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Convergence to stable laws in Mallows distance for mixing sequences of random variables

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Abstract. Convergence in Mallows distance is of particular interest when heavy-tailed distributions are considered. For $1 \le \alpha < 2$, it constitutes an alternative technique to derive Central Limit type theorems for non-Gaussian α-stable laws. In this note, for properly stabilized martingale sums and sequences of ϕ-mixing random variables, we establish Mallows convergence to stable laws. Sufficient conditions are presented in the setting of familiar Lindeberg-like conditions and extend earlier results for the independent case.

1 Introduction

For $\alpha > 0$, the Mallows (1972) α-distance between two cumulative distribution functions F_X and F_Y is given by

$$d_\alpha(F_X, F_Y) = \inf_{(X,Y)} \big(E(|X - Y|^\alpha)\big)^{1/\alpha}, \tag{1}$$

where the infimum is taken over all random vectors (X, Y) with marginal distributions F_X and F_Y, respectively $X \sim F_X$ and $Y \sim F_Y$. The Mallows distance, also known as the Wasserstein metric, satisfies the metric relations

$$d_\alpha^\alpha(F_X, F_Y) \le d_\alpha^\alpha(F_X, F_Z) + d_\alpha^\alpha(F_Z, F_Y), \qquad 0 < \alpha \le 1, \tag{2}$$

and

$$d_\alpha(F_X, F_Y) \le d_\alpha(F_X, F_Z) + d_\alpha(F_Z, F_Y), \qquad \alpha \ge 1. \tag{3}$$

The connection between convergence in Mallows distance and the convergence in distribution ($\xrightarrow{d}$) was established by Bickel and Freedman (1981): for $\alpha \ge 1$ and for distribution functions F_0 and $\{F_n\}_{n\ge1}$ satisfying $\int |x|^\alpha \, dF_j(x) < \infty$, $j = 0, 1, 2, \ldots,$ we have

$$d_\alpha(F_n, F_0) \underset{n}{\to} 0 \quad \Leftrightarrow \quad F_n \xrightarrow{d} F_0 \quad \text{and} \quad \int |x|^\alpha \, dF_n(x) \underset{n}{\to} \int |x|^\alpha \, dF_0(x). \tag{4}$$

This fact was used by Johnson and Samworth (2005) to establish Central Limit type theorems for stable laws. And was further explored by Barbosa and

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Dorea (2009) to derive Lindeberg-like conditions for sequences of independent random variables $X_1, X_2, \ldots$:

$$\frac{1}{n}\sum_{j=1}^{n} E\{|X_j - Y_j|^\alpha 1(|X_j - Y_j| > bn^{(2-\alpha)/(2\alpha)})\} \to 0 \qquad \forall b > 0, \tag{5}$$

where $Y_1, Y_2, \ldots$ are independent copies of an α-stable random variable Y. More specifically, for $\alpha \geq 1$ and $S_n = X_1 + \cdots + X_n$, there exist stabilizing constants $\{c_n\}$ such that $d_\alpha(F_n, F_Y) \underset{n}{\to} 0$, where $Y \sim F_Y$ and $\frac{S_n - c_n}{n^{1/\alpha}} \sim F_n$.

In this note, for $1 < \alpha < 2$, we extend this result for a martingale setting and for mixing sequences of random variables $\{X_n\}_{n\geq 1}$ satisfying

$$|P(A \cap B) - P(A)P(B)| \leq \phi(n)P(A)P(B) \qquad \text{with } \phi(n) \downarrow 0 \tag{6}$$

for all $k \geq 1$, all $A \in \mathcal{F}_k = \sigma(X_1, \ldots, X_k)$ and all $B \in \mathcal{F}_{k+n}^\infty = \sigma(X_{k+n}, X_{k+n+1}, \ldots)$. Theorems 1 and 2 exhibit Lindeberg-like conditions that assure the desired convergence.

2 Preliminaries and auxiliary results

First, we state some properties for stable distributions [see, e.g., Samorodnitsky and Taqqu (2000)]. For $0 < \alpha \leq 2$, we say that Y is an α-stable random variable, or, equivalently possesses an α-stable distribution, if for any $n \geq 2$, there are a positive number a_n and a real number d_n such that

$$Y_1 + Y_2 + \cdots + Y_n \stackrel{d}{=} a_n Y + d_n, \tag{7}$$

where $Y_1, Y_2, \ldots, Y_n$ are independent copies of Y and $\stackrel{d}{=}$ stands for equality in distribution.

