45	1.03	9.77	4.91	2.39	0.96	9.84	4.67	2.32	1.01	9.83	4.91	2.57	1.14
	S_K	13.61	8.69	6.53	5.23	10.63	5.77	3.27	1.86	10.17	5.01	2.64	1.34	9.94	5.03	2.69	1.27
	S_C	13.68	8.73	6.56	5.23	10.66	5.78	3.28	1.87	10.19	5.02	2.66	1.34	9.95	5.06	2.69	1.27
0.0	S	10.16	5.26	3.43	2.35	9.13	4.48	2.42	1.26	9.20	4.34	2.26	0.92	9.32	4.20	2.04	0.83
	S_B	9.92	4.81	2.26	0.87	9.79	4.97	2.70	1.03	9.68	4.75	2.48	0.86	9.77	4.73	2.27	0.82
	S_K	11.53	6.43	3.92	2.55	10.07	5.27	3.01	1.35	9.76	4.82	2.55	0.94	9.80	4.77	2.31	0.86
	S_C	11.61	6.47	3.94	2.56	10.11	5.31	3.02	1.35	9.77	4.85	2.56	0.94	9.80	4.77	2.31	0.87
0.3	S	11.09	7.16	5.48	4.70	10.06	5.20	2.93	1.77	9.22	4.38	2.34	1.08	9.77	4.62	2.31	1.06
	S_B	8.51	4.08	2.07	0.83	9.87	5.03	2.37	0.89	9.86	4.60	2.43	0.86	10.28	4.99	2.49	1.01
	S_K	12.49	8.03	6.08	4.88	10.83	6.03	3.34	1.86	10.11	4.85	2.69	1.15	10.36	5.09	2.59	1.09
	S_C	12.57	8.04	6.14	4.90	10.85	6.05	3.35	1.87	10.13	4.89	2.69	1.17	10.38	5.10	2.59	1.09
0.6	S	22.55	18.63	16.96	16.17	15.20	10.59	8.53	7.40	12.12	7.55	5.42	4.31	10.37	5.58	3.43	2.33
	S_B	8.59	4.22	2.24	1.05	9.49	4.67	2.40	0.98	9.32	4.77	2.33	0.88	9.27	4.41	2.12	0.78
	S_K	23.78	19.43	17.48	16.40	16.10	11.25	9.00	7.60	12.76	8.25	5.79	4.38	10.83	5.98	3.68	2.36
	S_C	23.81	19.47	17.49	16.40	16.11	11.28	9.03	7.61	12.76	8.28	5.80	4.39	10.83	5.99	3.68	2.36
0.9	S	58.57	56.29	55.16	54.51	50.99	48.27	47.07	46.50	45.18	42.16	40.39	39.58	39.55	36.15	34.40	33.63
	S_B	5.62	3.14	1.70	0.82	5.84	3.01	1.51	0.81	6.79	3.77	1.85	0.87	6.92	3.36	1.55	0.61
	S_K	59.32	56.85	55.45	54.59	51.58	48.74	47.25	46.55	45.57	42.57	40.64	39.67	39.96	36.40	34.61	33.67
	S_C	59.37	56.87	55.48	54.60	51.61	48.76	47.27	46.56	45.58	42.57	40.65	39.67	39.96	36.40	34.61	33.68

Table 7 *Estimated power (%) for S, S_B, S_K and S_C,* MA(1) *model,* $\theta^{(0)} = -0.3$ *is the interest,* $\sigma^2 = 1$ *fixed*

