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EQUATIONS FOR FRAME WAVELETS IN $L^2(\mathbb{R}^2)$

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Abstract. A finite solution to a system of equations will generate a single function normalized tight frame wavelet (Parseval's frame wavelet) with compact support associated with a 2×2 expansive integral matrix whose determinant is either 2 or -2 in $L^2(\mathbb{R}^2)$.

1. Introduction

In this article, $\mathbb{R}^2$ will be the 2-dimensional Euclidean space, and $\mathbb{C}^2$ will be the 2-dimensional complex Euclidean space. We will use notation $\vec{t}, \vec{s}, \vec{r}, \vec{\xi}, \vec{\eta}$ for vectors in $\mathbb{R}^2$ or $\mathbb{C}^2$. We will use the notation $\vec{t}_1 \circ \vec{t}_2$ for the standard inner product of two vectors $\vec{t}_1$ and $\vec{t}_2$. For a vector $\vec{\xi} = \binom{\xi_1}{\xi_2}$ in $\mathbb{C}^2$, its real part is $\mathfrak{Re}(\vec{\xi}) \equiv \binom{\mathfrak{Re}(\xi_1)}{\mathfrak{Re}(\xi_2)}$, and its imaginary part is $\mathfrak{Im}(\vec{\xi}) \equiv \binom{\mathfrak{Im}(\xi_1)}{\mathfrak{Im}(\xi_2)}$. The measure μ will be the Lebesgue measure on $\mathbb{R}^2$, and $L^2(\mathbb{R}^2)$ will be the Hilbert space of all square integrable functions on $\mathbb{R}^2$. A (countable) set of elements $\{\psi_i : i \in \Lambda\}$ in $L^2(\mathbb{R}^2)$ is called a *normalized tight frame* of $L^2(\mathbb{R}^2)$ if

$$\sum_{i\in\Lambda} \left|\langle f, \psi_i\rangle\right|^2 = \|f\|^2, \quad \forall f \in L^2(\mathbb{R}^2). \tag{1.1}$$

It is well known in the literature (see [4]) that the equation (1.1) is equivalent to

$$f = \sum_{i\in\Lambda} \langle f, \psi_i\rangle \psi_i, \quad \forall f \in L^2(\mathbb{R}^2). \tag{1.2}$$

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Let $\mathbb{Z}^2$ be the integer lattice in $\mathbb{R}^2$. For a vector $\vec{\ell} \in \mathbb{Z}^2$, the *translation operator* $T_{\vec{\ell}}$ is defined as

$$(T_{\vec{\ell}} f)(\vec{t}) \equiv f(\vec{t} - \vec{\ell}), \quad \forall f \in L^2(\mathbb{R}^2).$$

A square matrix is called *expansive* if all of its eigenvalues have absolute values greater than 1. Let A be a 2×2 expansive integral matrix with eigenvalues $\{\lambda_1, \lambda_2\}$. The norm of the linear transformation A on $\mathbb{R}^2$ (or $\mathbb{C}^2$) will be $\|A\| \equiv \max\{|\beta_1|, |\beta_2|\}$. For two vectors $\vec{t}_1, \vec{t}_2$ in the Euclidean space $\mathbb{R}^2$, we have $\vec{t}_1 \circ A\vec{t}_2 = A^\tau \vec{t}_1 \circ \vec{t}_2$, where A^τ is the transpose matrix of A. We define operator U_A as

$$(U_A f)(\vec{t}) \equiv \left(\sqrt{|\det(A)|}\right) f(A\vec{t}), \quad \forall f \in L^2(\mathbb{R}^2).$$

This is a unitary operator on $L^2(\mathbb{R}^2)$. In particular, for an expansive matrix A with $|\det(A)| = 2$, we will use D_A to denote U_A and we will call it a *dilation operator*.

Definition 1.1. Let A be an expansive integral matrix with $|\det(A)| = 2$. A function $\psi \in L^2(\mathbb{R}^2)$ is called a *normalized tight frame wavelet*, or NTFW, associated with A, if the set

$$\{D_A^n T_{\vec{\ell}} \psi, n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^2\} \tag{1.3}$$

constitutes a normalized tight frame of $L^2(\mathbb{R}^2)$.

Remark 1.2. The function ψ is called a *single function* NTFW since the frame set (1.1) is generated by one function ψ. An NTFW is not necessarily a unit vector in $L^2(\mathbb{R}^2)$ unless it is an orthonormal wavelet. By definition, an element $\psi \in L^2(\mathbb{R}^2)$ is an NTFW if and only if

$$\|f\|^2 = \sum_{n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^2} \left|\langle f, D_A^n T_{\vec{\ell}} \psi \rangle\right|^2, \quad \forall f \in L^2(\mathbb{R}^2). \tag{1.4}$$

By (1.2), this is equivalent to

$$f(\vec{t}) = \sum_{n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^2} \langle f, D_A^n T_{\vec{\ell}} \psi \rangle D_A^n T_{\vec{\ell}} \psi(\vec{t}), \quad \forall f \in L^2(\mathbb{R}^2), \vec{t} \in \mathbb{R}^2 \text{ a.e.} \tag{1.5}$$

The literature on wavelet theory is very rich. Many authors have made significant contributions to the theory, so much so that it is hard to make a short list. But the following works are closely related to the present article.

Q. Gu and D. Han [7] proved that, if an integral expansive matrix associates with single function *orthogonal wavelets* with *multi-resolution analysis* (MRA), then the absolute value of the matrix determinant must be 2. These orthogonal wavelets are special single function normalized tight frame wavelets. In this paper we will construct single function normalized tight frame wavelets with compact support associated with expansive integral matrices with determinant ± 2 in $L^2(\mathbb{R}^2)$. We do not find many examples of normalized frame wavelets (orthogonal wavelets) associated with integral matrices constructed by using traditional methods. The reader can find these examples in [10] by J. Lagarias and Y. Wang, and in [9] by I. A. Krishtal, B. D. Robinson, G. L. Weiss, and E. N. Wilson for

Haar-type orthonormal wavelets (and hence with compact support) in $L^2(\mathbb{R}^2)$ and for higher dimensions. In [2], E. Belogay and Y. Wang constructed examples of wavelets with compact support and with properties of high smoothness in $L^2(\mathbb{R}^2)$. Other wavelets in higher dimensions with noncompact supports were also constructed in [5] by Larson, Speegle, and the author. The goal of the present paper is to prove that every solution to the system of equations (3.1) will generate filters for normalized tight frame wavelets. In applications, we just need to solve the systems of equations for filters. Some computational methods, including the Monte Carlo method (see [8]), are capable of serving this purpose. Compared with [10], our methods appear to be more constructive. Moreover, we address single-function Parseval wavelets, including orthogonal wavelets, and our methods here provide a foundation for further work on frame wavelets with normal properties, such as the wavelets in [2].

We will follow the classical method for constructing such frame wavelets as provided by I. Daubechies in [6], that is, from the filter function m_0 to the scaling function φ and then to the wavelet function ψ. To construct the filter function m_0, we start with the system of equations (3.1). The system of equations (3.1) is a generalization of W. Lawton's system of equations for frame wavelets in $L^2(\mathbb{R})$ (see [11]).

The scaling function φ in this paper is not necessarily orthogonal, and so the related MRA system constructed should be similar to the *frame multiresolution analysis* (FMRA) by J. Benedetto and S. Li in [3], and it is also related to the *general multiresolution analysis* (GMRA) by L. Baggett, H. Medina, and K. Merrill [1]. We provide some examples in Section 7.

2. Reduction theorems

In [2], E. Belogay and Y. Wang also proved that every expansive 2×2 integral matrix with $|\det(A)| = 2$ can be expressed in the form SBS^{-1}, where S is a 2×2 integral matrix with $|\det(S)| = 1$ and B is one of the six matrices listed below:

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \\ \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}. \tag{2.1}$$

Proposition 2.1. *Let A be a 2×2 expansive integral matrix with $|\det(A)| = 2$. Then there is a 2×2 integral matrix S with $|\det(S)| = 1$ such that SAS^{-1} is one of the following six matrices:*

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \\ \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}. \tag{2.2}$$

Proof. This statement is an immediate consequence of the list (2.1) by E. Belogay and Y. Wang [2] and the following calculation:

$$\begin{bmatrix}-1 & 1\\ 2 & -3\end{bmatrix}\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}\begin{bmatrix}-1 & 1\\ 2 & -3\end{bmatrix}^{-1} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},$$
$$\begin{bmatrix}1 & -1\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & 2\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & -1\end{bmatrix}^{-1} = \begin{bmatrix}1 & -3\\ 1 & -1\end{bmatrix},$$
$$\begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & 2\\ -1 & 1\end{bmatrix}\begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}^{-1} = \begin{bmatrix}-1 & 2\\ -2 & 2\end{bmatrix},$$
$$\begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & -2\\ 1 & -1\end{bmatrix}\begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}^{-1} = \begin{bmatrix}1 & -2\\ 2 & -2\end{bmatrix}.$$ □

Lemma 2.2. *Let A be a 2×2 expansive integral matrix with $|\det(A)| = 2$. For a 2×2 integral matrix S of $|\det(S)| = 1$, assume $B = S^{-1}AS$. Then*

$$U_S T_{\vec{\ell}} U_S^{-1} = T_{S^{-1}\vec{\ell}}, \quad \forall \vec{\ell} \in \mathbb{Z}^2; \tag{2.3}$$
$$U_S D_A^n U_S^{-1} = D_B^n, \quad \forall n \in \mathbb{Z}. \tag{2.4}$$

Proof. Let $h \in L^2(\mathbb{R}^2)$. By definition, $U_S U_{S^{-1}} h(\vec{t}) = U_S h(S^{-1}\vec{t}) = h(SS^{-1}\vec{t}) = h(\vec{t})$, and so $U_S U_{S^{-1}} = I$. Similarly, we have $U_{S^{-1}} U_S = I$. Therefore, $U_S^{-1} = U_{S^{-1}}$. For $\vec{\ell} \in \mathbb{Z}^d$, we have

$$\begin{aligned}U_S T_{\vec{\ell}} U_S^{-1} h(\vec{t}) &= U_S T_{\vec{\ell}} U_{S^{-1}} h(\vec{t})\\ &= U_S T_{\vec{\ell}} h(S^{-1}\vec{t})\\ &= U_S h\big(S^{-1}(\vec{t} - \vec{\ell})\big)\\ &= U_S h(S^{-1}\vec{t} - S^{-1}\vec{\ell})\\ &= h(S^{-1}S\vec{t} - S^{-1}\vec{\ell})\\ &= h(\vec{t} - S^{-1}\vec{\ell})\\ &= T_{S^{-1}\vec{\ell}} h(\vec{t}),\end{aligned}$$

and so we have equation (2.3). Also, we have $U_S D_A U_S^{-1} h(\vec{t}) = \sqrt{2} h(S^{-1}AS\vec{t}) = D_B h(\vec{t})$, and so

$$\begin{aligned}U_S D_A U_S^{-1} &= D_B,\\ U_S D_A^{-1} U_S^{-1} &= (U_S D_A U_S^{-1})^{-1} = D_B^{-1}.\end{aligned}$$

This implies that, for all $n \in \mathbb{N}$,

$$\begin{aligned}U_S D_A^n U_S^{-1} &= (U_S D_A U_S^{-1})^n = D_B^n;\\ U_S D_A^{-n} U_S^{-1} &= (U_S D_A U_S^{-1})^{-n} = D_B^{-n}.\end{aligned}$$

This proves equation (2.4). □

Theorem 2.3. *Let A be a 2×2 expansive integral matrix with $|\det(A)| = 2$, and let S be a 2×2 integral matrix with the property that $|\det(S)| = 1$. Let $B \equiv S^{-1}AS$. Assume that a function ψ_A is a normalized tight frame wavelet* *associated with the matrix A. Then the function $\eta_B \equiv U_S\psi_A$ is a normalized tight frame wavelet associated with the matrix B.*

Proof. By assumption and Lemma 2.2, we have $B = S^{-1}AS, D_B = U_B = U_{S^{-1}AS} = U_S U_A U_{S^{-1}} = U_S D_A U_S^{-1}$. Let $f \in L^2(\mathbb{R}^2)$. We have

$$U_S^{-1} f = \sum_{n\in\mathbb{Z},\vec{\ell}\in\mathbb{Z}^2} \langle U_S^{-1} f, D_A^n T_{\vec{\ell}}\psi_A\rangle D_A^n T_{\vec{\ell}}\psi_A.$$

Since U_S is a unitary operator, we have

$$\begin{aligned} f &= \sum_{n\in\mathbb{Z},\vec{\ell}\in\mathbb{Z}^2} \langle f, U_S D_A^n T_{\vec{\ell}}\psi_A\rangle U_S D_A^n T_{\vec{\ell}}\psi_A \\ &= \sum_{n\in\mathbb{Z},\vec{\ell}\in\mathbb{Z}^2} \langle f, U_S D_A^n U_S^{-1} U_S T_{\vec{\ell}} U_S^{-1} U_S\psi_A\rangle U_S D_A^n U_S^{-1} U_S T_{\vec{\ell}} U_S^{-1} U_S\psi_A \\ &= \sum_{n\in\mathbb{Z},\vec{\ell}\in\mathbb{Z}^2} \langle f, D_B^n T_{S^{-1}\vec{\ell}}\eta_B\rangle D_B^n T_{S^{-1}\vec{\ell}}\eta_B \\ &= \sum_{n\in\mathbb{Z},\vec{\ell}\in S^{-1}\mathbb{Z}^2} \langle f, D_B^n T_{\vec{\ell}}\eta_B\rangle D_B^n T_{\vec{\ell}}\eta_B. \end{aligned}$$

Since S is an integral matrix with $|\det(S)| = 1$, we have $\mathbb{Z}^2 = S\mathbb{Z}^2 = S^{-1}\mathbb{Z}^2$, and so we have

$$f = \sum_{n\in\mathbb{Z},\vec{\ell}\in\mathbb{Z}^2} \langle f, D_B^n T_{\vec{\ell}}\eta_B\rangle D_B^n T_{\vec{\ell}}\eta_B.$$

□

For $f, g \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, the Fourier transform and Fourier inverse transform are defined as

$$\begin{aligned} (\mathcal{F}f)(\vec{s}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\vec{s}\circ\vec{t}} f(\vec{t})\, d\vec{t} = \widehat{f}(\vec{s}), \\ (\mathcal{F}^{-1}g)(\vec{t}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\vec{s}\circ\vec{t}} g(\vec{s})\, d\vec{s} = \check{g}(\vec{t}). \end{aligned}$$

The set $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$, and the operator $\mathcal{F}$ extends to a unitary operator on $L^2(\mathbb{R}^2)$ which is still called the Fourier transform. For an operator V on $L^2(\mathbb{R}^2)$, we will write $\mathcal{F}V\mathcal{F}^{-1} \equiv \widehat{V}$. We will use the following formulas in this paper.

Lemma 2.4. *Let A be a 2×2 expansive integral matrix. Then*

$$\begin{aligned} T_{\vec{\ell}} D_A &= D_A T_{A\vec{\ell}}, \\ \widehat{T}_{\vec{\ell}} &= M_{e^{-i\vec{s}\circ\vec{\ell}}}, \\ \widehat{D}_A &= U_{(A^{-1})^\tau} = U_{(A^\tau)^{-1}} = D_{A^\tau}^{-1} = D_{A^\tau}^*, \end{aligned}$$

where $M_{e^{-i\vec{s}\circ\vec{\ell}}}$ is the multiplication operator by $e^{-i\vec{s}\circ\vec{\ell}}$. Operators $T_{\vec{\ell}}, D_A, \mathcal{F}$, and $M_{e^{-i\vec{s}\circ\vec{\ell}}}$ are unitary operators acting on $L^2(\mathbb{R}^2)$.

Remark 2.5. For the translation operator $T_{A^{-J}\vec{\ell}}$, where vector $A^{-J}\vec{\ell}$ is in the refined lattice $A^{-J}\mathbb{Z}^2$, we have $T_{A^{-J}\vec{\ell}}D_A^J = D_A^J T_{\vec{\ell}}$. We also have $\widehat{D_A^J\widehat{\varphi}(\vec{t}\,)} = \frac{1}{\sqrt{2^J}}\widehat{\widehat{\varphi}((A^\tau)^{-J}\vec{t}\,)}$. We will need this in the proof of Lemma 5.7. We leave these to the reader to verify using the same method as in the proof of Lemma 2.4.

