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INTRINSIC ATOMIC AND MOLECULAR DECOMPOSITIONS OF HARDY–MUSIELAK–ORLICZ SPACES

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ABSTRACT. We introduce the Hardy type space for Musielak–Orlicz spaces. It includes several existing Hardy type spaces such as the Hardy–Orlicz spaces and the Hardy spaces with variable exponents. Furthermore, we develop an atomic decomposition such that the size condition just relies on the norms of Musielak–Orlicz spaces. This gives us a nature extension of the molecular decompositions to the Hardy type space for Musielak–Orlicz spaces.

1. INTRODUCTION

In this article we aim to establish the atomic and molecular decompositions for the Hardy spaces built on the Musielak–Orlicz spaces.

Once Fefferman, Stein, and Weiss introduced the classical Hardy spaces H^p, they became one of the most important function spaces in analysis. Recently, there have been several extensions of classical Hardy spaces. We have the Hardy–Orlicz spaces which are the Orlicz space version of the classical Hardy spaces (see [18], [17], [28], [35]). The Hardy spaces with variable exponents were introduced and studied in [27] and [30]. The atomic decomposition for weighted Hardy spaces with variable exponents is developed in [14].

We also have the atomic decompositions for the Hardy–Morrey spaces (see [11], [16]). The reader may also consult [31] and [37] for some detailed studies on the Triebel–Lizorkin–Morrey spaces which are generalizations of the Hardy–Morrey spaces.

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The atomic decomposition of the Hardy–Morrey spaces with variable exponents is given in [12]. For the study of some generalizations of Morrey spaces, the reader is referred to [10], [13], [25], and [26]. Furthermore, the Hardy spaces built on the Musielak–Orlicz spaces were given in [15], [19]–[21], and [36].

The Musielak–Orlicz spaces [24], [5, Sections 2.3–2.8] provide a framework for the study of function spaces including the weighted Lebesgue spaces, the Orlicz spaces, and the Lebesgue spaces with a variable exponent. Notice that the Musielak–Orlicz spaces used in [19] and [20] do not cover the Lebesgue spaces with a variable exponent.

The studies of the Hardy spaces mentioned above give us a motivation to establish a unified family of Hardy spaces built on general Musielak–Orlicz spaces so that the Hardy spaces with variable exponents and the Hardy–Musielak–Orlicz spaces introduced in [19] and [20] are included under this new family of Hardy spaces.

Other than a new unified family of Hardy spaces, we also introduce a new atomic decomposition adapted to the Hardy–Musielak–Orlicz spaces. We call it the intrinsic atomic decompositions.

Before we describe the intrinsic atomic decomposition, let us review the atomic decomposition for the classical Hardy spaces. We are particularly interested in the size condition. Recall that the size condition imposed on the atom a with $\operatorname{supp} a \subset 3B$, where Q is ball, for the atomic decompositions of the classical Hardy spaces H^p is given by $\|a\|_{L^q} \leq |Q|^{\frac{1}{q}-\frac{1}{p}} = \|\chi_Q\|_{L^q}^{1-\frac{q}{p}}$, $1<q<\infty$, where χ_Q is the characteristic function of Q.

On the other hand, in view of [27, Definition 1.4], the size condition for the atoms associated with the atomic decompositions for the Hardy spaces with variable exponents $H^{p(\cdot)}$ is given by $\|a\|_{L^q} \leq \frac{|Q|^{\frac{1}{q}}}{\|\chi_Q\|_{L^{p(\cdot)}}} = \frac{\|\chi_Q\|_{L^q}}{\|\chi_Q\|_{L^{p(\cdot)}}}$ where $\operatorname{supp} a \subset Q$. The reader is referred to [27], [3], and [5] for the definitions and properties for Hardy spaces with variable exponents and Lebesgue spaces with variable exponents, respectively.

We see that the size condition for the atom a for $H^{p(\cdot)}$ involves the L^q norms of a and χ_Q. In this paper, as a special case of the general result for the Hardy–Musielak–Orlicz spaces, we present another atomic decomposition for $H^{p(\cdot)}$ where the size condition for the corresponding atom is given by

$$\|a\|_{L^{q(\cdot)}} \leq \frac{\|\chi_Q\|_{L^{q(\cdot)}}}{\|\chi_Q\|_{L^{p(\cdot)}}} = \|\chi_Q\|_{L^{p(\cdot)}}^{\frac{1}{r}-1}, \tag{1.1}$$

where $q(\cdot) = rp(\cdot)$ for some sufficient large $r>1$. Note that $\|\chi_B\|_{L^{q(\cdot)}} = \|\chi_B\|_{L^{p(\cdot)}}^{\frac{1}{r}}$.

The size condition (1.1) does not involve the L^q norm, it just relies on the quasinorms $\|\cdot\|_{L^{rp(\cdot)}}$ and $\|\cdot\|_{L^{p(\cdot)}}$. Using the terminology from the geometry of quasi-Banach spaces [22, Vol. II, pp. 53–54] and [29, Section 2.2], $L^{rp(\cdot)}$ is the r-convexification of $L^{p(\cdot)}$.

Therefore, we call the atomic decompositions for the Hardy spaces with variable exponents using the atoms satisfying the size condition (1.1) the *intrinsic atomic decomposition*.

The condition (1.1) does not only give us a more naturally adapted size condition for the atomic decompositions for the Hardy–Musielak–Orlicz spaces, but it also offers a straightforward generalization for the size condition on the molecules used in the molecular decompositions. Notice that even for the classical Hardy spaces, we also have atomic decompositions with atoms defined in terms of Banach function spaces (see [7], [9]).

Recall that the molecular decomposition for the classical Hardy spaces was introduced by Coifman, Taibleson, and Weiss [2], [33]. The molecular decompositions for the classical Hardy spaces provide some profound applications on function spaces, especially on the boundedness of operators on the classical Hardy spaces (see [2], [23], [33]).

The intrinsic atomic and molecular decompositions established in this paper offer us a mapping property for the singular integral operator. Roughly speaking, we find that if a singular integral operator T satisfies some mild condition on its Schwartz kernel and is bounded on some Musielak–Orlicz spaces, then it is also bounded on the corresponding Hardy–Musielak–Orlicz spaces. For the precise statement of this mapping property, the reader is referred to Section 4.

This article has the following organization. We present the definitions and some properties of the Musielak–Orlicz spaces in Section 2. The Hardy–Musielak–Orlicz space is introduced in Section 3. We also establish the intrinsic atomic decompositions of the Hardy–Musielak–Orlicz spaces in that section. The molecular decomposition for the Hardy–Musielak–Orlicz spaces is given in Section 4. We also present an application of the intrinsic atomic and molecular decompositions on singular integral operators in Section 4.

2. Musielak–Orlicz spaces

The classical Musielak–Orlicz space was introduced in [24]. In this section, in order to prepare for the atomic decomposition, we present some properties for the Musielak–Orlicz spaces. We especially obtain the Fefferman–Stein vector-valued maximal inequalities for the Musielak–Orlicz spaces.

Let $B(x,r)=\{y\in\mathbb{R}^n : |x-y|<r\}$ denote the open ball with center $x\in\mathbb{R}^n$ and radius $r>0$. Let $\mathbb{B}=\{B(x,r): x\in\mathbb{R}^n, r>0\}$. For any $B\in\mathbb{B}$, let x_B and r_B be its center and radius. Let S be the set of simple functions, and let L^1_{loc} be the set of locally integrable functions.

For any $0<q<\infty$ and Lebesgue measurable function $\phi:\mathbb{R}^n\times[0,\infty)\to[0,\infty)$, write $\phi_q(x,t)=\phi(x,t^q)$, $\forall x\in\mathbb{R}^n$ and $t>0$. The following definition is modified from [5, Definition 2.3.1].