		$\theta^{(1)}$						$\theta^{(1)}$					
n		−0.9	−0.6	0.0	0.3	0.6	0.9	−0.9	−0.6	0.0	0.3	0.6	0.9
20				$\alpha = 10\%$						$\alpha = 5\%$			
	S	41.44	19.68	36.44	68.97	83.63	87.14	20.14	7.05	26.66	58.47	75.46	80.13
	S_B	47.93	24.22	39.48	70.65	82.18	85.06	28.00	11.15	30.39	61.38	75.17	78.64
	S_K	48.05	24.38	39.66	71.74	85.52	88.85	28.30	11.28	30.64	62.69	79.15	83.26
	S_C	48.21	24.54	39.77	71.82	85.56	88.89	28.64	11.44	30.80	62.78	79.23	83.35
				$\alpha = 2.5\%$						$\alpha = 1\%$			
	S	6.68	1.97	18.78	48.29	66.57	72.12	0.66	0.17	11.60	36.07	54.05	60.22
	S_B	13.14	4.14	22.50	52.25	66.75	70.98	2.60	0.69	14.81	40.27	54.76	59.59
	S_K	13.39	4.22	22.82	53.97	71.57	76.57	2.69	0.71	15.14	42.48	60.96	66.86
	S_C	13.66	4.40	22.97	54.14	71.80	76.73	2.84	0.74	15.35	42.83	61.19	67.04
30				$\alpha = 10\%$						$\alpha = 5\%$			
	S	67.61	34.59	47.77	83.97	94.56	96.27	46.83	17.47	37.26	76.19	90.98	93.45
	S_B	70.99	37.81	49.74	82.77	86.96	84.97	53.18	21.82	39.95	75.38	82.91	81.32
	S_K	71.13	37.90	49.99	85.41	95.18	96.69	53.33	21.90	40.22	78.45	92.11	94.22
	S_C	71.23	37.98	50.03	85.46	95.21	96.71	53.53	22.04	40.26	78.53	92.15	94.23
				$\alpha = 2.5\%$						$\alpha = 1\%$			
	S	26.69	7.39	28.61	67.95	85.83	89.13	8.92	1.85	19.55	56.50	77.46	82.02
	S_B	34.00	10.36	31.24	67.57	77.83	76.53	13.93	3.23	22.01	55.91	68.25	68.37
	S_K	34.19	10.43	31.58	71.09	88.14	90.86	14.03	3.26	22.46	60.42	80.58	84.95
	S_C	34.42	10.52	31.64	71.17	88.20	90.88	14.21	3.29	22.60	60.51	80.72	85.03
40				$\alpha = 10\%$						$\alpha = 5\%$			
	S	84.04	47.33	56.90	92.76	98.51	99.08	67.89	29.07	45.71	88.50	96.86	97.96
	S_B	85.83	50.04	58.32	88.29	83.55	78.79	71.86	32.52	47.61	84.22	80.64	76.23
	S_K	85.84	50.07	58.56	93.31	98.70	99.19	71.92	32.57	47.89	89.82	97.27	98.22
	S_C	85.89	50.16	58.59	93.32	98.70	99.19	71.98	32.63	47.93	89.87	97.29	98.23
				$\alpha = 2.5\%$						$\alpha = 1\%$			
	S	49.26	16.06	36.75	82.68	94.27	96.36	25.42	5.40	26.06	73.39	89.59	92.88
	S_B	54.65	19.49	38.95	78.22	76.98	73.21	31.24	7.34	28.37	68.41	70.45	67.11
	S_K	54.78	19.57	39.29	84.60	95.17	96.86	31.39	7.39	28.74	76.20	91.02	93.94
	S_C	54.87	19.62	39.30	84.67	95.22	96.86	31.56	7.42	28.78	76.24	91.02	93.95

We consider $H:\phi = \phi^{(0)}$ against $A:\phi = \phi^{(0)}$, with θ and σ^2 as a nuisance parameters. For this case, the score function is given by

$$\begin{aligned} U(\phi, \theta, \sigma^2) = -\frac{1}{2}\bigg\{ & \frac{2n\phi}{(1-\phi^2)} + \operatorname{tr}(\mathbf{G}^{-1}\mathbf{G}_\phi) \\ & - \frac{1}{\sigma^2}\mathbf{Y}^\top[2\phi\mathbf{G}^{-1} + (1-\phi^2)\mathbf{G}^{-1}\mathbf{G}_\phi\mathbf{G}^{-1}]\mathbf{Y}\bigg\}, \end{aligned} \tag{4.3}$$