Proof. We have

$$\begin{aligned}
(\widehat{T}_{\vec{\ell}}\widehat{f})(\vec{s}\,) &= (\mathcal{F}T_{\vec{\ell}}\mathcal{F}^{-1}\mathcal{F}f)(\vec{s}\,)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\vec{s}\circ\vec{t}} f(\vec{t}-\vec{\ell}\,)\,d\vec{t}\\
&= e^{-i\vec{s}\circ\vec{\ell}}\cdot\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\vec{s}\circ\vec{u}} f(\vec{u}\,)\,d\vec{u}\\
&= e^{-i\vec{s}\circ\vec{\ell}}\cdot\widehat{f}(\vec{s}\,)\\
&= (M_{e^{-i\vec{s}\circ\vec{\ell}}}\widehat{f})(\vec{s}\,).
\end{aligned}$$

Here the substitution $\vec{u} = \vec{t} - \vec{\ell}$ is used. Next, we have

$$\begin{aligned}
(\widehat{D}_A\widehat{f})(\vec{s}\,) &= (\mathcal{F}D_Af)(\vec{s}\,)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\vec{s}\circ\vec{t}}\cdot\sqrt{2}f(A\vec{t}\,)\,d\mu\\
&= \frac{1}{\sqrt{2}}\cdot\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\vec{s}\circ(A^{-1}\vec{u}\,)} f(\vec{u}\,)\,d\nu\\
&= \frac{1}{\sqrt{2}}\cdot\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i(A^{-1})^\tau\vec{s}\circ\vec{u}} f(\vec{u}\,)\,d\nu.
\end{aligned}$$

Here substitutions $\vec{u} = A\vec{t}$ and $d\nu = 2\,d\mu$ are used, and so we have

$$\begin{aligned}
(\widehat{D}_A\widehat{f})(\vec{s}\,) &= \sqrt{\det\bigl((A^{-1})^\tau\bigr)}\cdot\widehat{f}\bigl((A^{-1})^\tau\vec{s}\,\bigr)\\
&= (U_{(A^{-1})^\tau}\widehat{f})(\vec{s}\,).
\end{aligned}$$

This implies that

$$\widehat{D}_A = U_{(A^{-1})^\tau} = U_{(A^\tau)^{-1}} = D_{A^\tau}^{-1} = D_{A^\tau}^*.$$

Also, we have

$$\begin{aligned}
T_{\vec{\ell}}D_Af(\vec{t}\,) &= \sqrt{2}T_{\vec{\ell}}f(A\vec{t}\,)\\
&= \sqrt{2}f\bigl(A(\vec{t}-\vec{\ell}\,)\bigr)\\
&= \sqrt{2}f(A\vec{t}-A\vec{\ell}\,)\\
&= D_AT_{A\vec{\ell}}f(\vec{t}\,),\\
T_{\vec{\ell}}D_A &= D_AT_{A\vec{\ell}}.
\end{aligned}$$

□

The integral lattice $\mathbb{Z}^2$ is an abelian group under vector addition. The subset $(2\mathbb{Z})^2$ is a subgroup. For a fixed 2×2 integral matrix A with $|\det(A)| = 2$, the two sets $A\mathbb{Z}^2$ and $A^\tau\mathbb{Z}^2$ are proper subgroups of $\mathbb{Z}^2$ containing $(2\mathbb{Z})^2$. The two quotient groups $\frac{A\mathbb{Z}^2}{(2\mathbb{Z})^2}$ and $\frac{A^\tau\mathbb{Z}^2}{(2\mathbb{Z})^2}$ are two proper subgroups of the quotient group

$\frac{\mathbb{Z}^2}{(2\mathbb{Z})^2}$ which has 4 elements, $\{\binom{0}{0}+(2\mathbb{Z})^2, \binom{0}{1}+(2\mathbb{Z})^2, \binom{1}{0}+(2\mathbb{Z})^2, \binom{1}{1}+(2\mathbb{Z})^2\}$. If the two elements of the subgroup $\frac{A\mathbb{Z}^2}{(2\mathbb{Z})^2}$ are $\vec{0}+(2\mathbb{Z})^2, \vec{s}+(2\mathbb{Z})^2$, we will call $\{\vec{0}, \vec{s}\}$ the *generators* for $A\mathbb{Z}^2$. We define the generators for $A^\tau\mathbb{Z}^2$ in a similar way: $A\mathbb{Z}^2 = A^\tau\mathbb{Z}^2$ if and only if they have the same generators (in the four elements).

Proposition 2.6. *Let A be one of the six matrices in (2.2) as in Proposition 2.1. Then there exist vectors $\vec{\ell}_A$ and $\vec{q}_A$ in $\mathbb{Z}^2$ with the following properties:*

(1) $\mathbb{Z}^2 = A^\tau\mathbb{Z}^2 \uplus (\vec{\ell}_A + A^\tau\mathbb{Z}^2)$;
(2) $\vec{q}_A \circ A^\tau\mathbb{Z}^2 \subseteq 2\mathbb{Z}$ *and* $\vec{q}_A \circ (\vec{\ell}_A + A^\tau\mathbb{Z}^2) \subseteq 2\mathbb{Z}+1$;
(3) $A^\tau\vec{q}_A \in (2\mathbb{Z})^2$;
(4) $A\mathbb{Z}^2 = A^\tau\mathbb{Z}^2$.

Remark 2.7. Equation (4) is not true in general. Let A be $\left[\begin{smallmatrix}0 & -2\\ 1 & -1\end{smallmatrix}\right]$, which is in the list (2.1). Then $A^\tau = \left[\begin{smallmatrix}0 & 1\\ -2 & -1\end{smallmatrix}\right]$. It is left to the reader to verify that $\{\binom{0}{0}, \binom{1}{1}\}$ is the generator for $A^\tau\mathbb{Z}^2$, while the generator for $A\mathbb{Z}^2$ is $\{\binom{0}{0}, \binom{0}{1}\}$, and so $A\mathbb{Z}^2 \neq A^\tau\mathbb{Z}^2$.

Proof. 1. Let $A = \left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$. Then $A^\tau = \left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$. It is left to the reader to verify that $\{\binom{0}{0}, \binom{1}{1}\}$ is the generator for both $A\mathbb{Z}^2$ and $A^\tau\mathbb{Z}^2$. Therefore, we have equation (4), $A\mathbb{Z}^2 = A^\tau\mathbb{Z}^2$,[1] since

$$A\mathbb{Z}^2 = \left(\begin{pmatrix}0\\0\end{pmatrix} + (2\mathbb{Z})^2\right) \uplus \left(\begin{pmatrix}1\\1\end{pmatrix} + (2\mathbb{Z})^2\right),$$
$$A^\tau\mathbb{Z}^2 = \left(\begin{pmatrix}0\\0\end{pmatrix} + (2\mathbb{Z})^2\right) \uplus \left(\begin{pmatrix}1\\1\end{pmatrix} + (2\mathbb{Z})^2\right).$$

This also implies that

$$\mathbb{Z}^2 \backslash A^\tau\mathbb{Z}^2 = \left(\begin{pmatrix}1\\0\end{pmatrix} + (2\mathbb{Z})^2\right) \uplus \left(\begin{pmatrix}0\\1\end{pmatrix} + (2\mathbb{Z})^2\right) = \begin{pmatrix}1\\0\end{pmatrix} + A^\tau\mathbb{Z}^2$$

since $\{(\binom{0}{0}+(2\mathbb{Z})^2), (\binom{1}{1}+(2\mathbb{Z})^2), (\binom{1}{0}+(2\mathbb{Z})^2), (\binom{0}{1}+(2\mathbb{Z})^2)\}$ is a partition of $\mathbb{Z}^2$, and so the vector $\vec{\ell}_A \equiv \binom{1}{0}$ satisfies the equation

$$\mathbb{Z}^2 = A^\tau\mathbb{Z}^2 \uplus (\vec{\ell}_A + A^\tau\mathbb{Z}^2).$$

Define $\vec{q}_A \equiv \binom{1}{1}$. It is left to the reader to verify that $q_A \circ A^\tau\mathbb{Z}^2$ are even numbers and $\vec{q}_A \circ (\vec{\ell}_A + A^\tau\mathbb{Z}^2)$ are odd numbers since $\vec{q}_A \circ \vec{\ell}_A = 1$. Finally, $A^\tau\vec{q}_A = \left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]\binom{1}{1} = \binom{2}{0} \in (2\mathbb{Z})^2$. This proves property (3).

2. We list all six matrices in the list (2.2) and their corresponding $\vec{\ell}_A$ and $\vec{q}_A$ in the next table. The reader may verify, as we did in part 1, the equations in (1), (2), and (3) are satisfied. Also, it is left to the reader to verify that, for each matrix in the six cases, the generators for $A\mathbb{Z}^2$ and $A^\tau\mathbb{Z}^2$ are the same, and so

[1] In this example, $A = A^\tau$. We do not assume this condition in general.

we have now established the equation in (4):

$$
\begin{array}{cccccc}
A & A^\tau & \text{gen of } A^\tau\mathbb{Z}^2 & \vec{\ell}_A & \vec{q}_A & A^\tau\vec{q}_A \\[1ex]
\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} & \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} & \begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}2\\0\end{pmatrix} \\[1ex]
\begin{bmatrix}1 & -3\\ 1 & -1\end{bmatrix} & \begin{bmatrix}1 & 1\\ -3 & -1\end{bmatrix} & \begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}2\\-4\end{pmatrix} \\[1ex]
\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} & \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} & \begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}0\\2\end{pmatrix} \\[1ex]
\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix} & \begin{bmatrix}-1 & 1\\ -1 & -1\end{bmatrix} & \begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}0\\-2\end{pmatrix} \\[1ex]
\begin{bmatrix}-1 & 2\\ -2 & 2\end{bmatrix} & \begin{bmatrix}-1 & -2\\ 2 & 2\end{bmatrix} & \begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}-2\\2\end{pmatrix} \\[1ex]
\begin{bmatrix}1 & -2\\ 2 & -2\end{bmatrix} & \begin{bmatrix}1 & 2\\ -2 & -2\end{bmatrix} & \begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}2\\-2\end{pmatrix}
\end{array}
\tag{2.5}
$$

$\square$

Two subsets $\mathcal{G}_1, \mathcal{G}_2$ of $\mathbb{R}^2$ are said to be 2-*translation equivalent*, or $\mathcal{G}_1 \overset{2}{\sim} \mathcal{G}_2$, if there exists a mapping Θ from $\mathcal{G}_1$ onto $\mathcal{G}_2$ with the property that

$$\Theta(\vec{t}\,) - \vec{t} \in (2\mathbb{Z})^2, \quad \vec{t} \in \mathcal{G}_1 \text{ a.e.}$$

Proposition 2.8. *Let A be one of the six expansive matrices in Proposition 2.6, and let $\vec{q}_A$ be the corresponding vector related to A and $\Gamma_0 \equiv [-1,1]^2$. Then there are two measurable sets Γ_1 and Γ_2 such that*

$$\begin{aligned} \Gamma_1 &\overset{2}{\sim} \Gamma_0; \\ \Gamma_2 &\overset{2}{\sim} \Gamma_0; \\ A^\tau\Gamma_0 &\overset{2}{\sim} \Gamma_1 \mathbin{\dot\cup} (\vec{q}_A + \Gamma_2). \end{aligned}$$

Corollary 2.9. *Let $\Gamma_\pi \equiv \pi\Gamma_0 = [-\pi,\pi]^2$, and let $h(\vec{\xi}\,)$ be a 2π-periodical continuous function on $\mathbb{R}^2$. Then*

$$\int_{A^\tau\Gamma_\pi} h(\vec{\xi}\,)\, d\mu = \int_{\Gamma_\pi} h(\vec{\xi}\,)\, d\mu + \int_{\Gamma_\pi + \pi\vec{q}_A} h(\vec{\xi}\,)\, d\mu. \tag{2.6}$$

Proof. 1. For any matrix A in the collection $\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}-1 & -1\\ 1 & -1\end{smallmatrix}\right]$, $A^\tau\Gamma_0$ has the same vertices of $\{\left(\begin{smallmatrix}2\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\-2\end{smallmatrix}\right)\}$. By the table in the proof of Proposition 2.6, the above three matrices share the same vector $\vec{q}_A = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and $\Gamma_0 \subset A^\tau\Gamma_0$ (Figure 1, left). It is enough to discuss only one of the three cases.

Consider $A = \left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$. Let $\Gamma_1 \equiv \Gamma_0$ and $\Gamma_2 \equiv A^\tau\Gamma_0 \backslash \Gamma_1$. Notice that $\Gamma_2 + \vec{q}_A$ (Figure 1, middle) is a disjoint union of eight triangles $\{\triangle_k, k = 1, 2, \ldots, 8\}$. The

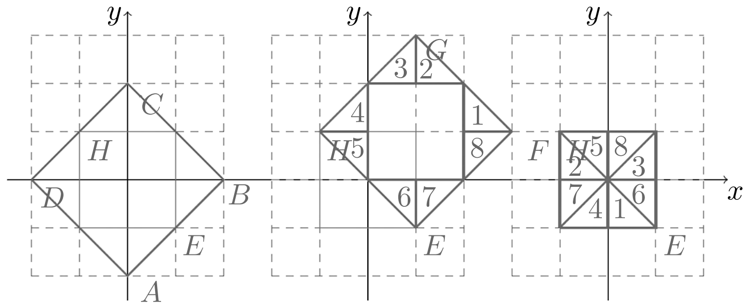

FIGURE 1. $A^{\tau}\Gamma_0$.

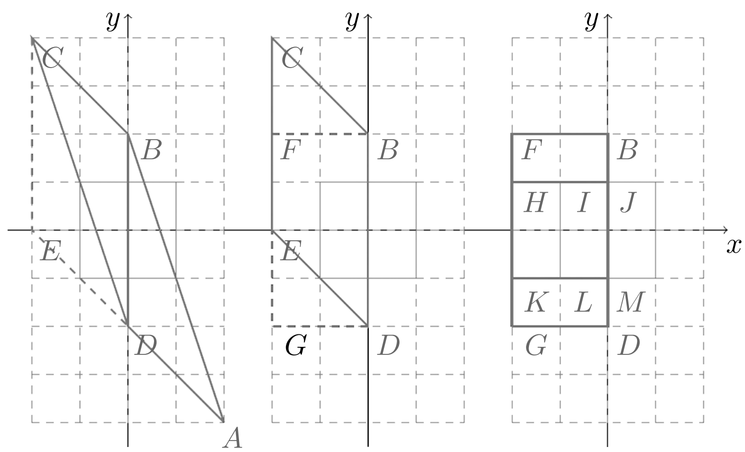

FIGURE 2. $A^{\tau}\Gamma_0$.

following new triangles $\{\triangle'_k, k = 1, 2, \dots, 8\}$ form a partition for Γ_0 modulus zero measure sets (Figure 1, right).

We have $\triangle'_1 = \triangle_1 + \left(\begin{smallmatrix}-2\\-2\end{smallmatrix}\right); \triangle'_2 = \triangle_2 + \left(\begin{smallmatrix}-2\\-2\end{smallmatrix}\right); \triangle'_3 = \triangle_3 + \left(\begin{smallmatrix}0\\-2\end{smallmatrix}\right); \triangle'_4 = \triangle_4 + \left(\begin{smallmatrix}0\\-2\end{smallmatrix}\right); \triangle'_5 = \triangle_5; \triangle'_6 = \triangle_6; \triangle'_7 = \triangle_7 + \left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right); \triangle'_8 = \triangle_8 + \left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right)$. This proves that $\vec{q}_A + \Gamma_2 \overset{2}{\sim} \Gamma_0$.

2. Let $A = \left[\begin{smallmatrix}1&-3\\1&-1\end{smallmatrix}\right]$. Then $A^{\tau} = \left[\begin{smallmatrix}1&1\\-3&-1\end{smallmatrix}\right]$ and $\vec{q}_A = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$. $A^{\tau}\Gamma_0$ is the parallelogram $ABCD$ (Figure 2, left). It is 2-translation equivalent to parallelogram $BCED$ (Figure 2, middle), which is the disjoint union of $\triangle ABD + \left(\begin{smallmatrix}-2\\2\end{smallmatrix}\right)$ and $\triangle CBD$. The parallelogram $BCED$ is 2-translation equivalent to rectangle $BFGD$ since $\triangle DGE = \triangle BFC + \left(\begin{smallmatrix}0\\-4\end{smallmatrix}\right)$. Now let Γ_1 be the square $MJHK$. It is 2-translation equivalent to Γ_0 since $\Gamma_0 = MJIL \uplus (LIHK + \left(\begin{smallmatrix}2\\0\end{smallmatrix}\right))$. Let $\Gamma_2 \equiv \square JBFH \uplus \square DMKG$. Thus $((\square JBFH + \left(\begin{smallmatrix}0\\-2\end{smallmatrix}\right)) \uplus \square DMKG) + \vec{q}_A = \Gamma_0$.