Definition 2.1. A function $\phi:\mathbb{R}^n\times[0,\infty)\to[0,\infty)$ is a generalized quasi-Φ-function if

(1) for any $x\in\mathbb{R}^n$, $\phi(x,\cdot)$ is nondecreasing, left-continuous with $\phi(x,0)=0$, $\lim_{t\to0^+}\phi(x,t)=0$, and $\lim_{t\to\infty}\phi(x,t)=\infty$ for all $x\in\mathbb{R}^n$,
(2) for any $t\in[0,\infty)$, $\phi(\cdot,t)$ is a Lebesgue measurable function,
(3) there exists a $r>1$ such that $\phi_r(x,\cdot)$ is convex.

We say that ϕ is a generalized Φ-function if $\phi(x,\cdot)$ is convex and ϕ is a generalized quasi-Φ-function.

In order to define the Hardy–Musielak–Orlicz spaces, we need to use the generalized quasi-Φ-function instead of the generalized Φ-function. For example, the classical Hardy space is defined via the function $\psi_p(t) = t^p$, $0 < p < 1$, which is not convex.

Definition 2.2. Let ϕ be a generalized quasi-Φ-function. The Musielak–Orlicz space L^ϕ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L^\phi} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \phi\Big(x, \frac{|f(x)|}{\lambda}\Big)\, dx \leq 1\Big\} < \infty.$$

For any $R > 0$, L^ϕ_R consists of all Lebesgue measurable functions f satisfying $\|f\|_{L^\phi_R} = \|f\chi_{B(0,R)}\|_{L^\phi} < \infty$ where $\chi_{B(0,R)}$ is the characteristic function of $B(0,R)$.

The reader is referred to [5, Definition 2.3.11] for the definition of the Musielak–Orlicz space associated with the generalized Φ-function. We find that

$$\|f\|_{L^{\phi_s}} = \big\||f|^s\big\|^{1/s}_{L^\phi}. \tag{2.1}$$

Therefore, L^{ϕ_s} is the $\frac{1}{s}$th power of L^ϕ (the s-convexification of L^ϕ) (see [29, Section 2.2] or [22, Vol. II, pp. 53–54]).

According to item (3) of Definition 2.1 and [5, Theorem 2.3.13], for any generalized quasi-Φ-function ϕ, L_{ϕ_r} is a Banach space, and therefore (2.1) assures that L^ϕ is a quasi-Banach space.

We recall another important notion for the generalized Φ-function from [5, Definition 2.6.1].

Definition 2.3. For any generalized Φ-function ϕ, the conjugate function ϕ^* is defined by

$$\phi^*(x,u) = \sup_{t\geq 0}\big(tu - \phi(x,t)\big).$$

In view of [5, Corollary 2.6.3], we have $\phi = (\phi^*)^*$. Next, we present the Hölder inequality for the pair L^ϕ and L^{ϕ^*}.

Lemma 2.4 (Hölder inequality)**.** *Let ϕ be a generalized Φ-function. We have*

$$\int_{\mathbb{R}^n} \big|f(x)g(x)\big|\, dx \leq 2\|f\|_{L^\phi}\|g\|_{L^\phi}.$$

For the proof of the above lemma, the reader is referred to [5, Lemma 2.6.5].

Corollary 2.5. *Let ϕ be a generalized Φ-function. If for any $R > 0$, $S \subset L^{\phi^*}_R$, then, for any $f \in L^\phi$,*

$$\|f\|_{L^\phi} \leq \sup_{g\in S, \|g\|_{L^{\phi^*}}\leq 1} \int_{\mathbb{R}^n} \big|f(x)g(x)\big|\, dx \leq 2\|f\|_{L^\phi}.$$

The reader may consult [5, Corollary 2.7.5 and Remark 2.7.6] for the proof of the above result.

Definition 2.6. A generalized Φ-function ϕ is called *proper* if L^ϕ is a Banach function space.

The reader is referred to [1, Chapter 1, Definition 1.1] for the definition of a Banach function space. In addition, it is shown in [5, p. 61] that ϕ is proper if and only if $S \subset L^{\phi} \cap (L^{\phi})'$.

It is easy to see that if ϕ is proper, then, for any $s \geq 1$, ϕ_s is also proper.

Definition 2.7. For any Banach function space X, the associated space X' consists of all $f \in \mathcal{M}$ such that

$$\|f\|_{X'} = \sup\Big\{ \int_{\mathbb{R}^n} f(x)g(x)\, dx : \|g\|_X \leq 1 \Big\} < \infty.$$

The associated space X' is a Banach function space (see [1, Chapter 1, Theorem 2.2, and Definition 2.3]). We now recall an important result for L^{ϕ} from [5, Theorem 2.7.12].

Theorem 2.8. *Let ϕ be a proper generalized Φ-function. Then $L^{\phi^*} = (L^{\phi})'$ and $(L^{\phi^*})' = L^{\phi}$. In addition, $(L^{\phi})'' = L^{\phi}$ with $\|\cdot\|_{L^{\phi}} = \|\cdot\|_{(L^{\phi})''}$.*

We now present a generalization of the notion of the proper generalized Φ-function. The main motivation for this generalization is on the weighted Lebesgue spaces. The weighted Lebesgue space $L^p(\omega)$ is not necessarily a Banach function space, as even ω belongs to the Muckenhoupt A_p class, since the characteristic function of an unbounded Lebesgue measurable set E with $|E| < \infty$ does not necessarily belong to $L^p(\omega)$. Therefore, we need to extend the notion of proper so that the weighted Lebesgue spaces are included in our study. The following definition is inspired by [5, Definition 2.7.8].

Definition 2.9. A generalized Φ-function ϕ is said to be *semiproper* if for any $R > 0$, $S \subset L^{\phi}_R \cap L^{\phi^*}_R$.

When $\omega \in A_p$, $1 \leq p < \infty$, the definition of the Muckenhoupt A_p class [32, Chapter V] guarantees that $\phi_\omega(x,t) = t^p\omega(x)$ is semiproper.

Definition 2.10. For any generalized Φ-function ϕ, we write $\phi \in \mathbb{M}$ if ϕ is a semiproper generalized Φ-function and the Hardy–Littlewood maximal operator M is bounded on L^{ϕ}.

The above definition is a special case from Banach function spaces (see [9, Definition 2.1]). When $\phi \in \mathbb{M}$ or $\phi^* \in \mathbb{M}$, we have the following result.

Lemma 2.11. *Let ϕ be a generalized Φ-function. If $\phi \in \mathbb{M}$ or $\phi^* \in \mathbb{M}$, then there exist constants $C_1, C_2 > 0$ such that*

$$C_1|B| \leq \|\chi_B\|_{L^{\phi}}\|\chi_B\|_{L^{\phi^*}} \leq C_2|B|, \quad \forall B \in \mathbb{B}. \tag{2.2}$$

Proof. In view of the fact that $\phi = (\phi^*)^*$, it suffices to assume that $\phi \in \mathbb{M}$. The Hölder inequality on L^{ϕ} yields the left-hand side inequality in (2.2).

For any $B \in \mathbb{B}$, we consider the projection $(P_B g)(y) = (\frac{1}{|B|}\int_B |g(x)|\, dx)\chi_B(y)$. There exists a constant $C > 0$ independent of $R > 0$ such that, for any $B \in \mathbb{B}$, $P_B(f) \leq CM(f)$; hence, $\sup_B \|P_B\|_{L^{\phi}_R \to L^{\phi}_R} \leq C\|M\|_{L^{\phi} \to L^{\phi}}$.

Furthermore, as L^{ϕ}_R is a Banach function space, Theorem 2.8 guarantees that $(L^{\phi}_R)' = L^{\phi^*}_R$. Thus the definition of associate space ensures that there exists a constant $C > 0$ such that, for any $R > 0$,

$$\begin{aligned}\|\chi_B\|_{L^{\phi^*}_R}\|\chi_B\|_{L^{\phi}_R} &= \sup\Big\{\Big|\int_B g(x)\,dx\Big|\|\chi_B\|_{L^{\phi}_R} : g \in L^{\phi}_R, \|g\|_{L^{\phi}_R} \le 1\Big\} \\ &\le C|B|\end{aligned} \tag{2.3}$$

for some $C > 0$ independent of $R > 0$ and $B \in \mathbb{B}$.