Proposition 1. *If Y has an α-stable distribution then*:

(a) $E(|Y|^{\alpha'}) < \infty$ *for $0 < \alpha' < \alpha$ and in (7) we can take $a_n = n^{1/\alpha}$.*

(b) *If $\alpha > 1$ there exists a real number μ (shift parameter) such that $\mu = EY$ and $d_n = \mu(n - n^{1/\alpha})$.*

A key point to our proofs is the use of moment inequalities. In Proposition 2, we gather some needed moment inequalities that include Bahr–Essen's inequality for independent random variables and Burkholder's martingale inequality [see, e.g., de la Peña (1990) or Hall and Heyde (1980)].

Proposition 2. (a) *For random variables $\xi_1, \xi_2, \ldots$ and $S_n = \xi_1 + \cdots + \xi_n$ we have*

$$E\{|S_n|^\alpha\} \leq \sum_{j=1}^{n} E\{|\xi_j|^\alpha\}, \qquad 0 < \alpha \leq 1, \tag{8}$$

and

$$E\{|S_n|^\alpha\} \le n^{\alpha-1} \sum_{j=1}^{n} E\{|\xi_j|^\alpha\}, \qquad \alpha \ge 1. \tag{9}$$

(b) *If, in addition,* $\xi_1, \xi_2, \ldots$ *are independent then*

$$E\{|S_n - E(S_n)|^\alpha\} \le 2 \sum_{j=1}^{n} E\{|\xi_j - E(\xi_j)|^\alpha\}, \qquad 1 < \alpha \le 2. \tag{10}$$

(c) *And, if* $\{S_n\}$ *is a martingale then there exists a constant* $C(\alpha) > 0$ *such that*

$$E\{|S_n|^\alpha\} \le C(\alpha) E\left\{\left[\sum_{j=1}^{n} \xi_j^2\right]^{\alpha/2}\right\}, \qquad \alpha > 1. \tag{11}$$

Next, we derive the corresponding inequality for ϕ-mixing sequences.

Lemma 1. *Let* $\{\xi_n\}_{n\ge1}$ *be a sequence of* ϕ*-mixing random variables satisfying* (6) *and assume that* $E(\xi_n) = 0$ *and* $\sup_n E(|\xi_n|) \le M < \infty$. *Then, for* $\alpha > 1$, *there exists a constant* $C(\alpha) > 0$ *such that for all* $0 < m_n < n$, $k_n = [n/m_n]$ *and all* $b_n > 0$ *we have*

$$\begin{aligned} E\{|S_n|^\alpha\} \le C(\alpha)\Big\{ & m_n^\alpha b_n^\alpha k_n^{\alpha/2} + (m_n k_n \phi(m_n) M)^\alpha \\ & + m_n^{\alpha-1} \sum_{j=1}^{n} E\big[|\xi_j|^\alpha 1\big(|\xi_j| > b_n - \phi(m_n)M\big)\big]\Big\}. \end{aligned} \tag{12}$$

Proof. (a) Let $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$ and let $l_n = n - k_n m_n$. Define

$$U_i(j) = \xi_{im_n+j} - E\big(\xi_{im_n+j} | \mathcal{F}_{(i-1)m_n+j}\big) \tag{13}$$

and

$$V_i(j) = E\big(\xi_{im_n+j} | \mathcal{F}_{(i-1)m_n+j}\big). \tag{14}$$

Since ξ_{im_n+j} is $\mathcal{F}^{\infty}_{im_n+j}$-measurable we have from (6) and Roussas and Ioannides (1987),

$$|V_i(j)| \le \phi(m_n) E(|\xi_{im_n+j}|) \le \phi(m_n) M \qquad \text{a.s.} \tag{15}$$

(a.s.: almost surely).

(b) Write

$$S_n = \sum_{j=1}^{m_n} \xi_j + \sum_{j=1}^{m_n} \sum_{i=1}^{k_n-1} U_i(j) + \sum_{j=1}^{m_n} \sum_{i=1}^{k_n-1} V_i(j) + \sum_{l=1}^{l_n} \xi_{k_n m_n + l}. \tag{16}$$

Since $\alpha > 1$, from (9) we derive

$$E|S_n|^\alpha \le 4^{\alpha-1}[E|A_n|^\alpha + E|B_n|^\alpha + E|C_n|^\alpha + E|D_n|^\alpha], \tag{17}$$

$$E|A_n|^\alpha \le m_n^{\alpha-1}\sum_{j=1}^{m_n} E|\xi_j|^\alpha \quad \text{and} \quad E|D_n|^\alpha \le l_n^{\alpha-1}\sum_{l=1}^{l_n} E|\xi_{k_n m_n+l}|^\alpha. \tag{18}$$