3. For any matrix A in the collection $\left[\begin{smallmatrix}-1&2\\-2&2\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1&-2\\2&-2\end{smallmatrix}\right]$, its corresponding $A^{\tau}\Gamma_0$ has the same vertices $\{\left(\begin{smallmatrix}3\\-4\end{smallmatrix}\right), \left(\begin{smallmatrix}-3\\4\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)\}$ and the same vector $\vec{q}_A = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. It is enough to discuss only one of the cases.

Let $A = \left[\begin{smallmatrix}1&-2\\2&-2\end{smallmatrix}\right]$. Then, by Proposition 2.8, $A^{\tau} = \left[\begin{smallmatrix}1&2\\-2&-2\end{smallmatrix}\right]$ and $\vec{q}_A = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. As in Figure 3 (left), $A^{\tau}\Gamma_0$ is the parallelogram $ABCD$. Then we have

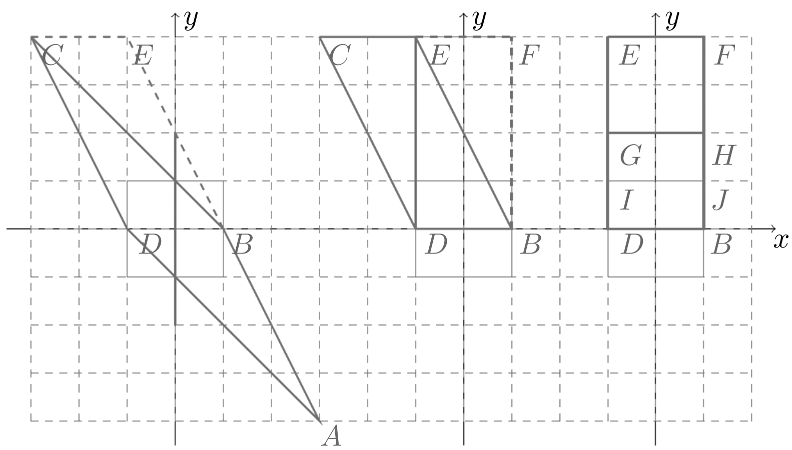

FIGURE 3. $A^\tau\Gamma_0$.

$$\begin{aligned}A^\tau\Gamma_0 &\overset{2}{\sim} \triangle BCD \,\dot\cup\, \left(\triangle ABD + \begin{pmatrix}-2\\4\end{pmatrix}\right)\\ &\overset{2}{\sim} \triangle BED \,\dot\cup\, \left(\triangle DEC + \begin{pmatrix}2\\0\end{pmatrix}\right)\\ &= \square BFED,\end{aligned}$$

and so we have shown $A^\tau\Gamma_0 \overset{2}{\sim} \square BFED$.

Let $\Gamma_1 \equiv \square BHGD$ and $\Gamma_2 \equiv \square HFEG$. We have $\Gamma_1 \overset{2}{\sim} \Gamma_0$ since

$$\begin{aligned}\square BHGD &= \square BJID \,\dot\cup\, \square JHGI\\ &\overset{2}{\sim} \square BJID \,\dot\cup\, \left(\square JHGI + \begin{pmatrix}0\\-2\end{pmatrix}\right)\\ &= \Gamma_0.\end{aligned}$$

Also, since $\square HFEG + \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \Gamma_0 + \left(\begin{smallmatrix}0\\4\end{smallmatrix}\right)$, we have $\Gamma_2 + \vec{q}_A \overset{2}{\sim} \Gamma_0$. $\square$

3. LAWTON'S EQUATIONS AND FILTER FUNCTION

Throughout the rest of this paper, A will be one of the six matrices as stated in list (2.2). Let $N_0 \in \mathbb{N}$ and $\mathcal{S} = \{h_{\vec{n}} : \vec{n} \in \mathbb{Z}^2\}$ be a complex solution to the system of equations

$$\begin{cases}\sum_{\vec{n}\,\in\mathbb{Z}^2} h_{\vec{n}}\,\overline{h_{\vec{n}\,+\vec{k}}} = \delta_{\vec{0}\,\vec{k}}\,, & \vec{k} \in A^\tau\mathbb{Z}^2,\\ \sum_{\vec{n}\,\in\mathbb{Z}^2} h_{\vec{n}} \;= \sqrt{2}\end{cases} \tag{3.1}$$

with the property that $h_{\vec{n}} = 0$ for all $\vec{n} \in \mathbb{Z}^2\backslash[-N_0, N_0]^2$. Let us denote $\Lambda_0 \equiv \mathbb{Z}^2 \cap [-N_0, N_0]^2$. Here δ is Kronecker's notation. We will call the system of equations (3.1) *Lawton's system of equations for normalized frame wavelets in* 2*D*, or *Lawton's equations*.

Define

$$m_0(\vec{t}\,) = \frac{1}{\sqrt{2}} \sum_{\vec{n}\,\in\mathbb{Z}^2} h_{\vec{n}}\, e^{-i\vec{n}\circ\vec{t}} = \frac{1}{\sqrt{2}} \sum_{\vec{n}\,\in\Lambda_0} h_{\vec{n}}\, e^{-i\vec{n}\circ\vec{t}}, \quad \vec{t} \in \mathbb{C}^2. \tag{3.2}$$

This is a finite sum and $m_0(0) = 1$. It is a 2π-periodic trigonometric polynomial function in the sense that $m_0(\vec{t}\,) = m_0(\vec{t} + \pi\vec{t}_0), \forall \vec{t}_0 \in (2\mathbb{Z})^2$.

Proposition 3.1. *Let A be an expansive 2×2 integral matrix, and let $\vec{q}_A$ be as stated in Proposition 2.6. Let m_0 be defined as in (3.2). Then m_0 satisfies*

$$\left|m_0(\vec{t}\,)\right|^2 + \left|m_0(\vec{t} + \pi\vec{q}_A)\right|^2 = 1, \quad \forall \vec{t} \in \mathbb{R}^2. \tag{3.3}$$

Remark 3.2. By Proposition 3.1, we have $|m_0(\vec{t}\,)| \le 1$ for all $\vec{t} \in \mathbb{R}^2$. Also, (3.3) may not hold for $\vec{t} \in \mathbb{C}^2$ in general.

Proof. We have[2]

$$\begin{aligned}
&\left|m_0(\vec{t}\,)\right|^2 + \left|m_0(\vec{t} + \pi\vec{q}_A)\right|^2 \\
&\quad = \frac{1}{2}\Big|\sum_{\vec{m}\,\in\mathbb{Z}^2} h_{\vec{m}}\, e^{-i\vec{m}\circ\vec{t}}\Big|^2 + \frac{1}{2}\Big|\sum_{\vec{m}\,\in\mathbb{Z}^2} h_{\vec{m}}\, e^{-i\vec{m}\circ(\vec{t}+\pi\cdot\vec{q}_A)}\Big|^2 \\
&\quad = \frac{1}{2}\Big[\sum_{\vec{m}\,\in\mathbb{Z}^2,\vec{n}\,\in\mathbb{Z}^2} h_{\vec{m}}\,\overline{h_{\vec{n}}}\, e^{-i(\vec{m}\,-\vec{n}\,)\circ\vec{t}} + \sum_{\vec{m}\,\in\mathbb{Z}^2,\vec{n}\,\in\mathbb{Z}^2} (-1)^{(\vec{m}\,-\vec{n}\,)\circ\vec{q}_A} h_{\vec{m}}\,\overline{h_{\vec{n}}}\, e^{-i(\vec{m}\,-\vec{n}\,)\circ\vec{t}}\Big] \\
&\quad = \frac{1}{2}\Big[\sum_{\vec{m}\,\in\mathbb{Z}^2,\vec{k}\,\in\mathbb{Z}^2} h_{\vec{m}}\,\overline{h_{\vec{m}\,+\vec{k}}}\, e^{i\vec{k}\circ\vec{t}} + \sum_{\vec{m}\,\in\mathbb{Z}^2,\vec{k}\,\in\mathbb{Z}^2} (-1)^{-\vec{k}\circ\vec{q}_A} h_{\vec{m}}\,\overline{h_{\vec{m}\,+\vec{k}}}\, e^{i\vec{k}\circ\vec{t}}\Big].
\end{aligned}$$

Here $\vec{n}$ is replaced by $\vec{m} + \vec{k}$.

By Proposition 2.6, $\vec{k} \circ \vec{q}_A$ is odd when $\vec{k} \in (\ell_A + A^\tau\mathbb{Z}^2)$. In the second sum, the terms $(-1)^{-\vec{k}\circ\vec{q}_A} h_{\vec{m}}\,\overline{h_{\vec{m}\,+\vec{k}}}\, e^{i\vec{k}\circ\vec{t}}$ cancel terms $h_{\vec{m}}\,\overline{h_{\vec{m}\,+\vec{k}}}\, e^{i\vec{k}\circ\vec{t}}$ in the first sum. The term $\vec{k} \circ \vec{q}_A$ is even when $\vec{k} \in A^\tau\mathbb{Z}^2$, and so by definition of $\{h_{\vec{t}}\}$ we have

$$\begin{aligned}
\left|m_0(\vec{t}\,)\right|^2 + \left|m_0(\vec{t} + \pi\vec{q}_A)\right|^2 &= \sum_{\vec{m}\in\mathbb{Z}^2,\vec{k}\in A^\tau\mathbb{Z}^2} h_{\vec{m}}\overline{h_{\vec{m}+\vec{k}}} e^{i\vec{k}\circ\vec{t}} \\
&= \sum_{\vec{k}\in A^\tau\mathbb{Z}^2}\Big(\sum_{\vec{m}\in\mathbb{Z}^2} h_{\vec{m}}\overline{h_{\vec{m}+\vec{k}}}\Big) e^{i\vec{k}\circ\vec{t}} = \sum_{\vec{k}\in A^\tau\mathbb{Z}^2} \delta_{\vec{0}\vec{k}} e^{i\vec{k}\circ\vec{t}} = 1. \qquad \square
\end{aligned}$$

4. The frame scaling function

Define

$$g(\vec{\xi}\,) = \frac{1}{2\pi}\prod_{j=1}^{\infty} m_0\big((A^\tau)^{-j}\vec{\xi}\,\big), \quad \forall \vec{\xi} \in \mathbb{R}^2 \qquad \text{and} \tag{4.1}$$

$$\varphi = \mathcal{F}^{-1} g. \tag{4.2}$$

In this section we will prove that g and φ are well-defined $L^2(\mathbb{R}^2)$ functions. We will also prove that, in the extended domain $\mathbb{C}^2$, g is an entire function and φ has a compact support in $\mathbb{R}^2$. We will call φ the *scaling function*.

[2]In the calculation, the infinite sum is always converging since there are only finitely many nonzero terms.

For $z \in \mathbb{C}$, define

$$v(z) = \begin{cases} \frac{e^z-1}{z}, & z \neq 0, \\ 1, & z = 0. \end{cases} \tag{4.3}$$

The function $v(z)$ is an entire function on $\mathbb{C}$.

We will need the following inequality in the proofs of Lemma 4.8 and Proposition 4.9.

Lemma 4.1. *We have*

$$|e^{-iz} - 1| \leq \min\big(2, |z|\big), \quad \forall z \in \mathbb{C}, \mathfrak{Im}(z) \leq 0.$$

Proof. Let $z = a + ib$ with $b = \mathfrak{Im}(z) \leq 0$, and so we have

$$|e^{-iz} - 1| \leq 1 + |e^{-iz}| \leq 1 + e^b \leq 2. \tag{4.4}$$

Next we will show that

$$|e^{-iz} - 1| \leq |z|, \quad \forall b \leq 0.$$

We have

$$\begin{aligned} |e^{-iz} - 1|^2 &= e^{2b} - 2e^b \cos a + 1 \\ &= (e^b - 1)^2 + 2e^b(1 - \cos a). \end{aligned}$$

Since $e^b > 1+b, \forall b \neq 0$, when $b < 0$, $b^2 = (-b)^2 > (1-e^b)^2$. Also, $2e^b(1-\cos a) \leq 2(1 - \cos a) = 4\sin^2 \frac{a}{2} \leq a^2$, and so

$$|e^{-iz} - 1|^2 \leq b^2 + a^2 = |z|^2.$$

This proves the inequality. $\square$

Lemma 4.2. *Let A be an expansive integral matrix with $|\det(A)| = 2$, let Ω be a bounded closed region in $\mathbb{C}^2$, and let $d_j(\vec{\xi}) \equiv m_0((A^\tau)^{-j}\vec{\xi}) - 1$. Then*

$$\big|d_j(\vec{\xi})\big| \leq C_\Omega \big\|(A^\tau)^{-1}\big\|^j, \quad \forall j \in \mathbb{N}, \vec{\xi} \in \Omega, \tag{4.5}$$

for some constant $C_\Omega > 0$.

Proof. By definition of $h_{\vec{n}}$, we have

$$\begin{aligned} \big|d_j(\vec{\xi})\big| &= \big|m_0\big((A^\tau)^{-j}\vec{\xi}\big) - 1\big| \\ &= \Big|\frac{1}{\sqrt{2}} \sum_{\vec{n} \in \mathbb{Z}^2} h_{\vec{n}}\, e^{-i\vec{n}\circ(A^\tau)^{-j}\vec{\xi}} - 1\Big| \\ &= \Big|\frac{1}{\sqrt{2}} \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}}\, (e^{-i\vec{n}\circ(A^\tau)^{-j}\vec{\xi}} - 1)\Big| \\ &= \Big|\frac{1}{\sqrt{2}} \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}}\, v\big(-i\vec{n} \circ (A^\tau)^{-j}\vec{\xi}\big)\big[-i\vec{n} \circ (A^\tau)^{-j}\vec{\xi}\big]\Big| \\ &\leq \frac{1}{\sqrt{2}} \sum_{\vec{n} \in \Lambda_0} \big|v\big(-i\vec{n} \circ (A^\tau)^{-j}\vec{\xi}\big)\big| \cdot \big|-i\vec{n} \circ (A^\tau)^{-j}\vec{\xi}\big| \end{aligned}$$

since $|h_n| \leq 1$ by Remark 3.2.

For $\vec{n} \in \Lambda_0, \vec{\xi} \in \Omega$, we have $|\vec{\xi}| \leq M_\Omega$ for some $M_\Omega > 0$, and

$$\left|-i\vec{n} \circ (A^\tau)^{-j}\vec{\xi}\right| \leq \sqrt{2}N_0 \cdot M_\Omega \cdot \left\|(A^\tau)^{-1}\right\|^j = C_1\left\|(A^\tau)^{-1}\right\|^j \leq C_1,$$

where $C_1 \equiv \sqrt{2}N_0 M_\Omega$. Let C_2 be the finite least upper bound for a continuous function $|v(z)|, |z| \leq C_1$.

Then we have

$$\left|d_j(\vec{\xi})\right| \leq \frac{1}{\sqrt{2}}(2N_0+1)^2 C_1 C_2 \left\|(A^\tau)^{-1}\right\|^j.$$

Therefore,

$$\left|d_j(\vec{\xi})\right| \leq C_\Omega \left\|(A^\tau)^{-1}\right\|^j, \tag{4.6}$$

where $C_\Omega \equiv \frac{1}{\sqrt{2}}(2N_0+1)^2 C_1 C_2$. □

Proposition 4.3. *The function $g(\vec{\xi})$ is an entire function on $\mathbb{C}^2$.*

Remark 4.4. By (3.3) and the definition of g, it is clear that the function g is bounded on $\mathbb{R}^2$.

Proof. For $J \in \mathbb{N}$, define

$$g_J(\vec{\xi}) = \frac{1}{2\pi}\prod_{j=1}^{J} m_0\big((A^\tau)^{-j}\vec{\xi}\big), \quad \forall \vec{\xi} \in \mathbb{C}^2. \tag{4.7}$$

It is clear that g_J is an entire function. We have

$$\begin{aligned} g_J(\vec{\xi}) &= \frac{1}{2\pi}\prod_{j=1}^{J} m_0\big((A^\tau)^{-j}\vec{\xi}\big) \\ &= \frac{1}{2\pi}\prod_{j=1}^{J}\big(1 + d_j(\vec{\xi})\big). \end{aligned}$$

By Lemma 4.2, $\sum |d_j(\vec{\xi})|$ converges uniformly on the bounded region Ω, and the product $\prod_{j=0}^{\infty}(1 + |d_j(\vec{\xi})|)$ converges uniformly on Ω. This implies that g is the uniform limit of a sequence of entire functions g_J. By the Morera theorem, g is an entire function on $\mathbb{C}^2$. □

Proposition 4.5. *The functions g and φ are in $L^2(\mathbb{R}^2)$.*

Proof. We will use Γ_π to denote $[-\pi, \pi]^2$. For $J \in \mathbb{N}$, we define on $\mathbb{R}^2$

$$M_J(\vec{\xi}) = \begin{cases} \prod_{j=1}^{J} |m_0((A^\tau)^{-j}\vec{\xi})|^2, & \text{if } \vec{\xi} \in (A^\tau)^{J+1}\Gamma_\pi, \\ 0, & \text{if } \vec{\xi} \in \mathbb{R}^2 \backslash (A^\tau)^{J+1}\Gamma_\pi. \end{cases}$$

Since A is expansive, A^τ is expansive. We have $\lim M_J(\vec{\xi}) = 4\pi^2|g(\vec{\xi})|^2, \vec{\xi} \in \mathbb{R}^2$. To prove the proposition, by Fatou's lemma, it suffices to show that $\{\int_{\mathbb{R}^2} M_J(\vec{\xi})\, d\vec{\xi}, J \in \mathbb{N}\}$ is a bounded sequence.