For any given $B \in \mathbb{B}$, pick $R > 0$ such that $B \subset B(0, R)$. Then $\|\chi_B\|_{L^{\phi^*}_R} = \|\chi_B\|_{L^{\phi^*}}$ and $\|\chi_B\|_{L^{\phi}_R} = \|\chi_B\|_{L^{\phi}}$. Therefore, the right-hand side inequality of (2.2) follows from (2.3). □

The following is the basic assumption imposed on ϕ for the study of the atomic decomposition of the Hardy–Musielak–Orlicz spaces.

Definition 2.12. Let $1 \le s < \infty$ and $0 < v \le 1$ and ϕ be a generalized quasi-Φ-function. We write $\phi \in \mathbb{H}_{s,v}$ if

(1) $(\phi^*_s)_v$ is a semiproper generalized Φ-function where $\phi^*_s = (\phi_s)^*$, and
(2) $(\phi^*_s)_v \in \mathbb{M}$.

As $(\phi^*_s)_v$ is a semiproper generalized Φ-function, [5, Remark 2.7.11] asserts that, for any $R > 0$, $L^{(\phi^*_s)_v}_R$ is a Banach function space. Since $0 < v \le 1$, $L^{\phi^*_s}_R$ is also a Banach function space; hence, $S \subset L^{\phi^*_s}_R$, and [5, Corollary 2.7.9] guarantees that ϕ_s is semiproper.

Therefore, we have

$$\|\chi_B\|_{L^{\phi_s}} < \infty, \quad \forall B \in \mathbb{B}. \tag{2.4}$$

Lemma 2.13. *Let $1 \le s < \infty$. For any $0 < v \le t \le 1$, we have $\mathbb{H}_{s,v} \subseteq \mathbb{H}_{s,t}$.*

Proof. As $(\phi^*_s)_v$ is a semiproper generalized Φ-function, for any $R > 0$, $L^{(\phi^*_s)_v}_R$ is a Banach function space. Consequently, $L^{(\phi^*_s)_t}_R = (L^{(\phi^*_s)_v}_R)^{t/v}$ is also a Banach function space. Thus $(\phi^*_s)_t$ is a semiproper generalized Φ-function.

Furthermore, the Hardy–Littlewood maximal operator is bounded on $(L^{\phi^*_s})^v$. In view of Jensen's inequality, for any $0 < v \le t$, we have $M(f) \le (M(|f|^{t/v}))^{v/t}$, $\forall f \in L^1_{\mathrm{loc}}$. Therefore,

$$\begin{aligned}\big\|M(f)\big\|_{L^{(\phi^*_s)_t}} &\le \big\|\big(M\big(|f|^{t/v}\big)\big)^{v/t}\big\|_{L^{(\phi^*_s)_t}} = \big\|\big(M\big(|f|^{t/v}\big)\big)\big\|^{v/t}_{L^{(\phi^*_s)_v}} \\ &\le C\big\|\big(|f|^{t/v}\big)\big\|^{v/t}_{L^{(\phi^*_s)_v}} = C\|f\|_{L^{(\phi^*_s)_t}}\end{aligned}$$

for some $C > 0$; that is, $(\phi^*_s)_t \in \mathbb{M}$. Thus $\mathbb{H}_{s,v} \subseteq \mathbb{H}_{s,t}$. □

We introduce some indices in the following. They are used to define the indices appearing in the intrinsic atomic decomposition.

Definition 2.14. For any $\phi \in \bigcup_{1 \le s < \infty} \mathbb{H}_{s,1}$, write

$$s_\phi = \inf\{s \ge 1 : \phi^*_s \in \mathbb{M}\}. \tag{2.5}$$

Define $\mathbb{H} = \bigcup_{\substack{1 \le s < \infty \\ 0 < v < 1}} \mathbb{H}_{s,v}$. For any $\phi \in \mathbb{H}$, write

$$\mathbb{S}_\phi = \big\{ s : s \ge 1, (\phi_s^*)_v \in \mathbb{M} \text{ for some } 0 < v < 1 \big\}. \tag{2.6}$$

For any fixed $s \in \mathbb{S}_\phi$, define $v_\phi^s = \inf\{v : (\phi_s^*)_v \in \mathbb{M}\}$.

Roughly speaking, the index v_ϕ^s is used to measure the "left-openness" of the boundedness of the Hardy–Littlewood maximal operator on $L^{\phi_s^*}$.

The Jensen inequality shows that, for any $v > v_\phi^s$, $(\phi_s^*)_v \in \mathbb{M}$. Thus $s \in \mathbb{S}_\phi$ implies $s > s_\phi$.

For any $0 < p < \infty$, let $\psi_p(t) = t^p$. To compute s_{ψ_p}, we find that, for any $s \ge 1$, we have $(\psi_p)_s(t) = \psi_{ps}(t) = t^{ps}$. Therefore, $(\psi_p)_s^*(t) = t^{(ps)^*}$ when $ps > 1$.

Thus,

$$s_{\psi_p} = 1/p, \qquad \mathbb{S}_{\psi_p} = [1/p, \infty) \qquad \text{and} \qquad \nu_{\psi_p}^s = 1 - \frac{1}{ps} \tag{2.7}$$

whenever $ps > 1$.

Later in the atomic decomposition of the Hardy–Musielak–Orlicz spaces, we see that the indices s_{ψ_p} and $\nu_{\psi_p}^s$ are used to generate some well-known indices in the atomic decompositions such as the order of the vanishing moment conditions.

Next, we establish the Fefferman–Stein vector-valued maximal inequalities on the Musielak–Orlicz spaces in the following.

Theorem 2.15. *Let ϕ be a generalized Φ-function. If $\phi \in \bigcup_{1 < s < \infty} \mathbb{H}_{s,1}$, then, for any $1 \le s_\phi < \beta < \infty$, we have*

$$\big\| \|\{Mf_i\}_{i\in\mathbb{N}}\|_{l_q} \big\|_{L^{\phi_\beta}} \le C \big\| \|\{f_i\}_{i\in\mathbb{N}}\|_{l_q} \big\|_{L^{\phi_\beta}} \tag{2.8}$$

for some $C > 0$. In particular, we have

$$\|Mf\|_{L^{\phi_\beta}} \le C \|f\|_{L^{\phi_\beta}}. \tag{2.9}$$

Proof. As $\phi \in \mathbb{H}_{t,1}$ for some $0 < t < \infty$, there exists a s such that $s_\phi < s < \beta$ and $\phi_s^* \in \mathbb{M}$. Since ϕ_s^* is semiproper, Theorem 2.8 shows that, for any $R > 0$, $(L_R^{\phi_s})' = L_R^{\phi_s^*}$ is a Banach function space. Furthermore, $\phi_s^* \in \mathbb{M}$ assures that the Hardy–Littlewood maximal operator is bounded on $(L_R^{\phi_s})'$. Therefore, there exists a constant $C > 0$ independent of R such that $\|Mf\|_{L^{\phi_s^*}} \le C\|f\|_{L^{\phi_s^*}}$. Consequently,

$$\|Mf\|_{(L_R^{\phi_s})'} = \|Mf\|_{L^{\phi_s^*}} \le C\|f\|_{L^{\phi_s^*}} = C\|f\|_{(L_R^{\phi_s})'}, \quad \forall f \in (L_R^{\phi_s})'. \tag{2.10}$$

The constant in (2.10) is independent of $R > 0$, and the Hardy–Littlewood maximal operator is bounded on $L^1(\omega)$ for any $\omega \in A_1$. Therefore, by using $L^{\phi_\beta} = (L^{\phi_s})^{\beta/s}$ and $\beta > s$, the proof of the extrapolation theorem for general Banach function spaces (see [4, Corollary 4.8]) guarantees that (2.8) is valid for any bounded Lebesgue measurable functions with supp $f_i \subset B(0, R)$,

$$\big\| \|\{Mf_i\}_{i\in\mathbb{N}}\|_{l_q} \big\|_{L^{\phi_\beta}} \le C \big\| \|\{f_i\}_{i\in\mathbb{N}}\|_{l_q} \big\|_{L^{\phi_\beta}},$$

where $C > 0$ is independent of $R > 0$.