From (15) we have

$$\begin{aligned} E|C_n|^\alpha &\le m_n^{\alpha-1}(k_n-1)^{\alpha-1}\sum_{j=1}^{m_n}\sum_{i=1}^{k_n-1} E|V_i(j)|^\alpha \\ &\le (m_n k_n \phi(m_n) M)^\alpha. \end{aligned} \tag{19}$$

(c) Fix j and let $Z_k(j) = \sum_{i=1}^k U_i(j)$. Since $E(|\xi_{im_n+j}|) \le M$ we have $E(|U_i(j)|) \le 2M$ and $E(|Z_k(j)|) < \infty$. From (13) we have $E\{Z_{k+1}(j)| \mathcal{F}_{(k-1)m_n+j}\} = Z_k(j)$ a.s. Thus $\{Z_k(j), \mathcal{F}_{km_n+j}\}$ is a martingale. From inequality (11) there exists $C'(\alpha) > 0$ such that for $j = 1, \dots, m_n$ we have

$$\begin{aligned} &E\left|\sum_{i=1}^{k_n-1} U_i(j)\right|^\alpha \\ &\quad\le C'(\alpha) E\left\{\left[\sum_{i=1}^{k_n-1} U_i^2(j)\right]^{\alpha/2}\right\} \\ &\quad\le C'(\alpha) E\left\{\left[(k_n-1)b_n^2 + \sum_{i=1}^{k_n-1} U_i^2(j) 1\big(|U_i(j)| > b_n\big)\right]^{\alpha/2}\right\} \\ &\quad\le C'(\alpha)\left[(k_n-1)^{\alpha/2} b_n^\alpha + \sum_{i=1}^{k_n-1} E\big\{|\xi_{im_n+j}|^\alpha 1\big(|\xi_{im_n+j}| > b_n - \phi(m_n)M\big)\big\}\right], \end{aligned}$$

where for the last inequalities we have used (8), (13), (15) and the fact that

$$1\big(|U_i(j)| > b_n\big) \le 1\big(|\xi_{im_n+j}| > b_n - \phi(m_n)M\big). \tag{20}$$

It follows that

$$\begin{aligned} E|B_n|^\alpha \le C'(\alpha)\Bigg[& m_n k_n^{\alpha/2} b_n^\alpha \\ & + m_n^{\alpha-1}\sum_{j=1}^{m_n}\sum_{i=1}^{k_n-1} E\big\{|\xi_{im_n+j}|^\alpha \\ & \times 1\big(|\xi_{im_n+j}| > b_n - \phi(m_n)M\big)\big\}\Bigg]. \end{aligned} \tag{21}$$

(d) Using the inequality (18) and arguing as above we obtain

$$E|A_n|^\alpha \le m_n^{\alpha-1}\left[m_n b_n^\alpha + \sum_{j=1}^{m_n} E\{|\xi_j|^\alpha 1(|\xi_j| > b_n - \phi(m_n)M)\}\right] \tag{22}$$

and

$$E|D_n|^\alpha \le l_n^{\alpha-1}\left[l_n b_n^\alpha + \sum_{l=1}^{l_n} E\{|\xi_{k_n m_n + j}|^\alpha 1(|\xi_{k_n m_n + j}| > b_n - \phi(m_n)M)\}\right]. \tag{23}$$

Finally, from (17) and (19)–(23) we get (12) with $C(\alpha) = 4^{\alpha-1}C'(\alpha)$. □

3 Martingale sums and mixing sequences

For $1 < \alpha < 2$ we seek conditions under which there exit constants $\{c_n\}$ such that

$$\lim_{n\to\infty} d_\alpha(F_n, F_Y) = 0, \qquad \frac{S_n^{(X)} - c_n}{n^{1/\alpha}} \sim F_n, \tag{24}$$

where $S_n^{(X)} = X_1 + \cdots + X_n$ and F_Y is an α-stable distribution.