Table 8 *Estimated rejection rates for S, S_B, S_K, S_C,* ARMA(1, 1) *model, ϕ is the interest, $\theta = 0.3$ and $\sigma^2 = 1$ are the nuisance*

		$n = 20$				$n = 30$				$n = 40$				$n = 50$			
$\phi^{(0)}$	$\alpha \rightarrow$	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%
−0.9	S	6.21	4.28	3.12	2.01	5.91	4.10	3.00	1.96	6.61	4.54	3.15	2.02	6.54	4.37	3.29	2.24
	S_B	9.86	6.63	4.91	2.80	8.58	5.23	3.48	0.59	9.50	5.52	3.13	0.00	8.81	4.61	2.19	0.00
	S_K	14.67	10.44	8.92	7.22	12.11	8.44	6.81	5.08	12.58	8.58	6.43	3.05	11.55	7.55	5.51	2.34
	S_C	18.36	13.39	10.83	9.31	14.11	9.89	7.87	6.34	14.12	9.48	7.36	5.17	12.63	8.21	6.03	3.27
−0.6	S	5.64	2.71	1.50	0.67	6.52	3.03	1.56	0.77	8.02	4.00	2.03	0.89	8.26	3.67	1.79	0.72
	S_B	8.73	4.66	2.51	1.00	9.25	4.39	2.35	1.02	9.93	5.34	2.78	1.05	10.00	4.86	2.34	0.86
	S_K	8.97	4.95	2.76	1.30	9.37	4.48	2.47	1.13	10.08	5.50	2.92	1.18	10.04	4.90	2.37	0.93
	S_C	9.22	5.12	2.94	1.38	9.47	4.54	2.56	1.20	10.14	5.59	2.94	1.23	10.11	4.92	2.40	0.94
−0.3	S	6.07	2.36	1.12	0.39	7.44	3.12	1.19	0.30	8.35	3.77	1.76	0.66	8.62	3.82	1.84	0.63
	S_B	8.45	3.89	1.96	0.94	9.29	4.51	2.08	0.71	10.09	4.84	2.46	1.08	9.86	4.73	2.35	0.95
	S_K	8.50	4.02	2.04	1.02	9.36	4.55	2.15	0.71	10.11	4.87	2.47	1.09	9.87	4.75	2.36	0.98
	S_C	8.54	4.10	2.13	1.04	9.39	4.56	2.16	0.74	10.18	4.90	2.48	1.10	9.90	4.79	2.36	0.99
0.3	S	5.25	1.58	0.42	0.10	7.13	2.93	1.16	0.32	7.92	3.38	1.44	0.41	8.46	3.57	1.68	0.47
	S_B	8.18	3.52	1.38	0.39	9.13	4.43	2.11	0.70	9.54	4.72	2.27	0.90	9.81	4.70	2.34	0.81
	S_K	8.30	3.62	1.42	0.43	9.19	4.51	2.17	0.77	9.56	4.72	2.29	0.91	9.89	4.72	2.35	0.82
	S_C	7.24	3.11	1.23	0.34	7.93	3.83	1.91	0.73	8.46	4.19	1.97	0.77	8.42	4.01	2.00	0.78
0.6	S	5.00	1.87	0.83	0.26	6.41	2.61	1.04	0.37	7.30	2.96	1.22	0.38	8.26	3.62	1.53	0.51
	S_B	8.78	4.29	2.29	1.09	9.25	4.59	2.29	0.91	9.77	4.62	2.30	0.91	10.44	5.05	2.42	0.90
	S_K	9.01	4.59	2.44	1.20	9.44	4.70	2.43	0.98	9.86	4.70	2.35	0.96	10.51	5.14	2.47	0.96
	S_C	9.16	4.72	2.62	1.34	9.37	4.81	2.52	1.07	9.80	4.71	2.34	0.97	10.48	5.08	2.51	0.94
0.9	S	5.01	3.05	1.69	0.72	6.21	3.67	2.16	1.19	5.47	3.41	2.30	1.29	6.02	3.79	2.41	1.26
	S_B	11.31	7.55	5.51	3.50	11.48	6.97	4.86	1.94	9.60	5.39	3.12	0.46	9.61	5.43	3.20	0.11
	S_K	14.32	9.70	7.90	5.98	13.86	9.24	6.87	4.83	11.71	7.16	5.07	3.22	11.09	6.90	4.95	1.84
	S_C	17.46	12.51	9.78	8.12	16.03	10.82	8.13	6.25	13.08	8.25	5.84	3.78	12.06	7.45	5.43	2.99