Finally, by applying Fatou's lemma (see [5, Theorem 2.3.17(d)]) on $f_i^k = f_i \chi_{\{x : |f_i(x)| \le k, |x| \le k\}}$, $k \in \mathbb{N}$, we establish (2.8). □

We introduce another new index in the following.

Definition 2.16. For any $\phi \in \mathbb{H}$, define

$$d_\phi = \sup\{s(1-\nu_\phi^s) : s \in \mathbb{S}_\phi\}.$$

For instance, in view of (2.7), we have $d_{\psi_p} = 1/p$, $0 < p \le 1$. By using this index, we establish the following property which is crucial for the atomic and molecular decompositions.

Roughly speaking, this index is used to measure the "dilation property" for the characteristic function of $B \in \mathbb{B}$ under the quasinorm $\|\cdot\|_{L^\phi}$.

Proposition 2.17. *Let $\phi \in \mathbb{H}$. For any $d < d_\phi$, there exists constant $C_2 > 0$ such that, for any $x_0 \in \mathbb{R}^n$ and $r > 0$, we have*

$$C_2 2^{jnd} \le \frac{\|\chi_{B(x_0,2^j r)}\|_{L^\phi}}{\|\chi_{B(x_0,r)}\|_{L^\phi}}, \quad \forall j \in \mathbb{N}. \tag{2.11}$$

Proof. In view of the definition of d_ϕ, for any $d < d_\phi$ there exist $1 \le s < \infty$ and $0 < t < 1$ such that $d < s(1-t)$ and $\phi \in \mathbb{H}_{s,t}$.

For any $B = B(x_0, r) \in \mathbb{B}$ and $j \in \mathbb{N}$, we have a constant $C > 0$ such that

$$C2^{-jn} \le \mathrm{M}(\chi_B)(x) \tag{2.12}$$

when $x \in B(x_0, 2^j r)$, $j \in \mathbb{N}$. Thus,

$$2^{-jn}\|\chi_{B(x_0,2^j r)}\|_{L^{(\phi_s^*)_t}} \le C\big\|\mathrm{M}(\chi_B)\big\|_{L^{(\phi_s^*)_t}} \le C\|\chi_B\|_{L^{(\phi_s^*)_t}};$$

that is,

$$2^{-jnt}\|\chi_{B(x_0,2^j r)}\|_{L^{\phi_s^*}} \le C\big\|\mathrm{M}(\chi_B)\big\|_{L^{\phi_s^*}} \le C\|\chi_B\|_{L^{\phi_s^*}}. \tag{2.13}$$

Since $\phi_s^* \in \mathbb{M}$, Lemma 2.11 provides constants $D_1, D_2 > 0$ such that, for any $B \in \mathbb{B}$,

$$D_1|B| \le \|\chi_B\|_{L^{\phi_s}}\|\chi_B\|_{L^{\phi_s^*}} \le D_2|B|. \tag{2.14}$$

Therefore, (2.13), (2.14), $d < s(1-t)$, and $\|\chi_B\|_{L^{\phi_s}} = \|\chi_B\|_{L^\phi}^{1/s}$ yield (2.11). $\square$

We also have an upper estimate for (2.11). Since we do not need that estimate in this paper, for brevity, we refer the reader to [10, Proposition 2.5] for details.

3. Atomic decomposition

In this section, we define the Hardy–Musielak–Orlicz spaces and establish the corresponding intrinsic atomic decomposition.

Let $\mathcal{S}$ and $\mathcal{S}'$ denote the classes of tempered functions and Schwartz distributions, respectively. Let $\mathcal{P}$ denote the class of polynomials in $\mathbb{R}^n$. A Schwartz distribution $f \in \mathcal{S}'$ is a bounded tempered distribution if $\psi * f \in L^\infty(\mathbb{R}^n)$ for any $\psi \in \mathcal{S}$.

For any $j \in \mathbb{Z}$ and $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, $Q_{j,k} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : k_i \le 2^j x_i \le k_i + 1, i = 1, 2, \dots, n\}$. We write x_Q, $|Q|$, and $l(Q)$ to be the center of Q, the Lebesgue measure of Q, and the side length of Q, respectively. We denote the set of dyadic cubes $\{Q_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ by $\mathcal{Q}$.

For any $N \in \mathbb{N}$ and $\varphi \in \mathcal{S}$, define $\mathfrak{N}_N(\varphi) = \sup_{x\in\mathbb{R}^n}(1+|x|)^N \sum_{|\gamma|\leq N+1} |\partial^\gamma \varphi(x)|$. Write $\mathcal{F}_N = \{\varphi \in \mathcal{S} : \mathfrak{N}_N(\varphi) \leq 1\}$. For any $t > 0$ and $\varphi \in \mathcal{S}$, write $\varphi_t(x) = t^{-n}\varphi(x/t)$. For any $f \in \mathcal{S}'$, the grand maximal function of f is given by

$$(\mathcal{M}f)(x) = \sup_{\varphi\in\mathcal{F}_N} \sup_{t>0} \big|(\varphi_t * f)(x)\big|$$

(see [32, Chapter III, (2)]).

The grand maximal function depends on N. For simplicity, we use the abused notion $\mathcal{M}$. We are now ready to define the Hardy–Musielak–Orlicz spaces.

Definition 3.1. Let $\phi \in \mathbb{H}$. The Hardy–Musielak–Orlicz spaces H^ϕ consist of all bounded $f \in \mathcal{S}'$ satisfying $\|f\|_{H^\phi} = \|\mathcal{M}f\|_{L^\phi} < \infty$.

We introduce the atoms used in the intrinsic atomic decomposition for H^ϕ.

Definition 3.2. Let $\phi \in \mathbb{H}$, and let $p > s_\phi$. For any $N \in \mathbb{N}$, a family of measurable functions $\{a_B\}_{B\in\mathbb{B}}$ is called a (ϕ, p, N)-*atomic family* if

$$\operatorname{supp} a_B \subseteq 3B, \quad \forall B \in \mathbb{B}, \tag{3.1}$$

$$\int_{\mathbb{R}^n} x^\gamma a_B(x)\, dx = 0, \quad \forall \gamma \in \mathbb{N}^n \text{ with } 0 \leq |\gamma| \leq N, \tag{3.2}$$

$$\|a_B\|_{L^{\phi_p}} \leq \|\chi_B\|_{L^\phi}^{\frac{1}{p}-1}. \tag{3.3}$$

We call a_B an atom supported in B.

In the size condition (3.3), it only involves the quasinorms $\|\cdot\|_{L^\phi}$ and $\|\cdot\|_{L^{\phi_p}}$.

Similar to the atomic decomposition of the classical Hardy spaces, the intrinsic atomic decomposition for H^ϕ consists of two parts, namely, the decomposition theorem and the reconstruction theorem. We now present the decomposition part of the intrinsic atomic decomposition for H^ϕ.

Theorem 3.3. *Let $\phi \in \mathbb{H}$. For any $s \in \mathbb{S}_\phi$, $f \in H^\phi$, and any positive integer N, there exist a (ϕ, s, N)-atomic family, $\{a_B\}_{B\in\mathbb{B}}$, and a sequence $t = \{t_B\}_{B\in\mathbb{B}}$ such that*

$$f = \sum_{B\in\mathbb{B}} t_B a_B \tag{3.4}$$

converges in $\mathcal{S}'$ and

$$\Big\|\sum_{B\in\mathbb{B}} \Big(\frac{|t_B|}{\|\chi_B\|_{L^\phi}}\Big)^\theta \chi_B\Big\|_{L^{\phi_{1/\theta}}}^{\frac{1}{\theta}} \leq C\|f\|_{H^\phi}, \quad \forall 0 < \theta < \infty$$

for some $C > 0$.