From (1) and Proposition 1 we can replace F_Y by the distribution of

$$\frac{S_n^{(Y)} - (n - n^{1/\alpha})EY}{n^{1/\alpha}}, \tag{25}$$

where $S_n^{(Y)} = Y_1 + \cdots + Y_n$ and $Y_1, \dots, Y_n$ are independent copies of Y. It follows that, by taking

$$c_n = \sum_{j=1}^{n} E(X_j - Y_j) - (n - n^{1/\alpha})EY \tag{26}$$

the Mallows convergence (24) can be assured if

$$\frac{1}{n}E\{|S_n|^\alpha\} = \frac{1}{n}E\left\{\left|\sum_{j=1}^{n}[(X_j - Y_j) - E(X_j - Y_j)]\right|^\alpha\right\} \underset{n}{\to} 0. \tag{27}$$

As in the independence case Lindeberg-like condition will be considered.

Condition 1. *Let $1 < \alpha < 2$ and let Y be an α-stable random variable. For Y_1, $Y_2, \dots$ independent copies of Y and $\{Y_n\}_{n\ge1}$ independent of $\{X_n\}_{n\ge1}$, assume that*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} E\{|X_j - Y_j|^\alpha 1(|X_j - Y_j| > bn^{(2-\alpha)/(2\alpha)})\} = 0 \qquad \forall b > 0. \tag{28}$$

Theorem 1. *Assume that Condition 1 holds and that* $\{S_n^{(X)}, \mathcal{F}_n\}$, $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, *is a martingale. Then* (24) *holds with* c_n *given by* (26). *Moreover,*

$$\frac{S_n^{(X)} - c_n}{n^{1/\alpha}} \xrightarrow{d} Y. \tag{29}$$

Proof. (a) First, note that it is enough to prove (27). The arguments above show that Mallows convergence (24) follows from (27). Also, since

$$E|(X_j - Y_j) - E(X_j - Y_j)|^\alpha \le 2^\alpha E|(X_j - Y_j)|^\alpha \tag{30}$$

it follows that the pair $\{(X_n - EX_n)\}$ and $\{(Y_n - EY_n)\}$ satisfies (28). And we may take $E(X_j - Y_j) = 0$.

(b) Let $E(X_j - Y_j) = 0$. Since $\{Y_n\}_{n\ge1}$ is independent of $\{X_n\}_{n\ge1}$ then for

$$S_n = S_n^{(X)} - S_n^{(Y)} \quad \text{and} \quad \mathcal{G}_n = \sigma((X_1, Y_1), \dots, (X_n, Y_n)). \tag{31}$$

We have a martingale $\{S_n, \mathcal{G}_n\}$. Let $Z_j = X_j - Y_j$. From inequality (11) there exists $C(\alpha) > 0$ such that

$$\begin{aligned} E|S_n|^\alpha &\le C(\alpha) E\left\{\left[\sum_{j=1}^n Z_j^2\right]^{\alpha/2}\right\} \\ &\le C(\alpha) E\left\{\left[b^2 n^{2/\alpha} + \sum_{j=1}^n Z_j^2 1\big(|Z_j| > bn^{(2-\alpha)/(2\alpha)}\big)\right]^{\alpha/2}\right\}. \end{aligned}$$

Since $\alpha/2 \le 1$ we have from (8)

$$E|S_n|^\alpha \le C(\alpha)\left[b^\alpha n + \sum_{j=1}^n E\{|Z_j|^\alpha 1\big(|Z_j| > bn^{(2-\alpha)/(2\alpha)}\big)\}\right]. \tag{32}$$

Then we obtain (27) by letting $b \to 0$. To prove (29) we make use of (4). Note that $E|X_n| < \infty$, $E|Y| < \infty$ and $d_1(F_n, F_Y) \le d_\alpha(F_n, F_Y)$. Since $d_\alpha(F_n, F_Y) \to 0$ result follows. □

Example 1. This example illustrates that Condition 1 cannot be weakened even in the independent case. Let Y be α-stable, $1 < \alpha < 2$, and assume that $Y \stackrel{d}{=} -Y$ (symmetrical distribution). For $Y_1, Y_2, \dots$ independent copies of Y and $\theta = \frac{2-\alpha}{2\alpha}$ define

$$X_j = Y_j + j^\theta 1(Y_j \ge 0) - j^\theta 1(Y_j < 0).$$

Then, for $1 \le j \le n$ we have

$$|X_j - Y_j| = j^\theta \le n^\delta \qquad \text{for } \delta > \frac{2-\alpha}{2\alpha}.$$