where $\mathbf{G}_\phi = \frac{\partial \mathbf{G}}{\partial \phi}$. The Fisher information matrix $\mathbf{K}$ is

$$\mathbf{K} = \begin{pmatrix} m^{\phi\phi} & m^{\phi\theta} & m^{\phi\sigma^2} \\ m^{\phi\theta} & m^{\theta\theta} & m^{\theta\sigma^2} \\ m^{\phi\sigma^2} & m^{\theta\sigma^2} & m^{\sigma^2\sigma^2} \end{pmatrix}, \tag{4.4}$$

where $\mathbf{G}_\theta = \frac{\partial \mathbf{G}}{\partial \theta}$, $\bar{\mathbf{G}}_\phi = \mathbf{G}_\phi \mathbf{G}^{-1}$, $\bar{\mathbf{G}}_\theta = \mathbf{G}_\theta \mathbf{G}^{-1}$ and $m^{\phi\phi} = \frac{n(1+\phi^2)}{(1-\phi^2)^2} - \frac{n}{1-\phi^2} + \frac{2\phi}{1-\phi^2} \operatorname{tr}(\bar{\mathbf{G}}_\phi) + \frac{1}{2} \operatorname{tr}(\bar{\mathbf{G}}_\phi \bar{\mathbf{G}}_\phi)$, $m^{\phi\theta} = \frac{\phi}{1-\phi^2} \operatorname{tr}(\bar{\mathbf{G}}_\theta) + \frac{1}{2} \operatorname{tr}(\bar{\mathbf{G}}_\theta \bar{\mathbf{G}}_\phi)$, $m^{\phi\sigma^2} = \frac{1}{2\sigma^2(1-\phi^2)} \times [2n\phi + (1-\phi^2) \operatorname{tr}(\bar{\mathbf{G}}_\phi)]$, $m^{\theta\theta} = \frac{1}{2} \operatorname{tr}(\bar{\mathbf{G}}_\theta \bar{\mathbf{G}}_\theta)$, $m^{\theta\sigma^2} = \frac{1}{2\sigma^2} \operatorname{tr}(\bar{\mathbf{G}}_\theta)$, $m^{\sigma^2\sigma^2} = \frac{n}{2\sigma^4}$.

The numerical results are shown in the Tables 8 and 9.

4.4 The numerical results

We now describe the simulation study to investigate the performance of the corrected score statistics, S_B, S_K and S_C. This performance is measured by the estimated rejection rate and the power of the test.

Table 9 *Estimated power (%) for S, S_B, S_K and S_C,* ARMA(1, 1) *model,* $\phi^{(0)} = -0.3$ *is the interest,* $\theta = 0.3$ *and* $\sigma^2 = 1$ *fixed*