We recall a crucial supporting result for the atomic decomposition [32, Chapter III, Section 2.1]. We use the presentation given in [12, Proposition 5.4] and [28, Lemma 4.7]. For any $d \in \mathbb{N}$, let $\mathcal{P}_d$ denote the class of polynomials in $\mathbb{R}^n$ of degree less than or equal to d.

Proposition 3.4. *Let $d \in \mathbb{N}$, and let $\sigma > 0$. For any $f \in \mathcal{S}'$, there exist $g \in \mathcal{S}'$, $\{b_k\}_{k\in\mathbb{N}} \subset \mathcal{S}'$, a collection of cubes $\{Q_k\}_{k\in\mathbb{N}}$, and a family of smooth functions with compact supports $\{\eta_k\}$ such that*

(1) *$f = g + b$, where $b = \sum_{k\in\mathbb{N}} b_k$,*
(2) *the family $\{Q_k\}_{k\in\mathbb{N}}$ has bounded intersection property and $\bigcup_{k\in\mathbb{N}} Q_k = \{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > \sigma\}$,*
(3) *$\operatorname{supp} \eta_k \subset Q_k$, $0 \le \eta_k \le 1$, and $\sum_{k\in\mathbb{N}} \eta_k = \chi_{\{x\in\mathbb{R}^n:(\mathcal{M}f)(x)>\sigma\}}$,*
(4) *the tempered distribution g satisfies*

$$(\mathcal{M}g)(x) \le C(\mathcal{M}f)(x)\chi_{\{y\in\mathbb{R}^n:(\mathcal{M}f)(y)\le\sigma\}}(x) + C\sigma\sum_{k\in\mathbb{N}}\frac{l(Q_k)^{n+d+1}}{(l(Q_k)+|x-x_k|)^{n+d+1}},$$

where x_k denotes the center of the cube Q_k,
(5) *the tempered distribution b_k is given by $b_k = (f - c_k)\eta_k$, where $c_k \in \mathcal{P}_d$ satisfying $\langle f - c_k, q\cdot\eta_k\rangle = 0$, $\forall q \in \mathcal{P}_d$, and*

$$(\mathcal{M}b_k)(x) \le C(\mathcal{M}f)(x)\chi_{Q_k}(x) + C\sigma\frac{l(Q_k)^{n+d+1}}{|x-x_k|^{n+d+1}}\chi_{\mathbb{R}^n\setminus Q_k}(x) \tag{3.5}$$

for some $C > 0$.

We present some folklore facts about b and g given in Proposition 3.4.

Lemma 3.5. *Let $\phi \in \mathbb{H}$, and let $f \in H^\phi$. The distribution g given in Proposition 3.4 is locally integrable.*

Proof. We first show that $\mathcal{M}g \in L^1_{\mathrm{loc}}$. In view of item (4) of Proposition 3.4 and the fact that

$$\frac{Cl^n}{(l+|x-y|)^n} \le (M\chi_{B(y,l)})(x) \tag{3.6}$$

for some $C > 0$ independent of $x, y \in \mathbb{R}^n$ and $l > 0$, it suffices to show that $F = \sum_{k\in\mathbb{N}}(M\chi_{Q_k})^{\frac{n+d+1}{n}} \in L^1_{\mathrm{loc}}$.

The definition of s_ϕ assures that there exists $s_\phi < r$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{\phi_r^*}$.

For any $B \in \mathbb{B}$, by [6, Chapter II, Theorem 2.12], we have

$$\begin{aligned}\int_B |F(x)|\,dx &\le \sum_{k\in\mathbb{N}}\int_{\mathbb{R}^n}\big(M\chi_{Q_k}(x)\big)^{\frac{n+d+1}{n}}\chi_B(x)\,dx\\ &\le C\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{N}}\chi_{Q_k}(x)\Big)(M\chi_B)(x)\,dx.\end{aligned}$$

By using the bounded intersection property for $\{Q_k\}$, we find that

$$\int_B |F(x)|\,dx \le C\int_{\mathbb{R}^n}\chi_{\{y\in\mathbb{R}^n:(\mathcal{M}f)(y)>\sigma\}}(x)(M\chi_B)(x)\,dx.$$

The Hölder inequality yields

$$\begin{aligned}\int_B |F(x)|\,dx &\le C\|\chi_{\{x\in\mathbb{R}^n:(\mathcal{M}f)(x)>\sigma\}}\|_{L^{\phi_r}}\|M\chi_B\|_{L^{\phi_r^*}}\\ &\le C\|\chi_{\{x\in\mathbb{R}^n:(\mathcal{M}f)(x)>\sigma\}}\|_{L^\phi}^{1/r}\|\chi_B\|_{L^{\phi_r^*}}\\ &\le C\sigma^{-1/r}\|\mathcal{M}f\|_{L^\phi}^{1/r}\|\chi_B\|_{L^{\phi_r^*}} < \infty;\end{aligned}$$

that is, $F \in L^1_{\mathrm{loc}}$, and hence $\mathcal{M}g \in L^1_{\mathrm{loc}}$. By using the idea from [32, Chapter III, 2.3.3], we now prove that $g \in L^1_{\mathrm{loc}}$.

For any $B \in \mathbb{B}$, let A_B be the spaces of finite Borel measures on B. A_B is the dual of the space of continuous functions on B and $\mathcal{M}g \in L^1_{\mathrm{loc}} \subset A_B$. Taking an approximate of identity Ψ, we have $|\Psi_i * g| \leq C\mathcal{M}g$ for some $C > 0$ and $\Psi_i * g \to g$ in $\mathcal{S}'$.

The Banach–Alaoglou theorem assures that there exists a subsequence of $\Psi_i * g$ converging weakly to a measure $d\mu \in A_B$. Since $|\Psi_i * g| \leq C\mathcal{M}g$, we find that $d\mu = h\,dx$ is absolutely continuous with $\int_B |h(x)|\,dx < \infty$, and hence $g = h$. Therefore, $g \in L^1_{\mathrm{loc}}$. □

Proposition 3.6. *Let $\phi \in \mathbb{H}$, and let $f \in H^\phi$. If $d \geq [ns_\phi - n]$, then the distributions b and g given in Proposition 3.4 belong to H^ϕ.*

Proof. Since $g = f - b$, it suffices to show that $b \in H^\phi$. Let $O = \{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > \sigma\} = \bigcup_{k\in\mathbb{N}} Q_k$. The definition of s_ϕ assures the existence of $r > s_\phi$ such that

$$s_\phi < r < \frac{[ns_\phi - n] + n + 1}{n} \leq \frac{d + n + 1}{n}, \tag{3.7}$$

and the Hardy–Littlewood maximal operator M is bounded on $L^{\phi_r^*}$.

