



Banach J. Math. Anal. 12 (2018), no. 4, 843–872

<https://doi.org/10.1215/17358787-2017-0056>

ISSN: 1735-8787 (electronic)

<http://projecteuclid.org/bjma>

## PARTIAL ACTIONS OF $C^*$ -QUANTUM GROUPS

FRANZISKA KRAKEN, PAULA QUAST, and THOMAS TIMMERMANN<sup>\*</sup>

Communicated by M. Skeide

**ABSTRACT.** Partial actions of groups on  $C^*$ -algebras and the closely related actions and coactions of Hopf algebras have received much attention in recent decades. They arise naturally as restrictions of their global counterparts to noninvariant subalgebras, and the ambient enveloping global (co)actions have proven useful for the study of associated crossed products. In this article, we introduce the partial coactions of  $C^*$ -bialgebras, focusing on  $C^*$ -quantum groups, and we prove the existence of an enveloping global coaction under mild technical assumptions. We also show that partial coactions of the function algebra of a discrete group correspond to partial actions on direct summands of a  $C^*$ -algebra, and we relate partial coactions of a compact or its dual discrete  $C^*$ -quantum group to partial coactions or partial actions of the dense Hopf subalgebra. As a fundamental example, we associate to every discrete  $C^*$ -quantum group a quantum Bernoulli shift.

### 1. Introduction

Partial actions of groups on spaces and on  $C^*$ -algebras were gradually introduced in [14], [15], and [21], with more recent study of associated crossed products shedding new light on the inner structure of many interesting  $C^*$ -algebras (see [16] for a comprehensive introduction and overview). In the purely algebraic setting, the corresponding notion of a partial action or a partial coaction of a Hopf algebra on an algebra was introduced in [12].

---

Copyright 2018 by the Tusi Mathematical Research Group.

Received Jun. 23, 2017; Accepted Sep. 1, 2017.

First published online Jan. 30, 2018.

\*Corresponding author.

2010 *Mathematics Subject Classification*. Primary 46L55; Secondary 16T20.

*Keywords.*  $C^*$ -algebra, partial action, quantum group, Hopf algebra, globalization.

Naturally, such partial (co)actions arise by restricting global (co)actions to noninvariant subspaces or ideals, and in these cases, all the tools that are available for the study of the global situation can be applied to the study of the partial one. Therefore, it is highly desirable to know, given a partial group action or a partial Hopf algebra (co)action, whether it can be identified with some restriction of a global one, whether there exists a minimal global one—called a *globalization*—and whether the latter, if it exists, can be constructed explicitly. For partial actions of groups on locally compact Hausdorff spaces, such a globalization can always be constructed, but the underlying space need no longer be Hausdorff (see [1], [2]). As a consequence, partial actions of groups on  $C^*$ -algebras cannot always be identified with the restriction of a global action (see [1]). In the purely algebraic setting, partial (co)actions of Hopf algebras always have a globalization (see [4], [5]; see also [3], [6], [13]).

In this article, we introduce partial coactions of  $C^*$ -bialgebras, in particular, of  $C^*$ -quantum groups, on  $C^*$ -algebras, and relate them to the partial (co)actions discussed above. In the case of the function algebra of a discrete group, partial coactions correspond to partial actions of groups where for every group element, the associated domain of definition is a direct summand of the total  $C^*$ -algebra, and these are precisely the partial actions for which existence of a globalization can be proven. If the  $C^*$ -bialgebra is a discrete  $C^*$ -quantum group, then every partial coaction gives rise to a partial action of the Hopf algebra of matrix coefficients of the dual compact quantum group. Finally, in the case of a compact  $C^*$ -quantum group, partial coactions restrict, under a natural condition, to partial coactions of the Hopf algebra of matrix coefficients on a dense subalgebra.

Partial coactions appear naturally as restrictions of ordinary coactions to ideals or, more generally, to  $C^*$ -subalgebras that are weakly invariant in a suitable sense. An identification of a partial coaction with such a restriction will be called a *dilation* of the partial coaction. The main result of this article is the existence and a construction of a *minimal dilation*, also called a *globalization*, under mild assumptions. We follow the approach for coactions of Hopf algebras in [5], but face new technical difficulties. To deal with these, we assume that the  $C^*$ -algebra of the quantum group under consideration has the slice map property, which follows, for example, from nuclearity (see [31]), and is automatic if the quantum group is discrete. Briefly, the main result can be summarized as follows.

**Theorem.** *Let  $(A, \Delta)$  be a  $C^*$ -quantum group, where  $A$  has the slice map property. Then every injective, weakly continuous, regular partial coaction of  $(A, \Delta)$  has a minimal dilation and the latter is unique up to isomorphism.*

Presently, we cannot tell whether this slice map assumption is just convenient or genuinely necessary.

Parts of the results in this article were obtained in the first and second authors' Master's theses. In forthcoming articles, we plan to study crossed products for partial coactions, and partial corepresentations of  $C^*$ -bialgebras.

The article is organized as follows. In Section 2, we recall background on  $C^*$ -quantum groups, strict  $*$ -homomorphisms and the slice map property. In Section 3, we introduce partial coactions of  $C^*$ -bialgebras, and discuss a few desirable properties like weak and strong continuity. In Section 4, we show that partial actions of a discrete group  $\Gamma$  on a  $C^*$ -algebra correspond to counital partial coactions of the function algebra  $C_0(\Gamma)$  if and only if the domains of definition are direct summands of the  $C^*$ -algebra. In Section 5, we relate partial coactions of compact and discrete  $C^*$ -quantum groups to coactions and actions of the Hopf algebra of matrix elements of the compact quantum group. In Section 6, we show how partial coactions arise from global ones by restriction, and discuss the closely related notion of weak or strong morphisms between partial coactions. In Section 7, we construct for every discrete quantum group a quantum Bernoulli shift and obtain, by restriction, a partial coaction that is initial in a suitable sense. In Section 8, we consider the situation where a partial coaction can be identified with the restriction of a global coaction, and study a few preliminary properties of such identifications. Finally, in Section 9, we prove the main result stated above.

## 2. Preliminaries

Let us fix some notation and recall some background.

**2.1. Conventions and notation.** Given a locally compact Hausdorff space  $X$ , we denote by  $C_b(X)$  and  $C_0(X)$  the  $C^*$ -algebra of continuous functions that are bounded or vanish at infinity, respectively. For a subset  $F$  of a normed space  $E$ , we denote by  $[F] \subseteq E$  its closed linear span. Given a  $C^*$ -algebra  $A$ , we denote by  $A^*$  the space of bounded linear functionals on  $A$ , by  $M(A)$  the multiplier algebra, and by  $1_A \in M(A)$  the unit of  $M(A)$ . Given a Hilbert space  $K$ , we denote by  $1_K$  the identity on  $H$ .

Let  $A$  and  $B$  be  $C^*$ -algebras. A  $*$ -homomorphism  $\varphi: A \rightarrow M(B)$  is called *nondegenerate* if  $[\varphi(A)B] = B$ . Each nondegenerate  $*$ -homomorphism  $\varphi: A \rightarrow M(B)$  extends uniquely to a unital  $*$ -homomorphism from  $M(A)$  to  $M(B)$ , which we denote by  $\phi$  again. By a *representation* of a  $C^*$ -algebra  $A$  on a Hilbert space  $H$  we mean a  $*$ -homomorphism  $\pi: A \rightarrow \mathcal{B}(H)$ . All tensor products of  $C^*$ -algebras will be minimal ones.

We write  $\sigma$  for the tensor flip isomorphism  $A \otimes B \rightarrow B \otimes A$ ,  $a \otimes b \mapsto b \otimes a$ .

**2.2.  $C^*$ -bialgebras and  $C^*$ -quantum groups.** A  $C^*$ -bialgebra is a  $C^*$ -algebra  $A$  with a nondegenerate  $*$ -homomorphism  $\Delta: A \rightarrow M(A \otimes A)$ , called the *comultiplication*, that is coassociative in the sense that  $(\Delta \otimes \text{id}_A) \circ \Delta = (\text{id}_A \otimes \Delta) \circ \Delta$ . It satisfies the *cancellation conditions* if

$$[\Delta(A)(1_A \otimes A)] = A \otimes A = [(A \otimes 1_A)\Delta(A)]. \quad (2.1)$$

Given a  $C^*$ -bialgebra  $(A, \Delta)$ , the dual space  $A^*$  is an algebra with respect to the convolution product defined by  $v\omega := (v \otimes \omega) \circ \Delta$ . A *counit* for a  $C^*$ -bialgebra  $(A, \Delta)$  is a character  $\varepsilon$  on  $A$  satisfying  $(\varepsilon \otimes \text{id}_A) \circ \Delta = \text{id}_A = (\text{id}_A \otimes \varepsilon) \circ \Delta$ . If it exists, such a counit is a unit in the algebra  $A^*$  and thus unique. A *morphism* of

$C^*$ -bialgebras  $(A, \Delta_A)$  and  $(B, \Delta_B)$  is a nondegenerate  $*$ -homomorphism  $f: A \rightarrow M(B)$  satisfying  $\Delta_B \circ f = (f \otimes f) \circ \Delta_A$ .

A  $C^*$ -quantum group is a  $C^*$ -bialgebra that arises from a well-behaved multiplicative unitary as follows (see [27], [28], [32]). Suppose that  $H$  is a Hilbert space and that  $W \in \mathcal{B}(H \otimes H)$  is a multiplicative unitary (see [8]) that is manageable or modular (see [32], [27]). Then the spaces

$$A := [(\omega \otimes \text{id}_H)W : \omega \in \mathcal{B}(H)_*] \quad \text{and} \quad \hat{A} := [(\text{id}_H \otimes \omega)W : \omega \in \mathcal{B}(H)_*]$$

are separable, nondegenerate  $C^*$ -subalgebras of  $\mathcal{B}(H)$ , the unitary  $W$  is a multiplier of  $\hat{A} \otimes A \subseteq \mathcal{B}(H \otimes H)$ , and the formulas

$$\Delta(a) = W(a \otimes 1_H)W^*, \quad \hat{\Delta}(\hat{a}) = \sigma(W^*(1_H \otimes \hat{a})W) \quad (2.2)$$

define comultiplications on  $A$  and  $\hat{A}$ , respectively, such that  $(A, \Delta)$  and  $(\hat{A}, \hat{\Delta})$  become  $C^*$ -bialgebras. A  $C^*$ -bialgebra  $(A, \Delta)$  is a  $C^*$ -quantum group if it arises from a modular multiplicative unitary  $W$  as above.

Let  $(A, \Delta)$  be a  $C^*$ -quantum group arising from a unitary  $W$  as above. Denote by  $\Sigma$  the flip on  $H \otimes H$ . Then also the *dual*  $\widehat{W} := \Sigma W^* \Sigma$  of  $W$  is a modular or manageable multiplicative unitary and the associated  $C^*$ -quantum group is  $(\hat{A}, \hat{\Delta})$ . The latter only depends on  $(A, \Delta)$  and not on the choice of  $W$ , and is called the *dual* of  $(A, \Delta)$ . The images of  $W$  and  $\widehat{W}$  in  $M(\hat{A} \otimes A)$  or  $M(A \otimes \hat{A})$ , respectively, do not depend on the choice of  $W$  but only on  $(A, \Delta)$ . We call them the *reduced bicharacters* of  $(A, \Delta)$  and  $(\hat{A}, \hat{\Delta})$  and denote them by  $W^A$  and  $\widehat{W}^A$ , respectively. We will need an *anti-Heisenberg pair* for  $(A, \Delta)$ , which consists of nondegenerate, faithful representations  $\pi$  of  $A$  and  $\hat{\pi}$  of  $\hat{A}$  on a Hilbert space  $K$  such that the unitary

$$V := (\text{id}_A \otimes \hat{\pi})(\widehat{W}^A) \in M(A \otimes \hat{\pi}(\hat{A})), \quad (2.3)$$

regarded as an element of  $M(A \otimes \mathcal{K}(K))$ , satisfies

$$V(1_A \otimes \pi(a))V^* = (\text{id}_A \otimes \pi)\Delta(a) \quad \text{for all } a \in A \quad (2.4)$$

(see [22, Section 3] and [25, Section 3.1]).

Every locally compact quantum group or, more precisely, every reduced  $C^*$ -algebraic quantum group in the sense of Kustermans and Vaes [18], is a  $C^*$ -quantum group.

We will use regularity of  $C^*$ -quantum groups, which was studied for multiplicative unitaries in [8] and for reduced  $C^*$ -algebraic quantum groups in [9, Section 5(b)]. We follow the approach of [24, Definition 5.37] and call a  $C^*$ -quantum group  $(A, \Delta)$  *regular* if its reduced bicharacter satisfies  $[(\hat{A} \otimes 1_A)W^A(1_{\hat{A}} \otimes A)] = \hat{A} \otimes A$  in  $M(\hat{A} \otimes A)$ . This is equivalent to the condition  $[(1_{\hat{A}} \otimes A)W^A(\hat{A} \otimes 1_A)] = \hat{A} \otimes A$  (see [24, proof of Corollary 5.39]). For the unitary (2.3), this translates into

$$[(1_A \otimes \hat{\pi}(\hat{A}))V(A \otimes 1_{\hat{\pi}(\hat{A})})] = A \otimes \hat{\pi}(\hat{A}) \quad \text{in } M(A \otimes \hat{\pi}(\hat{A})). \quad (2.5)$$

In [24], this condition is referred to as *weak regularity*. However, every reduced  $C^*$ -algebraic quantum group  $(A, \Delta)$  is regular in the above sense if and only if it is regular in the sense of [9, Section 5(b)]. One implication is contained in [8, Proposition 3.6], and the other follows easily from [9, Proposition 5.6].

A *compact  $C^*$ -quantum group* is, by definition, a unital  $C^*$ -bialgebra  $\mathbb{G} = (A, \Delta)$  that satisfies the cancellation conditions, and is indeed a weakly regular  $C^*$ -quantum group (see [33]). Associated to such a compact quantum group is a rigid  $C^*$ -tensor category of unitary finite-dimensional corepresentations (see [23]). We denote by  $\text{Irr}(\mathbb{G})$  the equivalence classes of irreducible corepresentations. Their matrix elements span a dense Hopf subalgebra  $\mathcal{O}(\mathbb{G})$ . The dual  $(\hat{A}, \hat{\Delta})$  is called a *discrete  $C^*$ -quantum group*, and the underlying  $C^*$ -algebra  $\hat{A}$  is a direct sum of matrix algebras, indexed by  $\text{Irr}(\mathbb{G})$ . We also denote the underlying  $C^*$ -algebra  $\hat{A}$  of  $\hat{\mathbb{G}}$  by  $C_0(\hat{\mathbb{G}})$ .

**2.3. Strict  $*$ -homomorphisms of  $C^*$ -algebras.** Recall from [19, Section 5, Corollary 5.7] that a  $*$ -homomorphism  $\pi: B \rightarrow M(C)$  is *strict* if it is strictly continuous on the unit ball, and that in that case, it extends to a  $*$ -homomorphism  $M(B) \rightarrow M(C)$  that is strictly continuous on the unit ball. We denote this extension by  $\pi$  again. Using this extension, we define the composition of strict  $*$ -homomorphisms, which evidently is strict again. Hence,  $C^*$ -algebras with strict  $*$ -homomorphisms form a category.

Recall that a *corner* of a  $C^*$ -algebra  $B$  is a  $C^*$ -subalgebra of the form  $pBp$  for some projection  $p \in M(B)$ .

Strict  $*$ -homomorphisms are just nondegenerate  $*$ -homomorphisms in the usual sense from the domain to a corner of the target. Indeed, if  $\pi: B \rightarrow M(C)$  is a strict  $*$ -homomorphism, then  $p := \pi(1_B) \in M(C)$  is a projection,  $pCp \subseteq C$  is a corner, and the corestriction  $\pi: B \rightarrow M(pCp)$  is nondegenerate. Conversely, given a corner  $C_0 \subseteq C$  and a nondegenerate  $*$ -homomorphism  $\pi: B \rightarrow M(C_0)$ , we get a strict extension  $M(B) \rightarrow M(C_0)$ , a natural strict map  $M(C_0) \rightarrow M(C)$  [11, II.7.3.14], and the composition is a strict  $*$ -homomorphism.