We have

$$\begin{aligned}\int_{\mathbb{R}^2} M_J(\vec{\xi})\, d\vec{\xi} &= \int_{(A^\tau)^{J+1}\Gamma_\pi} \prod_{k=1}^{J} \left|m_0\big((A^\tau)^{-k}\vec{\xi}\big)\right|^2 d\vec{\xi} \\ &= \int_{(A^\tau)^{J}(A^\tau\Gamma_\pi)} \left|m_0\big((A^\tau)^{-J}\vec{\xi}\big)\right|^2 \\ &\qquad \cdot \prod_{k=1}^{J-1} \left|m_0\big((A^\tau)^{-k}\vec{\xi}\big)\right|^2 d\vec{\xi}.\end{aligned}$$

Using $\vec{\eta} \equiv (A^\tau)^{-J}\vec{\xi}$, by Proposition 2.8, we have

$$\begin{aligned}\int_{\mathbb{R}^2} M_J(\vec{\xi})\, d\vec{\xi} &= \left|\det\big((A^\tau)^J\big)\right| \int_{A^\tau\Gamma_\pi} \left|m_0(\vec{\eta})\right|^2 \cdot \prod_{m=1}^{J-1} \left|m_0\big((A^\tau)^m\eta\big)\right|^2 d\vec{\eta} \\ &= \left|\det\big((A^\tau)^J\big)\right| \Big(\int_{\Gamma_\pi} \left|m_0(\vec{\eta})\right|^2 \cdot \prod_{m=1}^{J-1} \left|m_0\big((A^\tau)^m\eta\big)\right|^2 d\vec{\eta} \\ &\qquad + \int_{\Gamma_\pi + \pi\vec{q}_A} \left|m_0(\vec{\eta})\right|^2 \cdot \prod_{m=1}^{J-1} \left|m_0\big((A^\tau)^m\eta\big)\right|^2 d\vec{\eta}\Big) \\ &= \left|\det\big((A^\tau)^J\big)\right| \Big(\int_{\Gamma_\pi} \left|m_0(\vec{\eta})\right|^2 \cdot \prod_{m=1}^{J-1} \left|m_0\big((A^\tau)^m\eta\big)\right|^2 d\vec{\eta} \\ &\qquad + \int_{\Gamma_\pi} \left|m_0(\vec{\eta} - \pi\vec{q}_A)\right|^2 \cdot \prod_{m=1}^{J-1} \left|m_0\big((A^\tau)^m\eta - \pi(A^\tau)^m\vec{q}_A\big)\right|^2 d\vec{\eta}\Big).\end{aligned}$$

Since m_0 is 2π-periodical, and by Proposition 2.6(3), we have $A^\tau\vec{q}_A \in (2\mathbb{Z})^2$, and so $\pi(A^\tau)^m\vec{q} \in \pi(2\mathbb{Z})^2$. By Corollary 2.9, we have

$$\begin{aligned}\int_{\mathbb{R}^2} M_J(\vec{\xi})\, d\vec{\xi} &= \left|\det\big((A^\tau)^J\big)\right| \Big(\int_{\Gamma_\pi} \left|m_0(\vec{\eta})\right|^2 \cdot \prod_{m=1}^{J-1} \left|m_0\big((A^\tau)^m\eta\big)\right|^2 d\vec{\eta} \\ &\qquad + \int_{\Gamma_\pi} \left|m_0(\vec{\eta} + \pi\vec{q}_A)\right|^2 \cdot \prod_{m=1}^{J-1} \left|m_0\big((A^\tau)^m\eta\big)\right|^2 d\vec{\eta}\Big) \\ &= \left|\det\big((A^\tau)^J\big)\right| \int_{\Gamma_\pi} \big(\left|m_0(\vec{\eta})\right|^2 + \left|m_0(\vec{\eta} + \pi\vec{q}_A)\right|^2\big) \\ &\qquad \cdot \prod_{m=1}^{J-1} \left|m_0\big((A^\tau)^m\eta\big)\right|^2 d\vec{\eta}.\end{aligned}$$

By equation (3.3) and then using the substitution $\vec{\xi} \equiv (A^\tau)^J \vec{\eta}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} M_J(\vec{\xi})\, d\vec{\xi} &= \left|\det\big((A^\tau)^J\big)\right| \int_{\Gamma_\pi} \prod_{m=1}^{J-1} \left|m_0\big((A^\tau)^m \eta\big)\right|^2 d\vec{\eta} \\ &= \int_{(A^\tau)^J \Gamma_\pi} \prod_{k=1}^{J-1} \left|m_0\big((A^\tau)^{-k}\vec{\xi}\big)\right|^2 d\vec{\xi} \\ &= \int_{\mathbb{R}^2} M_{J-1}(\vec{\xi})\, d\vec{\xi}.\end{aligned}$$

This proves that the sequence $\int_{\mathbb{R}^2} M_J(\vec{\xi})\, d\vec{\xi}$, $J \in \mathbb{N}$ is a constant sequence. Therefore, g is square integrable on $\mathbb{R}^2$. By the Plancherel theorem, the function φ, which is the Fourier inverse transform of g, is in $L^2(\mathbb{R}^2)$. $\square$

Next, we will prove that the scaling function φ has a compact support in $\mathbb{R}^2$. We will need the following Schwartz's Paley–Wiener theorem.

Theorem 4.6 (Schwartz's Paley–Wiener theorem)**.** *An entire function F on $\mathbb{C}^d, d \in \mathbb{N}$, is the Fourier transform of a distribution with compact support in $\mathbb{R}^d$ if and only if there are some constants C, N, and B such that*

$$\left|F(\vec{\xi})\right| \le C\big(1 + |\vec{\xi}|\big)^N e^{B|\mathfrak{Im}(\vec{\xi})|}, \quad \forall \vec{\xi} \in \mathbb{C}^d. \tag{4.8}$$

The distribution is supported on the closed ball of center $\vec{0}$ and radius B.

Remark 4.7. In our current situation, $d = 2$ and the distribution φ is a regular $L^2(\mathbb{R}^2)$ function, as we proved in Proposition 4.5.

Lemma 4.8. *There exist constants B_0, C_0 such that, for all $j \in \mathbb{N}, \vec{\xi} \in \mathbb{C}^2$,*

$$\left|m_0\big((A^\tau)^{-j}\vec{\xi}\big)\right| \le e^{B_0\|(A^\tau)^{-1}\|^j|\mathfrak{Im}(\vec{\xi})|}\big(1 + C_0 \min\big(1, \left\|(A^\tau)^{-1}\right\|^j |\vec{\xi}|\big)\big).$$

Proof. Let $j \in \mathbb{N}, \vec{\xi} \in \mathbb{C}^2$ and $(A^\tau)^{-j}\vec{\xi} = \binom{\xi_1}{\xi_2} \in \mathbb{C}^2$. Define $\vec{\ell}_{\vec{\xi}} = \binom{\ell_1}{\ell_2} \in \mathbb{Z}^2$ by

$$\ell_m = \begin{cases} -N_0, & \text{if } \mathfrak{Im}(\xi_m) \le 0; \\ N_0, & \text{if } \mathfrak{Im}(\xi_m) > 0; \end{cases} \quad m = 1, 2.$$

Then $\mathfrak{Im}((\vec{n} - \vec{\ell}_{\vec{\xi}}) \circ ((A^\tau)^{-j}\vec{\xi})) \le 0$ for $\vec{n} \in \Lambda_0$. We have $|(\vec{n} - \vec{\ell}_{\vec{\xi}})| \le 2\sqrt{2}N_0, \forall \vec{n} \in \Lambda_0$. We denote $B_0 \equiv 4\sqrt{2}N_0$. It is clear that $|\vec{\ell}_{\xi}| \le \frac{B_0}{2}$ and $|(\vec{n} - \vec{\ell}_{\vec{\xi}})| \le \frac{B_0}{2}$. By Lemma 4.1, we have

$$|e^{-i(\vec{n} - \vec{\ell}_{\vec{\xi}})\circ((A^\tau)^{-j}\vec{\xi})} - 1| \le \min\big(2, \left|(\vec{n} - \vec{\ell}_{\vec{\xi}}) \circ \big((A^\tau)^{-j}\vec{\xi}\big)\right|\big), \quad \forall \vec{n} \in \Lambda_0.$$

Then we have

$$|e^{-i(\vec{n} - \vec{\ell}_{\vec{\xi}})\circ((A^\tau)^{-j}\vec{\xi})} - 1| \le \min\big(2, B_0\left\|(A^\tau)^{-1}\right\|^j|\vec{\xi}|\big), \quad \forall \vec{n} \in \Lambda_0. \tag{4.9}$$

We also have

$$\begin{aligned}|e^{-i\vec{\ell}_{\vec{\xi}}\circ((A^\tau)^{-j}\vec{\xi})}| &= e^{\vec{\ell}_{\vec{\xi}}\circ((A^\tau)^{-j}\mathfrak{Im}(\vec{\xi}))} \\ &\le e^{|\vec{\ell}_{\vec{\xi}}|\|(A^\tau)^{-1}\|^j|\mathfrak{Im}(\vec{\xi})|}.\end{aligned}$$

This implies

$$|e^{-i\vec{\ell}_{\vec{\xi}}\circ((A^\tau)^{-j}\vec{\xi})}| \le e^{B_0\|(A^\tau)^{-1}\|^j|\mathfrak{Im}(\vec{\xi})|}. \tag{4.10}$$

Since

$$\begin{aligned} m_0\big((A^\tau)^{-j}\vec{\xi}\big) &= \sum_{\vec{n}\in\mathbb{Z}^2}\frac{1}{\sqrt{2}}h_{\vec{n}}\,e^{-i\vec{n}\circ((A^\tau)^{-j}\vec{\xi})} \\ &= e^{-i\vec{\ell}_{\vec{\xi}}\circ((A^\tau)^{-j}\vec{\xi})}\sum_{\vec{n}\in\Lambda_0}\frac{1}{\sqrt{2}}h_{\vec{n}}\,e^{-i(\vec{n}-\vec{\ell}_{\vec{\xi}})\circ((A^\tau)^{-j}\vec{\xi})} \\ &= e^{-i\vec{\ell}_{\vec{\xi}}\circ((A^\tau)^{-j}\vec{\xi})}\Big(1+\sum_{\vec{n}\in\Lambda_0}\frac{1}{\sqrt{2}}h_{\vec{n}}\,(e^{-i(\vec{n}-\vec{\ell}_{\vec{\xi}})\circ((A^\tau)^{-j}\vec{\xi})}-1)\Big), \end{aligned}$$

by (4.9) and (4.10), we obtain

$$\begin{aligned} \big|m_0\big((A^\tau)^{-j}\vec{\xi}\big)\big| &\le |e^{-i\vec{\ell}_{\vec{\xi}}\circ((A^\tau)^{-j}\vec{\xi})}|\Big(1+\sum_{\vec{n}\in\Lambda_0}\frac{1}{\sqrt{2}}|h_{\vec{n}}|\cdot\big|(e^{-i(\vec{n}-\vec{\ell}_{\vec{\xi}})\circ((A^\tau)^{-j}\vec{\xi})}-1)\big|\Big) \\ &\le e^{B_0\|(A^\tau)^{-1}\|^j|\mathfrak{Im}(\vec{\xi})|}\Big(1+\frac{1}{\sqrt{2}}(2N_0+1)^2\min\big(2,B_0\big\|(A^\tau)^{-1}\big\|^j|\vec{\xi}|\big)\Big) \\ &\le e^{B_0\|(A^\tau)^{-1}\|^j|\mathfrak{Im}(\vec{\xi})|}\big(1+C_0\min\big(1,\big\|(A^\tau)^{-1}\big\|^j|\vec{\xi}|\big)\big), \end{aligned}$$

where $C_0\equiv\max(\sqrt{2}(2N_0+1)^2,\frac{B_0}{\sqrt{2}}(2N_0+1)^2)$. □

Proposition 4.9. *The scaling function φ is an $L^2(\mathbb{R}^2)$ function with compact support.*

Proof. Let $\vec{\xi}\in\mathbb{R}^2,\vec{\xi}\neq\vec{0}$.[3] By Schwartz's Paley–Wiener theorem, it suffices to prove that the function g satisfies the inequality (4.8).

We write $\beta=\|(A^\tau)^{-1}\|^{-1}$. Since A is expansive, $\beta\in(1,\infty)$. We have

$$\prod_{j=1}^{\infty}e^{B_0\|(A^\tau)^{-1}\|^j|\mathfrak{Im}(\vec{\xi})|}=e^{B|\mathfrak{Im}(\vec{\xi})|},$$

where $B\equiv\sum_{j=1}^{\infty}\frac{B_0}{\beta^j}$. Then, by Lemma 4.8, we have

$$\begin{aligned} \big|g(\vec{\xi})\big| &= \Big|\frac{1}{2\pi}\prod_{j=1}^{\infty}m_0\big((A^\tau)^{-j}\vec{\xi}\big)\Big| \\ &\le \frac{1}{2\pi}\prod_{j=1}^{\infty}e^{B_0\|(A^\tau)^{-1}\|^j|\mathfrak{Im}(\vec{\xi})|}\big(1+C_0\min\big(1,\big\|(A^\tau)^{-1}\big\|^j|\vec{\xi}|\big)\big) \\ &\le \frac{1}{2\pi}e^{B|\mathfrak{Im}(\vec{\xi})|}\prod_{j=1}^{\infty}\big(1+C_0\min\big(1,\big\|(A^\tau)^{-1}\big\|^j|\vec{\xi}|\big)\big) \\ &= \frac{1}{2\pi}e^{B|\mathfrak{Im}(\vec{\xi})|}\prod_{j=1}^{\infty}\Big(1+C_0\min\Big(1,\frac{|\vec{\xi}|}{\beta^j}\Big)\Big). \end{aligned}$$

[3]The case when $\vec{\xi}=\vec{0}$ is trivial. We omit it.

On the other hand, the sequence $\{\beta^j\}$ is monotonically increasing to $+\infty$. Let $I_j \equiv [\beta^j, \beta^{j+1}), j \in \mathbb{N}$, and $I_0 \equiv (0, \beta)$. The set of intervals $\{I_j, j \geq 0\}$ is a partition of $(0, \infty)$. Then $|\vec{\xi}| \in I_{j_0}$ for some integer $j_0 \geq 0$. We have

$$\begin{aligned}(1+C_0)^{j_0} &= \beta^{j_0 \log_\beta(1+C_0)} \\ &\leq |\vec{\xi}|^{\log_\beta(1+C_0)} \\ &\leq \big(1+|\vec{\xi}|\big)^N,\end{aligned}$$

where N is the smallest natural number no less than $\log_\beta(1+C_0)$. This is a constant related to A and N_0 only, and so we have

$$\begin{aligned}\big|g(\vec{\xi})\big| &\leq \frac{1}{2\pi}(1+C_0)^{j_0} e^{B|\mathfrak{Im}(\vec{\xi})|} \prod_{j=j_0+1}^{\infty}\Big(1+C_0\min\Big(1, \frac{|\vec{\xi}|}{\beta^j}\Big)\Big) \\ &\leq \big(1+|\vec{\xi}|\big)^N e^{B|\mathfrak{Im}(\vec{\xi})|}\cdot\Big(\frac{1}{2\pi}\prod_{j=j_0+1}^{\infty}\Big(1+C_0\min\Big(1, \frac{|\vec{\xi}|}{\beta^j}\Big)\Big)\Big).\end{aligned}$$

Now, since $|\vec{\xi}| \in I_{j_0} = [\beta^{j_0}, \beta^{j_0+1})$, $\frac{|\vec{\xi}|}{\beta^{j_0+1}} < 1$. We have

$$\begin{aligned}&\frac{1}{2\pi}\prod_{j=j_0+1}^{\infty}\Big(1+C_0\min\Big(1, \frac{|\vec{\xi}|}{\beta^j}\Big)\Big) \\ &\quad= \frac{1}{2\pi}\prod_{j=j_0+1}^{\infty}\Big(1+C_0\frac{|\vec{\xi}|}{\beta^{j_0+1}}\cdot\frac{1}{\beta^{j-(j_0+1)}}\Big) \\ &\quad\leq \frac{1}{2\pi}\prod_{k=0}^{\infty}\Big(1+\frac{C_0}{\beta^k}\Big) \\ &\quad\leq \frac{1}{2\pi}e^{\sum\frac{C_0}{\beta^k}}.\end{aligned}$$

Denote $C \equiv \frac{1}{2\pi}e^{\sum\frac{C_0}{\beta^k}}$. This is a constant decided by the matrix A.