In view of (3.5) and (3.6), for any $h \in L^{\phi_r^*}$ with $\|h\|_{L^{\phi_r^*}} \leq 1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\mathcal{M}b)(x)^{1/r}\big|h(x)\big|\,dx \\
&\quad\leq C\int_{\mathbb{R}^n} \sum_{k\in\mathbb{N}} (\mathcal{M}f)(x)^{1/r}\big|h(x)\big|\chi_{Q_k}(x)\,dx \\
&\qquad + C\sigma^{1/r}\int_{\mathbb{R}^n} \big|h(x)\big| \sum_{k\in\mathbb{N}} \left(\frac{l(Q_k)^{n+d+1}\chi_{\mathbb{R}^n\setminus Q_k}(x)}{(l(Q_k) + |x - x_k|)^{n+d+1}}\right)^{1/r} dx \\
&\quad\leq C\int_O (\mathcal{M}f)(x)^{1/r}\big|h(x)\big|\,dx + C\sigma^{1/r}\sum_{k\in\mathbb{N}} \int_{\mathbb{R}^n} (M\chi_{Q_k})(x)^{(n+d+1)/rn}\big|h(x)\big|\,dx.
\end{aligned}$$

By using [6, Chapter II, Theorem 2.12], we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} (M\chi_{Q_k})(x)^{(n+d+1)/rn}\big|h(x)\big|\,dx &\leq C\int_{\mathbb{R}^n} \chi_{Q_k}(x)^{(n+d+1)/rn}(Mh)(x)\,dx \\
&= C\int_{\mathbb{R}^n} \chi_{Q_k}(x)(Mh)(x)\,dx \\
&= C\int_{Q_k} (Mh)(x)\,dx
\end{aligned}$$

because $\frac{n+d+1}{rn} > 1$.

Lemma 2.4, the bounded intersection property satisfied by $\{Q_k\}_{k\in\mathbb{N}}$, and (3.9) assure that

$$\begin{aligned}
\int_{\mathbb{R}^n} (\mathcal{M}b)(x)^{1/r}\big|h(x)\big|\,dx &\leq C\int_O (\mathcal{M}f)(x)^{1/r}(Mh)(x)\,dx \\
&\leq C\big\|\chi_O(\mathcal{M}f)^{1/r}\big\|_{L^{\phi_r}}\|Mh\|_{L^{\phi_r^*}}.
\end{aligned}$$

Since M is bounded on $L^{\phi_r^*}$ and $\|h\|_{L^{\phi_r^*}} \le 1$, we obtain

$$\int_{\mathbb{R}^n} (\mathcal{M}b)(x)^{1/r} \big|h(x)\big|\, dx \le C \big\|\chi_O(\mathcal{M}f)^{1/r}\big\|_{L^{\phi_r}} \|h\|_{L^{\phi_r^*}} \le C \big\|\chi_O(\mathcal{M}f)^{1/r}\big\|_{L^{\phi_r}}.$$

By taking supremum over those $h \in L^{\phi_r^*}$ with $\|h\|_{L^{\phi_r^*}} \le 1$, Proposition 2.5 yields

$$\|\mathcal{M}b\|_{L^\phi}^{1/r} = \big\|(\mathcal{M}b)^{1/r}\big\|_{L^{\phi_r}} \le C\big\|\chi_O(\mathcal{M}f)^{1/r}\big\|_{L^{\phi_r}} = C\|\chi_O\mathcal{M}f\|_{L^\phi}^{1/r} < \infty; \quad (3.8)$$

that is, $b \in H^\phi$. □

To consider the density of $H^\phi \cap L^1_{\mathrm{loc}}$ in H^ϕ, we use the notion of the absolutely continuous quasinorm (see [8, Definition 2.4]).

Definition 3.7. Let $\phi \in \mathbb{H}$. We say that $f \in H^\phi$ has an absolutely continuous quasinorm if $\|\chi_{E_j}\mathcal{M}f\|_{L^\phi} \downarrow 0$ whenever $\{E_j\}_{j=1}^\infty$ are Lebesgue measurable sets and $E_j \downarrow \emptyset$.

We say that H^ϕ has an absolutely continuous quasinorm provided that any $f \in H^\phi$ has an absolutely continuous quasinorm.

Corollary 3.8. *Let $\phi \in \mathbb{H}$. If $f \in H^\phi$ has an absolutely continuous quasinorm, then there exist a family of locally integrable functions $\{g^j\}_{j=1}^\infty \subset H^\phi \cap L^1_{\mathrm{loc}}$ such that $\lim_{j\to\infty} \|f - g^j\|_{H^\phi} = 0$. Furthermore, if H^ϕ has an absolutely continuous quasinorm, then $H^\phi \cap L^1_{\mathrm{loc}}$ is dense in H^ϕ.*

Proof. Suppose that $f \in H^\phi$ has an absolutely continuous quasinorm. Let b^j, g^j be the distributions given in Proposition 3.4 corresponding to $\sigma = 2^j$, $j \in \mathbb{Z}$. Let $O^j = \{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > 2^j\}$. We have $O_j \downarrow \emptyset$ as $j \to \infty$.

Therefore, (3.8) shows that

$$\lim_{j\to\infty} \|b^j\|_{H^\phi} = \lim_{j\to\infty} \|\mathcal{M}b^j\|_{L^\phi} \le C \lim_{j\to\infty} \|\chi_{O^j}\mathcal{M}f\|_{L^\phi} = 0.$$

Since $g^j \in H^\phi \cap L^1_{\mathrm{loc}}$, we find that $\lim_{j\to\infty} \|f - g^j\|_{H^\phi} = \lim_{j\to\infty} \|b^j\|_{H^\phi} = 0$.

Obviously, if H^ϕ has an absolutely continuous quasinorm, then $H^\phi \cap L^1_{\mathrm{loc}}$ is dense in H^ϕ. □

In view of [3, Theorems 2.58 and 2.62 and p. 73], the Lebesgue space with variable exponent $L^{p(\cdot)}$ has absolutely continuous norm if and only if $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty$. Thus, if $q : \mathbb{R}^n \to (0, \infty]$ satisfies $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} q(x) = \infty$, then H^ϕ with $\phi(x,t) = t^{q(x)}$ does not have absolutely continuous norm.

We are now ready to prove Theorem 3.3. The proof follows the idea from [32, Chapter III, Section 2].

Proof of Theorem 3.3. For any $s \in \mathbb{S}_\phi$ with $s > s_\phi$, let $d_s = [ns - n]$. Notice that we have $s_\phi < s < \frac{d_s+n+1}{n}$. It suffices to establish Theorem 3.3 for an (ϕ, s, d)-atomic family with $d > d_s$.

For any $f \in H^\phi$, by applying Proposition 3.4 with d, $\sigma = 2^j$, $j \in \mathbb{Z}$, we have $f = g^j + b^j$ with $b^j = \sum_{k\in\mathbb{N}} b_k^j$. The b_k^j are supported in the cubes Q_k^j where these cubes satisfy

$$\bigcup_{k\in\mathbb{N}} Q_k^j = \left\{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > 2^j\right\} = O^j. \tag{3.9}$$

Let $\{\eta_k^j\}$ be the family of smooth functions given in item (3) of Proposition 3.4 for the collection of cube $\{Q_k^j\}$.

Item (4) of Proposition 3.4 ensures that

$$\begin{aligned} c\left|\varphi * g^j(x)\right| &\le (\mathcal{M}g^j)(x) \\ &\le C(\mathcal{M}f)(x)\chi_{\{y\in\mathbb{R}^n:(\mathcal{M}f)(y)\le 2^j\}}(x) + C2^j \sum_{k\in\mathbb{N}} \frac{l(Q_k^j)^{n+d+1}}{(l(Q_k^j)+|x-x_k^j|)^{n+d+1}} \\ &\le C2^j \end{aligned}$$

for some constants $c, C > 0$, where x_k^j is the center of Q_k^j. Therefore, $g^j \to 0$ in $\mathcal{S}'$ as $j \to -\infty$.