This description of strict  $*$ -homomorphisms immediately implies that the minimal tensor product of strict morphisms is a strict morphism again, and that an embedding of  $C^*$ -algebras  $B \hookrightarrow C$  is a strict  $*$ -homomorphism if and only if  $B$  is a nondegenerate  $C^*$ -subalgebra of a corner of  $C$ . We will call such embeddings *strict*.

In the commutative case, partial morphisms correspond to partially defined continuous maps with a clopen domain of definition. Indeed, let  $X$  and  $Y$  be locally compact Hausdorff spaces. Then every continuous map  $F$  from a clopen subset  $D \subseteq Y$  to  $X$  induces a strict  $*$ -homomorphism  $F^*: C_0(X) \rightarrow M(C_0(Y)) = C_b(Y)$  defined by

$$(F^*(f))(y) = 0 \quad \text{if } y \notin D, \quad (F^*(f))(y) = f(F(y)) \quad \text{if } y \in D.$$

Conversely, if  $\pi: C_0(X) \rightarrow M(C_0(Y))$  is a strict  $*$ -homomorphism, then  $\pi(1_X)$  is the characteristic function of a clopen subset  $D \subseteq Y$  and the corestriction  $\pi: C_0(X) \rightarrow M(C_0(D))$  is the pullback along a continuous function  $F: D \rightarrow X$ .

**2.4. The slice map property.** In Sections 8 and 9, we need the following property. A  $C^*$ -algebra  $A$  has the *slice map property* if, for every  $C^*$ -algebra  $B$  and every  $C^*$ -subalgebra  $C \subseteq B$ , every  $x \in B \otimes A$  satisfying  $(\text{id} \otimes \omega)(x) \in C$  for all  $\omega \in A^*$  lies in  $C \otimes A$  (see [31]). This property holds if  $A$  is nuclear, or, more generally, if  $A$  has the completely bounded approximation property or the strong operator approximation property (see [30] for a survey). In particular, this condition holds whenever  $(A, \Delta)$  is a discrete quantum group, or, more generally, whenever  $(A, \Delta)$  is a reduced  $C^*$ -algebraic quantum group whose dual is amenable (see [10, Theorem 3.3]).

### 3. Partial coactions of $C^*$ -bialgebras

The definition of a partial coaction given for Hopf algebras in [12] carries over to  $C^*$ -bialgebras as follows.

*Definition 3.1.* A *partial coaction* of a  $C^*$ -bialgebra  $(A, \Delta)$  on a  $C^*$ -algebra  $C$  is a strict  $*$ -homomorphism  $\delta: C \rightarrow M(C \otimes A)$  satisfying the following conditions:

- (1)  $\delta(C)(1_C \otimes A) \subseteq C \otimes A$ ;
- (2)  $\delta$  is partially coassociative in the sense that

$$(\delta \otimes \text{id}_A)\delta(c) = (\delta(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)\delta(c) \quad (3.1)$$

for all  $c \in C$ , or, equivalently, the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\delta} & M(C \otimes A) \\ \downarrow \delta & & \downarrow \delta \otimes \text{id} \\ M(C \otimes A) & \xrightarrow{(\delta(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)\delta} & M(C \otimes A \otimes A) \end{array} \quad (3.2)$$

Let  $\delta$  be a partial coaction of a  $C^*$ -bialgebra  $(A, \Delta)$  on a  $C^*$ -algebra  $C$ . For every functional  $\omega \in A^*$  and every multiplier  $T \in M(C)$ , we define a multiplier

$$\omega \triangleright T := (\text{id}_C \otimes \omega)\delta(T) \in M(C),$$

where we use the fact that we can write  $\omega = av$  or  $\omega = v'a'$  with  $a, a' \in A$  and  $v, v' \in A^*$  by Cohen's factorization theorem.

Let  $c \in C$  and  $\omega \in A^*$ . Then conditions (1) and (2) in Definition 3.1 imply that  $\omega \triangleright c \in C$  and that

$$\begin{aligned} \delta(\omega \triangleright c) &= (\text{id}_C \otimes \text{id}_A \otimes \omega)(\delta \otimes \text{id}_A)\delta(c) \\ &= \delta(1_C)(\text{id}_C \otimes \text{id}_A \otimes \omega)(\text{id}_C \otimes \Delta)\delta(c). \end{aligned} \quad (3.3)$$

In particular, for every character  $\chi \in A^*$ ,

$$\chi \triangleright (\omega \triangleright c) = (\chi \triangleright 1_C)(\text{id}_C \otimes (\chi \otimes \omega)\Delta)\delta(c) = (\chi \triangleright 1_C)(\chi \omega \triangleright c). \quad (3.4)$$

The following conditions on a partial coaction are straightforward generalizations of the corresponding conditions on coactions, and they will play an equally important role in the following.

*Definition 3.2.* We say that a partial coaction  $\delta$  of a  $C^*$ -bialgebra  $(A, \Delta)$  on a  $C^*$ -algebra  $C$

- satisfies the *Podleś condition* if  $[\delta(C)(1_C \otimes A)] = [\delta(1_C)(C \otimes A)]$ ,
- is *weakly continuous* if  $[A^* \triangleright C] = C$ ,
- is *counital* if  $(A, \Delta)$  has a counit  $\varepsilon$  and  $(\text{id}_C \otimes \varepsilon) \circ \delta = \text{id}$ .

*Remark 3.3.* If  $\delta$  is a partial coaction as above and  $X \subseteq A^*$  is a subset that separates the points of  $A$ , then a standard application of the Hahn–Banach theorem shows that  $[X \triangleright C] = [A^* \triangleright C]$ .

Every counital partial coaction evidently is weakly continuous. A coaction satisfying the Podleś condition is automatically weakly continuous, and is usually called (*strongly*) *continuous*. For partial coactions, this implication no longer holds in general, and so we avoid this terminology.

**Lemma 3.4.** *Let  $\delta$  be a partial coaction of a  $C^*$ -bialgebra  $(A, \Delta)$  on a  $C^*$ -algebra  $C$  that satisfies the Podleś condition. Then*

- (1)  $\delta$  is weakly continuous if and only if  $[(A^* \triangleright 1_C)C] = C$ ,
- (2)  $\delta$  is counital if and only if  $(A, \Delta)$  has a counit  $\varepsilon$  and  $\varepsilon \triangleright 1_C = 1_C$ .

*Proof.* (1) By assumption, the closed linear span of all elements of the form  $a\omega \triangleright c = (\text{id}_C \otimes \omega)(\delta(c)(1_C \otimes a))$ , where  $\omega \in A^*$ ,  $a \in A$ , and  $c \in C$ , is equal to the closed linear span of all elements of the form  $(\text{id}_C \otimes \omega)(\delta(1_C)(c \otimes a)) = (a\omega \triangleright 1_C)c$ . Now, use Cohen’s factorization theorem.

(2) If  $\varepsilon \triangleright 1_C = 1_C$ , then elements of the form  $a\varepsilon \triangleright c$ , where  $a \in A$  and  $c \in C$ , are linearly dense in  $C$ , and for every  $\omega \in A^*$  and  $c \in C$ , (3.4) implies that  $\varepsilon \triangleright (\omega \triangleright c) = 1_C \cdot (\omega \triangleright c)$ .  $\square$

For regular reduced  $C^*$ -algebraic quantum groups, weakly continuous coactions automatically satisfy the Podleś condition (see [9, Proposition 5.8]). More generally, we show the following.

**Proposition 3.5.** *Let  $(A, \Delta)$  be a regular  $C^*$ -quantum group. Then every weakly continuous partial coaction of  $(A, \Delta)$  satisfies the Podleś condition.*

*Proof.* We proceed in a manner similar to that in the proof of [9, Proposition 5.8], and we use an anti-Heisenberg pair  $(\pi, \hat{\pi})$  for  $(A, \Delta)$  on some Hilbert space  $K$  and the unitary  $V$  in (2.3).

Let  $\delta$  be a weakly continuous partial coaction of  $(A, \Delta)$  on a  $C^*$ -algebra  $C$ . By (3.3) and Remark 3.3,

$$\begin{aligned} & [\delta(C)(1_C \otimes A)] \\ &= [\delta(\omega \circ \pi \triangleright C)(1_C \otimes A) : \omega \in \mathcal{B}(K)_*] \\ &= [\delta(1_C) \cdot (\text{id}_C \otimes \text{id}_A \otimes \omega \circ \pi)((\text{id}_C \otimes \Delta)(\delta(C))) \cdot (1_C \otimes A) : \omega \in \mathcal{B}(K)_*]. \end{aligned}$$

To shorten the notation, let  $\delta_\pi := (\text{id}_C \otimes \pi) \circ \delta$ . We use the relations (2.4), (2.5), and  $[\hat{\pi}(\hat{A})\mathcal{B}(K)_*] = \mathcal{B}(K)_*$ , and we find that

$$\begin{aligned} & [(\text{id}_C \otimes \text{id}_A \otimes \omega \circ \pi)((\text{id}_C \otimes \Delta)(\delta(C))(1_C \otimes A \otimes 1_A)) : \omega \in \mathcal{B}(K)_*] \\ &= [(\text{id}_C \otimes \text{id}_A \otimes \omega)(V_{23}\delta_\pi(C)_{13}V_{23}^*(A \otimes \hat{\pi}(\hat{A}))_{23}) : \omega \in \mathcal{B}(K)_*] \\ &= [(\text{id}_C \otimes \text{id}_A \otimes \omega)(V_{23}\delta_\pi(C)_{13}(A \otimes \hat{\pi}(\hat{A}))_{23}) : \omega \in \mathcal{B}(K)_*] \end{aligned}$$

$$\begin{aligned}
&= [(\text{id}_C \otimes \text{id}_A \otimes \omega)((1_A \otimes \hat{\pi}(\hat{A}))_{23} V_{23} (A \otimes 1_K)_{23} \delta_\pi(C)_{13}) : \omega \in \mathcal{B}(K)_*] \\
&= [(\text{id}_C \otimes \text{id}_A \otimes \omega)((A \otimes \hat{\pi}(\hat{A}))_{23} \delta_\pi(C)_{13}) : \omega \in \mathcal{B}(K)_*] \\
&= [A^* \triangleright C] \otimes A,
\end{aligned}$$

whence  $[\delta(C)(1_C \otimes A)] = [\delta(1_C)(C \otimes A)]$ .  $\square$

Partial coactions on  $\mathbb{C}$  correspond to certain projections, as shown in the following lemma.

**Lemma 3.6.** *Partial coactions of a  $C^*$ -bialgebra  $(A, \Delta)$  on  $\mathbb{C}$  correspond bijectively to projections  $p \in M(A)$  satisfying*

$$(p \otimes 1_A)\Delta(p) = p \otimes p. \quad (3.5)$$

*Proof.* Projections  $p \in M(A)$  correspond to strict  $*$ -homomorphisms  $\delta: \mathbb{C} \rightarrow M(\mathbb{C} \otimes A) \cong M(A)$  via  $p = \delta(1)$ , and under this correspondence,  $(\delta \otimes \text{id}_A)\delta(\lambda) = \lambda \otimes p \otimes p$  and  $(\delta(1) \otimes 1_A)(\text{id}_{\mathbb{C}} \otimes \Delta)(\delta(\lambda)) = \lambda \otimes (p \otimes 1_A)\Delta(p)$ .  $\square$

Note that if  $(A, \Delta)$  is cocommutative, for example, if  $A = C^*(G)$  or  $A = C_r^*(G)$  for a locally compact group  $G$ , then (3.5) just means that  $p$  is *group-like* in the sense that  $(p \otimes 1_A)\Delta(p) = p \otimes p = (1_A \otimes p)\Delta(p)$ . Group-like projections were also studied in connection with idempotent states (see [17, Section 2]). Elementary examples related to groups are as follows.

*Example 3.7.* Let  $G$  be a locally compact group.

- (1) Consider the  $C^*$ -bialgebra  $(C_0(G), \Delta)$ . A projection  $p \in M(C_0(G))$  is just the characteristic function of a clopen subset  $H \subseteq G$ , and it satisfies (3.5) if and only if  $p(g)p(gg') = p(g)p(g')$  for all  $g, g' \in G$ , that is, if and only if  $H \subseteq G$  is a subgroup. Thus, partial coactions of  $(C_0(G), \Delta)$  on  $\mathbb{C}$  correspond to open subgroups of  $G$ .
- (2) Consider the reduced group  $C^*$ -bialgebra  $(C_r^*(G), \Delta)$ . For every finite normal subgroup  $N \subseteq G$ , the sum  $p = \sum_{g \in N} \lambda_g$  is a central projection in  $M(C_r^*(G))$  satisfying (3.5), where  $\lambda_g$  denotes the left translation by  $g \in G$ . (More information on group-like projections in  $C_r^*(G)$  and  $C^*(G)$  can be found in [20, Proposition 7.6] and [29].)

Every central projection satisfying (3.5) gives rise to a quotient  $C^*$ -bialgebra  $(A_p, \Delta_p)$  of  $(A, \Delta)$  whose coactions can be regarded as partial coactions of  $(A, \Delta)$ .

**Lemma 3.8.** *Suppose that  $(A, \Delta)$  is a  $C^*$ -bialgebra with a central projection  $p \in M(A)$  satisfying (3.5). Let  $A_p = pA$ , and define  $\Delta_p: A_p \rightarrow M(A_p \otimes A_p)$  by  $a \mapsto (p \otimes p)\Delta(a)$ . Then  $(A_p, \Delta_p)$  is a  $C^*$ -bialgebra, the map  $A \rightarrow A_p$ ,  $a \mapsto pa$ , is a morphism of  $C^*$ -bialgebras, and every coaction of  $(A_p, \Delta_p)$  can be regarded as a partial coaction of  $(A, \Delta)$ .*

*Proof.* All of these assertions are easily verified. For example, if  $\delta$  is a coaction of  $(A_p, \Delta_p)$  on a  $C^*$ -algebra  $C$ , then for all  $c \in C$ ,

$$\begin{aligned}
(\delta(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)\delta(c) &= (1_C \otimes p \otimes 1_A)(1_C \otimes \Delta)((1_C \otimes p)\delta(c)) \\
&= (1_C \otimes p \otimes p)(\text{id}_C \otimes \Delta)\delta(c)
\end{aligned}$$

$$\begin{aligned}
&= (1_C \otimes \Delta_p)\delta(c) \\
&= (\text{id}_C \otimes \Delta)\delta(c).
\end{aligned}$$

□

*Example 3.9.* Let  $\mathbb{G} = (A, \Delta)$  be a discrete quantum group, so that  $A$  is a  $c_0$ -sum of matrix algebras indexed by  $\text{Irr}(\hat{\mathbb{G}})$ . Consider a central projection  $p \in M(A)$  supported on  $\mathcal{J} \subseteq \text{Irr}(\hat{\mathbb{G}})$ . Then  $(p \otimes 1)\Delta(p) = p \otimes p$  if and only if the following condition holds:

$$\text{If } \alpha \in \mathcal{J}, \beta, \gamma \in \text{Irr}(\hat{\mathbb{G}}) \text{ and } \alpha \otimes \beta \text{ contains } \gamma, \text{ then } \beta \in \mathcal{J} \text{ if and only if } \gamma \in \mathcal{J}. \quad (3.6)$$

If  $(A_p, \Delta_p)$  is a discrete quantum subgroup of  $(A, \Delta)$ , then  $\mathcal{J}$  is closed under taking duals and summands of tensor products, and then Frobenius duality implies (3.6). Conversely, suppose that (3.6) holds. Taking  $\gamma = \alpha$ , we see that  $\mathcal{J}$  contains the trivial representation, and taking this for  $\gamma$ , we see that  $\mathcal{J}$  contains the dual of  $\alpha$ . Thus, finite sums of representations in  $\mathcal{J}$  form a rigid tensor subcategory, and  $(A_p, \Delta_p)$  is a discrete quantum subgroup of  $(A, \Delta)$ .