		$\phi^{(1)}$						$\phi^{(1)}$					
n		−0.9	−0.6	0.0	0.3	0.6	0.9	−0.9	−0.6	0.0	0.3	0.6	0.9
20		$\alpha = 10\%$						$\alpha = 5\%$					
	S	60.37	24.39	5.50	4.75	14.37	44.84	55.59	16.07	1.22	1.69	7.28	33.95
	S_B	59.46	28.24	8.46	7.76	18.95	50.03	55.24	20.49	3.02	3.35	11.52	42.12
	S_K	62.20	28.41	8.60	7.92	19.18	50.26	58.39	20.63	3.14	3.45	11.80	42.50
	S_C	62.24	28.55	8.36	6.79	18.24	50.33	58.51	20.91	3.13	2.89	11.21	42.77
		$\alpha = 2.5\%$						$\alpha = 1\%$					
	S	50.65	10.49	0.27	0.53	3.32	23.24	43.44	5.86	0.05	0.15	1.05	11.85
	S_B	50.91	14.36	0.84	1.39	6.88	33.72	44.46	9.27	0.22	0.49	3.24	23.33
	S_K	54.59	14.61	0.87	1.49	7.13	34.39	49.03	9.44	0.23	0.54	3.45	24.03
	S_C	54.74	14.86	0.93	1.29	6.90	34.93	49.31	9.72	0.27	0.48	3.54	25.03
30		$\alpha = 10\%$						$\alpha = 5\%$					
	S	79.72	35.05	8.59	6.90	21.66	65.49	76.99	25.21	3.36	3.02	12.63	56.50
	S_B	68.81	38.33	10.78	8.97	25.56	68.61	65.87	28.89	5.02	4.24	16.62	61.21
	S_K	80.30	38.39	10.87	9.03	25.74	68.71	78.11	28.97	5.08	4.28	16.85	61.32
	S_C	80.31	38.51	10.44	7.78	24.65	68.81	78.11	29.11	4.93	3.81	16.24	61.54
		$\alpha = 2.5\%$						$\alpha = 1\%$					
	S	73.96	17.62	1.18	1.03	6.91	46.44	69.34	10.58	0.15	0.27	2.51	33.29
	S_B	62.51	21.43	2.20	2.05	10.40	53.65	57.09	14.20	0.66	0.68	5.36	43.15
	S_K	75.75	21.61	2.24	2.07	10.60	53.90	71.85	14.28	0.68	0.69	5.53	43.55
	S_C	75.79	21.78	2.17	1.88	10.24	54.19	71.94	14.40	0.69	0.62	5.33	44.00
40		$\alpha = 10\%$						$\alpha = 5\%$					
	S	90.38	44.24	11.16	6.97	27.02	77.83	88.87	33.35	4.90	3.12	16.62	70.62
	S_B	67.16	46.56	13.05	8.89	30.22	79.74	64.94	36.38	6.61	4.18	20.05	73.84
	S_K	90.60	46.62	13.09	8.95	30.32	79.79	89.42	36.45	6.67	4.18	20.16	73.92
	S_C	90.61	46.74	12.63	7.72	29.59	79.78	89.43	36.50	6.52	3.65	19.74	73.97
		$\alpha = 2.5\%$						$\alpha = 1\%$					
	S	87.23	24.78	2.00	1.32	10.03	62.75	84.75	16.23	0.34	0.41	4.50	50.82
	S_B	62.12	28.09	3.15	1.97	13.14	67.50	58.38	19.60	0.87	0.78	7.16	57.90
	S_K	87.89	28.17	3.22	1.98	13.30	67.68	85.91	19.66	0.90	0.79	7.32	58.11
	S_C	87.92	28.24	3.17	1.76	13.03	67.82	85.95	19.84	0.94	0.71	7.25	58.33

For simulation of size, we set the length of the series as $n = 20$, 30, 40 and 50 and for simulation of power, the first three values of n. The true values of the parameters ϕ and/or θ for the simulation of the size were taken as −0.9, −0.6, −0.3, 0.0, 0.3, 0.6 and 0.9, assuming that $\sigma^2 = 1$. For the power of the AR(1) and MA(1) models, we considered $\phi^{(0)}$ or $\theta^{(0)}$ as −0.3, $\sigma^2 = 1$ and $\phi^{(1)}$ or $\theta^{(1)}$ were set at −0.9, −0.6, 0.0, 0.3, 0.6 and 0.9. For ARMA(1, 1) the parameter θ was fixed as 0.3 in both simulations.

We report the results for four different nominal significance levels, namely: $\alpha = 0, 10$, 0.05, 0.025 and 0.01. The results are presented in Tables 2–9. Entries are percentages. The simulations were performed using the R programming environment and all results are based on 10,000 replications.

For AR(1) model, considering ϕ as the parameter of interest (Table 2), simulations reveal that the score test underrejects the null hypothesis more frequently than expected, except for $|\phi|$ near to 1 and small nominal levels. For these values of ϕ, S_K and S_C overreject and, considering $\alpha = 0.01$, S_B never rejects the null hypothesis. Overall, S_B has the best performance. Considering ϕ far from the extremes of the interval $[-1, 1]$, we note that the three corrected statistics have similar performance and the corrections are very effective in pushing the rejection rates of the modified statistics toward to the nominal levels.