Next, we show that $b^j \to 0$ in $\mathcal{S}'$ as $j \to \infty$. By item (5) of Proposition 3.4, we find that, for any $B = B(z, 1) \in \mathbb{B}$, $z \in \mathbb{R}^n$,

$$\begin{aligned} &\int_B (\mathcal{M}b^j)(x)^{1/s}\,dx \\ &\quad\le C\int_B \sum_{k\in\mathbb{N}} (\mathcal{M}f)(x)^{1/s}\chi_{Q_k^j}(x)\,dx \\ &\qquad + C2^{j/s}\int_B \sum_{k\in\mathbb{N}} \Big(\frac{l(Q_k^j)^{n+d+1}\chi_{\mathbb{R}^n\setminus Q_k^j}(x)}{(l(Q_k^j)+|x-x_k^j|)^{n+d+1}}\Big)^{1/s}\,dx \\ &\quad\le C\int_{B\cap O^j} (\mathcal{M}f)(x)^{1/s}\,dx + C2^{j/s}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^n} \chi_B(x)(M\chi_{Q_k^j})(x)^{(n+d+1)/sn}\,dx. \end{aligned}$$

By using [6, Chapter II, Theorem 2.12], we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} (M\chi_{Q_k^j})(x)^{(n+d+1)/sn}\chi_B(x)\,dx &\le C\int_{\mathbb{R}^n} \chi_{Q_k^j}(x)^{(n+d+1)/sn}(M\chi_B)(x)\,dx \\ &= C\int_{\mathbb{R}^n} \chi_{Q_k^j}(x)(M\chi_B)(x)\,dx \\ &= C\int_{Q_k^j} (M\chi_B)(x)\,dx \end{aligned}$$

because $(n + d_s + 1)/sn > 1$.

Consequently, the above inequalities, (3.9), and the bounded intersection property satisfied by $\{Q_k^j\}_{k\in\mathbb{N}}$ yield

$$\begin{aligned} \int_B (\mathcal{M}b^j)(x)^{1/s}\,dx &\le C\int_{O^j} (\mathcal{M}f)(x)^{1/s}(M\chi_B)(x)\,dx \\ &\le C\int_{O^j} (\mathcal{M}f)(y)^{1/s}\big(1+|z-y|\big)^{-n}\,dy. \end{aligned}$$

By using Lemma 2.11 and the Hölder inequality for the pair L^{ϕ_s} and $L^{\phi_s^*}$, we find that

$$\begin{aligned}&\int_{\mathbb{R}^n}(\mathcal{M}f)(y)^{1/s}\big(1+|z-y|\big)^{-n}\,dy\\&\quad\le C\sum_{k=0}^{\infty}2^{-kn}\int_{\mathbb{R}^n}(\mathcal{M}f)(y)^{1/s}\chi_{B^k}(y)\,dy\\&\quad\le C\sum_{k=0}^{\infty}\frac{1}{|B(z,2^k)|}\big\|(\mathcal{M}f)(y)^{1/s}\big\|_{L^{\phi_s}}\|\chi_{B(z,2^k)}\|_{L^{\phi_s^*}}\\&\quad\le C\sum_{k=0}^{\infty}\frac{1}{\|\chi_{B(z,2^k)}\|_{L^{\phi_s}}}\|\mathcal{M}f\|_{L^\phi}^{1/s},\end{aligned}$$

where $B^k=B(z,2^k)\backslash B(z,2^{k-1})$ when $k\ge 1$ and $B^0=B(z,1)$.

Proposition 2.17 gives

$$\int_{\mathbb{R}^n}(\mathcal{M}f)(y)^{1/s}\big(1+|z-y|\big)^{-n}\,dy\le C\|\mathcal{M}f\|_{L^\phi}^{1/s}.$$

In view of the fact that $O^j\downarrow\emptyset$, the dominated convergence theorem yields that, for any fixed $z\in\mathbb{R}^n$,

$$\begin{aligned}&\lim_{j\to\infty}\int_{B(z,1)}(\mathcal{M}b^j)(x)^{1/s}\,dx\\&\quad\le C\lim_{j\to\infty}\int_{O^j}(\mathcal{M}f)(y)^{1/s}\big(1+|z-y|\big)^{-n}\,dy=0.\end{aligned}\tag{3.10}$$

For any $\psi\in\mathcal{S}$ and $z\in\mathbb{R}^n$, we have

$$\big|b^j*\psi(z)\big|^{1/s}\le\inf_{|y-z|\le1}\sup_{x\in\mathbb{R}^n:|y-x|\le1}\big|(\psi*b^j)(x)\big|^{1/s}\le C\int_{B(z,1)}\mathcal{M}(b^j)(y)^{1/s}\,dy$$

for some $C>0$ independent of j. Thus (3.10) yields that $\lim_{j\to\infty}b_j=0$ in $\mathcal{S}'$.

Therefore, for any $f\in H^\phi$ there exists a sequence of locally integrable functions $\{g^j\}\subset L^1_{\mathrm{loc}}$ such that $g^j\to f$ in $\mathcal{S}'$. Consequently, to obtain the atomic decomposition, it suffices to assume that $f\in H^\phi\cap L^1_{\mathrm{loc}}$.

The convergence of g^j and b^j guarantees that

$$f=\sum_{j\in\mathbb{Z}}(g^{j+1}-g^j)\quad\text{in }\mathcal{S}'.\tag{3.11}$$

Moreover, item (5) of Proposition 3.4 gives

$$g^{j+1}-g^j=b^{j+1}-b^j=\sum_{k\in\mathbb{N}}\big((f-c_k^{j+1})\eta_k^{j+1}-(f-c_k^j)\eta_k^j\big),$$

where $c_k^j\in\mathcal{P}_d$ satisfies $\int_{\mathbb{R}^n}(f(x)-c_k^j(x))q(x)\eta_k^j(x)\,dx=0$, $\forall q\in\mathcal{P}_d$.

Consequently, we have $f=\sum_{j,k}A_k^j$, where

$$A_k^j=(f-c_k^j)\eta_k^j-\sum_{l\in\mathbb{N}}(f-c_l^{j+1})\eta_l^{j+1}\eta_k^j+\sum_{l\in\mathbb{N}}c_{k,l}\eta_l^{j+1}$$

and $c_{k,l} \in \mathcal{P}_d$ fulfills

$$\int_{\mathbb{R}^n} \big((f(x) - c_l^{j+1}(x))\eta_k^j(x) - c_{k,l}(x)\big)q(x)\eta_l^{j+1}(x)\,dx = 0, \quad \forall q \in \mathcal{P}_d.$$

Define $a_k^j = \lambda_{j,k}^{-1} A_k^j$ and $\lambda_{j,k} = c2^j \|\chi_{Q_k^j}\|_{L^\phi}$, where c is a constant determined by the family $\{A_k^j\}_{j,k}$. The important fact is that the constant c is independent of j and k (see [32, pp. 108–109]).

The proof for the classical Hardy space (see [32, Chapter III, Section 2]) assures that a_k^j satisfies (3.1), (3.2), and $\|a_k^j\|_{L^\infty} \le C\|\chi_{Q_k^j}\|_{L^\phi}^{-1}$.

Furthermore, we have $\chi_{3Q_j^k} \le CM(\chi_{Q_j^k})$ for some $C > 0$ independent of j, k. Thus, (2.9) shows that, for any fixed $\beta > s_\phi$,

$$\|\chi_{3Q_j^k}\|_{L^\phi} = \|\chi_{3Q_j^k}\|_{L^{\phi_\beta}}^\beta \le C\big\|M(\chi_{Q_j^k})\big\|_{L^{\phi_\beta}}^\beta \le C\|\chi_{Q_j^k}\|_{L^\phi}.$$

Therefore, $\|a_k^j\|_{L^{\phi_s}} \le C\|\chi_{3Q_j^k}\|_{L^{\phi_s}}\|\chi_{Q_j^k}\|_{L^\phi}^{-1} = \|\chi_{Q_j^k}\|_{L^\phi}^{\frac{1}{s}-1}$. The definition of Q_k^j and the finite intersection property of the family $\{Q_k^j\}_{k\in\mathbb{N}}$ yield that, for any $0 < \theta < \infty$,

$$\sum_{k\in\mathbb{N}} \Big(\frac{|\lambda_{j,k}|}{\|\chi_{Q_k^j}\|_{L^\phi}}\Big)^\theta \chi_{Q_k^j}(x) \le C2^{\theta j}\chi_{O^j}(x);$$

that is,

$$\sum_{j,k} \Big(\frac{|\lambda_{j,k}|}{\|\chi_{Q_k^j}\|_{L^\phi}}\Big)^\theta \chi_{Q_k^j}(x) \le C\sum_{j\in\mathbb{Z}} 2^{\theta j}\chi_{O^j}(x) \le C(\mathcal{M}f)(x)^\theta.$$