#### 4. The relation to partial actions of groups

We now relate partial actions of a (discrete) group  $\Gamma$  to counital partial coactions of the  $C^*$ -bialgebra  $C_0(\Gamma)$ . Recall that a *partial action* of  $\Gamma$  on a  $C^*$ -algebra  $C$  is a family  $(D_g)_{g \in \Gamma}$  of closed ideals of  $C$  together with a family  $(\theta_g)_{g \in \Gamma}$  of isomorphisms  $\theta_g: D_{g^{-1}} \rightarrow D_g$  such that (see [16], [21])

- (G1)  $D_e = C$  and  $\theta_e = \text{id}_C$ , where  $e \in \Gamma$  denotes the unit,
- (G2)  $\theta_{g^{-1}}\theta_g\theta_h = \theta_{g^{-1}}\theta_{gh}$  and  $\theta_g\theta_h\theta_{h^{-1}} = \theta_{gh}\theta_{h^{-1}}$  for all  $g, h \in \Gamma$  as partially defined maps.

We show that partial coactions of  $C_0(\Gamma)$  correspond to partial actions of  $\Gamma$  as above, where each ideal  $D_g$  is a direct summand, and adopt the following terminology.

*Definition 4.1.* A *disconnected partial action* of  $\Gamma$  on a  $C^*$ -algebra  $C$  is given by a family  $(p_g)_{g \in \Gamma}$  of central projections in  $M(C)$  and a family  $(\theta_g)_{g \in \Gamma}$  of isomorphisms  $\theta_g: p_{g^{-1}}C \rightarrow p_gC$  such that  $((p_gC)_{g \in \Gamma}, (\theta_g)_{g \in \Gamma})$  is a partial action.

*Remark 4.2.*

- (1) Let  $X$  be a locally compact Hausdorff space. Then partial actions of  $\Gamma$  on  $C_0(X)$  correspond bijectively to partial actions of  $\Gamma$  on  $X$  (see [16, Corollary 11.6]), and a partial action on  $C_0(X)$  is disconnected if and only if for every group element  $g \in \Gamma$ , the domain of definition of its action on  $X$  is not only open but also closed. This condition also implies that the partial action on  $X$  admits a globalization that is Hausdorff (see [16, Proposition 5.7]).
- (2) A partial action of  $\Gamma$  on an algebra  $C$  admits a globalization if and only if for every group element  $g \in \Gamma$ , its domain of definition is not just a two-sided ideal of  $C$  but also unital, that is, a direct summand (see [16, Theorem 6.13]).

We denote by  $C_b(\Gamma; C)$  the  $C^*$ -algebra of norm-bounded  $C$ -valued functions on  $\Gamma$ , and we identify this  $C^*$ -algebra with a subalgebra of  $M(C \otimes C_0(\Gamma))$  in the canonical way. For each  $g \in \Gamma$ , we denote by  $\text{ev}_g \in C_0(\Gamma)^*$  the evaluation at  $g$ .

**Proposition 4.3.** *Let  $\Gamma$  be a group, and let  $C$  be a  $C^*$ -algebra.*

(1) *Let  $\delta$  be a counital partial coaction of  $C_0(\Gamma)$  on  $C$ . Then the projections*

$$p_g := \text{ev}_g \triangleright 1_C$$

*are central and the maps  $\theta_g: p_{g^{-1}}C \rightarrow p_gC$  given by*

$$\theta_g(c) := \text{ev}_g \triangleright c$$

*form a disconnected partial action of  $\Gamma$  on  $C$ .*

(2) *Let  $((p_g)_{g \in \Gamma}, (\theta_g)_{g \in \Gamma})$  be a disconnected partial action of  $\Gamma$  on  $C$ . Then the map*

$$\delta: C \rightarrow C_b(\Gamma; C) \hookrightarrow M(C \otimes C_0(\Gamma))$$

*defined by*

$$(\delta(c))(g) := \theta_g(p_{g^{-1}}c) \quad (c \in C, g \in \Gamma)$$

*is a counital partial coaction of  $C_0(\Gamma)$  on  $C$ .*

*Proof.* (1) For each  $g \in \Gamma$ , the map  $\Theta_g: C \rightarrow C$  given by  $c \mapsto \text{ev}_g \triangleright c$  is a strict endomorphism. Since  $\delta$  is counital,  $\Theta_e$  is the identity on  $C$ . Let  $g, h \in \Gamma$ . Then by (3.4),

$$\Theta_g(\Theta_h(c)) = (\text{ev}_g \triangleright 1_C)(\text{ev}_g \text{ev}_h \triangleright c) = p_g \Theta_{gh}(c); \quad (4.1)$$

in particular,

$$\Theta_g(p_h) = p_g p_{gh}, \quad \Theta_g(\Theta_{g^{-1}}(c)) = p_g c, \quad \Theta_{g^{-1}}(\Theta_g(c)) = p_{g^{-1}}c. \quad (4.2)$$

Since  $\Theta_g \circ \Theta_{g^{-1}}$  is a  $*$ -homomorphism, the second equation implies that  $p_g c = c p_g$  for all  $c \in C$ ; that is,  $p_g$  is central and  $D_g := p_g C$  is a direct summand of  $C$ . The second and third equations imply that  $\Theta_g$  and  $\Theta_{g^{-1}}$  restrict to mutually inverse isomorphisms

$$D_{g^{-1}} \xrightleftharpoons[\theta_{g^{-1}}]{\theta_g} D_g.$$

It remains to show that  $\theta_{g^{-1}}\theta_{gh} = \theta_{g^{-1}}\theta_g\theta_h$ . But the relations (4.1) and (4.2) imply that

$$(\Theta_{g^{-1}} \circ \Theta_{gh})(c) = p_{g^{-1}}\Theta_{gh}(c) = (\Theta_{g^{-1}} \circ \Theta_g \circ \Theta_h)(c)$$

for all  $c \in C$ , and that the compositions  $\theta_{g^{-1}}\theta_{gh}$  and  $\theta_{g^{-1}}\theta_g\theta_h$  have the domain

$$\Theta_{h^{-1}g^{-1}}(p_g)C = p_{h^{-1}g^{-1}}p_{h^{-1}}C = \Theta_{h^{-1}}(p_{g^{-1}})C.$$

(2) For each  $g \in \Gamma$ , denote by  $\delta_g \in C_0(\Gamma)$  the characteristic function of  $\{g\} \subset \Gamma$ . Then

$$\delta(c)(1_C \otimes \Delta_g) = \theta_g(p_g c) \otimes \Delta_g \quad (g \in \Gamma, c \in C).$$

We conclude that  $\delta(C)(1_C \otimes C_0(\Gamma))$  is contained in  $C \otimes C_0(\Gamma)$ , and that  $\delta$  corestricts to a nondegenerate  $*$ -homomorphism from  $C$  to  $q(C \otimes C_0(\Gamma))$ , where  $q = \sum_{g \in \Gamma} p_g \otimes \Delta_g$ , so that  $\delta$  is strict. To verify that  $\delta$  is partially coassociative, it suffices to check that for all  $g, h \in \Gamma$  and  $c \in C$ , the element

$$(\text{id}_C \otimes \text{ev}_g \otimes \text{ev}_h)(\delta \otimes \text{id}_A)\delta(c) = \theta_g(p_{g^{-1}}\theta_h(p_{h^{-1}}c))$$

is equal to the element

$$(\text{id}_C \otimes \text{ev}_g \otimes \text{ev}_h)((\delta(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)\delta(c)) = \theta_g(p_{g^{-1}})\theta_{gh}(p_{h^{-1}g^{-1}}c),$$

and this follows easily from the definition of a partial action.  $\square$

The following example shows that the correspondence between partial coactions of  $C_0(\Gamma)$  and partial actions of  $\Gamma$  does not easily extend from groups to inverse semigroups.

*Example 4.4.* Denote by  $\Gamma$  the inverse semigroup consisting of the  $2 \times 2$  matrices

$$0, \quad v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad v^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad vv^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad v^*v = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

with matrix multiplication as composition. Then  $C(\Gamma)$  is a  $C^*$ -bialgebra with respect to the transpose  $\Delta$  of the multiplication. For  $x \in \Gamma$ , define  $\delta_x \in C(\Gamma)$  by  $y \mapsto \delta_{x,y}$ . Then, for example,

$$\Delta(\delta_{v^*v}) = \delta_{v^*} \otimes \Delta_v + \delta_{v^*v} \otimes \Delta_{v^*v}, \quad \Delta(\delta_v) = \delta_{vv^*} \otimes \Delta_v + \delta_v \otimes \Delta_{v^*v}.$$

Now, the  $*$ -homomorphism

$$\delta: \mathbb{C}^2 \rightarrow \mathbb{C}^2 \otimes C(\Gamma), \quad (\alpha, \beta) \mapsto (\alpha, 0) \otimes \Delta_{v^*v} + (0, \alpha) \otimes \Delta_v$$

is a partial coaction. Indeed, for all  $\alpha, \beta \in \mathbb{C}$ ,

$$(\delta \otimes \text{id}_{C(\Gamma)})\delta((\alpha, \beta)) = (\alpha, 0) \otimes \Delta_{v^*v} \otimes \Delta_{v^*v} + (0, \alpha) \otimes \Delta_v \otimes \Delta_{v^*v}$$

is equal to the product of

$$\delta((1, 0)) \otimes 1_{C(\Gamma)} = (1, 0) \otimes \Delta_{v^*v} \otimes 1_{C(\Gamma)} + (0, 1) \otimes \Delta_v \otimes 1_{C(\Gamma)}$$

with

$$\begin{aligned} (\text{id}_{\mathbb{C}^2} \otimes \Delta)\delta((\alpha, \beta)) &= (\alpha, 0) \otimes (\delta_{v^*v} \otimes \Delta_{v^*} + \delta_{v^*} \otimes \Delta_{vv^*}) \\ &\quad + (0, \alpha) \otimes (\delta_v \otimes \Delta_{v^*v} + \delta_{vv^*} \otimes \Delta_v). \end{aligned}$$

But the maps  $\Theta_w := (\text{id} \otimes \text{ev}_w) \circ \delta$ , where  $w \in \Gamma$ , are given by

$$\Theta_0 = \Theta_{v^*} = \Theta_{vv^*} = 0, \quad \Theta_v((\alpha, \beta)) = (0, \alpha), \quad \Theta_{v^*v}((\alpha, \beta)) = (\alpha, 0);$$

in particular,  $\Theta_v \Theta_{v^*} \Theta_v = 0$  and  $\Theta_v \Theta_{v^*v} = \Theta_v$ .

## 5. Partial coactions of $C^*$ -quantum groups that are discrete or compact

Let  $\mathbb{G} = (A, \Delta)$  be a compact  $C^*$ -quantum group, and denote by  $\mathcal{O}(\mathbb{G}) \subseteq A$  the dense Hopf subalgebra of matrix elements of finite-dimensional corepresentations. We now relate partial (co)actions of  $\mathbb{G}$  and of the discrete dual  $\hat{\mathbb{G}}$  to partial coactions and partial actions of the Hopf algebra  $\mathcal{O}(\mathbb{G})$ , respectively. Note that  $(A, \Delta)$  and  $(\hat{A}, \hat{\Delta})$  are regular, so that weakly continuous partial coactions automatically satisfy the Podleś condition by Proposition 3.5.

Recall that a *partial action* of a Hopf algebra  $H$  on a unital algebra  $C$  is a map

$$H \otimes C \rightarrow C, \quad h \otimes c \mapsto h \triangleright c$$

satisfying the following conditions (see [12]):

$$(H1) \quad 1_H \triangleright c = c \text{ for all } c \in C,$$

$$(H2) \quad h \triangleright (cd) = (h_{(1)} \triangleright c)(h_{(2)} \triangleright d) \text{ for all } h \in H \text{ and } c, d \in C,$$

$$(H3) \quad h \triangleright (k \triangleright c) = (h_{(1)} \triangleright 1_C)(h_{(2)} k \triangleright c) \text{ for all } h, k \in H \text{ and } c \in C;$$

and that such a partial action is *symmetric* if additionally (see [7])

$$(H4) \quad h \triangleright (k \triangleright c) = (h_{(1)} k \triangleright c)(h_{(2)} \triangleright 1_C) \text{ for all } h, k \in H \text{ and } c \in C.$$

If additionally  $h \triangleright 1_C = \varepsilon(h)$  for all  $h \in H$ , we have a genuine action; in that case, (H3) and (H4) reduce to  $h \triangleright (k \triangleright c) = hk \triangleright c$ .

Recall that the  $C^*$ -algebra  $\hat{A}$  of the discrete  $C^*$ -quantum group  $\hat{\mathbb{G}}$  is a  $c_0$ -direct sum of matrix algebras  $\hat{A}_\alpha$  indexed by  $\alpha \in \text{Irr}(\mathbb{G})$ . The Hopf algebra  $\mathcal{O}(\mathbb{G})$  can be identified with the subspace of all functionals  $\omega \in \hat{A}^*$  that vanish on  $\hat{A}_\alpha$  for all but finitely many  $\alpha \in \text{Irr}(\mathbb{G})$ , and then

$$\Delta(\omega)(\hat{a} \otimes \hat{b}) = \omega(\hat{a}\hat{b}) \quad \text{and} \quad (v\omega)(\hat{a}) = (v \otimes \omega)(\hat{a})$$

for all  $v, \omega \in \mathcal{O}(\mathbb{G})$  and  $\hat{a}, \hat{b} \in \hat{A}$ .

**Theorem 5.1.** *Let  $\mathbb{G} = (A, \Delta)$  be a compact quantum group, and let  $\delta$  be a counital partial coaction of the discrete dual  $\hat{\mathbb{G}} = (\hat{A}, \hat{\Delta})$  on a unital  $C^*$ -algebra  $C$ . Then the formula*

$$v \otimes c \mapsto v \triangleright c = (\text{id}_C \otimes v)(\delta(c)) \quad (v \in \mathcal{O}(\mathbb{G}), c \in C)$$

*defines a symmetric partial action of the Hopf algebra  $\mathcal{O}(\mathbb{G})$  on  $C$ .*

*Proof.* Condition (H1) holds because the unit of  $\mathcal{O}(\mathbb{G})$ , regarded as a functional on  $\hat{A}$ , is the counit. Let  $v, \omega \in \mathcal{O}(\mathbb{G})$  and  $c, d \in C$ . Choose central projections  $p, q \in \hat{A}$  such that  $v(p\hat{a}) = v(\hat{a})$ ,  $\omega(\hat{a}) = \omega(q\hat{a})$ , and  $v_{(1)}(\hat{a})v_{(2)}(\hat{b}) = v_{(1)}(p\hat{a})v_{(2)}(q\hat{b})$  for all  $\hat{a}, \hat{b} \in \hat{A}$ . Then

$$v \triangleright cd = (\text{id}_C \otimes v)((1_C \otimes p)\delta(c)\delta(d)(1_C \otimes p)).$$

Since  $(1_C \otimes p)\delta(c)$  and  $\delta(d)(1_C \otimes p)$  are contained in the tensor product of  $C$  with the finite-dimensional  $C^*$ -algebra  $p\hat{A} + q\hat{A}$ , this expression is equal to

$$(\text{id} \otimes v_{(1)})((1 \otimes p)\delta(c)) \cdot (\text{id} \otimes v_{(2)})(\delta(d)(1 \otimes p)) = (v_{(1)} \triangleright c)(v_{(2)} \triangleright d).$$

Thus, condition (H2) is satisfied. Likewise,

$$\begin{aligned} v \triangleright (\omega \triangleright c) &= (\text{id}_C \otimes v \otimes \omega)((\delta \otimes \text{id}_{\hat{A}})\delta(c)) \\ &= (\text{id}_C \otimes v \otimes \omega)((1_C \otimes p \otimes q)(\delta(1_C) \otimes 1_{\hat{A}})(\text{id}_C \otimes \hat{\Delta})(\delta(c))), \end{aligned}$$

and a similar argument as above shows that this expression is equal to

$$(\text{id}_C \otimes v_{(1)})(\delta(1_C))(\text{id}_C \otimes (v_{(2)} \otimes \omega) \circ \hat{\Delta})(\delta(c)) = (v_{(1)} \triangleright 1_C) \cdot (v_{(2)} \omega \triangleright c).$$

Therefore, condition (H3) holds as well, and a similar argument proves (H4).  $\square$

Next, we consider partial coactions of the compact  $C^*$ -quantum group  $(A, \Delta)$ , and relate them to partial coactions of the Hopf algebra  $\mathcal{O}(\mathbb{G})$ . Recall that a *partial coaction* of a Hopf algebra  $H$  on a unital algebra  $C$  is a homomorphism

$$\delta: C \mapsto C \otimes H$$

satisfying the following conditions (see [12]):

- (CH1)  $(\delta \otimes \text{id}_H)(\delta(c)) = (\delta(1_C) \otimes 1_H) \cdot (\text{id}_C \otimes \Delta_H)(\delta(c))$  for all  $c \in C$ , and
- (CH2)  $(\text{id}_C \otimes \varepsilon_H)(\delta_0(c)) = c$  for all  $c \in C$ .