We have (28) satisfied for $\delta > \frac{2-\alpha}{2\alpha}$ in place of $\frac{2-\alpha}{2\alpha}$. Since Y is symmetrical, by Proposition 1, we can take $d_n = 0$ and $Y_1 + Y_2 + \cdots + Y_n \stackrel{d}{=} n^{1/\alpha} Y$. By (26) we have $c_n = 0$. We will show that $\frac{S_n}{n^{1/\alpha}} \stackrel{d}{\to} Y$ does not hold and hence $d_\alpha(F_{S_n/n^{1/\alpha}}, F_Y) \underset{n}{\not\to} 0$. To see this, let $\xi_n = X_n - Y_n$. Then $P(\xi_n \leq 0) = P(\xi_n \geq 0) = \frac{1}{2}$,

$$\sum_{j=1}^{n} E\{1(|\xi_j| > n^{1/\alpha})\} = 0$$

and

$$\frac{1}{n^{2/\alpha}} \sum_{j=1}^{n} E\{\xi_j^2 1(|\xi_j| \leq n^{1/\alpha})\} \approx \frac{1}{2\theta + 1}\left(1 - \frac{1}{n^{2/\alpha}}\right) \underset{n}{\not\to} 0.$$

By Theorem 5.2.3 from Chung (1974) we cannot have $\frac{\xi_1 + \cdots + \xi_n}{n^{1/\alpha}} \underset{n}{\to} 0$ a.s. It follows that, $\frac{S_n}{n^{1/\alpha}} \stackrel{d}{\not\to} Y$.

Theorem 2. *Assume that Condition 1 holds and that* $\{X_n\}$ *is a* ϕ*-mixing sequence satisfying* (6) *with* $\sum_{n\geq 1} n^\gamma \phi(n) < \infty\ \forall \gamma$. *In addition, assume that* $\sup_n E|X_n| \leq M < \infty$ *and that for some* $\varepsilon > 0$ *and some* $0 < \delta < \frac{2-\alpha}{2\alpha}$ *we have satisfied*

$$\lim_{n\to\infty} \frac{1}{n^{1-\varepsilon}} \sum_{j=1}^{n} E\{|X_j - Y_j|^\alpha 1(|X_j - Y_j| > bn^\delta)\} = 0 \qquad \forall b > 0. \tag{33}$$

Then (24) *and* (29) *hold.*

Proof. (a) Since $\alpha > 1$ we have $E(Y_n) < \infty$. From the independence of $\{Y_n\}$ and $\{X_n\}$ we have for $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$

$$E\{Y_{n+m} - E(Y_{n+m})|\mathcal{F}_n\} = 0 \qquad \forall n, \forall m. \tag{34}$$

As in the proof of Theorem 1 we may assume that $E(X_j - Y_j) = 0$. It follows that

$$\begin{aligned} &\left|E\{X_{n+m} - Y_{n+m}|\mathcal{F}_n\}\right| \\ &\quad = \left|E\{X_{n+m} - E(X_{n+m}) - [Y_{n+m} - E(Y_{n+m})]|\mathcal{F}_n\}\right| \\ &\quad = |E\{X_{n+m}|\mathcal{F}_n\}| \leq \phi(m)M. \end{aligned} \tag{35}$$

(b) From Lemma 1 and (35) we can write

$$\begin{aligned} \frac{E|S_n|^\alpha}{n} \leq C(\alpha)\Bigg[&\frac{m_n^\alpha b_n^\alpha k_n^{\alpha/2}}{n} + \frac{(m_n k_n \phi(m_n) M)^\alpha}{n} \\ &+ \frac{m_n^{\alpha-1}}{n} \sum_{j=1}^{n} E\{|X_j - Y_j|^\alpha 1\big(|X_j - Y_j| > b_n - \phi(m_n)M\big)\}\Bigg]. \end{aligned}$$

Let $b_n = bn^\delta$, $m_n = [n^\beta] \simeq n^\beta$ and $k_n = [n/n^\beta] \simeq n^{1-\beta}$, where $\beta = \varepsilon \wedge (2[\frac{2-\alpha}{2\alpha} - \delta]) > 0$. Since $\alpha[\frac{1}{2} + \frac{\beta}{2} + \delta] \le \frac{1}{\alpha}$ we have

$$\frac{m_n^\alpha b_n^\alpha k_n^{\alpha/2}}{n} = \frac{b^\alpha n^{\alpha[1/2+\beta/2+\delta]}}{n} \to 0 \qquad \text{as } n \to \infty. \tag{36}$$

Also, since $\sum_{k\ge 1} k^{(1/\beta)(1-1/\alpha)}\phi(k) < \infty$ we have

$$\frac{(m_n k_n \phi(m_n) M)^\alpha}{n} = \big(m_n^{(1/\beta)(1-1/\alpha)}\phi(m_n)M\big)^\alpha \underset{n}{\to} 0. \tag{37}$$