If the interest is to test σ^2 with ϕ as a nuisance parameter (Table 3), the corrections improve the results; the rejections rate are near to the real levels, S_K and S_C have similar and superior performance compared to S_B.

For MA(1) model, testing θ, comments are the same as for AR(1) model (Table 5). Figures in Table 6 reveal important information when the interest is to test σ^2. The original score statistic S and the corrected statistics S_K and S_C are largely liberal; their rejection rates are very large. The performance of S_B is much better than the performance of the other statistics, although it had the tendency of underrejecting the null hypothesis.

For ARMA(1, 1) model (Table 8), if $\phi^{(0)}$ is fixed as -0.9 or 0.9, S_K and S_C overreject the hypothesis and S_B presents rates less distant to the true values. Otherwise, S_C and S_K have the best performance and, if $\phi^{(0)}$ is -0.6 or 0.6, S_C is slightly superior and, if $\phi^{(0)}$ is -0.3 or 0.3, S_K is slightly superior.

For the simulation of power of the AR(1) model (Table 4), we can notice, as expected, that the power increases when $\phi^{(1)}$ moves away from the fixed value for $\phi^{(0)}$. The corrected statistic S_B presents power strongly smaller than the others statistics, for $\phi^{(1)} = -0.9$, and all values considered for the nominal level and sample size, decreasing in some cases. This can occur because S_B is not monotone. The power is larger for the corrected statistics than the original one, showing superior values for S_K and S_C, which are similar for $n = 40$, S_C being slightly superior for $n = 20$ and 30.

For the MA(1) model (Table 7), comments are about the same, however S_B does not present smaller values for $\theta^{(1)}$ equal to -0.9. The best performance is due to S_C statistic. The values of the power function are smaller than the others obtained for the AR(1) model.

Finally, for ARMA(1, 1) model (Table 9), the conclusions are the same as for AR(1) model, but the performance of S_B is better when $\phi^{(1)} = -0.9$.

Clearly, the corrections have less impact as the sample size increases.

5 Conclusions

We can observe from the simulations that the results are coherent with what is expected, concerning the fixed parameters under H, the length of the series and the adopted significance levels.

For the estimated rejection rate, we observe that is clear the discrimination of results concerning stationarity and nonstationarity, invertibility and noninvertibility, small and large values of n, nominal levels smaller and bigger than 5%.

The corrected statistics have a better performance than the original statistic S. We also notice the following behavior about the rejection rate ($\tilde{p}$) of the corrected statistics:

$$\widetilde{p}_{S_B} < \widetilde{p}_{S_K} < \widetilde{p}_{S_C}.$$

Concerning the estimated power, $\widetilde{\pi}$, it is possible to conclude more clearly the good performance of the corrected statistics compared with the original statistic, with the remark that if $n \to \infty$, $\theta \to 1$ and $\alpha \to 0$, S has a larger power than S_B and if $|\theta| = 0.6$ and $\alpha = 1\%$ the corrections do not have effect on the power, comparing to S. Lastly it can be observed that in general

$$\widetilde{\pi}_S < \widetilde{\pi}_{S_B} < \widetilde{\pi}_{S_K} < \widetilde{\pi}_{S_C}.$$

Acknowledgments

The first author was supported by Project DIUC 204.014.014-1.0 University of Concepción, FONDECYT (Chile) through the Grant N^o1010213 and CNPq (Brazil). The second author was partially supported by the FAPESP Grant 2003/10105-2 and the third author by FAPESP Grant 2004/15304-6. The authors are grateful to two anonymous referees and to Gauss M. Cordeiro and Elisete C. Q. Aubin for comments that led to the improvement of the paper.

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B. M. Lagos
University of Concepción
Concepción
Chile

P. A. Morettin
L. P. Barroso
University of São Paulo
São Paulo
Brazil