**Theorem 5.2.** *Let  $\delta$  be a partial coaction of a compact  $C^*$ -quantum group  $\mathbb{G} = (A, \Delta)$  on a unital  $C^*$ -algebra  $C$ . Then the following conditions are equivalent:*

- (1)  $\delta$  is weakly continuous,  $\delta(1_C)$  lies in the algebraic tensor product  $C \otimes \mathcal{O}(\mathbb{G})$ , and  $(\text{id}_C \otimes \varepsilon)(\delta(1_C)) = 1_C$ , where  $\varepsilon$  denotes the counit of  $\mathcal{O}(\mathbb{G})$ ;
- (2)  $\delta$  restricts to a partial coaction of  $\mathcal{O}(\mathbb{G})$  on a unital dense  $*$ -subalgebra  $C_0$  of  $C$ .

*Proof.* Denote by  $\mathcal{O}(\hat{\mathbb{G}}) \subseteq \hat{A}$  the algebraic direct sum of the matrix algebras  $\hat{A}_\alpha$  associated to all  $\alpha \in \text{Irr}(\mathbb{G})$ , and recall that we can canonically identify  $\mathcal{O}(\mathbb{G})$  with a subspace of  $A^*$ .

(1)  $\Rightarrow$  (2): By Remark 3.3, the subspace  $C_0 = \mathcal{O}(\hat{\mathbb{G}}) \triangleright C$  of  $C$  is dense. We show that  $C_0 \subseteq C$  is a subalgebra. Let  $c, d \in C$  and  $v, \omega \in \mathcal{O}(\mathbb{G})$ . Then

$$(v \triangleright c)(\omega \triangleright d) = (\text{id}_C \otimes v \otimes \omega)(\delta(c)_{12}\delta(d)_{13}),$$

where we use the leg notation on  $\delta(c)$  and  $\delta(d)$ . Now, we find finitely many  $v'_i, \omega'_i \in \mathcal{O}(\hat{\mathbb{G}})$  such that

$$v(a)\omega(b) = \sum_i (v'_i \otimes \omega'_i)((a \otimes 1_A)\Delta(b))$$

for all  $a, b \in A$ , and then

$$\begin{aligned} (v \triangleright c)(\omega \triangleright d) &= \sum_i (\text{id}_C \otimes v'_i \otimes \omega'_i)((\delta(c) \otimes 1_A)(\text{id} \otimes \Delta)(\delta(d))) \\ &= \sum_i (\text{id}_C \otimes v'_i \otimes \omega'_i)(\delta \otimes \text{id}_A)((c \otimes 1_A)\delta(d)) \\ &= \sum_i v'_i \triangleright (c(\omega'_i \triangleright d)) \in C_0. \end{aligned}$$

Next, we show that  $\delta(C_0)$  is contained in the algebraic tensor product  $C \otimes \mathcal{O}(\hat{\mathbb{G}})$ . Let  $\omega \in \mathcal{O}(\hat{\mathbb{G}})$  and  $c \in C$ . Since  $\mathcal{O}(\mathbb{G})$  has a basis of elements  $(u_{i,j}^\alpha)_{\alpha,i,j}$  satisfying  $\Delta(u_{i,j}^\alpha) = \sum_k u_{ik}^\alpha \otimes u_{kj}^\alpha$  (see [33, Proposition 5.1]), we can find finitely many  $v_1, \dots, v_n \in \mathcal{O}(\hat{\mathbb{G}})$  and  $a_1, \dots, a_n \in \mathcal{O}(\mathbb{G})$  such that

$$(\text{id}_A \otimes \omega)(\Delta(b)) = \sum_{i=1}^n v_i(b)a_i$$

for all  $b \in \mathcal{O}(\mathbb{G})$ , and then

$$\begin{aligned} \delta(\omega \triangleright c) &= (\text{id}_C \otimes \text{id}_A \otimes \omega)(\delta \otimes \text{id}_A)\delta(c) \\ &= \delta(1_C)(\text{id}_C \otimes (\text{id}_A \otimes \omega)\Delta)\delta(c) \\ &= \delta(1_C) \cdot \sum_{i=1}^n (v_i \triangleright c) \otimes a_i \end{aligned}$$

lies in the algebraic tensor product of  $C$  with  $\mathcal{O}(\mathbb{G})$ . Using a basis for  $\mathcal{O}(\hat{\mathbb{G}})$  consisting of functionals  $(\phi_{i,j}^\alpha)_{\alpha,i,j}$  such that  $\phi_{i,j}^\alpha(u_{k,l}^\beta) = \delta_{\alpha,\beta}\delta_{i,k}\delta_{j,l}$  (see [33, Section 6]), we see that  $\delta(C_0)$  is contained in the algebraic tensor product  $C_0 \otimes \mathcal{O}(\mathbb{G})$ .

To finish the proof, note that with  $\omega, c$  as above, (3.4) implies that

$$\varepsilon \triangleright (\omega \triangleright c) = (\text{id}_C \otimes \varepsilon)(\delta(1_C)) \cdot (\omega \triangleright c) = \omega \triangleright c.$$

(2) $\Rightarrow$ (1): Since  $C_0 \subseteq C$  is dense, the unit of  $C_0$  has to be  $1_C$ , whence  $\delta(1_C)$  lies in the algebraic tensor product  $C \otimes \mathcal{O}(\mathbb{G})$  and  $(\text{id} \otimes \varepsilon)\delta(1_C) = 1_C$ . To prove weak continuity, we show that for every  $c \in C_0$ , there exists some  $\omega \in A^*$  such that  $\omega \triangleright c = c$ . So, take  $c \in C_0$ , and write  $\delta(c) = \sum_{i=1}^n d_i \otimes a_i$  with  $d_i \in C_0$  and  $a_i \in \mathcal{O}(\mathbb{G})$ . By the Hahn–Banach theorem, the restriction of  $\varepsilon$  to the finite-dimensional subspace of  $A$  spanned by  $a_1, \dots, a_n$  extends to a bounded linear functional  $\omega \in A^*$  that satisfies  $\omega \triangleright c = \varepsilon \triangleright c = c$ .  $\square$

## 6. Restriction

Like partial actions of groups and partial (co)actions of Hopf algebras, partial coactions of  $C^*$ -bialgebras can be obtained from nonpartial ones by restriction.

*Definition 6.1.* Let  $\delta_B$  be a partial coaction of a  $C^*$ -bialgebra  $(A, \Delta)$  on a  $C^*$ -algebra  $B$ . We call a  $C^*$ -subalgebra  $C \subseteq B$  *weakly invariant* if

$$\delta_B(C)(C \otimes A) \subseteq C \otimes A,$$

and *strongly invariant* if the embedding  $C \hookrightarrow B$  is strict and  $\delta_B(C) \subseteq M(C \otimes A) \subseteq M(B \otimes A)$ .

Note here that if the embedding  $C \hookrightarrow B$  is strict, then the embedding  $C \otimes A \hookrightarrow B \otimes A$  is strict as well and extends to an embedding  $M(C \otimes A) \hookrightarrow M(B \otimes A)$ .

*Remark 6.2.*

- (1) Every ideal  $C \subseteq B$  is weakly invariant but not necessarily strongly invariant.

- (2) A corner  $C \subseteq B$  is strongly invariant if and only if  $1_C \in M(C) \subseteq M(B)$  is strongly invariant in the sense that

$$\delta_B(1_C) = \delta_B(1_C)(1_C \otimes 1_A),$$

as one can easily check. If one thinks of elements of  $M(B)$  and  $M(B \otimes A)$  as  $2 \times 2$  matrices with respect to the Peirce decomposition  $B = 1_C B + (1_B - 1_C)B$ , then strong invariance of  $C$  means that  $\delta_B(C)$  is contained in the upper left corner, while weak invariance of  $C$  means that the off-diagonal part of  $\delta_B(C)$  vanishes.

*Example 6.3.* Suppose that  $\delta_B$  is the partial coaction corresponding to a disconnected partial action  $((p_g)_{g \in \Gamma}, (\theta_g)_{g \in \Gamma})$  of a discrete group  $\Gamma$  on a  $C^*$ -algebra  $B$  as in Proposition 4.3, and assume that  $C \subseteq B$  is a direct summand. Then  $C$  is automatically weakly invariant, but strongly invariant if and only if  $\theta_g(p_{g^{-1}}C) \subseteq C$  for all  $g \in \Gamma$ .

Evidently, partial coactions can be restricted to strongly invariant  $C^*$ -subalgebras. Restriction to weakly invariant  $C^*$ -subalgebras is a bit more delicate unless the embedding of the  $C^*$ -subalgebra is strict.

**Proposition 6.4.** *Let  $\delta_B$  be a partial coaction of a  $C^*$ -bialgebra  $(A, \Delta)$  on a  $C^*$ -algebra  $B$ , and let  $C \subseteq B$  be a weakly invariant  $C^*$ -subalgebra. Then we have the following:*

- (1)  $\delta_B$  restricts to a  $*$ -homomorphism  $\delta_C: C \rightarrow M(C \otimes A)$ ,
- (2) if the embedding  $C \hookrightarrow B$  is strict, then the composition of  $\delta_C$  with the embedding of  $M(C \otimes A)$  into  $M(B \otimes A)$  is strict and

$$\delta_C(c) = \delta_B(c)(1_C \otimes 1_A) \quad (c \in C),$$

- (3) if  $\delta_C$  is strict, then it is a partial coaction of  $(A, \Delta)$  on  $C$ .

*Proof.* (1) This follows immediately from the definition.

(2) Suppose that the embedding  $C \hookrightarrow B$  is strict. Then so is its composition with  $\delta_B$  and hence also  $\delta_C$ . To prove the formula given for  $\delta_C(c)$ , choose a bounded approximate unit  $(u_\nu)_\nu$  for  $C$ , and note that  $\delta_C(c)(u_\nu \otimes 1_A) = \delta_B(c)(u_\nu \otimes 1_A)$  converges strictly to  $\delta_C(c)$  in  $M(C \otimes A)$  and to  $\delta_B(c)(1_C \otimes 1_A)$  in  $M(B \otimes A)$ .

(3) Let  $(u_\nu)_\nu$  be as above, and let  $c, c' \in C$ . Then by definition of  $\delta_C$ ,

$$\begin{aligned} & (c' \otimes 1_A \otimes 1_A) \cdot (\delta_C \otimes \text{id}_A)(\delta_C(c)(u_\nu \otimes 1_A)) \\ &= (c' \otimes 1_A \otimes 1_A) \cdot (\delta_C \otimes \text{id}_A)(\delta_B(c)(u_\nu \otimes 1_A)) \\ &= (c' \otimes 1_A \otimes 1_A) \cdot (\delta_B \otimes \text{id}_A)(\delta_B(c)) \cdot (\delta_C(u_\nu) \otimes 1_A) \\ &= (\text{id}_C \otimes \Delta)((c' \otimes 1_A)\delta_B(c)) \cdot (\delta_C(u_\nu) \otimes 1_A) \\ &= (c' \otimes 1_A \otimes 1_A) \cdot (\text{id}_C \otimes \Delta)(\delta_C(c)) \cdot (\delta_C(u_\nu) \otimes 1_A). \end{aligned}$$

Since  $c' \in C$  was arbitrary, we can conclude that

$$(\delta_C \otimes \text{id}_A)(\delta_C(c)(u_\nu \otimes 1_A)) = (\text{id}_C \otimes \Delta)(\delta_C(c)) \cdot (\delta_C(u_\nu) \otimes 1_A).$$

As  $\nu$  tends to infinity,  $\delta_C(c)(u_\nu \otimes 1_A)$  converges strictly to  $\delta_C(c)$ , and since  $\delta_C$  and hence also  $\delta_C \otimes \text{id}_A$  are strict, the left-hand side converges to  $(\delta_C \otimes \text{id}_A)\delta_C(c)$  and the right-hand side converges to  $(\text{id}_C \otimes \Delta)(\delta_C(c))(\delta_C(1_C) \otimes 1_A)$ .  $\square$

*Remark 6.5.*

- (1) As a corollary, a (partial) coaction on a  $C^*$ -algebra  $C$  restricts to a partial coaction on every direct summand of  $C$  because every direct summand is weakly invariant by Remark 6.2(1).
- (2) The restriction  $\delta_C$  can be strict without the embedding  $C \hookrightarrow B$  being strict; for example, this is the case if  $\delta_B$  is the trivial coaction  $b \mapsto b \otimes 1_A$  and  $C \subseteq B$  is a closed ideal that is not a direct summand.

*Example 6.6.* Let  $\mathbb{G} = (A, \Delta)$  be a discrete quantum group so that  $A$  is a  $c_0$ -sum of matrix algebras  $A_\alpha$  with  $\alpha \in \text{Irr}(\hat{\mathbb{G}})$ . Then for every subset  $\mathcal{J} \subseteq \text{Irr}(\hat{\mathbb{G}})$ , the restriction of  $\Delta$  to the  $c_0$ -sum  $A_J := \bigoplus_{\alpha \in J} A_\alpha$  yields a partial coaction. But if  $\mathcal{J}$  is nontrivial, then  $A_J$  is not strongly invariant: if  $\alpha \notin \mathcal{J}$  and  $\gamma \in \mathcal{J}$ , then  $\alpha \otimes (\alpha^\dagger \otimes \gamma)$ , where  $\alpha^\dagger$  denotes the dual of  $\alpha$ , contains  $\gamma$ , and hence  $\Delta(A_\gamma)(A_\alpha \otimes 1) \neq 0$ .

Closely related to the concept of restriction is the notion of a morphism of partial coactions.

*Definition 6.7.* Let  $\delta_B$  and  $\delta_C$  be partial coactions of a  $C^*$ -bialgebra  $(A, \Delta)$  on  $C^*$ -algebras  $B$  and  $C$ , respectively. A *strong morphism* from  $\delta_C$  to  $\delta_B$  is a strict  $*$ -homomorphism  $\pi: C \rightarrow M(B)$  satisfying

$$(\pi \otimes \text{id}_A)\delta_C(c) = \delta_B(\pi(c)) \quad (c \in C).$$

A *weak morphism* from  $\delta_C$  to  $\delta_B$  is a  $*$ -homomorphism  $\pi: C \rightarrow M(B)$  satisfying

$$(\pi \otimes \text{id}_A)(\delta_C(c)(c' \otimes a)) = \delta_B(\pi(c))(\pi(c') \otimes a) \quad (c, c' \in C, a \in A).$$

We call such a weak or strong morphism  $\pi$  *proper* if  $\pi(C) \subseteq B$ .

Evidently, partial coactions with strong morphisms or with proper weak morphisms as above form categories.

*Remark 6.8.*

- (1) Clearly,  $\pi$  is a strong or a weak morphism if and only if

$$\pi(\omega \triangleright c) = \omega \triangleright \pi(c) \quad \text{or} \quad \pi(\omega \triangleright c)\pi(c') = (\omega \triangleright \pi(c))\pi(c'), \quad (6.1)$$

respectively, for all  $\omega \in A^*$  and  $c, c' \in C$ .