Combining the above argument, we have

$$\big|g(\vec{\xi})\big| \leq C\big(1+|\vec{\xi}|\big)^N e^{B|\mathfrak{Im}(\vec{\xi})|}. \qquad \square$$

5. Normalized tight frame wavelet function

In this section, we will construct a normalized tight frame wavelet function ψ associated with the scaling function φ. By definition (4.1) and Lemma 2.4, we have

$$\begin{aligned}\widehat{\varphi}(\vec{s}) = g(\vec{s}) &= m_0\big((A^\tau)^{-1}\vec{s}\big)\cdot\frac{1}{2\pi}\prod_{j=2}^{\infty}m_0\big((A^\tau)^{-j}\vec{\xi}\big) \\ &= m_0\big((A^\tau)^{-1}\vec{s}\big)g\big((A^\tau)^{-1}\vec{s}\big) \\ &= \frac{1}{\sqrt{2}}\sum_{\vec{n}\in\Lambda_0}h_{\vec{n}}\,e^{-i\vec{n}\circ(A^\tau)^{-1}\vec{s}}g\big((A^\tau)^{-1}\vec{s}\big)\end{aligned}$$

$$= \sum_{\vec{n}\in\Lambda_0} h_{\vec{n}}\, \widehat{T}_{A^{-1}\vec{n}}\, \widehat{D}_A g(\vec{s}\,)$$
$$= \sum_{\vec{n}\in\Lambda_0} h_{\vec{n}}\, \widehat{D}_A \widehat{T}_{\vec{n}}\, \widehat{\varphi}(\vec{s}\,).$$

Taking the Fourier inverse transform on two sides, we have

$$\varphi = \sum_{\vec{n}\in\Lambda_0} h_{\vec{n}}\, D_A T_{\vec{n}}\, \varphi, \qquad \text{or} \tag{5.1}$$

$$\varphi(\vec{t}\,) = \sqrt{2}\sum_{\vec{n}\in\Lambda_0} h_{\vec{n}}\, \varphi(A\vec{t} - \vec{n}\,), \quad \vec{t}\in\mathbb{R}^2. \tag{5.2}$$

Define

$$\sigma_A(\vec{n}\,) = \begin{cases} 0, & \vec{n}\in A\mathbb{Z}^2, \\ 1, & \vec{n}\notin A\mathbb{Z}^2. \end{cases}$$

Remark 5.1. It is clear that we have $\sigma_A(\vec{u} + A\vec{v}\,) = \sigma_A(\vec{u}\,), \forall \vec{u}, \vec{v} \in \mathbb{Z}^2$. By Proposition 2.6(4), $A\mathbb{Z}^2 = A^\tau\mathbb{Z}^2$, and so we have $\sigma_A(\vec{n}\,) = 0$ if and only if $\vec{n} \in A^\tau\mathbb{Z}^2$. Furthermore, we have $\sigma_A(\vec{\ell}_A) = 1$ and $\sigma_A(\vec{\ell}_A - \vec{n}\,) = 1 - \sigma_A(\vec{n}\,), \forall \vec{n}\in\mathbb{Z}^2$.

Definition 5.2. Define a function ψ on $\mathbb{R}^2$ by

$$\psi = \sum_{\vec{n}\in\mathbb{Z}^2} (-1)^{\sigma_A(\vec{n}\,)}\overline{h_{\vec{\ell}_A-\vec{n}}}\, D_A T_{\vec{n}}\, \varphi, \qquad \text{or, equivalently,} \tag{5.3}$$

$$\psi(\vec{t}\,) = \sqrt{2}\sum_{\vec{n}\in\mathbb{Z}^2} (-1)^{\sigma_A(\vec{n}\,)}\overline{h_{\vec{\ell}_A-\vec{n}}}\,\varphi(A\vec{t} - \vec{n}\,), \quad \forall\vec{t}\in\mathbb{R}^2. \tag{5.4}$$

In this section we will prove that the function ψ is a normalized tight frame wavelet associated with the expansive matrix A. It is clear that the function ψ has a compact support since the scaling function φ has a compact support and the sum in the definition for ψ has only finite nonzero terms. For $J\in\mathbb{Z}$ and $f\in L^2(\mathbb{R}^2)$, define

$$I_J \equiv \sum_{\vec{k}\in\mathbb{Z}^2} \langle f, D_A^J T_{\vec{k}}\, \varphi\rangle D_A^J T_{\vec{k}}\, \varphi; \qquad F_J \equiv \sum_{\vec{k}\in\mathbb{Z}^2} \langle f, D_A^J T_{\vec{k}}\, \psi\rangle D_A^J T_{\vec{k}}\, \psi.$$

Lemma 5.3. *Let $f\in L^2(\mathbb{R}^2)$. Then*

$$I_{J+1} = I_J + F_J, \quad \forall J\in\mathbb{Z}. \tag{5.5}$$

Proof. 1. The case $J=0$. By equation (5.1), Definition 5.2, and Lemma 2.4, we have

$$I_0 = \sum_{\vec{k}\in\mathbb{Z}^2} \langle f, T_{\vec{k}}\, \varphi\rangle T_{\vec{k}}\, \varphi$$
$$= \sum_{\vec{k}\in\mathbb{Z}^2} \Big\langle f, T_{\vec{k}} \sum_{\vec{p}\in\mathbb{Z}^2} h_{\vec{p}}\, D_A T_{\vec{p}}\, \varphi\Big\rangle T_{\vec{k}} \sum_{\vec{q}\in\mathbb{Z}^2} h_{\vec{q}}\, D_A T_{\vec{q}}\, \varphi$$
$$= \sum_{\vec{p}\in\mathbb{Z}^2}\sum_{\vec{q}\in\mathbb{Z}^2}\sum_{\vec{k}\in\mathbb{Z}^2} \overline{h_{\vec{p}}}\, h_{\vec{q}}\, \langle f, D_A T_{\vec{p}+A\vec{k}}\, \varphi\rangle D_A T_{\vec{q}+A\vec{k}}\, \varphi,$$

$$\begin{aligned}
F_0 &= \sum_{\vec{k}\in\mathbb{Z}^2}\langle f, T_{\vec{k}}\,\psi\rangle T_{\vec{k}}\,\psi \\
&= \sum_{\vec{k}\in\mathbb{Z}^2}\Big\langle f, T_{\vec{k}}\sum_{\vec{p}\in\mathbb{Z}^2}(-1)^{\sigma_A(\vec{p})}\overline{h_{\vec{\ell}_A-\vec{p}}}D_A T_{\vec{p}}\,\varphi\Big\rangle T_{\vec{k}}\sum_{\vec{q}\in\mathbb{Z}^2}(-1)^{\sigma_A(\vec{q})}\overline{h_{\vec{\ell}_A-\vec{q}}}D_A T_{\vec{q}}\,\varphi \\
&= \sum_{\vec{p}\in\mathbb{Z}^2}\sum_{\vec{q}\in\mathbb{Z}^2}\sum_{\vec{k}\in\mathbb{Z}^2}(-1)^{\sigma_A(\vec{p})+\sigma_A(\vec{q})}h_{\vec{\ell}_A-\vec{p}}\overline{h_{\vec{\ell}_A-\vec{q}}}\langle f, D_A T_{\vec{p}+A\vec{k}}\,\varphi\rangle D_A T_{\vec{q}+A\vec{k}}\,\varphi.
\end{aligned}$$

Using the substitutions $\vec{m}\equiv\vec{p}+A\vec{k}$ and $\vec{n}\equiv\vec{q}+A\vec{k}$, we have, by Remark 5.1,

$$\begin{aligned}
I_0 &= \sum_{\vec{m},\vec{n}\in\mathbb{Z}^2}\sum_{\vec{k}\in\mathbb{Z}^2}\overline{h_{\vec{m}-A\vec{k}}}h_{\vec{n}-A\vec{k}}\langle f, D_A T_{\vec{m}}\,\varphi\rangle D_A T_{\vec{n}}\,\varphi, \\
F_0 &= \sum_{\vec{m},\vec{n}\in\mathbb{Z}^2}\sum_{\vec{k}\in\mathbb{Z}^2}(-1)^{\sigma_A(\vec{m}-A\vec{k})+\sigma_A(\vec{n}-A\vec{k})}h_{\vec{\ell}_A-\vec{m}+A\vec{k}}\overline{h_{\vec{\ell}_A-\vec{n}+A\vec{k}}}\langle f, D_A T_{\vec{m}}\varphi\rangle D_A T_{\vec{n}}\varphi \\
&= \sum_{\vec{m},\vec{n}\in\mathbb{Z}^2}\sum_{\vec{k}\in\mathbb{Z}^2}(-1)^{\sigma_A(\vec{m})+\sigma_A(\vec{n})}h_{\vec{\ell}_A-\vec{m}+A\vec{k}}\overline{h_{\vec{\ell}_A-\vec{n}+A\vec{k}}}\langle f, D_A T_{\vec{m}}\,\varphi\rangle D_A T_{\vec{n}}\,\varphi.
\end{aligned}$$

We will use the notation

$$\begin{aligned}
\alpha_{\vec{m},\vec{n}} &\equiv \sum_{\vec{k}\in\mathbb{Z}^2}\overline{h_{\vec{m}-A\vec{k}}}h_{\vec{n}-A\vec{k}} \\
&= \sum_{\vec{\ell}\in A\mathbb{Z}^2}\overline{h_{\vec{m}+\vec{\ell}}}h_{\vec{n}+\vec{\ell}} = \sum_{\vec{\ell}\in\vec{n}+A\mathbb{Z}^2}\overline{h_{\vec{\ell}+(\vec{m}-\vec{n})}}h_{\vec{\ell}}, \\
\beta_{\vec{m},\vec{n}} &\equiv \sum_{\vec{k}\in\mathbb{Z}^2}(-1)^{\sigma_A(\vec{m})+\sigma_A(\vec{n})}h_{\vec{\ell}_A-\vec{m}+A\vec{k}}\overline{h_{\vec{\ell}_A-\vec{n}+A\vec{k}}} \\
&= (-1)^{\sigma_A(\vec{m})+\sigma_A(\vec{n})}\sum_{\vec{\ell}\in\vec{\ell}_A-\vec{m}+A\mathbb{Z}^2}\overline{h_{\vec{\ell}+(\vec{m}-\vec{n})}}h_{\vec{\ell}}.
\end{aligned}$$

If $\vec{m}-\vec{n}\in A\mathbb{Z}^2$, then $\sigma(\vec{m})=\sigma_A(\vec{n})$, $(-1)^{\sigma_A(\vec{m})+\sigma_A(\vec{n})}=1$, and $(\vec{\ell}_A-\vec{m}+A^\tau\mathbb{Z}^2)\cup(\vec{n}+A^\tau\mathbb{Z}^2)=\mathbb{Z}^2$. By Lawton's equations (3.1), we have

$$\begin{aligned}
\alpha_{\vec{m},\vec{n}}+\beta_{\vec{m},\vec{n}} &= \sum_{\vec{\ell}\in\vec{n}+A\mathbb{Z}^2}\overline{h_{\vec{\ell}+(\vec{m}-\vec{n})}}h_{\vec{\ell}} + \sum_{\vec{\ell}\in\vec{\ell}_A-\vec{m}+A\mathbb{Z}^2}\overline{h_{\vec{\ell}+(\vec{m}-\vec{n})}}h_{\vec{\ell}} \\
&= \sum_{\vec{\ell}\in\mathbb{Z}^2}\overline{h_{\vec{\ell}+(\vec{m}-\vec{n})}}h_{\vec{\ell}} = \delta_{\vec{m},\vec{n}}.
\end{aligned}$$

If $\vec{m}-\vec{n}\in\vec{\ell}_A+A\mathbb{Z}^2$, then exactly one element of $\vec{m}$ and $\vec{n}$ is in $A\mathbb{Z}^2$ and the other one is in $\vec{\ell}_A+A\mathbb{Z}^2$. Then $(-1)^{\sigma_A(\vec{m})+\sigma_A(\vec{n})}=-1$ and $(\vec{\ell}_A-\vec{m}+A^\tau\mathbb{Z}^2)=(\vec{n}+A^\tau\mathbb{Z}^2)$; hence,

$$\begin{aligned}
\alpha_{\vec{m},\vec{n}}+\beta_{\vec{m},\vec{n}} &= \sum_{\vec{\ell}\in\vec{n}+A^\tau\mathbb{Z}^2}\overline{h_{\vec{\ell}+(\vec{m}-\vec{n})}}h_{\vec{\ell}} - \sum_{\vec{\ell}\in\vec{\ell}_A-\vec{m}+A^\tau\mathbb{Z}^2}\overline{h_{\vec{\ell}+(\vec{m}-\vec{n})}}h_{\vec{\ell}} \\
&= 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned} I_0 + F_0 &= \sum_{\vec{m},\vec{n}\in\mathbb{Z}^2} (\alpha_{\vec{m},\vec{n}} + \beta_{\vec{m},\vec{n}})\langle f, D_A T_{\vec{m}}\varphi\rangle D_A T_{\vec{n}}\varphi \\ &= \sum_{\vec{m},\vec{n}\in\mathbb{Z}^2} \delta_{\vec{m},\vec{n}}\langle f, D_A T_{\vec{m}}\varphi\rangle D_A T_{\vec{n}}\varphi \\ &= \sum_{\vec{k}\in\mathbb{Z}^2} \langle f, D_A T_{\vec{k}}\varphi\rangle D_A T_{\vec{k}}\varphi \\ &= I_1. \end{aligned}$$

This is

$$\sum_{\vec{k}\in\mathbb{Z}^2} \langle f, D_A T_{\vec{k}}\varphi\rangle D_A T_{\vec{k}}\varphi = \sum_{\vec{k}\in\mathbb{Z}^2} \langle f, T_{\vec{k}}\varphi\rangle T_{\vec{k}}\varphi + \sum_{\vec{k}\in\mathbb{Z}^2} \langle f, T_{\vec{k}}\psi\rangle T_{\vec{k}}\psi. \tag{5.6}$$

2. The general case. Let $f \in L^2(\mathbb{R}^2)$. We replace f by $(D_A^*)^J f$ in equation (5.6), where D_A^* is the unitary operator dual to D_A. Then we have

$$\sum_{\vec{k}\in\mathbb{Z}^2} \big\langle (D_A^J)^* f, D_A T_{\vec{k}}\varphi\big\rangle D_A T_{\vec{k}}\varphi = \sum_{\vec{k}\in\mathbb{Z}^2} \big\langle (D_A^J)^* f, T_{\vec{k}}\varphi\big\rangle T_{\vec{k}}\varphi + \sum_{\vec{k}\in\mathbb{Z}^2} \big\langle (D_A^J)^* f, T_{\vec{k}}\psi\big\rangle T_{\vec{k}}\psi.$$

Apply D_A^J to both sides of the equation. By using $\langle (D_A^J)^* f, h\rangle = \langle f, D_A^J h\rangle$, we obtain the desired general equality (5.5). $\square$

In the rest of this section, we will establish the main result of this paper. We state Theorem 5.4 first. We will complete the proof through lemmas and propositions. For $f \in L^2(\mathbb{R}^2)$ and $J \in \mathbb{Z}$, we will use the following notation:

$$\begin{aligned} L_J(f) &\equiv \sum_{\vec{\ell}\in\mathbb{Z}^2} \left|\langle f, D_A^J T_{\vec{\ell}}\varphi\rangle\right|^2; \qquad \text{in particular,} \\ L_0(f) &= \sum_{\vec{\ell}\in\mathbb{Z}^2} \left|\langle f, T_{\vec{\ell}}\varphi\rangle\right|^2. \end{aligned}$$

For a positive number ρ we define functions f_ρ and $f_{\overline{\rho}}$ by $\widehat{f_\rho} \equiv \widehat{f}\cdot\chi_{\{|\vec{t}|\le\rho\}}$ and $\widehat{f_{\overline{\rho}}} \equiv \widehat{f}\cdot\chi_{\{|\vec{t}|>\rho\}}$, respectively. Here χ is the characteristic function. Then we have $f = f_\rho + f_{\overline{\rho}}$. Also, it is clear that $\|f\|^2 = \|\widehat{f}\|^2 = \|f_\rho\|^2 + \|f_{\overline{\rho}}\|^2$, $\lim_{\rho\to\infty}\|f_\rho\|^2 = \|f\|^2$, and $\lim_{\rho\to\infty}\|f_{\overline{\rho}}\|^2 = 0$.