From (33) and the facts

$$\big(b_n - \phi(m_n)M\big) \ge bn^\delta \quad \text{and} \quad \frac{m_n^{\alpha-1}}{n} \le \frac{m_n^\beta}{n} \le \frac{1}{n^{1-\varepsilon}} \tag{38}$$

we get

$$\frac{m_n^{\alpha-1}}{n}\sum_{j=1}^n E\big(|X_j - Y_j|^\alpha 1(|X_j - Y_j| > bn^\delta)\big) \underset{n}{\to} 0. \tag{39}$$

Thus, (24) and (29) follow. □

Remark 1. (a) As shown in the proof of Theorem 1, the hypothesis of $\{X_n\}$ and $\{Y_n\}$ being independent were used to establish

$$\begin{aligned} &E\big\{\big(X_{n+1} - E(X_{n+1})\big) - \big(Y_{n+1} - E(Y_{n+1})\big)|(X_1, Y_1), \dots, (X_n, Y_n)\big\} \\ &\quad = 0 \qquad \text{a.s.} \end{aligned} \tag{40}$$

The independency hypothesis can be dropped if this is assumed.

(b) Similarly, for Theorem 2 the independence assumption was used to derive the bound (35). It can be replaced by a weaker one,

$$\begin{aligned} &\big|E\{X_{n+m} - Y_{n+m}|(X_1, Y_1), \dots, (X_n, Y_n)\} - E\{X_{n+m} - Y_{n+m}\}\big| \\ &\quad \le \phi(m)M \qquad \text{a.s.} \end{aligned} \tag{41}$$

(c) The condition $\sum_{n\ge 1} n^\gamma \phi(n) < \infty$ on the mixing fuction is satisfied for any geometric function $\phi(n) = \beta\rho^n$ with $0 < \rho < 1$. Also, any stationary ergodic Markov chains, $\{X_n\}_{n\ge 1}$, satisfy mixing conditions of the type

$$|P(A \cap B) - P(A)P(B)| \le \phi(n)P(A) \qquad \text{with } \phi(n) \downarrow 0 \tag{42}$$

for all $k \ge 1$, all $A \in \mathcal{F}_k = \sigma(X_1, \dots, X_k)$, all $B \in \mathcal{F}_{k+n}^\infty = \sigma(X_{k+n}, X_{k+n+1}, \dots)$ and $\phi(n) = \beta\rho^n$ for some $\beta > 0$ and $0 < \rho < 1$ [cf. Roussas and Ioannides (1987)]. As of now, it is unclear whether similar moment bounds, as in Lemma 1, can be shown for Markov chains. For some applications, that includes estimation of ruin probability for risk processes with dependent claim sizes, we refer the reader to Ferreira (2009).

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References

Barbosa, E. G. and Dorea, C. C. Y. (2009). A note on the Lindeberg condition for convergence to stable laws in Mallows distance. *Bernoulli* **15** 922–924.

Bickel, P. J. and Freedman, D. A. (1981). Some asymptotic theory for the bootstrap. *The Annals of Statistics* **9** 1196–1217. MR0630103

Chung, K. L. (1974). *A Course in Probability Theory*, 2nd ed. Academic Press, New York. MR0346858

de la Peña, V. H. (1990). Bounds on the expectation of functions of martingales and sums of positive r.v.'s in terms of norms of sums of independent random variables. *Proceedings of the American Mathematical Society* **108** 233–239. MR0990432

Ferreira, D. B. (2009). Mallows distance for estimating ruin probability of classical risk processes. Ph.D. thesis, Dept. Matematica, Universidade de Brasilia, Brazil (in Portuguese).

Hall, P. and Heyde, C. C. (1980). *Martingale Limit Theory and Its Applications*. Academic Press, New York. MR0624435

Johnson, O. and Samworth, R. (2005). Central limit theorem and convergence to stable laws in Mallows distance. *Bernoulli* **11** 829–845. MR2172843

Mallows, C. L. (1972). A note on asymptotic joint normality. *The Annals of Mathematical Statistics* **43** 508–515. MR0298812

Roussas, G. G. and Ioannides, D. (1987). Moment inequalities for mixing sequences of random variables. *Stochastic Analysis and Applications* **5** 61–120. MR0882698

Samorodnitsky, G. and Taqqu, M. S. (2000). *Stable Non-Gaussian Random Processes*. CRC Press, Boca Raton, FL.

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