- (2) If  $\pi$  is a weak or a strong morphism and proper, then its image is weakly or strongly invariant, respectively.
- (3) Suppose that  $\delta_B$  is a partial coaction of  $(A, \Delta)$  on a  $C^*$ -algebra  $B$  and that  $C \subseteq B$  is a  $C^*$ -subalgebra that is weakly or strongly invariant. If the embedding  $C \hookrightarrow B$  is strict, then this embedding is a weak or a strong morphism with respect to the restriction of  $\delta_B$  to  $C$  defined above.

Let us look at the special case of partial coactions associated to disconnected partial group actions.

**Proposition 6.9.** *Let  $B$  and  $C$  be two  $C^*$ -algebras with disconnected partial actions  $((p_g)_g, (\beta_g)_g)$  and  $((q_g)_g, (\gamma_g)_g)$ , respectively, of a discrete group  $\Gamma$ . With respect to the associated partial coactions of  $C_0(\Gamma)$ , a strict  $*$ -homomorphism  $\pi: B \rightarrow M(C)$  is a strong morphism if and only if*

$$\pi(p_g) = q_g \pi(1_C) \quad \text{and} \quad \pi \circ \beta_g \subseteq \gamma_g \circ \pi \quad \text{for all } g \in \Gamma, \quad (6.2)$$

and a weak morphism if and only if

$$\begin{aligned} \pi(1_C) \gamma_g(q_{g^{-1}} \pi(1_C)) &= \pi(p_g) = \gamma_g(\pi(p_{g^{-1}})) \quad \text{and} \\ \pi \circ \beta_g &\subseteq \gamma_g \circ \pi \quad \text{for all } g \in \Gamma. \end{aligned} \quad (6.3)$$

*Proof.* Denote the partial coactions by  $\delta_B$  and  $\delta_C$ .

(1) Suppose that  $\pi$  is a strong morphism. Then the definition of  $\delta_B$  and  $\delta_C$  implies that

$$(\pi \circ \beta_g)(p_{g^{-1}} b) = (\pi \otimes \text{ev}_g) \delta_B(b) = (\text{id}_C \otimes \text{ev}_g) \delta_C(\pi(b)) = \gamma_g(q_{g^{-1}} \pi(b)) \quad (6.4)$$

for all  $g \in \Gamma$  and  $b \in B$ . Taking  $b = 1_C$  or  $b = p_{g^{-1}}$ , we conclude that

$$\gamma_g(q_{g^{-1}} \pi(p_{g^{-1}})) = \pi(p_g) = \gamma_g(q_{g^{-1}} \pi(1_C)),$$

and in particular, that  $\pi(p_g)q_g = \pi(p_g)$ . We use this relation on the left-hand side above, apply  $\gamma_{g^{-1}}$ , and get  $\pi(p_g) = q_g \pi(1_C)$ . Moreover,  $\pi(p_g B) \subseteq q_g C$ , and (6.4) implies that  $\pi \circ \beta_g \subseteq \gamma_g \circ \pi$ .

Conversely, the first relation in (6.2) implies that  $q_{g^{-1}} \pi(1_C - p_{g^{-1}}) = 0$ , whence both sides in (6.4) are zero for all  $b \in (1 - p_{g^{-1}})B$ , and the second relation in (6.2) implies that (6.4) holds for all  $b \in p_{g^{-1}}B$ . Combined, (6.2) implies that  $(\pi \otimes \text{id}) \delta_B = \delta_C \circ \pi$ .

(2) Suppose that  $\pi$  is a strict weak morphism. As in (1), we find that

$$(\pi \circ \beta_g)(p_{g^{-1}} b) = \pi(1_C) \gamma_g(q_{g^{-1}} \pi(b)) \quad (6.5)$$

for all  $g \in \Gamma$  and  $b \in B$ , and arguments similar to those in (1) yield the first equation in (6.3). Now, we apply  $\gamma_{g^{-1}}$  to this relation and find that

$$\gamma_{g^{-1}}(\pi(p_g)) = \gamma_{g^{-1}}(q_g \pi(1_C)) \pi(1_C) = \pi(p_{g^{-1}}).$$

In particular, this relation and (6.5) imply the second relation in (6.3).

Conversely, (6.3) implies that both sides of (6.5) coincide for all  $b \in p_{g^{-1}}B$ , and that for all  $b \in (1_C - p_{g^{-1}})B$ ,

$$\pi(1_C) \gamma_g(q_{g^{-1}} \pi(1_C - p_{g^{-1}})) = \pi(p_g) - \pi(p_g) = 0,$$

whence both sides of (6.5) are zero for all  $b \in (1_C - p_{g^{-1}})B$ . But this implies that  $(\pi \otimes \text{id}) \circ \delta_B = (\pi(1_C) \otimes 1_A) (\delta_C \circ \pi)$ .  $\square$

## 7. The Bernoulli shift of a discrete quantum group

The Bernoulli shift of a discrete group  $\Gamma$  is its action on the power set  $\mathcal{P}(\Gamma)$ , which we identify with the infinite product  $\{0, 1\}^\Gamma$ , by left translation. Restriction to the subsets containing the unit  $e_\Gamma$  yields an important example of a partial action. To a discrete quantum group, we now associate a quantum Bernoulli shift and obtain, by restriction, a partial coaction that is initial in a natural sense.

The space  $\{0, 1\}^\Gamma \cong \mathcal{P}(\Gamma)$  parameterizes all maps from  $\Gamma$  to  $\{0, 1\}$  or, equivalently, all subsets of  $\Gamma$ , which correspond to projections in  $M(C_0(\Gamma))$ . Given a discrete quantum group  $\mathbb{G}$ , it is natural to define its quantum power set as a universal quantum family of maps from  $\mathbb{G}$  to  $\{0, 1\}$  in the sense of [26] or, equivalently, as the unital  $C^*$ -algebra  $C$  that comes with a universal projection in  $M(C \otimes C_0(\mathbb{G}))$ . However, we need an additional commutativity assumption.

Let  $\mathbb{G} = (C_0(\mathbb{G}), \Delta)$  be a discrete  $C^*$ -quantum group with counit  $\varepsilon$  and compact dual  $\hat{\mathbb{G}}$ .

*Definition 7.1.* Let  $C$  be a  $C^*$ -algebra. We call a projection  $p \in M(C \otimes C_0(\mathbb{G}))$  *admissible* if in  $M(C \otimes C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ ,

$$(p \otimes 1) \cdot (\text{id} \otimes \Delta)(p) = (\text{id} \otimes \Delta)(p) \cdot (p \otimes 1). \quad (7.1)$$

*Remark 7.2.* For every partial coaction  $\delta$  of  $C_0(\mathbb{G})$  on a  $C^*$ -algebra  $C$ , the projection  $\delta(1_C) \in M(C \otimes C_0(\mathbb{G}))$  is admissible.

**Proposition 7.3.** *Let  $\mathbb{G}$  be a discrete  $C^*$ -quantum group. Then there exists a unital  $C^*$ -algebra  $C(\mathbb{B}_\mathbb{G})$  with an admissible projection  $p \in M(C(\mathbb{B}_\mathbb{G}) \otimes C_0(\mathbb{G}))$  that is universal in the following sense: for every  $C^*$ -algebra  $C$  with an admissible projection  $q \in M(C(\mathbb{B}_\mathbb{G}) \otimes C_0(\mathbb{G}))$ , there exists a unique unital  $*$ -homomorphism  $\pi: C(\mathbb{B}_\mathbb{G}) \rightarrow M(C)$  such that  $q = (\pi \otimes \text{id})(p)$ .*

*Proof.* Write  $C_0(\mathbb{G}) \cong \bigoplus_\alpha I_\alpha$ , where  $\alpha$  varies in  $\text{Irr}(\hat{\mathbb{G}})$  and each  $I_\alpha$  is a matrix algebra. Choose matrix units  $(e_{ij}^\alpha)_{i,j}$  for each  $I_\alpha$ . Denote by  $C(\mathbb{B}_\mathbb{G})$  the universal unital  $C^*$ -algebra with generators 1 and  $(p_{ij}^\alpha)_{\alpha,i,j}$  satisfying the following relations:

- (1) the finite sum  $p^\alpha := \sum_{i,j} p_{ij}^\alpha \otimes e_{ij}^\alpha$  is a projection for every  $\alpha \in \text{Irr}(\hat{\mathbb{G}})$ ,
- (2)  $(p^\alpha \otimes 1)(\text{id} \otimes \Delta)(p^\beta) = (\text{id} \otimes \Delta)(p^\beta)(p^\alpha \otimes 1)$  for all  $\alpha, \beta \in \text{Irr}(\hat{\mathbb{G}})$ .

Then the sum  $p = \sum_\alpha p^\alpha \in M(C(\mathbb{B}_\mathbb{G}) \otimes C_0(\mathbb{G}))$  converges strictly because each summand  $p^\alpha$  lies in a different summand of  $C(\mathbb{B}_\mathbb{G}) \otimes C_0(\mathbb{G}) \cong \bigoplus_\alpha (C(\mathbb{B}_\mathbb{G}) \otimes I_\alpha)$  and has norm at most 1. By (1) and (2), this  $p$  is an admissible projection, and by construction,  $C(\mathbb{B}_\mathbb{G})$  has the desired universal property.  $\square$

We denote by  $C_0(\mathbb{B}_\mathbb{G}^\times) \subset C(\mathbb{B}_\mathbb{G})$  the nonunital  $C^*$ -subalgebra generated by all  $p_{i,j}^\alpha$ 's.

*Example 7.4.* In the case in which  $\mathbb{G}$  is a classical discrete group  $\Gamma$ , we can identify  $C(\mathbb{B}_\Gamma)$  with  $C(\{0, 1\}^\Gamma)$ . Indeed, in that case,  $\text{Irr}(\hat{\Gamma})$  can be identified with  $\Gamma$  so that  $C(\mathbb{B}_\Gamma)$  is generated by 1 and a family of projections  $p^\gamma$ , where  $\gamma \in \Gamma$ . Denote by  $\delta_\gamma \in C_0(\Gamma)$  the Dirac delta function at  $\gamma \in \Gamma$ . Then  $p = \sum_\gamma p^\gamma \otimes \Delta_\gamma$  and the

admissibility condition takes the form

$$\left[ \sum_{\gamma} p^{\gamma} \otimes \Delta_{\gamma} \otimes 1, \sum_{\gamma, \gamma'} p^{\gamma \gamma'} \otimes \Delta_{\gamma} \otimes \Delta_{\gamma'} \right] = 0$$

or, equivalently,  $[p^{\gamma}, p^{\gamma''}] = 0$  for all  $\gamma, \gamma'' \in \Gamma$ . Thus,  $C(\mathbb{B}_{\Gamma})$  is commutative. Therefore, the map that sends  $p^{\gamma}$  to the projection of  $\{0, 1\}^{\Gamma}$  onto the  $\gamma$ th component induces an isomorphism  $C(\mathbb{B}_{\Gamma}) \cong C(\{0, 1\}^{\Gamma})$ . Under this isomorphism, the  $C^*$ -subalgebra  $C_0(\mathbb{B}_{\mathbb{G}}^{\times})$  corresponds to  $C_0(\mathcal{P}(\Gamma) \setminus \{\emptyset\})$ .

The quantum space  $\mathbb{B}_{\mathbb{G}}$  comes with a natural action of  $\mathbb{G}$ , as follows.

**Proposition 7.5.** *There exists a unique coaction  $\delta$  of  $C_0(\mathbb{G})$  on  $C(\mathbb{B}_{\mathbb{G}})$  such that*

$$(\delta \otimes \text{id})(p) = (\text{id} \otimes \Delta)(p). \quad (7.2)$$

*This coaction is counital and restricts to a coaction on  $C_0(\mathbb{B}_{\mathbb{G}}^{\times})$ .*

*Proof.* The projection  $q := (\text{id} \otimes \Delta)(p) \in M((C(\mathbb{B}_{\mathbb{G}}) \otimes C_0(\mathbb{G})) \otimes C_0(\mathbb{G}))$  is admissible because

$$(\text{id} \otimes \text{id} \otimes \Delta)(q) = (\text{id} \otimes \Delta^{(2)})(p) = (\text{id} \otimes \Delta \otimes \text{id})(\text{id} \otimes \Delta)(p)$$

commutes with  $q \otimes 1 = (\text{id} \otimes \Delta \otimes \text{id})(p \otimes 1)$ . The universal property of  $p$  yields a unital  $*$ -homomorphism  $\delta: C(\mathbb{B}_{\mathbb{G}}) \rightarrow M(C(\mathbb{B}_{\mathbb{G}}) \otimes C_0(\mathbb{G}))$  such that  $(\delta \otimes \text{id})(p) = q = (\text{id} \otimes \Delta)(p)$ . We have  $(\delta \otimes \text{id})\delta = (\text{id} \otimes \Delta)\delta$  because by definition of  $\delta$ ,

$$\begin{aligned} ((\delta \otimes \text{id})\delta \otimes \text{id})(p) &= (\delta \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta)(p) \\ &= (\text{id} \otimes \text{id} \otimes \Delta)(\delta \otimes \text{id})(p) \\ &= (\text{id} \otimes \Delta \otimes \text{id})(\text{id} \otimes \Delta)(p) \\ &= ((\text{id} \otimes \Delta)\delta \otimes \text{id})(p). \end{aligned}$$

Next,  $((\text{id} \otimes \varepsilon)\delta \otimes \text{id})(p) = (\text{id} \otimes (\varepsilon \otimes \text{id})\Delta)(p) = p$  and hence  $(\text{id} \otimes \varepsilon)\delta = \text{id}$ . Finally, (7.2) implies that  $\delta(p_{ij}^{\alpha})(1 \otimes C_0(\mathbb{G})) \subseteq C_0(\mathbb{B}_{\mathbb{G}}^{\times}) \otimes C_0(\mathbb{G})$ .  $\square$

We will restrict the coaction  $\delta$  to the direct summand of  $C(\mathbb{B}_{\mathbb{G}})$  that is given by the following projection.

**Lemma 7.6.** *The projection  $p_{\varepsilon} := (\text{id} \otimes \varepsilon)(p) \in C(\mathbb{B}_{\mathbb{G}})$  is central and  $\delta(p_{\varepsilon}) = p$ .*

*Proof.* We apply  $\text{id} \otimes \varepsilon \otimes \text{id}$  to (7.1) and obtain  $(p_{\varepsilon} \otimes 1)p = p(p_{\varepsilon} \otimes 1)$ . Thus,  $p_{\varepsilon}$  commutes with  $(\text{id} \otimes \omega)(p) \in C(\mathbb{B}_{\mathbb{G}})$  for every  $\omega \in C_0(\mathbb{G})^*$  and hence with  $C(\mathbb{B}_{\mathbb{G}})$ . Moreover,

$$\delta(p_{\varepsilon}) = (\delta \otimes \varepsilon)(p) = (\text{id} \otimes \text{id} \otimes \varepsilon)(\delta \otimes \text{id})(p) = (\text{id} \otimes \text{id} \otimes \varepsilon)(\text{id} \otimes \Delta)(p) = p. \quad \square$$

*Example 7.7.* If  $\mathbb{G}$  is a classical group  $\Gamma$  (see Example 7.4), then  $\varepsilon$  is the evaluation at the unit  $e_{\Gamma}$  and  $p_{\varepsilon} = p^{e_{\Gamma}}$ . Therefore, restriction of the coaction  $\delta$  above to the direct summand  $p_{\varepsilon}C(\mathbb{B}_{\mathbb{G}})$  of  $C(\mathbb{B}_{\mathbb{G}})$  corresponds to restriction of the Bernoulli shift on  $\mathcal{P}(\Gamma)$  to the subsets containing the unit  $e_{\Gamma}$ .

We can now define the quantum analogue of the partial Bernoulli shift.