Theorem 5.4. *Let ψ be as defined in Definition 5.2. Then $\{D_A^n T_{\vec{\ell}}\psi, n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^2\}$ is a normalized tight frame for $\mathrm{L}^2(\mathbb{R}^2)$.*

Proof. Let $f \in L^2(\mathbb{R}^2)$. We will prove that

$$f = \sum_{n\in\mathbb{Z}} \sum_{\vec{\ell}\in\mathbb{Z}^2} \langle f, D_A^n T_{\vec{\ell}}\psi\rangle D_A^n T_{\vec{\ell}}\psi; \tag{5.7}$$

the convergence is in the $L^2(\mathbb{R}^2)$-norm.

By Lemma 5.3, we have $I_j - I_{j-1} = F_{j-1}$, $\forall j \in \mathbb{Z}$. Hence

$$\sum_{j=-J+1}^{J} F_j = I_J - I_{-J}, \quad \forall J \in \mathbb{Z}.$$

This implies that

$$\begin{aligned}
&\sum_{j=-J+1}^{J} \sum_{\vec{\ell} \in \mathbb{Z}^2} \langle f, D_A^j T_{\vec{\ell}}\psi\rangle D_A^j T_{\vec{\ell}}\psi \\
&\quad = \sum_{\vec{\ell} \in \mathbb{Z}^2} \langle f, D_A^J T_{\vec{\ell}}\varphi\rangle D_A^J T_{\vec{\ell}}\varphi - \sum_{\vec{\ell} \in \mathbb{Z}^2} \langle f, D_A^{-J} T_{\vec{\ell}}\varphi\rangle D^{-J} T_{\vec{\ell}}\varphi.
\end{aligned}$$

Taking the inner product of f with both sides of the equation, we have

$$\sum_{j=-J+1}^{J} \sum_{\vec{\ell} \in \mathbb{Z}^2} \left|\langle f, D_A^j T_{\vec{\ell}}\psi\rangle\right|^2 = L_J(f) - L_{-J}(f).$$

By Propositions 5.6 and 5.8, we have

$$\lim_{J\to+\infty} L_J(f) = \|f\|^2;$$
$$\lim_{J\to+\infty} L_{-J}(f) = 0.$$

Then we have

$$\sum_{j\in\mathbb{Z}} \sum_{\vec{\ell} \in \mathbb{Z}^2} \left|\langle f, D_A^j T_{\vec{\ell}}\psi\rangle\right|^2 = \|f\|^2, \quad \forall f \in L^2(\mathbb{R}^2).$$

□

To complete the proof of Theorem 5.4, we will prove Propositions 5.6 and 5.8 and the related lemmas. We first need the following.

Lemma 5.5. *Let $f \in L^2(\mathbb{R}^2)$. Then*

$$L_J(f) = \sum_{\vec{\ell} \in \mathbb{Z}^2} \left|\langle f, D_A^J T_{\vec{\ell}}\varphi\rangle\right|^2 \le (2B+1)^2\|\varphi\|^2\|f\|^2, \quad \forall J \in \mathbb{Z}; \tag{5.8}$$

$$\lim_{\rho\to\infty} \limsup_{J\to+\infty} L_J(f_{\overline{\rho}}) = 0. \tag{5.9}$$

Proof. By Proposition 4.9, the scaling function φ has a compact support. Let B be a natural number such that the set $[-B, B)^2$ contains the support of φ. We will write $E_0 \equiv [-\frac{1}{2}, \frac{1}{2})^2$, $E_B \equiv [-B - \frac{1}{2}, B + \frac{1}{2})^2$, and $\Lambda_B \equiv \mathbb{Z}^2 \cap [-B, B]^2$. For $\vec{n} \in \mathbb{Z}^2$, we have $\vec{n} = (2B+1)\vec{\ell} + \vec{d}, \vec{\ell} \in \mathbb{Z}^2, \vec{d} \in \Lambda_B$. Here $\vec{\ell}$ and $\vec{d} \in \Lambda_B$ are uniquely determined by $\vec{n}$. We have

$$\mathbb{Z}^2 = \bigcup_{d\in\Lambda_B} \bigcup_{\vec{\ell} \in \mathbb{Z}^2} (2B+1)\vec{\ell} + \vec{d}.$$

This is a disjoint union. Also, $\{E_B + (2B+1)\vec{\ell}, \vec{\ell} \in \mathbb{Z}^2\}$ is a partition of $\mathbb{R}^2$. Hence, for a fixed $\vec{d} \in \Lambda_B$, $\{E_B + (2B+1)\vec{\ell} + \vec{d}, \vec{\ell} \in \mathbb{Z}^2\}$ is a partition of $\mathbb{R}^2$. Note that the set $E_B + (2B+1)\vec{\ell} + \vec{d}$ contains the support for $T_{\vec{n}}\varphi$, where

$\vec{n} = (2B+1)\vec{\ell} + \vec{d}$. Then, for a fixed $\vec{d} \in \Lambda_B$, supports of functions in the set $\{T_{\vec{n}}\varphi, \vec{n} = (2B+1)\vec{\ell} + \vec{d}, \vec{\ell} \in \mathbb{Z}^2\}$ are disjoint. Then we have

$$\begin{aligned} L_0(f) &= \sum_{\vec{d}\in\Lambda_B} \sum_{\vec{\ell}\in\mathbb{Z}^2} \left|\langle f, T_{(2B+1)\vec{\ell}+\vec{d}}\varphi\rangle\right|^2 \\ &= \sum_{\vec{d}\in\Lambda_B} \sum_{\vec{\ell}\in\mathbb{Z}^2} \left|\int_{\mathbb{R}^2} \chi_{E_B+(2B+1)\vec{\ell}+\vec{d}}(\vec{t}) f(\vec{t}) T_{(2B+1)\vec{\ell}+\vec{d}}\varphi(\vec{t})\, d\mu\right|^2 \\ &\le \sum_{\vec{d}\in\Lambda_B} \left(\|\varphi\|^2 \sum_{\vec{\ell}\in\mathbb{Z}^2} \int_{E_B+(2B+1)\vec{\ell}+\vec{d}} \left|f(\vec{t})\right|^2 d\mu\right) \\ &\le \sum_{\vec{d}\in\Lambda_B} \|\varphi\|^2\|f\|^2 \le (2B+1)^2\|\varphi\|^2\|f\|^2, \end{aligned}$$

and so we have

$$\begin{aligned} L_J(f) &= \sum_{\vec{\ell}\in\mathbb{Z}^2} \left|\left\langle (D_A^J)^* f, T_{\vec{\ell}}\varphi\right\rangle\right|^2 \\ &\le (2B+1)^2\|\varphi\|^2\left\|(D_A^J)^* f\right\|^2 \\ &= (2B+1)^2\|\varphi\|^2\|f\|^2. \end{aligned}$$

Then we have (5.8). Since $\lim_{\rho\to\infty}\|f_{\bar{\rho}}\| = 0$, the equality (5.9) is an immediate consequence of the inequality (5.8) just proved. $\square$

Proposition 5.6. *Let $f \in \mathrm{L}^2(\mathbb{R}^2)$. Then*

$$\lim_{J\to+\infty} L_{-J}(f) = 0.$$

Proof. Let $f \in L^2(\mathbb{R}^2)$. We have

$$\begin{aligned} L_{-J}(f) &= \sum_{\vec{\ell}\in\mathbb{Z}^2} \left|\langle f, D_A^{-J} T_{\vec{\ell}}\varphi\rangle\right|^2 \\ &= \sum_{\vec{d}\in\Lambda_B} \sum_{\vec{\ell}\in\mathbb{Z}^2} \left|\langle f, D_A^{-J} T_{(2B+1)\vec{\ell}+\vec{d}}\varphi\rangle\right|^2 \\ &= \sum_{\vec{d}\in\Lambda_B} \sum_{\vec{\ell}\in\mathbb{Z}^2\setminus\{\vec{0}\}} \left|\langle f, D_A^{-J} T_{(2B+1)\vec{\ell}+\vec{d}}\varphi\rangle\right|^2 + \sum_{\vec{d}\in\Lambda_B} \left|\langle f, D_A^{-J} T_{\vec{d}}\varphi\rangle\right|^2. \end{aligned}$$

For each $\vec{d} \in \Lambda_B$, $\{E_B + (2B+1)\vec{\ell} + \vec{d}, \vec{\ell} \in \mathbb{Z}^2\}$ is a partition of $\mathbb{R}^2$. It is clear that $E_0 \subset E_B + \vec{d}$ and $(E_B + (2B+1)\vec{\ell} + \vec{d}) \cap E_0 = \emptyset, \forall \vec{\ell} \in \mathbb{Z}^2\setminus\{\vec{0}\}$. The support of the function $D_A^{-J} T_{\vec{\ell}}\varphi$ is contained in $A^J(E_B + \vec{\ell})$. We have

$$\begin{aligned} &\sum_{\vec{\ell}\in\mathbb{Z}^2\setminus\{\vec{0}\}} \left|\langle f, D_A^{-J} T_{(2B+1)\vec{\ell}+\vec{d}}\varphi\rangle\right|^2 \\ &= \sum_{\vec{\ell}\in\mathbb{Z}^2\setminus\{\vec{0}\}} \left|\int_{\mathbb{R}^2} \chi_{A^J(E_B+(2B+1)\vec{\ell}+\vec{d})} \cdot f \cdot \overline{D_A^{-J} T_{(2B+1)\vec{\ell}+\vec{d}}\varphi}\, d\mu\right|^2 \end{aligned}$$

$$\leq \sum_{\vec{\ell}\in\mathbb{Z}^2\setminus\{\vec{0}\}} \int_{A^J(E_B+(2B+1)\vec{\ell}+\vec{d})} |f|^2\,d\mu \cdot \|D_A^{-J}T_{(2B+1)\vec{\ell}+\vec{d}}\varphi\|^2$$
$$\leq \int_{\mathbb{R}^2\setminus A^J E_0} |f|^2\,d\mu \cdot \|\varphi\|^2.$$

Since A is expansive, $\lim_{J\to+\infty} A^J E_0 = \mathbb{R}^2$, $\lim_{J\to+\infty}\int_{\mathbb{R}^2\setminus A^J E_0}|f|^2\,d\mu = 0$, and so

$$\lim_{J\to+\infty}\sum_{\vec{d}\in\Lambda_B}\sum_{\vec{\ell}\in\mathbb{Z}^2\setminus\{\vec{0}\}} \left|\langle f, D_A^{-J}T_{(2B+1)\vec{\ell}+\vec{d}}\varphi\rangle\right|^2$$
$$\leq (2B+1)^2\|\varphi\|^2 \lim_{J\to+\infty}\int_{\mathbb{R}^2\setminus A^J E_0}|f|^2\,d\mu = 0.$$

To complete the proof of this proposition, we need to show that

$$\lim_{J\to+\infty}\sum_{\vec{d}\in\Lambda_B}\left|\langle f, D_A^{-J}T_{\vec{d}}\varphi\rangle\right|^2 = 0. \tag{5.10}$$

Let $f_N \equiv \chi_{[-N,N]^2}\cdot f$. Let $\varepsilon > 0$, and choose $N\in\mathbb{N}$ to be large such that $\|f - f_N\| \leq \frac{\varepsilon}{2\|\varphi\|}$. Then we have $|\langle f, D_A^{-J}T_{\vec{d}}\varphi\rangle| \leq |\langle f_N, D_A^{-J}T_{\vec{d}}\varphi\rangle| + \frac{\varepsilon}{2}$. Since

$$\begin{aligned}\left|\langle f_N, D_A^{-J}T_{\vec{d}}\varphi\rangle\right| &= \left|\langle D_A^J f_N, T_{\vec{d}}\varphi\rangle\right|\\ &= \left|\langle \chi_{A^{-J}[-N,N]^2} D_A^J f_N, T_{\vec{d}}\varphi\rangle\right|\\ &= \left|\langle D_A^J f_N, \chi_{A^{-J}[-N,N]^2} T_{\vec{d}}\varphi\rangle\right|,\end{aligned}$$

we have

$$\begin{aligned}\left|\langle f_N, D_A^{-J}T_{\vec{d}}\varphi\rangle\right| &\leq \|D_A^J f_N\|\cdot\sqrt{\int_{\mathbb{R}^2}|\chi_{A^{-J}[-N,N]^2}T_{\vec{d}}\varphi|^2\,d\mu}\\ &\leq \|f\|\cdot\sqrt{\int_{\mathbb{R}^2}|\chi_{A^{-J}[-N,N]^2}T_{\vec{d}}\varphi|^2\,d\mu}\cdot\|T_{\vec{d}}\varphi\|\\ &= \frac{(2N+1)\|f\|\|\varphi\|}{2^{\frac{J}{2}}}.\end{aligned}$$

When $J > 2\log_2\frac{2(2N+1)\|f\|\|\varphi\|}{\varepsilon}$, we have $\frac{(2N+1)\|f\|\|\varphi\|}{2^{\frac{J}{2}}} < \frac{\varepsilon}{2}$ and $|\langle f, D_A^{-J}T_{\vec{d}}\varphi\rangle| < \varepsilon$. Then

$$\lim_{J\to+\infty}\left|\langle f, D_A^{-J}T_{\vec{d}}\varphi\rangle\right|^2 = 0, \quad \vec{d}\in\Lambda_B.$$

Since Λ_B is a finite set, we have

$$\lim_{J\to+\infty}\sum_{\vec{d}\in\Lambda_B}\left|\langle f, D_A^{-J}T_{\vec{d}}\varphi\rangle\right|^2 = 0.$$

□

Lemma 5.7. *Let $f\in L^2(\mathbb{R}^2)$, and let $J\in\mathbb{Z}$. Then*

$$L_J(f) = (2\pi)^2\int_{\mathbb{R}^2}\sum_{\vec{\ell}\in\mathbb{Z}^2}\left(\widehat{f}(\vec{t})\overline{\widehat{f}\big(\vec{t}-2\pi(A^\tau)^J\vec{\ell}\big)}\widehat{\varphi}\big((A^\tau)^{-J}\vec{t}-2\pi\vec{\ell}\big)\overline{\widehat{\varphi}\big((A^\tau)^{-J}\vec{t}\big)}\right)d\vec{t}.$$

Proof. By Remark 2.5 after Lemma 2.6, we have $D_A^J T_{\vec{\ell}} = T_{A^{-J}\vec{\ell}} D_A^J$. Note the facts that the Fourier transform $\mathcal{F}$ is a unitary operator, $\widehat{D_A^J}\widehat{\varphi}(\vec{t}) = \frac{1}{\sqrt{2^J}}\widehat{\varphi}((A^\tau)^{-J}\vec{t})$, and $(A^{-J}\vec{\ell}) \circ \vec{t} = \vec{\ell} \circ ((A^\tau)^{-J}\vec{t})$. We have

$$\begin{aligned} L_J(f) &= \sum_{\vec{\ell} \in \mathbb{Z}^2} \left|\langle f, T_{A^{-J}\vec{\ell}} D_A^J \varphi\rangle\right|^2 \\ &= \sum_{\vec{\ell} \in \mathbb{Z}^2} \left|\langle \widehat{f}, \widehat{T}_{A^{-J}\vec{\ell}} \widehat{D}_A^J \widehat{\varphi}\rangle\right|^2 \\ &= \sum_{\vec{\ell} \in \mathbb{Z}^2} \left|\int_{\mathbb{R}^2} \widehat{f}(\vec{t}) \cdot e^{i(A^{-J}\vec{\ell})\circ\vec{t}} \cdot \overline{\widehat{D}_A^J \widehat{\varphi}(\vec{t})}\, d\vec{t}\right|^2 \\ &= \sum_{\vec{\ell} \in \mathbb{Z}^2} \left|\frac{1}{\sqrt{2^J}} \int_{\mathbb{R}^2} e^{i\vec{\ell}\circ((A^\tau)^{-J}\vec{t})} \cdot \widehat{f}(\vec{t}) \cdot \overline{\widehat{\varphi}\big((A^\tau)^{-J}\vec{t}\big)}\, d\vec{t}\right|^2. \end{aligned}$$