**Definition 7.8.** Let  $\mathbb{G}$  be a discrete  $C^*$ -quantum group, and write  $C(\mathbb{B}_{\mathbb{G}}^\varepsilon)$  for the direct summand  $p_\varepsilon C(\mathbb{B}_{\mathbb{G}})$  of  $C(\mathbb{B}_{\mathbb{G}})$ . Then the *partial Bernoulli action* of  $\mathbb{G}$  is the partial coaction  $\delta^\varepsilon$  of  $C_0(\mathbb{G})$  on  $C(\mathbb{B}_{\mathbb{G}}^\varepsilon)$  obtained as the restriction of the coaction  $\delta$  as in Proposition 6.4, that is,

$$\delta^\varepsilon(b) = \delta(b)(p_\varepsilon \otimes 1) \quad \text{for all } b \in C(\mathbb{B}_{\mathbb{G}}^\varepsilon).$$

Proposition 7.3 immediately implies the following.

**Corollary 7.9.** Let  $C$  be a  $C^*$ -algebra, and let  $q \in M(C \otimes C_0(\mathbb{G}))$  be an admissible projection such that  $(\text{id} \otimes \varepsilon)(q) = 1_C \in M(C)$ . Then there exists a unique unital  $*$ -homomorphism  $\pi: C(\mathbb{B}_{\mathbb{G}}^\varepsilon) \rightarrow M(C)$  such that  $q = (\pi \otimes \text{id})(p)$ .

The partial Bernoulli action is initial in the following sense.

**Proposition 7.10.** Let  $\delta_C$  be a counital partial coaction of  $C_0(\mathbb{G})$  on a  $C^*$ -algebra  $C$ . Then there exists a unique unital  $*$ -homomorphism  $\pi: C(\mathbb{B}_{\mathbb{G}}^\varepsilon) \rightarrow M(C)$  such that

$$(\pi \otimes \text{id})(p(p_\varepsilon \otimes 1)) = \delta_C(1_C), \tag{7.3}$$

and this  $\pi$  is a strong morphism of partial coactions, that is,  $(\pi \otimes \text{id}) \circ \delta^\varepsilon = \delta_C \circ \pi$ .

*Proof.* The projection  $\delta_C(1_C) \in M(C \otimes C_0(\mathbb{G}))$  is admissible and  $\delta_C$  is counital. Hence, Corollary 7.9 yields a unique unital  $*$ -homomorphism  $\pi: C(\mathbb{B}_{\mathbb{G}}^\varepsilon) \rightarrow M(C)$  such that  $(\pi \otimes \text{id})(p) = \delta_C(1_C)$ . We show that  $(\pi \otimes \text{id}) \circ \delta^\varepsilon = \delta_C \circ \pi$ . First, (7.2) and Lemma 7.6 imply that

$$\begin{aligned} (\delta^\varepsilon \otimes \text{id})(p(p_\varepsilon \otimes 1)) &= (\delta \otimes \text{id})(p(p_\varepsilon \otimes 1)) \cdot (p_\varepsilon \otimes 1) \\ &= (\text{id} \otimes \Delta)(p) \cdot (p \otimes 1) \cdot (p_\varepsilon \otimes 1 \otimes 1). \end{aligned}$$

We apply  $\pi \otimes \text{id} \otimes \text{id}$ , use (7.3), and find that

$$\begin{aligned} ((\pi \otimes \text{id}) \delta^\varepsilon \otimes \text{id})(p(p_\varepsilon \otimes 1)) &= (\pi \otimes \Delta)(p(p_\varepsilon \otimes 1)) \cdot (\pi \otimes \text{id})(p(p_\varepsilon \otimes 1)) \\ &= (\text{id} \otimes \Delta)(\delta_C(1_C)) \cdot (\delta_C(1_C) \otimes 1) \\ &= (\delta_C \otimes \text{id})\delta_C(1_C) \\ &= (\delta_C \circ \pi \otimes \text{id})(p(p_\varepsilon \otimes 1)). \end{aligned}$$

But this relation implies that  $(\pi \otimes \text{id}) \circ \delta^\varepsilon = \delta_C \circ \pi$ .  $\square$

We will study this partial Bernoulli shift in a forthcoming article. In particular, the partial coaction  $\delta^\varepsilon$  should give rise to a partial crossed product that can be regarded as a quantum counterpart to the partial group algebra of a discrete group (see [16, Section 10]) and as a  $C^*$ -algebraic counterpart to the Hopf algebroid  $H_{\text{par}}$  associated to a Hopf algebra  $H$  in [7].

## 8. Dilations

Let  $(A, \Delta)$  be a  $C^*$ -bialgebra. Given a partial coaction of  $(A, \Delta)$ , a natural and important question is whether it can be identified with the restriction of a coaction to a weakly invariant  $C^*$ -subalgebra as in Proposition 6.4.

*Definition 8.1.* Let  $\delta_C$  be a partial coaction of  $(A, \Delta)$  on a  $C^*$ -algebra  $C$ . A *dilation* of  $\delta_C$  consists of a  $C^*$ -algebra  $B$ , a coaction  $\delta_B$  of  $(A, \Delta)$  on  $B$ , and an embedding  $\iota: C \hookrightarrow B$  that is a weak morphism from  $\delta_C$  to  $\delta_B$  that satisfies

$$\delta_B(\iota(c))(\iota(c') \otimes a) = (\iota \otimes \text{id}_A)(\delta_C(c)(c' \otimes a)) \quad (c, c' \in C, a \in A).$$

*Example 8.2* (Disconnected partial actions of groups). Let  $C$  be a  $C^*$ -algebra with a disconnected partial action  $((p_g)_g, (\theta_g)_g)$  of a discrete group  $\Gamma$ , and consider the associated partial coaction  $\delta_C$  of  $C_0(\Gamma)$  as in Proposition 4.3.

A dilation of  $\delta_C$  is given by a  $C^*$ -algebra  $B$  with a coaction of  $C_0(\Gamma)$ , that is, by an action  $(\alpha_g)_{g \in \Gamma}$  of  $\Gamma$  on  $B$ , and an embedding  $C \hookrightarrow B$  that is a weak morphism. Suppose that this embedding is strict. By Proposition 6.9, it is a weak morphism if and only if

$$p_g = 1_C \alpha_g(1_C) \quad \text{and} \quad \theta_g = \alpha_g|_{p_g C} \quad (g \in \Gamma).$$

In particular,  $1_C$  commutes with  $\alpha_g(1_C)$  for each  $g \in \Gamma$ . We claim that our partial action coincides with the set-theoretic restriction  $((D_g)_g, (\alpha_g|_{D_g})_g)$  of  $\alpha$  to  $C$ , where  $D_g = \alpha_g(C) \cap C$  for each  $g \in \Gamma$ . Indeed, for every element  $c \in D_g$  with  $0 \leq c \leq 1_C$ , we have  $c \leq 1_C$  and  $\alpha_g^{-1}(c) \leq 1_C$ , whence  $c \leq \alpha_g(1_C)1_C = p_g$  and  $c \in p_g C$ . On the other hand, if  $c \in p_g C$ , then  $\alpha_g^{-1}(c) = \theta_{g^{-1}}(c) \in C$  and hence  $c \in \alpha_g(C) \cap C = D_g$ .

Conversely, suppose that  $\alpha$  is an action of  $\Gamma$  on a  $C^*$ -algebra  $B$  that contains  $C$  and suppose that  $\alpha$  is a dilation in the usual sense, so that  $C \subseteq B$  is an ideal,  $p_g C = \alpha_g(C) \cap C$ , and  $\theta_g = \alpha_g|_{p_g C}$  for each  $g \in \Gamma$ . If the embedding  $C \subseteq B$  is strict, then  $C$  is a direct summand, that is,  $C = 1_C B$ , and then  $\alpha_g(C) \cap C = \alpha_g(1_C)1_C$  for each  $g \in \Gamma$ , so that the coaction  $\delta_B$  corresponding to  $\alpha$  is a dilation of  $\delta_C$ .

The main question is, of course: Which partial coactions have a dilation? We start with a necessary condition.

*Definition 8.3.* We call a partial coaction  $\delta_C$  of  $(A, \Delta)$  on a  $C^*$ -algebra  $C$  *regular* if

$$(\text{id}_C \otimes \Delta)(\delta_C(C)) \cdot (1_C \otimes 1_A \otimes A) \subseteq M(C \otimes A) \otimes A. \quad (8.1)$$

*Example 8.4.*

- (1) Every coaction is easily seen to be regular.
- (2) The question of regularity arises only if  $C$  is nonunital because every partial coaction on a unital  $C^*$ -algebra is regular.
- (3) If  $A$  is a direct sum of matrix algebras, for example, if  $(A, \Delta)$  is a discrete quantum group, then every partial coaction of  $(A, \Delta)$  is regular.

Regularity is necessary for the existence of a dilation with a strict embedding.

**Lemma 8.5.** *If a partial coaction has a dilation  $(B, \delta_B, \iota)$ , where  $\iota$  is strict, then the partial coaction is regular.*

*Proof.* Suppose that  $\delta_C$  is a partial coaction of  $(A, \Delta)$  on a  $C^*$ -algebra  $C$  with a dilation  $(B, \delta_B, \iota)$ . It suffices to show that the product

$$(\iota \otimes \text{id}_A \otimes \text{id}_A)((\text{id}_C \otimes \Delta)\delta_C(C)) \cdot (1_B \otimes 1_A \otimes A)$$

lies in  $M(B \otimes A) \otimes A$ . Since  $\iota$  is a weak morphism, this product is equal to

$$(\text{id}_C \otimes \Delta)(\delta_B(\iota(C))) \cdot (\iota(1_C) \otimes 1_A \otimes A),$$

which by coassociativity of  $\delta_B$  can be rewritten as

$$(\delta_B \otimes \text{id}_A)(\delta_B(\iota(C))(1_B \otimes A)) \cdot (\iota(1_C) \otimes 1_A \otimes 1_A),$$

and this product lies in  $M(B \otimes A) \otimes A$  because  $\delta_B(\iota(C))(1_B \otimes A) \subseteq B \otimes A$ .  $\square$

If  $(A, \Delta)$  is a regular  $C^*$ -quantum group, for example, a compact one, and if  $\delta_C$  is weakly continuous, then regularity of  $\delta_C$  can be tested on the unit.

**Lemma 8.6.** *Let  $(A, \Delta)$  be a regular  $C^*$ -quantum group, and let  $\delta_C$  be a weakly continuous partial coaction of  $(A, \Delta)$  on a  $C^*$ -algebra  $C$  such that*

$$(\text{id}_C \otimes \Delta)(\delta_C(1_C)) \cdot (1_C \otimes 1_A \otimes A) \subseteq M(C \otimes A) \otimes A.$$

*Then  $\delta_C$  is regular.*

*Proof.* We use the same notation and a similar argument as in the proof of Proposition 3.5. By (3.3),

$$\begin{aligned} & (\text{id}_C \otimes \Delta)(\delta_C(\omega \triangleright c)) \\ &= (\text{id}_C \otimes \Delta)(\delta_C(1_C)) \cdot (\text{id}_C \otimes \text{id}_A \otimes \text{id}_A \otimes \omega)(\text{id}_C \otimes \Delta^{(2)})\delta_C(c) \end{aligned}$$

for all  $\omega \in A^*$  and  $c \in C$ , where  $\Delta^{(2)} = (\Delta \otimes \text{id}_A)\Delta = (\text{id}_A \otimes \Delta)\Delta$ . Since  $\delta_C$  is weakly continuous, we can conclude that  $[(\text{id}_C \otimes \Delta)\delta_C(C) \cdot (1_C \otimes 1_A \otimes A)]$  is equal to the product of  $(\text{id}_C \otimes \Delta)(\delta_C(1_C))$  with

$$[(\text{id}_C \otimes \text{id}_A \otimes \text{id}_A \otimes \omega)((\text{id}_C \otimes \Delta^{(2)})(\delta_C(C))(1_C \otimes 1_A \otimes A \otimes 1_A)) : \omega \in A^*].$$

Similarly as in the proof of Proposition 3.5, we rewrite this space in the form

$$\begin{aligned} & [(\text{id}_C \otimes \text{id}_A \otimes \text{id}_A \otimes \omega)(V_{34}(\text{id}_C \otimes (\text{id}_A \otimes \pi)\Delta)(\delta_C(C))_{124}V_{34}^*(A \otimes \hat{\pi}(\hat{A}))_{34}) : \\ & \quad \omega \in \mathcal{B}(K)_*] \\ &= [(\text{id}_C \otimes \text{id}_A \otimes \text{id}_A \otimes \omega)((A \otimes \hat{\pi}(\hat{A}))_{34}(\text{id}_C \otimes (\text{id}_A \otimes \pi)\Delta)(\delta_C(C))_{124}) : \\ & \quad \omega \in \mathcal{B}(K)_*] \\ &= [((\text{id}_C \otimes \text{id}_A \otimes \omega \circ \pi)(\text{id}_C \otimes \Delta)\delta_C(C)) \otimes A : \omega \in \mathcal{B}(K)_*] \\ &\subseteq M(C \otimes A) \otimes A. \end{aligned}$$

Summarizing, we find that

$$(\text{id}_C \otimes \Delta)\delta_C(C) \cdot (1_C \otimes 1_A \otimes A) \subseteq (\text{id}_C \otimes \Delta)\delta_C(1_C) \cdot (M(C \otimes A) \otimes A).$$

By assumption on  $\delta_C(1_C)$ , the right-hand side lies in  $M(C \otimes A) \otimes A$ .  $\square$

For partial actions of a group  $G$  on a space  $X$ , a canonical dilation can be constructed as a certain quotient of the product  $X \times G$  (see [1] or [16, Theorem 3.5, Proposition 5.5]). We now give a dual construction. Although this one will be improved upon in the next section, we include it for informational purposes (see also Example 8.8).

From now on, we almost always assume the  $C^*$ -algebra underlying our  $C^*$ -bialgebra to have the slice map property, which holds, for example, if it is nuclear (see Section 2.4).

**Proposition 8.7.** *Let  $\delta_C$  be an injective, regular partial coaction of a  $C^*$ -bialgebra  $(A, \Delta)$  on a  $C^*$ -algebra  $C$ , and suppose that  $A$  has the slice map property. Denote by  $C \boxtimes A \subseteq M(C \otimes A)$  the subset of all  $x$  satisfying the following conditions:*

- (1)  $[x, \delta_C(1_C)] = 0$ ,
- (2)  $(\delta_C \otimes \text{id}_A)(x) = (\delta_C(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)(x) = (\text{id}_C \otimes \Delta)(x)(\delta_C(1_C) \otimes 1_A)$ ,
- (3)  $x(1_C \otimes A)$  and  $(1_C \otimes A)x$  lie in  $C \otimes A$ ,
- (4)  $(\text{id}_C \otimes \Delta)(x)(1_C \otimes 1_A \otimes A)$  and  $(1_C \otimes 1_A \otimes A)(\text{id}_C \otimes \Delta)(x)$  lie in  $M(C \otimes A) \otimes A$ .

Then  $C \boxtimes A$  is a  $C^*$ -algebra,  $\text{id}_C \otimes \Delta$  restricts to a coaction of  $(A, \Delta)$  on  $C \boxtimes A$ , and  $(C \boxtimes A, \text{id}_C \otimes \Delta, \delta_C)$  is a dilation of  $\delta_C$ .

*Proof.* Clearly,  $C \boxtimes A$  is a  $C^*$ -algebra. It contains  $\delta_C(C)$  by (3.1) and regularity of  $\delta_C$ . Next, we need to show that

$$(\text{id}_C \otimes \Delta)(C \boxtimes A)(1_C \otimes 1_A \otimes A) \subseteq (C \boxtimes A) \otimes A.$$

Condition (4) implies that the left-hand side is contained in  $M(C \otimes A) \otimes A$ . Since  $A$  has the slice map property, it suffices to show that for every  $y \in C \boxtimes A$  and  $\omega \in A^*$ , the element

$$x := (\text{id}_C \otimes \text{id}_A \otimes \omega)(\text{id}_C \otimes \Delta)(y) = (\text{id}_C \otimes (\text{id}_A \otimes \omega)\Delta)(y)$$

lies in  $C \boxtimes A$  (i.e., it satisfies conditions (1)–(4) above). In cases (2)–(4), we only prove the first halves of the statements; the others follow similarly.

(1) The element  $x$  commutes with  $\delta_C(1_C)$  because  $(\text{id}_C \otimes \Delta)(y)$  commutes with  $(\delta_C(1_C) \otimes 1_A)$  by (2), applied to  $y$ .