Take the transform $d\vec{t} \equiv d(A^\tau)^J \vec{s} = 2^J\, d\vec{s}$. Note that $(A^\tau)^J \mathbb{R}^2 = \mathbb{R}^2$. We have

$$L_J(f) = 2^J \sum_{\vec{\ell} \in \mathbb{Z}^2} \left|\int_{\mathbb{R}^2} e^{i\vec{\ell}\circ\vec{s}} \cdot \widehat{f}\big((A^\tau)^J\vec{s}\big) \cdot \overline{\widehat{\varphi}(\vec{s})}\, d\vec{s}\right|^2.$$

Note the facts that the function $e^{i\vec{\ell}\circ\vec{s}}$ is 2π-periodic in $\vec{s}$ and that the set $\{\Gamma_\pi + 2\pi\vec{k}, \vec{k} \in \mathbb{Z}^2\}$ is a partition of $\mathbb{R}^2$, where Γ_π is $[-\pi,\pi)^2$. We have

$$\begin{aligned} L_J(f) &= 2^J \sum_{\vec{\ell} \in \mathbb{Z}^2} \Big|\sum_{\vec{k} \in \mathbb{Z}^2} \int_{\Gamma_\pi + 2\pi\vec{k}} e^{i\vec{\ell}\circ\vec{s}} \cdot \widehat{f}\big((A^\tau)^J\vec{s}\big) \cdot \overline{\widehat{\varphi}(\vec{s})}\, d\vec{s}\Big|^2 \\ &= 2^J \sum_{\vec{\ell} \in \mathbb{Z}^2} \Big|\sum_{\vec{k} \in \mathbb{Z}^2} \int_{\Gamma_\pi} e^{i\vec{\ell}\circ\vec{r}} \cdot \Big(\widehat{f}\big((A^\tau)^J\vec{r} - 2\pi(A^\tau)^J\vec{k}\big) \cdot \overline{\widehat{\varphi}(\vec{r} - 2\pi\vec{k})}\Big)\, d\vec{r}\Big|^2 \\ &= 2^J(2\pi)^2 \sum_{\vec{\ell} \in \mathbb{Z}^2} \Big|\int_{\Gamma_\pi} \frac{1}{2\pi} e^{i\vec{\ell}\circ\vec{r}} \\ &\quad \cdot \sum_{\vec{k} \in \mathbb{Z}^2} \Big(\widehat{f}\big((A^\tau)^J\vec{r} - 2\pi(A^\tau)^J\vec{k}\big) \cdot \overline{\widehat{\varphi}(\vec{r} - 2\pi\vec{k})}\Big)\, d\vec{r}\Big|^2, \end{aligned}$$

where we use the transform $\vec{r} = \vec{s} + 2\pi\vec{k}$ accordingly.

The set of functions $\{\frac{1}{2\pi}e^{i\vec{\ell}\circ\vec{t}}, \vec{\ell} \in \mathbb{Z}^2\}$ is an orthonormal basis for the Hilbert space $\mathcal{K} = L^2(\Gamma_\pi)$, the set of all square integrable 2π-periodical functions on $\mathbb{R}^2$. Denote

$$h(\vec{t}) \equiv \sum_{\vec{k} \in \mathbb{Z}^2} \Big(\widehat{f}\big((A^\tau)^J\vec{t} - 2\pi(A^\tau)^J\vec{k}\big) \cdot \overline{\widehat{\varphi}(\vec{t} - 2\pi\vec{k})}\Big).$$

Then, by the above calculation on $L_J(f)$ and Lemma 5.5, we have

$$\sum_{\vec{\ell} \in \mathbb{Z}^2} \Big|\int_{\Gamma_\pi} h(\vec{t}) \cdot \frac{1}{2\pi} e^{i\vec{\ell}\circ\vec{t}}\, d\vec{t}\Big|^2 = \frac{1}{2^J \cdot (2\pi)^2} \cdot L_J(f) < \infty.$$

This implies that $h \in \mathcal{K} = L^2(\Gamma_\pi)$ and

$$\|h\|_{\mathcal{K}}^2 = \sum_{\vec{\ell}\in\mathbb{Z}^2} \Big|\int_{\Gamma_\pi} h(\vec{t}\,)\cdot \frac{1}{2\pi} e^{i\vec{\ell}\circ\vec{t}}\,d\vec{t}\,\Big|^2,$$

where $\|\cdot\|_{\mathcal{K}}$ is the norm in $\mathcal{K}$. Therefore,

$$\begin{aligned} L_J(f) &= 2^J\cdot(2\pi)^2\cdot\|h\|_{\mathcal{K}}^2 \\ &= 2^J\cdot(2\pi)^2\int_{\Gamma_\pi}\Big|\sum_{\vec{k}\in\mathbb{Z}^2}\Big(\widehat{f}\big((A^\tau)^J\vec{t}-2\pi(A^\tau)^J\vec{k}\,\big)\cdot\overline{\widehat{\varphi}(\vec{t}-2\pi\vec{k}\,)}\Big)\Big|^2 d\vec{t} \\ &= (2\pi)^2\int_{(A^\tau)^J\Gamma_\pi}\Big|\sum_{\vec{k}\in\mathbb{Z}^2}\widehat{f}\big(\vec{s}-2\pi(A^\tau)^J\vec{k}\,\big)\cdot\overline{\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}-2\pi\vec{k}\,\big)}\Big|^2 d\vec{s}. \end{aligned}$$

Here we use a transform $\vec{t}\equiv(A^\tau)^{-J}\vec{s}, d\vec{t}=2^{-J}\,d\vec{s}$. Then we have

$$\begin{aligned} L_J(f) = (2\pi)^2\int_{(A^\tau)^J\Gamma_\pi}\sum_{\vec{k}\in\mathbb{Z}^2}\sum_{\vec{\ell}\in\mathbb{Z}^2}\Big(\widehat{f}\big(\vec{s}-2\pi(A^\tau)^J\vec{k}\,\big)\cdot\overline{\widehat{f}\big(\vec{s}-2\pi(A^\tau)^J\vec{\ell}\,\big)} \\ \cdot\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}-2\pi\vec{\ell}\,\big)\cdot\overline{\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}-2\pi\vec{k}\,\big)}\Big)\,d\vec{s}. \end{aligned}$$

In the second sum, replace $\vec{\ell}$ by $\vec{\ell}+\vec{k}$. Then we have

$$\begin{aligned} L_J(f) = (2\pi)^2\sum_{\vec{k}\in\mathbb{Z}^2}\int_{(A^\tau)^J\Gamma_\pi}\sum_{\vec{\ell}\in\mathbb{Z}^2}\Big(\widehat{f}\big(\vec{s}-2\pi(A^\tau)^J\vec{k}\,\big) \\ \cdot\overline{\widehat{f}\big(\vec{s}-2\pi(A^\tau)^J\vec{k}-2\pi(A^\tau)^J\vec{\ell}\,\big)} \\ \cdot\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}-2\pi\vec{k}-2\pi\vec{\ell}\,\big)\cdot\overline{\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}-2\pi\vec{k}\,\big)}\Big)\,d\vec{s}. \end{aligned}$$

Replacing $\vec{s}$ by $\vec{s}+2\pi(A^\tau)^J\vec{k}$, we have

$$\begin{aligned} L_J(f) = (2\pi)^2\sum_{\vec{k}\in\mathbb{Z}^2}\int_{(A^\tau)^J\Gamma_\pi+2\pi(A^\tau)^J\vec{k}}\sum_{\vec{\ell}\in\mathbb{Z}^2}\Big(\widehat{f}(\vec{s}\,)\cdot\overline{\widehat{f}\big(\vec{s}-2\pi(A^\tau)^J\vec{\ell}\,\big)} \\ \cdot\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}-2\pi\vec{\ell}\,\big)\cdot\overline{\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}\,\big)}\Big)\,d\vec{s}. \end{aligned}$$

Since $\{(A^\tau)^J\Gamma_\pi+2\pi(A^\tau)^J\vec{k}, \vec{k}\in\mathbb{Z}^2\}$ is a partition of $\mathbb{R}^2$, we have

$$\begin{aligned} L_J(f) = (2\pi)^2\int_{\mathbb{R}^2}\sum_{\vec{\ell}\in\mathbb{Z}^2}\Big(\widehat{f}(\vec{s}\,)\cdot\overline{\widehat{f}\big(\vec{s}-2\pi(A^\tau)^J\vec{\ell}\,\big)} \\ \cdot\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}-2\pi\vec{\ell}\,\big)\cdot\overline{\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}\,\big)}\Big)\,d\vec{s}. \end{aligned}$$

Lemma 5.7 is proved. □

Proposition 5.8. *We have*

$$\lim_{J\to+\infty} L_J(f) = \|f\|^2, \quad \forall f\in L^2(\mathbb{R}^2). \tag{5.11}$$

Proof. We denote

$$U_J(f) \equiv (2\pi)^2 \int_{\mathbb{R}^2} \widehat{f}(\vec{s}\,) \cdot \overline{\widehat{f}(\vec{s}\,)} \cdot \widehat{\varphi}\big((A^\tau)^{-J}\vec{s}\,\big) \cdot \overline{\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}\,\big)}\, d\vec{s}$$

$$= \int_{\mathbb{R}^2} \big|\widehat{f}(\vec{s}\,)\big|^2 \big|2\pi\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}\,\big)\big|^2\, d\vec{s},$$

$$V_J(f) \equiv (2\pi)^2 \int_{\mathbb{R}^2} \sum_{\vec{\ell}\in\mathbb{Z}^2\setminus\{\vec{0}\,\}} \Big(\widehat{f}(\vec{s}\,) \cdot \overline{\widehat{f}\big(\vec{s} - 2\pi(A^\tau)^J\vec{\ell}\,\big)}$$

$$\cdot\, \widehat{\varphi}\big((A^\tau)^{-J}\vec{s} - 2\pi\vec{\ell}\,\big) \cdot \overline{\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}\,\big)}\Big)\, d\vec{s}.$$

By Lemma 5.7, we have $L_J(f) = U_J(f) + V_J(f)$. It is enough to prove that

$$\lim_{J\to+\infty} U_J(f) = \|f\|^2 \qquad \text{and} \tag{5.12}$$

$$\lim_{J\to+\infty} V_J(f) = 0. \tag{5.13}$$

1. Recall that A^τ is expansive, and so $\lim_{J\to+\infty}(A^\tau)^{-J}\vec{s} = \vec{0}\,, \forall \vec{s} \in \mathbb{R}^2$. Also, by definition of g, Remark 3.2, and Lemma 4.3, $2\pi\widehat{\varphi}(\vec{0}\,) = 2\pi g(\vec{0}\,) = 1$, and g is continuous and bounded on $\mathbb{R}^2$. By the Lebesgue dominated convergence theorem, we have

$$\lim_{J\to+\infty} U_J(f) = \lim_{J\to+\infty} \int_{\mathbb{R}^2} \big|\widehat{f}(\vec{s}\,)\big|^2 \cdot \big|2\pi\widehat{\varphi}\big((A^\tau)^{-J}\vec{s}\,\big)\big|^2\, d\vec{s}$$

$$= \lim_{J\to+\infty} \int_{\mathbb{R}^2} \big|\widehat{f}(\vec{s}\,)\big|^2 \cdot \big|2\pi g\big((A^\tau)^{-J}\vec{s}\,\big)\big|^2\, d\vec{s}$$

$$= \|\widehat{f}\|^2 = \|f\|^2.$$

This proves (5.12).

2. Let $\rho \in \mathbb{R}^+$, and let Δ_ρ be the open ball with center $\vec{0}$ and radius ρ. In particular, Δ_1 is the open ball with center $\vec{0}$ and radius 1. Let χ_ρ and $\chi_{\overline{\rho}}$ be the characteristic functions of sets Δ_ρ and $\mathbb{R}^2\setminus\Delta_\rho$, respectively. Define f_ρ by $\widehat{f_\rho} \equiv \chi_\rho \widehat{f}$, and define $f_{\overline{\rho}}$ by $\widehat{f_{\overline{\rho}}} \equiv \chi_{\overline{\rho}}\widehat{f}$. Since the Fourier transform is linear, we have $f = f_\rho + f_{\overline{\rho}}$. Also, it is clear that $\|f\|^2 = \|\widehat{f}\|^2 = \|f_\rho\|^2 + \|f_{\overline{\rho}}\|^2$, $\lim_{\rho\to+\infty}\|f_\rho\|^2 = \|f\|^2$, and $\lim_{\rho\to+\infty}\|f_{\overline{\rho}}\|^2 = 0$.

Since A^τ is expansive, $\beta \equiv \|(A^\tau)^{-1}\|^{-1} > 1$. Denote $a \equiv \log_\beta(2\rho)$. Let J_ρ be the smallest natural number in the interval $(a, +\infty)$. When $J \geq J_\rho$, $(A^\tau)^J\Delta_1$ contains an open ball $\Delta_{2\rho}$. Since $\Delta_1 \cap \mathbb{Z}^2 = \{\vec{0}\,\}$, $2\pi(A^\tau)^J\Delta_1 \cap 2\pi(A^\tau)^J\mathbb{Z}^2 = \{\vec{0}\,\}$. Also, we have $\Delta_{2\rho} \subseteq (A^\tau)^J\Delta_1 \subseteq 2\pi(A^\tau)^J\Delta_1$. These facts imply that when $J \geq J_\rho$, the distance between $\vec{0}$ and $2\pi(A^\tau)^J\vec{\ell}$ is greater than 2ρ. Then, for each $\vec{\ell} \in \mathbb{Z}^2\setminus\{\vec{0}\,\}$, the support of $\widehat{f}_\rho(\vec{t}\,)$, which is Δ_ρ, and the support of $\widehat{f}_\rho(\vec{t} - 2\pi(A^\tau)^J\vec{\ell}\,)$, which is $\Delta_\rho + 2\pi(A^\tau)^J\vec{\ell}$, are disjoint. This implies that the product $\widehat{f}_\rho(\vec{t}\,)\overline{\widehat{f}_\rho(\vec{t} - 2\pi(A^\tau)^J\vec{\ell}\,)} \equiv 0$ when $J \geq J_\rho$. Therefore, we have

$$\lim_{J\to+\infty} V_J(f_\rho) = 0, \quad \forall \rho \in \mathbb{R}^+.$$

Together with (5.12), we have proved that

$$\lim_{J\to+\infty} L_J(f_\rho) = \|f_\rho\|^2, \quad \forall f \in L^2(\mathbb{R}^2), \forall \rho \in \mathbb{R}^+. \tag{5.14}$$

3. Let $D_\rho \equiv \sum_{\vec{\ell}\in\mathbb{Z}^2}(\langle f_\rho, D_A^J T_{\vec{\ell}}\varphi\rangle\overline{\langle f_{\overline{\rho}}, D_A^J T_{\vec{\ell}}\varphi\rangle} + \langle f_{\overline{\rho}}, D_A^J T_{\vec{\ell}}\varphi\rangle\overline{\langle f_\rho, D_A^J T_{\vec{\ell}}\varphi\rangle})$. Then

$$\begin{aligned}|D_\rho| &\le 2\sum_{\vec{\ell}\in\mathbb{Z}^2}\left|\langle f_\rho, D_A^J T_{\vec{\ell}}\varphi\rangle\right|\cdot\left|\langle f_{\overline{\rho}}, D_A^J T_{\vec{\ell}}\varphi\rangle\right| \\ &\le 2\sqrt{\sum_{\vec{\ell}\in\mathbb{Z}^2}\left|\langle f_\rho, D_A^J T_{\vec{\ell}}\varphi\rangle\right|^2}\cdot\sqrt{\sum_{\vec{\ell}\in\mathbb{Z}^2}\left|\langle f_{\overline{\rho}}, D_A^J T_{\vec{\ell}}\varphi\rangle\right|^2} \\ &= 2\sqrt{L_J(f_\rho)}\cdot\sqrt{L_J(f_{\overline{\rho}})}.\end{aligned}$$

By Lemma 5.5, we have

$$|D_\rho| \le 2(2B+1)^2\|\varphi\|^2\|f_\rho\|\|f_{\overline{\rho}}\|.$$

We have

$$\begin{aligned}L_J(f) - \|f\|^2 &= L_J(f_\rho + f_{\overline{\rho}}) - \|f\|^2 \\ &= \sum_{\vec{\ell}\in\mathbb{Z}^2}\langle f_\rho + f_{\overline{\rho}}, D_A^J T_{\vec{\ell}}\varphi\rangle\overline{\langle f_\rho + f_{\overline{\rho}}, D_A^J T_{\vec{\ell}}\varphi\rangle} - \|f\|^2 \\ &= L_J(f_\rho) - \|f\|^2 + L_J(f_{\overline{\rho}}) + D_\rho \\ &= \left(L_J(f_\rho) - \|f_\rho\|^2\right) - \|f_{\overline{\rho}}\|^2 + L_J(f_{\overline{\rho}}) + D_\rho.\end{aligned}$$

By (5.14), (5.9), and Lemma (5.8), we have

$$\limsup_{J\to+\infty}\left|L_J(f) - \|f\|^2\right| \le 0 + \|f_{\overline{\rho}}\|^2 + 0 + 2(2B+1)^2\|\varphi\|^2\|f_\rho\|\|f_{\overline{\rho}}\|, \quad \forall\rho\in\mathbb{R}^+.$$

The left-hand side contains no ρ. Let $\rho \to +\infty$. Since $\lim_{\rho\to+\infty}\|f_{\overline{\rho}}\| = 0$, we have

$$\lim_{J\to\infty} L_J(f) = \|f\|^2. \qquad \square$$

The proof of Theorem 5.4 is complete.

6. Conclusion

Let A_0 be a 2×2 expansive integral matrix with $|\det(A_0)| = 2$. We can construct normalized tight frame wavelets associated with A_0 in the following steps.

(1) Find a 2×2 integral matrix S with $|\det(S)| = 1$ and with the property that $SAS^{-1} = A_0$, where A is one of the six matrices in list (2.2) (Proposition 2.1).

(2) Solve the system of equations (see (3.1))

$$\begin{cases}\sum_{\vec{n}\in\mathbb{Z}^2} h_{\vec{n}}\overline{h_{\vec{n}+\vec{k}}} = \delta_{\vec{0}\,\vec{k}}, & \vec{k}\in A^\tau\mathbb{Z}^2, \\ \sum_{\vec{n}\in\mathbb{Z}^2} h_{\vec{n}} = \sqrt{2}\end{cases}$$

for a finite solution $\mathcal{S} = \{h_{\vec{n}} : \vec{n}\in\mathbb{Z}^2\}$; that is, the index set of nonzero terms $h_{\vec{n}}$ is included in the set $\Lambda_0 \equiv \mathbb{Z}^2 \cap [-N_0, N_0]^2$ for some natural number N_0.

(3) Define the filter function m_0 by (3.2):

$$m_0(\vec{t}) = \frac{1}{\sqrt{2}} \sum_{\vec{n} \in \mathbb{Z}^2} h_{\vec{n}}\, e^{-i\vec{n} \circ \vec{t}}.$$

(4) Define a function g by (4.1):

$$g(\vec{\xi}) = \frac{1}{2\pi} \prod_{j=1}^{\infty} m_0\big((A^\tau)^{-j}\vec{\xi}\big), \quad \forall \vec{\xi} \in \mathbb{R}^2.$$

The function g is an $L^2(\mathbb{R}^2)$-function (Proposition 4.5).

(5) Define the scaling function φ by (4.2):

$$\varphi = \mathcal{F}^{-1} g.$$

The scaling function φ is an $L^2(\mathbb{R}^2)$-function with compact support (Proposition 4.9).

(6) Let $\vec{\ell}_A$ be the vector as in Proposition 2.6. Define

$$\sigma_A(\vec{n}\,) = \begin{cases} 0, & \vec{n} \in A\mathbb{Z}^2, \\ 1, & \vec{n} \notin A\mathbb{Z}^2. \end{cases}$$

Define the wavelet function ψ_A on $\mathbb{R}^2$ by (5.3):

$$\psi_A = \sum_{\vec{n} \in \mathbb{Z}^2} (-1)^{\sigma_A(\vec{n}\,)} \overline{h_{\vec{\ell}_A - \vec{n}}}\, D_A T_{\vec{n}}\, \varphi.$$

This is a normalized tight frame wavelet with compact support associated with matrix A (Theorem 5.4).

(7) Define the wavelet function ψ by

$$\psi(\vec{t}) \equiv \psi_A(S\vec{t}), \quad \forall \vec{t} \in \mathbb{R}^2.$$

The function ψ is a normalized tight frame wavelet with compact support associated with the given matrix A_0 (Theorem 2.3).

7. Examples

Let A be one of the six matrices in Proposition 2.1. If we assume that Λ_0 contains only one element $\vec{0}$, then Lawton's equations have no solution since this will require that $h_{\vec{0}} = \pm 1$ and $\sqrt{2}$. Assuming that $\Lambda_0 = \{\vec{n}_0, \vec{n}_1\}, \vec{n}_0 = \vec{0}, \vec{n}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, we will have two cases.

Case (1). $\vec{n}_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \in A\mathbb{Z}^2$. In this case, the reduced Lawton system of equations (3.1) is

$$\begin{cases} h_{\vec{n}_0}^2 + h_{\vec{n}_1}^2 = 1, \\ h_{\vec{n}_0} \cdot h_{\vec{n}_1} = 0, \\ h_{\vec{n}_0} + h_{\vec{n}_1} = \sqrt{2}. \end{cases} \tag{7.1}$$

Apparently, this system of equations has no solution.

Case (2). $\vec{n}_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \notin A\mathbb{Z}^2$. The reduced Lawton system of equations (3.1) is

$$\begin{cases} h_{\vec{n}_0}^2 + h_{\vec{n}_1}^2 = 1, \\ h_{\vec{n}_0} + h_{\vec{n}_1} = \sqrt{2}. \end{cases} \tag{7.2}$$

The system has one solution $h_{\vec{n}_0} = h_{\vec{n}_1} = \frac{\sqrt{2}}{2}$. Then, for the first four matrices in Proposition 2.1,

$$\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}, \qquad \begin{bmatrix}1 & -3\\1 & -1\end{bmatrix}, \qquad \begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}, \qquad \begin{bmatrix}-1 & -1\\1 & -1\end{bmatrix} \tag{7.3}$$

according to the table in (2.5) $n_1 \notin A\mathbb{Z}^2 (= A^\tau \mathbb{Z}^2)$. Then, in the case that A is one of the four matrices, we have the solution. The wavelets created by the solutions are Haar wavelets. One of them is included in Example 7.1.

Example 7.1. Let

$$A = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}. \qquad \text{By the table in (2.5),} \qquad \vec{\ell}_A = \begin{bmatrix}1\\0\end{bmatrix}, \qquad \vec{q}_A = \begin{bmatrix}1\\1\end{bmatrix}.$$

$h_{\vec{n}_0} = h_{\vec{n}_1} = \frac{\sqrt{2}}{2}$ is the solution to the reduced Lawton's equations (7.2) as in the above discussion. Then the filter function (3.2) is reduced to

$$m_0(\vec{t}) \equiv \frac{1}{2} + \frac{1}{2}e^{-it_1}, \quad \vec{t} = \begin{bmatrix}t_1\\t_2\end{bmatrix} \in \mathbb{C}^2. \tag{7.4}$$

The two-scaling relation equation (5.1) is reduced to

$$\varphi_H = \frac{\sqrt{2}}{2} D_A (I + T_{\vec{n}_1}) \varphi_H. \tag{7.5}$$

The supports of φ_H and ψ_H are the same set Q_A, which is the parallelogram with vertices $\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\}$. The graph of the supports of φ_H and ψ_H is illustrated in Figure 4. The graph of φ_H and ψ_H is illustrated in Figure 5.

Example 7.2. In this example, we want to show that a solution to Lawton's equations will produce a known wavelet in the literature. Let

$$A = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}, \qquad S = \begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}, \qquad \text{and} \qquad A_0 = \begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}.$$

Then

$$A_0 = S^{-1} A S.$$

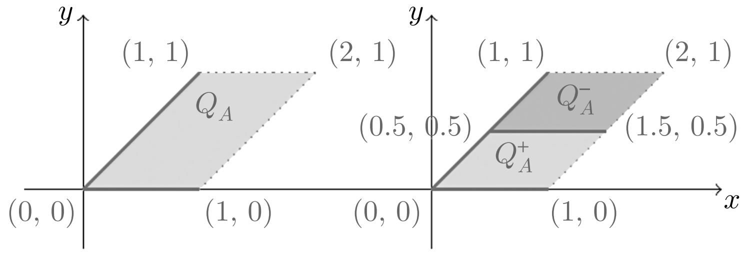

FIGURE 4. Supports of φ_H and ψ_H.

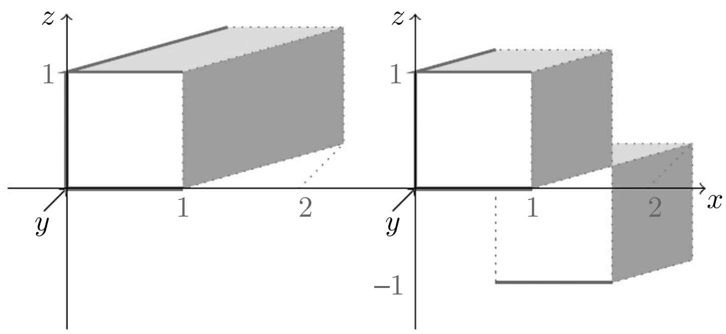

FIGURE 5. Graphs of φ_H and ψ_H.

We will construct a scaling function φ_A and related normalized tight frame wavelet ψ_A associated with matrix A. Then $U_S\psi_A$ will be a normalized tight frame wavelet associated with matrix A_0, and $U_S\varphi_A$ will be the scaling function for $U_S\psi_A$.

Assume that the support of the solution is Λ_0:

$$\Lambda_0 = \left\{ \begin{bmatrix} 0 \\ m \end{bmatrix}, m = 0, 1, \ldots, 7 \right\} \cup \left\{ \begin{bmatrix} 1 \\ m \end{bmatrix}, m = -1, 0, \ldots, 6 \right\}.$$

The reduced Lawton's system of equations related to Λ_0 associated with matrix A has the following 12 equations:

$$\begin{cases} \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}}^2 = 1, \\ \sum_{k=0}^{5} (h_{0,k} \cdot h_{0,(2+k)} + h_{1,(k-1)} \cdot h_{1,(k+1)}) = 0, \\ \sum_{k=0}^{3} (h_{0,k} \cdot h_{0,(4+k)} + h_{1,(k-1)} \cdot h_{1,(k+3)}) = 0, \\ \sum_{k=0}^{1} (h_{0,k} \cdot h_{0,(6+k)} + h_{1,(k-1)} \cdot h_{1,(k+5)}) = 0, \\ \sum_{k=0}^{7} h_{0,k} \cdot h_{1,(k-1)} = 0, \\ \sum_{k=0}^{5} h_{0,k} \cdot h_{1,(k+1)} = 0, \\ \sum_{k=0}^{3} h_{0,k} \cdot h_{1,(k+3)} = 0, \\ \sum_{k=0}^{1} h_{0,k} \cdot h_{1,(k+5)} = 0, \\ \sum_{k=0}^{5} h_{0,(k+2)} \cdot h_{1,(k-1)} = 0, \\ \sum_{k=0}^{3} h_{0,(k+4)} \cdot h_{1,(k-1)} = 0, \\ \sum_{k=0}^{1} h_{0,(k+6)} \cdot h_{1,(k-1)} = 0, \\ \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} = \sqrt{2}. \end{cases} \tag{7.6}$$

TABLE 1. A solution to equations (7.6).

$h_{0,7}$		0.014008991752812	
$h_{0,6}$	$h_{1,6}$	0.024264285477802	0.065527403135986
$h_{0,5}$	$h_{1,5}$	−0.118573529719665	0.113496791518999
$h_{0,4}$	$h_{1,4}$	0.003753698026408	0.489561273639764
$h_{0,3}$	$h_{1,3}$	0.195120084182308	0.777712940352809
$h_{0,2}$	$h_{1,2}$	−0.080300252489051	0.171377820183894
$h_{0,1}$	$h_{1,1}$	−0.090555546214041	−0.195280287797963
$h_{0,0}$	$h_{1,0}$	0.052282268983427	−0.019359715773096
	$h_{1,-1}$		0.011177337112703

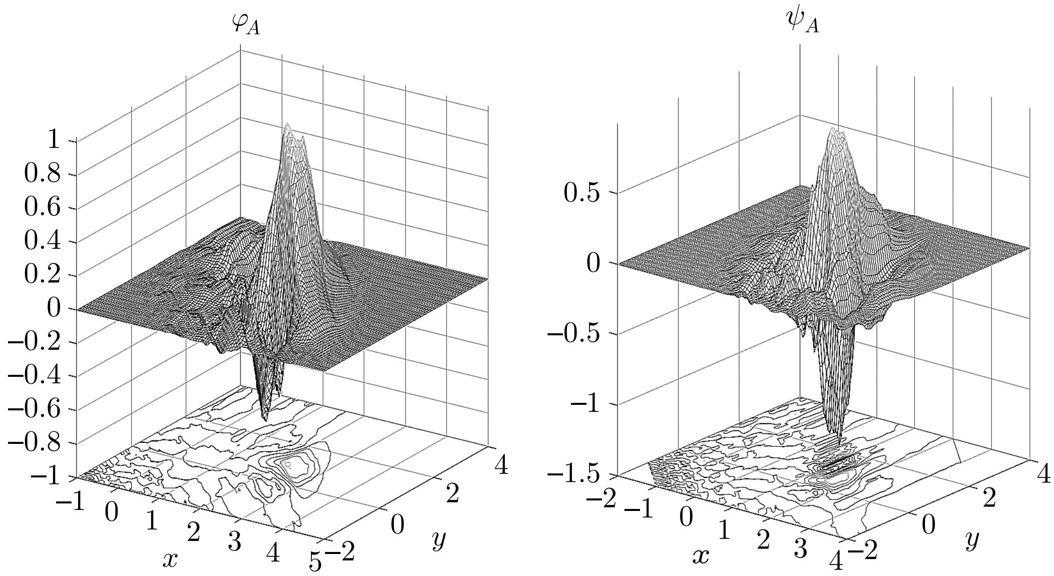

FIGURE 6. Graphs of φ_A and ψ_A.

Table 1 is a solution to equations (7.6). It is from [2, Table A.1, Solution 2], but we modified the data. The solution satisfies equations (7.6) within errors less than 10^{-13}.

Based on this solution, we obtain the corresponding two-scaling relation associated with A and $\{h_{\vec{n}}, \vec{n} \in \Lambda_0\}$:

$$\varphi_A = \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}}\, D_A T_{\vec{n}}\, \varphi_A.$$

Then we obtain the normalized tight frame wavelet function ψ_A and scaling function φ_A associated with A. The graphs of φ_A and ψ_A are illustrated in Figure 6.

Then $\psi_{A_0} \equiv U_S\psi_A$ and $\varphi_{A_0} \equiv U_S\varphi_A$ are the wavelet and scaling function associated with matrix A_0. The graphs of φ_{A_0} and ψ_{A_0} are illustrated in Figure 7. This φ_{A_0} is known as the scaling function "Resting Dog" (see Figure 5.2 in [2]).

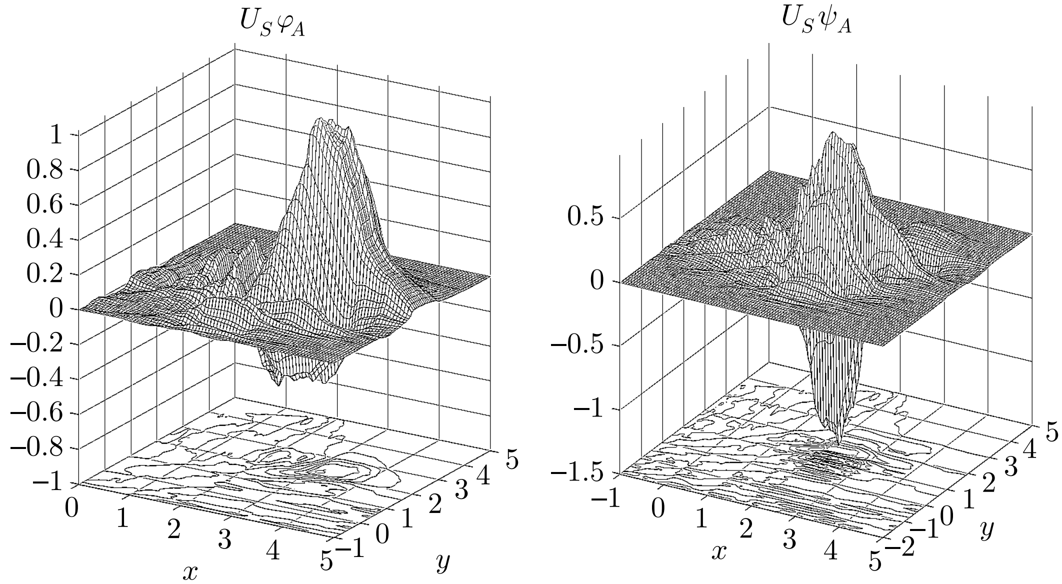

FIGURE 7. Graphs of φ_{A_0} and ψ_{A_0}.

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