(2) We use (1) for  $y$  and coassociativity of  $\Delta$  to see that

$$\begin{aligned} (\delta_C \otimes \text{id}_A)(x) &= (\text{id}_C \otimes \text{id}_A \otimes (\text{id}_A \otimes \omega)\Delta)(\delta_C \otimes \text{id}_A)(y) \\ &= (\text{id}_C \otimes \text{id}_A \otimes (\text{id}_A \otimes \omega)\Delta)((\delta_C(1) \otimes 1_A)(\text{id}_C \otimes \Delta)(y)) \\ &= (\delta_C(1_C) \otimes 1_A)(\text{id}_C \otimes (\text{id}_A \otimes \text{id}_A \otimes \omega)\Delta^{(2)})(y) \\ &= (\delta_C(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)(\text{id}_A \otimes (\text{id}_A \otimes \omega)\Delta)(y) \\ &= (\delta_C(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)(x). \end{aligned}$$

(3) Write  $\omega = av$  with  $a \in A$  and  $v \in A^*$  using Cohen's factorization theorem, and let  $a' \in A$ . Then

$$x(1_C \otimes a') = (\text{id}_C \otimes \text{id}_A \otimes v)((\text{id}_C \otimes \Delta)(y)(1_C \otimes a' \otimes a)).$$

We use the relation  $A \otimes A = [\Delta(A)(A \otimes A)]$  and condition (3) on  $y$  and find that  $x(1_C \otimes a')$  lies in  $C \otimes A$ , as desired.

(4) With  $a, a', v$  as above,

$$\begin{aligned} &(\text{id}_C \otimes \Delta)(x) \cdot (1_C \otimes 1_A \otimes a') \\ &= (\text{id}_C \otimes \text{id}_A \otimes \text{id}_A \otimes v)((\text{id}_C \otimes \Delta^{(2)})(y) \cdot (1_C \otimes 1_A \otimes a' \otimes a)). \end{aligned}$$

We use the relation  $A \otimes A = [\Delta(A)(A \otimes A)]$  again and find that

$$\begin{aligned} & (\text{id}_C \otimes \Delta^{(2)})(y) \cdot (1_C \otimes 1_A \otimes a' \otimes a) \\ & \in (\text{id}_C \otimes \text{id}_A \otimes \Delta)((\text{id}_C \otimes \Delta)(y) \cdot (1_C \otimes 1_A \otimes A)) \cdot (1_C \otimes 1_A \otimes A \otimes A). \end{aligned}$$

Condition (4), applied to  $y$ , implies that the expression above lies in  $M(C \otimes A) \otimes A \otimes A$ . Slicing the last factor with  $v$ , we get  $(\text{id}_C \otimes \Delta)(x) \cdot (1_C \otimes 1_A \otimes a') \in M(C \otimes A) \otimes A$ .  $\square$

*Example 8.8* (Case of a partial group action). Consider the partial coaction  $\delta_C$  associated to a disconnected partial action  $((p_g), (\theta_g)_g)$  of a discrete group  $\Gamma$  on a  $C^*$ -algebra  $C$ . Identify  $M(C \otimes C_0(\Gamma))$  with  $C_b(\Gamma; M(C))$ , and let  $f \in C_b(\Gamma; M(C))$ . Then conditions (1) and (4) in Proposition 8.7 are automatically satisfied by  $f$ , condition (3) is equivalent to  $f \in C_b(\Gamma; C)$ , and condition (2) corresponds to the invariance condition

$$\theta_g(p_{g^{-1}}f(h)) = p_g f(gh) \quad (g, h \in \Gamma).$$

In particular, if  $C = C_0(X)$  for some locally compact Hausdorff space  $X$ , then each  $p_g$  is the characteristic function of some clopen  $D_g \subseteq X$ , each  $\theta_g$  is the pull-back along some homeomorphism  $\alpha_{g^{-1}}: D_g \rightarrow D_{g^{-1}}$ , and the invariance condition above translates into

$$f(x, gh) = f(\alpha_{g^{-1}}(x), h) \quad (g, h \in \Gamma, x \in D_g),$$

so that  $f$  descends to the quotient space of  $X \times \Gamma$  with respect to the equivalence relation given by  $(x, gh) \sim (\alpha_{g^{-1}}(x), h)$  for all  $g, h \in \Gamma$  and  $x \in D_g$ . This space is, up to the reparameterization  $(x, g) \mapsto (g^{-1}, x)$ , the globalization of the partial action  $((D_g)_g, (\alpha_g)_g)$  of  $\Gamma$  on  $X$  (see [16, Theorem 3.5, Proposition 5.5]), and  $C_0(X) \boxtimes C_0(\Gamma)$  can be identified with a  $C^*$ -subalgebra of  $C_b((X \times \Gamma)/\sim)$ .

## 9. Minimal dilations

Among all dilations of a fixed partial coaction  $\delta_C$  of a  $C^*$ -bialgebra  $(A, \Delta)$ , we now single out a universal one, which we call the *globalization* of  $\delta_C$ . More precisely, we show that (1) every dilation of  $\delta_C$  contains one that is minimal in a natural sense, and (2) that all such minimal dilations are isomorphic. We need to assume, however, that  $\delta_C$  is regular and injective, that  $A$  has the slice map property, and, for (2), that  $(A, \Delta)$  is a  $C^*$ -quantum group.

*Definition 9.1.* Let  $\delta_C$  be a partial coaction of  $(A, \Delta)$  on a  $C^*$ -algebra  $C$ . We call a dilation  $(B, \delta_B, \iota)$  of  $\delta_C$  *minimal* if  $\iota(C)$  and  $A^* \triangleright \iota(C)$  generate  $B$  as a  $C^*$ -algebra.

*Remark 9.2.* Let  $(B, \delta_B, \iota)$  be a minimal dilation of a partial coaction  $\delta_C$  of  $(A, \Delta)$  on some  $C^*$ -algebra  $C$ . Then  $\iota(C) \subseteq B$  is an ideal because  $\iota(C)(A^* \triangleright \iota(C)) = \iota(C)\iota(A^* \triangleright C) \subseteq \iota(C)$  by (6.1). If, moreover,  $\iota$  is strict, then  $\iota(C)$  is a direct summand of  $B$ .

*Example 9.3.* If, in the situation above,  $(A, \Delta)$  is the  $C^*$ -bialgebra of functions on a discrete group  $\Gamma$ , then the coaction  $\delta_B$  corresponds to an action  $\alpha$  of  $\Gamma$  on  $B$ , and the dilation is minimal if and only if  $\sum_{g \in \Gamma} \alpha_g(\iota(C))$  generates  $B$  as a  $C^*$ -algebra.

*Example 9.4* (Partial Bernoulli shift). Let  $\mathbb{G} = (C_0(\mathbb{G}), \Delta)$  be a discrete  $C^*$ -quantum group. Denote by  $\delta$  the coaction of  $C_0(\mathbb{G})$  on  $C(\mathbb{B}_{\mathbb{G}})$  (see Section 7), denote by  $\delta^\varepsilon$  its restriction to a partial coaction on  $C(\mathbb{B}_{\mathbb{G}}^\varepsilon)$ , denote by  $\delta^\times$  the coaction of  $C_0(\mathbb{G})$  on  $C_0(\mathbb{B}_{\mathbb{G}}^\times)$  obtained as the restriction of  $\delta$  (see Proposition 7.5), and denote by  $\iota: C(\mathbb{B}_{\mathbb{G}}^\varepsilon) \hookrightarrow C_0(\mathbb{B}_{\mathbb{G}}^\times)$  the inclusion. Then  $(C_0(\mathbb{B}_{\mathbb{G}}^\times), \delta^\times, \iota)$  is a dilation of  $\delta^\varepsilon$  because  $\delta^\varepsilon$  is a restriction of  $\delta^\times$ , and this dilation is minimal. Indeed,  $\delta^\times(p_\varepsilon) = p$  by Lemma 7.6, whence  $C_0(\mathbb{G})^* \triangleright \iota(C_0(\mathbb{B}_{\mathbb{G}}^\varepsilon))$  contains  $p_{ij}^\alpha$  for every  $\alpha, i, j$ , and these elements generate  $C_0(\mathbb{B}_{\mathbb{G}}^\times)$ .

Every dilation contains a minimal one.

**Proposition 9.5.** *Let  $\delta_C$  be a partial coaction of  $(A, \Delta)$  on a  $C^*$ -algebra  $C$  with a dilation  $(B, \delta_B, \iota)$ , and suppose that  $A$  has the slice map property. Denote by  $B_0 \subseteq B$  the  $C^*$ -subalgebra generated by  $\iota(C)$  and  $A^* \triangleright \iota(C)$ .*

- (1) *The coaction  $\delta_B$  restricts to a coaction  $\delta_{B_0}$  on  $B_0$ , and  $(B_0, \delta_{B_0}, \iota)$  is a minimal dilation of  $\delta_C$ .*
- (2) *If  $(A, \Delta)$  is a regular  $C^*$ -quantum group and  $\delta_C$  is weakly continuous, then  $[A^* \triangleright \iota(C)] \subseteq B$  is a  $C^*$ -algebra. If additionally  $\iota$  is strict, then  $B_0 = [(A^* \triangleright \iota(C))(\mathbb{C}1_B + \mathbb{C}\iota(1_C))]$ .*

*Proof.* (1) To prove the first assertion, we only need to show that

$$\delta_B(\iota(C))(1_B \otimes A) \subseteq B_0 \otimes A \quad \text{and} \quad \delta_B(A^* \triangleright \iota(C))(1_B \otimes A) \subseteq B_0 \otimes A.$$

But for all  $c \in C, v, \omega \in A^*$ , both  $(\text{id} \otimes \omega)(\delta_B(\iota(c))) = \omega \triangleright \iota(c)$  and  $(\text{id} \otimes \omega)(\delta_B(v \triangleright \iota(c))) = \omega v \triangleright \iota(c)$  lie in  $B_0$ . Since  $A$  has the slice map property, the desired inclusions follow.

(2) We follow the proof of [9, Proposition 5.7], using the same notation and manipulations as in the proof of Proposition 3.5. To shorten the notation, let  $U := (\pi \otimes \text{id}_{\hat{\pi}(\hat{A})})(V)$  and  $\delta_\pi := (\text{id}_B \otimes \pi) \circ \delta_B \circ \iota$ . Then by (3.3),

$$\begin{aligned} & [A^* \triangleright \iota(C)] \\ &= [A^* \triangleright \iota(C(A^* \triangleright C))] \\ &= [(\text{id}_B \otimes v \otimes \omega)((\delta_B \otimes \text{id}_A)((C \otimes 1_A)\delta_C(C))) : v, \omega \in A^*] \\ &= [(\text{id}_B \otimes v \otimes \omega)((\delta_B \otimes \text{id}_A)((C \otimes 1_A)\delta_B(C))) : v, \omega \in A^*] \\ &= [(\text{id}_B \otimes v \otimes \omega \otimes \omega \circ \pi)((\delta_B(C) \otimes 1_A)(\text{id}_B \otimes \Delta)\delta_B(C)) : v, \omega \in \mathcal{B}(K)_*] \\ &= [(\text{id}_B \otimes v \otimes \omega)(\delta_\pi(C)_{12}U_{23}\delta_\pi(C)_{13}U_{23}^*) : v, \omega \in \mathcal{B}(K)_*] \\ &= [(\text{id}_B \otimes v \otimes \omega)(\delta_\pi(C)_{12}U_{23}\delta_\pi(C)_{13}(\pi(A) \otimes \hat{\pi}(\hat{A}))_{23}) : v, \omega \in \mathcal{B}(K)_*] \\ &= [(\text{id}_B \otimes v \otimes \omega)(\delta_\pi(C)_{12}(\pi(A) \otimes \hat{\pi}(\hat{A}))_{23}\delta_\pi(C)_{13}) : v, \omega \in \mathcal{B}(K)_*] \\ &= [(A^* \triangleright \iota(C))(A^* \triangleright \iota(C))]. \end{aligned}$$

Thus,  $[A^* \triangleright \iota(C)]$  is a  $C^*$ -algebra. If  $\iota$  is strict so that  $\iota(1_C)$  is well defined, then this  $C^*$ -algebra commutes with  $\iota(1_C)$ , and by (6.1) the product is  $[\iota(A^* \triangleright C)] = \iota(C)$ . This proves the last assertion concerning  $B_0$ .  $\square$

If we apply Proposition 9.5 to the canonical dilation  $(C \boxtimes A, \text{id}_C \otimes \Delta, \delta_C)$  constructed in Proposition 8.7, we obtain the following dilation.

**Theorem 9.6.** Let  $(A, \Delta)$  be a  $C^*$ -bialgebra, where  $A$  has the slice map property, and let  $\delta_C$  be an injective, regular partial coaction of  $(A, \Delta)$  on a  $C^*$ -algebra  $C$ . Denote by  $\mathfrak{G}(C) \subseteq M(C \otimes A)$  the  $C^*$ -subalgebra generated by

$$\{(\text{id}_C \otimes \text{id}_C \otimes \omega)(\text{id}_C \otimes \Delta)\delta_C(c) : \omega \in A^*, c \in C\} \quad \text{and} \quad \delta_C(C).$$

Then  $\text{id}_C \otimes \Delta$  restricts to a partial coaction on  $\mathfrak{G}(C)$ , and

$$\mathfrak{G}(\delta_C) := (\mathfrak{G}(C), \text{id}_C \otimes \Delta, \delta_C)$$

is a minimal dilation of  $\delta_C$ .

*Proof.* By an argument similar to that in the proof of Proposition 8.7, we only need to show that for every  $c \in C$  and  $v, \omega \in A^*$ , the elements

$$(\text{id}_C \otimes \text{id}_A \otimes v)((\text{id}_C \otimes \Delta)\delta_C(c))$$

and

$$(\text{id}_C \otimes \text{id}_A \otimes v)((\text{id}_C \otimes \Delta)((\text{id}_C \otimes \text{id}_A \otimes \omega)(\text{id}_C \otimes \Delta)\delta_C(c)))$$

lie in  $\mathfrak{G}(C)$ . In the first case, this is trivially true, and in the second case, one finds that the element is equal to  $d = (\text{id}_C \otimes \text{id}_A \otimes v\omega)((\text{id}_C \otimes \Delta)\delta_C(C)) \in \mathfrak{G}(C)$ .  $\square$

*Remark 9.7.* Suppose that  $(A, \Delta)$  and  $\delta_C$  are as above.

- (1) Be aware that  $\delta_C$  is strict as a map from  $C$  to  $M(C \otimes A)$ , but this does not imply that  $\delta_C$  is strict as a map from  $C$  to  $\mathfrak{G}(C)$ .
- (2) If  $\delta_C$  is weakly continuous, then (3.3) implies that  $\mathfrak{G}(C) \subseteq M(C \otimes A)$  is equal to the  $C^*$ -subalgebra generated by  $\{(\text{id}_C \otimes \text{id}_C \otimes \omega)(\text{id}_C \otimes \Delta)\delta_C(c) : \omega \in A^*, c \in C\}$  and  $\delta_C(1_C)$ .

*Example 9.8* (Case of a partial group action). Consider the partial coaction  $\delta_C$  associated to a disconnected partial action  $((p_g), (\theta_g)_g)$  of a discrete group  $\Gamma$  on a  $C^*$ -algebra  $C$ , and identify  $M(C \otimes C_0(\Gamma))$  with  $C_b(\Gamma; M(C))$ . In that case,  $\mathfrak{G}(C)$  is the  $C^*$ -algebra generated by all functions of the form

$$f_{c,h} = (\text{id}_C \otimes \text{id}_{C_0(\Gamma)} \otimes \text{ev}_h)(\text{id}_C \otimes \Delta)\delta_C(c) : g \mapsto \theta_{gh}(p_{h^{-1}g^{-1}}c),$$

where  $c \in C$  and  $g, h \in \Gamma$ . The action  $\rho$  of  $\Gamma$  corresponding to the coaction  $\text{id}_C \otimes \Delta$  is given by right translation of functions, whence  $\rho_{h'}(f_{c,h}) = f_{c,h'h}$  for all  $h' \in \Gamma$ .

We will use the following notion of a morphism between dilations.

*Definition 9.9.* Let  $\delta_C$  be a partial coaction of a  $C^*$ -bialgebra  $(A, \Delta)$  on some  $C^*$ -algebra  $C$ . A *morphism between dilations*  $\mathcal{B} = (B, \delta_B, \iota^B)$  and  $\mathcal{D} = (D, \delta_D, \iota^D)$  of  $\delta_C$  is a  $*$ -homomorphism  $\phi: B \rightarrow D$  satisfying

$$\begin{aligned} \phi(\iota^B(c)) &= \iota^D(c) \quad \text{and} \\ \delta_D(\phi(b))(1_D \otimes a) &= (\phi \otimes \text{id}_A)(\delta_B(b)(1_B \otimes a)) \end{aligned} \tag{9.1}$$

for all  $c \in C$ ,  $b \in B$ , and  $a \in A$ . Evidently, all dilations of a fixed partial coaction  $\delta_C$  form a category; we denote this category by  $\mathfrak{Dil}(\delta_C)$ .

*Remark 9.10.* The second equation in (9.1) is equivalent to the condition that  $\phi$  is a morphism of left  $A^*$ -modules; that is,  $\omega \triangleright \phi(b) = \phi(\omega \triangleright b)$  for all  $b \in B$  and  $\omega \in A^*$ .

If  $\delta_C$  is injective and regular, then the dilation  $\mathfrak{G}(\delta_C)$  is terminal among the minimal ones.

**Proposition 9.11.** *Let  $\delta_C$  be an injective, regular partial coaction of a  $C^*$ -bialgebra  $(A, \Delta)$  on a  $C^*$ -algebra  $C$ , let  $\mathcal{B} = (B, \delta_B, \iota)$  be a minimal dilation of  $\delta_C$ , and suppose that  $A$  has the slice map property. Then there exists a unique morphism  $\phi_{\mathcal{B}}$  from  $\mathcal{B}$  to  $\mathfrak{G}(\delta_C)$ , and on the level of  $C^*$ -algebras  $\phi_{\mathcal{B}}$  is surjective. For each  $b \in B$ , the image  $\phi_{\mathcal{B}}(b)$  is the restriction of  $\delta_B(b)$  to the ideal  $\iota(C) \otimes A \cong C \otimes A$  in  $B \otimes A$ .*

*Proof.* Uniqueness follows from the fact that  $B$  is generated by  $\iota(C)$  and  $A^* \triangleright \iota(C)$ .

To prove existence, define  $\phi_{\mathcal{B}}$  as in (3). Since  $\iota$  is a weak morphism,  $\phi_{\mathcal{B}} \circ \iota = \delta_C$ . The relation  $(\text{id}_B \otimes \Delta)\delta_B = (\delta_B \otimes \text{id}_A)\delta_B$  implies that

$$(\phi_{\mathcal{B}} \otimes \text{id}_A)(\delta_B(b)(b' \otimes a)) = (\text{id}_C \otimes \Delta)(\phi_{\mathcal{B}}(b))(\phi_{\mathcal{B}}(b') \otimes a)$$

for all  $b, b' \in B$  and  $a \in A$ ; in particular,

$$\phi_{\mathcal{B}}(\omega \triangleright \iota(c))\phi_{\mathcal{B}}(b) = (\text{id}_C \otimes \text{id}_A \otimes \omega)((\text{id}_C \otimes \Delta)\delta_C(C))\phi_{\mathcal{B}}(b)$$

for all  $c \in C$  and  $\omega \in C^*$ . Now, the definition of  $\mathfrak{G}(C)$  and minimality of  $\mathcal{B}$  imply that  $\phi_{\mathcal{B}}(B) = \mathfrak{G}(C)$ .  $\square$

If  $(A, \Delta)$  is a  $C^*$ -quantum group, then the morphism above is injective and hence an isomorphism. To show this, we use the following observation.

**Lemma 9.12.** *Let  $\delta_B$  be a coaction of a  $C^*$ -quantum group  $(A, \Delta)$  on a  $C^*$ -algebra  $B$ , and let  $b, b' \in M(B)$ . Then  $\delta_B(b)(b' \otimes 1_A) = 0$  if and only if  $(b \otimes 1_A)\delta_B(b') = 0$ .*

*Proof.* Choose a modular multiplicative unitary  $W$  for  $(A, \Delta)$  so that  $\Delta(a) = W(a \otimes 1)W^*$  for all  $a \in A$ . Then

$$\begin{aligned} (\delta_B \otimes \text{id}_A)(\delta_B(b)) \cdot (\delta_B(b') \otimes 1_A) &= (\text{id}_B \otimes \Delta)(\delta_B(b)) \cdot (\delta_B(b') \otimes 1_A) \\ &= W_{23}(\delta_B(b) \otimes 1_A)W_{23}^*(\delta_B(b') \otimes 1_A). \end{aligned}$$

Since  $\delta_B \otimes \text{id}_A$  is injective and  $W$  is unitary, we can conclude that  $\delta_B(b)(b' \otimes 1_A) = 0$  if

$$(\delta_B(b) \otimes 1_A)W_{23}^*(\delta_B(b') \otimes 1_A) = 0. \quad (9.2)$$

A similar argument shows that  $(b \otimes 1_A)\delta_B(b') = 0$  if and only if

$$(\delta_B(b) \otimes 1_A)W_{23}(\delta_B(b') \otimes 1_A) = 0. \quad (9.3)$$

Now, both (9.2) and (9.3) are equivalent to the condition  $\delta_B(b)(1_B \otimes \hat{A})\delta_B(b') = 0$ .  $\square$

We can now prove claim (2) stated in the introduction to Section 9.

**Proposition 9.13.** *Let  $\delta_C$  be an injective, regular partial coaction of a  $C^*$ -quantum group  $(A, \Delta)$  on a  $C^*$ -algebra  $C$ , suppose that  $A$  has the slice map property, and let  $\mathcal{B}$  be a minimal dilation of  $\delta_C$ . Then the morphism  $\phi_{\mathcal{B}}$  from  $\mathcal{B}$  to  $\mathfrak{G}(\delta_C)$  is an isomorphism.*

*Proof.* Write  $\mathcal{B} = (B, \delta_B, \iota)$ . It suffices to show that  $\phi_{\mathcal{B}}$  is injective on the level of  $C^*$ -algebras. On the direct summand  $\iota_C(C) \subseteq B$ , the morphism  $\phi_{\mathcal{B}}$  is given by  $\iota_C(c) \mapsto \delta_C(c)$  and hence injective. Since  $B$  is minimal, the direct summand  $(1_B - \iota(1_C))B$  of  $B$  is generated by  $(1_B - \iota(1_C))(A^* \triangleright \iota(C))$ . Given a nonzero  $b \in (1_B - \iota(1_C))B$ , we therefore find some  $c \in C$  such that  $\delta_B(\iota(c))(b \otimes 1_A)$  is nonzero, and then  $(\iota(c) \otimes 1_A)\delta_B(b)$  is nonzero by the lemma above, whence  $\phi_{\mathcal{B}}(b)$  is nonzero.  $\square$

**Corollary 9.14.** *Let  $(A, \Delta)$  be a  $C^*$ -quantum group, and suppose that  $A$  has the slice map property. Then all minimal dilations of an injective, regular partial coaction of  $(A, \Delta)$  are isomorphic.*

**Acknowledgment.** We thank the referees for helpful comments that improved the quality of our manuscript.

Timmermann's work was partially supported by the Deutsche Forschungsgemeinschaft (DFG) project SFB 878 "Groups, Geometry and Actions."

## References

1. F. Abadie, *Enveloping actions and Takai duality for partial actions*, J. Funct. Anal. **197** (2003), no. 1, 14–67. Zbl 1036.46037. MR1957674. DOI 10.1016/S0022-1236(02)00032-0. 844, 864
2. F. Abadie, *Sobre ações parciais, fibrados de Fell e grupóides*, preprint, Ph.D. dissertation, University of São Paulo, São Paulo, Brazil, 1999. 844
3. E. R. Alvares, M. M. S. Alves, and E. Batista, *Partial Hopf module categories*, J. Pure Appl. Algebra **217** (2013), no. 8, 1517–1534. Zbl 1280.16024. MR3030549. DOI 10.1016/j.jpaa.2012.11.008. 844
4. M. M. S. Alves and E. Batista, *Enveloping actions for partial Hopf actions*, Comm. Algebra **38** (2010), no. 8, 2872–2902. Zbl 1226.16022. MR2730285. DOI 10.1080/00927870903095582. 844
5. M. M. S. Alves and E. Batista, "Globalization theorems for partial Hopf (co)actions, and some of their applications" in *Groups, Algebras and Applications*, Contemp. Math. **537**, Amer. Math. Soc., Providence, 2011, 13–30. Zbl 1232.16020. MR2799089. 844
6. M. M. S. Alves, E. Batista, M. Dokuchaev, and A. Paques, *Globalization of twisted partial Hopf actions*, J. Aust. Math. Soc. **101** (2016), no. 1, 1–28. Zbl 06610375. MR3518320. DOI 10.1017/S1446788715000774. 844
7. M. M. S. Alves, E. Batista, and J. Vercruyse, *Partial representations of Hopf algebras*, J. Algebra **426** (2015), 137–187. Zbl 1370.16024. MR3301904. DOI 10.1016/j.jalgebra.2014.12.011. 854, 862
8. S. Baaj and G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de  $C^*$ -algèbres*, Ann. Sci. Éc. Norm. Supér. (4) **26** (1993), no. 4, 425–488. Zbl 0804.46078. MR1235438. 846, 847
9. S. Baaj, G. Skandalis, and S. Vaes, *Non-semi-regular quantum groups coming from number theory*, Comm. Math. Phys. **235** (2003), no. 1, 139–167. Zbl 1029.46113. MR1969723. DOI 10.1007/s00220-002-0780-6. 846, 847, 849, 867
10. E. Bédos and L. Tuset, *Amenability and co-amenable for locally compact quantum groups*, Int. J. Math. **14** (2003), no. 8, 865–884. Zbl 1051.46047. MR2013149. DOI 10.1142/S0129167X03002046. 848

11. B. Blackadar, *Operator Algebras: Theory of  $C^*$ -Algebras and von Neumann Algebras*, Encyclopaedia Math. Sci. **122**, Springer, Berlin, 2006. Zbl 1092.46003. MR2188261. 847
12. S. Caenepeel and K. Janssen, *Partial (co)actions of Hopf algebras and partial Hopf–Galois theory*, Comm. Algebra **36** (2008), no. 8, 2923–2946. Zbl 1168.16021. MR2440292. DOI 10.1080/00927870802110334. 843, 848, 854, 855
13. F. Castro, A. Paques, G. Quadros, and A. Santána, *Partial actions of weak Hopf algebras: Smash product, globalization and Morita theory*, J. Pure Appl. Algebra **219** (2015), no. 12, 5511–5538. Zbl 1331.16024. MR3390037. DOI 10.1016/j.jpaa.2015.05.031. 844
14. R. Exel, *Circle actions on  $C^*$ -algebras, partial automorphisms, and a generalized Pimsner–Voiculescu exact sequence*, J. Funct. Anal. **122** (1994), no. 2, 361–401. Zbl 0808.46091. MR1276163. DOI 10.1006/jfan.1994.1073. 843
15. R. Exel, *Twisted partial actions: A classification of regular  $C^*$ -algebraic bundles*, Proc. Lond. Math. Soc. (3) **74** (1997), no. 2, 417–443. Zbl 0874.46041. MR1425329. 843
16. R. Exel, *Partial dynamical systems, Fell bundles and applications*, Math. Surveys Monogr. **224**, Amer. Math. Soc., Providence, 2017. Zbl 06801030. MR3699795. 843, 851, 862, 864, 866
17. U. Franz and A. Skalski, *On idempotent states on quantum groups*, J. Algebra **322** (2009), no. 5, 1774–1802. Zbl 1176.43005. MR2543634. DOI 10.1016/j.jalgebra.2009.05.037. 850
18. J. Kustermans and S. Vaes, *Locally compact quantum groups*, Ann. Sci. Éc. Norm. Supér. (4) **33** (2000), no. 6, 837–934. Zbl 1034.46508. MR1832993. DOI 10.1016/S0012-9593(00)01055-7. 846
19. E. C. Lance, *Hilbert  $C^*$ -Modules: A Toolkit for Operator Algebraists*, London Math. Soc. Lecture Note Ser. **210**, Cambridge Univ. Press, Cambridge, 1995. Zbl 0822.46080. MR1325694. 847
20. M. B. Landstad and A. Van Daele, *Groups with compact open subgroups and multiplier Hopf  $*$ -algebras*, Expo. Math. **26** (2008), no. 3, 197–217. Zbl 1149.22005. MR2437092. DOI 10.1016/j.exmath.2007.10.004. 850
21. K. McClanahan, *K-theory for partial crossed products by discrete groups*, J. Funct. Anal. **130** (1995), no. 1, 77–117. Zbl 0867.46045. MR1331978. DOI 10.1006/jfan.1995.1064. 843, 851
22. R. Meyer, S. Roy, and S. L. Woronowicz, *Quantum group-twisted tensor products of  $C^*$ -algebras*, Int. J. Math. **25** (2014), no. 2, art. ID rnm1450019. Zbl 1315.46076. MR3189775. 846
23. S. Neshveyev and L. Tuset, *Compact Quantum Groups and Their Representation Categories*, Cours Spéc **20**, Soc. Math. France, Paris, 2013. Zbl 1316.46003. MR3204665. 847
24. S. Roy,  *$C^*$ -quantum groups with projection*, preprint, Ph.D. dissertation, University of Göttingen, Göttingen, Germany, 2013. 846, 847
25. S. Roy and T. Timmermann, *The maximal quantum group-twisted tensor product of  $C^*$ -algebras*, to appear in J. Noncommut. Geom., preprint, arXiv:1511.06100v3 [math.OA]. 846
26. P. M. Sołtan, *Quantum families of maps and quantum semigroups on finite quantum spaces*, J. Geom. Phys. **59** (2009), no. 3, 354–368. Zbl 1160.58007. MR2501746. 860
27. P. M. Sołtan and S. L. Woronowicz, *A remark on manageable multiplicative unitaries*, Lett. Math. Phys. **57** (2001), no. 3, 239–252. Zbl 1009.46037. MR1862455. 846
28. P. M. Sołtan and S. L. Woronowicz, *From multiplicative unitaries to quantum groups, II*, J. Funct. Anal. **252** (2007), no. 1, 42–67. Zbl 1134.46044. MR2357350. 846
29. M. Takesaki and N. Tatsuuma, *Duality and subgroups*, Ann. of Math. (2) **93** (1971), 344–364. Zbl 0201.45503. MR0281843. DOI 10.2307/1970778. 850
30. O. Uuye and J. Zacharias, *The Fubini product and its applications*, preprint, arXiv:1603.04527v1 [math.OA]. 848
31. S. Wassermann, *The slice map problem for  $C^*$ -algebras*, Proc. Lond. Math. Soc. (3) **32** (1976), no. 3, 537–559. Zbl 0321.46048. MR0410402. DOI 10.1112/plms/s3-32.3.537. 844, 848

32. S. L. Woronowicz, *From multiplicative unitaries to quantum groups*, Int. J. Math. **7** (1996), no. 1, 127–149. Zbl 0876.46044. MR1369908. DOI 10.1142/S0129167X96000086. 846
33. S. L. Woronowicz, “Compact quantum groups” in *Symétries Quantiques (Les Houches, 1995)*, North-Holland, Amsterdam, 1998, 845–884. Zbl 0997.46045. MR1616348. 847, 856

FB MATHEMATIK UND INFORMATIK, UNIVERSITY OF MUENSTER, EINSTEINSTR. 62, 48149 MUENSTER, GERMANY.

*E-mail address:* [franzi.kraken@googlemail.com](mailto:franzi.kraken@googlemail.com); [paula.quast@web.de](mailto:paula.quast@web.de);  
[timmermt@math.uni-muenster.de](mailto:timmermt@math.uni-muenster.de)