Applying the quasinorm $\|\cdot\|_{L^{\phi_{1/\theta}}}^{1/\theta}$ on both sides of the above inequality, we find that

$$\Big\|\sum_{j,k} \Big(\frac{|\lambda_{j,k}|}{\|\chi_{Q_k^j}\|_{L^\phi}}\Big)^\theta \chi_{Q_k^j}\Big\|_{L^{\phi_{1/\theta}}}^{\frac{1}{\theta}} \le C\|f\|_{H^\phi}, \quad 0 < \theta < \infty$$

for some $C > 0$ independent of f. □

In view of Corollary 3.8, we see that, whenever $f \in H^\phi$ has an absolutely continuous quasinorm, the decomposition of f given in (3.11) converges in H^ϕ. Therefore, the atomic decomposition obtained afterward also converges in H^ϕ. Thus, if $f \in H^\phi$ has an absolutely continuous quasinorm, then the atomic decomposition given in (3.4) converges in H^ϕ. Next, we have the reconstruction part of the intrinsic atomic decomposition of H^ϕ.

Theorem 3.9. *Let $\phi \in \mathbb{H}$. Suppose that*

(1) $0 < \theta \le 1$ *satisfies* $\frac{1}{\theta} \in \mathbb{S}_\phi$, *and*

(2) $q > \max(s_\phi, \theta s_\phi((v_\phi^{1/\theta})^{-1})')$.

Then, for any $(\phi, q, [ns_\phi - n])$-atomic family, $\{a_{B_j}\}_{B_j\in\mathbb{B}}$, and sequence of scalars $\{\lambda_j\}_{j\in\mathbb{N}}$ satisfying

$$\Big\|\sum_{j\in\mathbb{N}} \Big(\frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^\phi}}\Big)^\theta \chi_{B_j}\Big\|_{L^{\phi_{1/\theta}}}^{\frac{1}{\theta}} < \infty, \tag{3.12}$$

the series $f=\sum_{j\in\mathbb{N}}\lambda_j a_{B_j}$ converges in $\mathcal{S}'$ and $f\in H^\phi$ with

$$\|f\|_{H^\phi}\le C\Big\|\sum_{j\in\mathbb{N}}\Big(\frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^\phi}}\Big)^\theta\chi_{B_j}\Big\|_{L^{\phi_{1/\theta}}}^{\frac{1}{\theta}} \tag{3.13}$$

for some $C>0$ independent of f.

When $\phi=\psi_p$, $0<p\le 1$, we have $s_{\psi_p}=1/p$ and $1/\theta\in\mathbb{S}_\phi=[1/p,\infty)$. Therefore, $0<\theta\le p$ and $[ns_{\psi_p}-n]=[\frac{n}{p}-n]$. Moreover, if we pick $\theta=p$, then (3.12) becomes $(\sum_{j\in\mathbb{N}}|\lambda_j|^p)^{1/p}<\infty$, which is the well-known condition imposed on the coefficients of the atomic decomposition of the classical Hardy space H^p.

We find that $\nu_{\psi_p}^{1/\theta}=1-\frac{\theta}{p}$. Therefore, $(\nu_{\psi_p}^{1/\theta})^{-1}=\frac{p}{p-\theta}$ and $((\nu_{\psi_p}^{1/\theta})^{-1})'=\frac{p}{\theta}$; that is, $s_{\psi_p}((\nu_{\psi_p}^{1/\theta})^{-1})'=\frac{1}{p}\frac{p}{\theta}=\frac{1}{\theta}$, and hence $q>\theta s_{\psi_p}((\nu_{\psi_p}^{1/\theta})^{-1})'=1$.

Thus, when $\phi=\psi_p$, $0<p\le 1$, $H^\phi=H^p$ and the conditions (1) and (2) in Theorem 3.9 reduce to the well-known conditions for the atoms for the classical Hardy space H^p. Moreover, we find that those indices introduced in Section 2 are used to generate the well-known indices in the atomic decomposition of the classical Hardy spaces.

The following lemma can be considered as the generalization of the Jensen inequality to Musielak–Orlicz spaces.

Lemma 3.10. *Let $\phi\in\mathbb{H}$. If $s>s_\phi$, then, for any $1<\alpha<s/s_\phi$, there exists a constant $C>0$ such that, for any bounded measurable function g, we have*

$$\|\chi_B\|_{L^{\phi_s^*}}^{-1}\|g\chi_B\|_{L^{\phi_s^*}}\le C|B|^{-1/\alpha'}\|g\chi_B\|_{L^{\alpha'}},\quad \forall B\in\mathbb{B}. \tag{3.14}$$

Proof. The definition of s_ϕ assures that there exists a $r>\alpha$ such that $s_\phi<\frac{s}{r}<\frac{s}{\alpha}$ and $\phi^*_{s/r}\in\mathbb{M}$. Corollary 2.5 assures that

$$\begin{aligned}\|g\chi_B\|_{L^{\phi_s^*}}&\le C\sup_{\substack{f\in L^{\phi_s}\\ \|f\|_{L^{\phi_s}}\le 1}}\int_B\big|g(x)f(x)\big|\,dx\\ &\le C\sup_{\substack{f\in L^{\phi_s}\\ \|f\|_{L^{\phi_s}}\le 1}}\|g\chi_B\|_{L^{r'}}\|\chi_B f\|_{L^r}.\end{aligned}$$

Since $r<s/s_\phi$, $\phi^*_{s/r}$ is a proper generalized Φ-function. Therefore, we are allowed to apply the Hölder inequality for $L^{\phi_{s/r}}$ to obtain

$$\begin{aligned}\|g\chi_B\|_{L^{\phi_s^*}}&\le C\sup_{\substack{f\in L^{\phi_s}\\ \|f\|_{L^{\phi_s}}\le 1}}\|g\chi_B\|_{L^{r'}}\|\chi_B\|_{L^{\phi^*_{s/r}}}^{1/r}\big\|\chi_B|f|^r\big\|_{L^{\phi_{s/r}}}^{1/r}\\ &\le C\sup_{\substack{f\in L^{\phi_s}\\ \|f\|_{L^{\phi_s}}\le 1}}\|g\chi_B\|_{L^{r'}}\|\chi_B\|_{L^{\phi^*_{s/r}}}^{1/r}\|\chi_B f\|_{L^{\phi_s}} \qquad (3.15)\\ &\le C\|g\chi_B\|_{L^{r'}}\|\chi_B\|_{L^{\phi^*_{s/r}}}^{1/r}.\end{aligned}$$

As $s > s_\phi$, (2.9) assures that $\phi_s \in \mathbb{M}$. Therefore, we are allowed to apply Lemma 2.11 to L^{ϕ_s} and $L^{\phi^*_{s/r}}$. Consequently, for any $B \in \mathbb{B}$, we have

$$C_1|B| \le \|\chi_B\|_{L^{\phi_s}}\|\chi_B\|_{L^{\phi^*_s}} \le C_2|B|,$$
$$C_1|B|^{1/r} \le \|\chi_B\|^{1/r}_{L^{\phi_{s/r}}}\|\chi_B\|^{1/r}_{L^{\phi^*_{s/r}}} \le C_2|B|^{1/r}$$

for some $C_1, C_2 > 0$. As $\|\chi_B\|_{L^{\phi_s}} = \|\chi_B\|^{1/r}_{L^{\phi_{s/r}}}$, the above inequalities yield

$$C_1\|\chi_B\|^{1/r}_{L^{\phi^*_{s/r}}} \le |B|^{-1/r'}\|\chi_B\|_{L^{\phi^*_s}} \le C_2\|\chi_B\|^{1/r}_{L^{\phi^*_{s/r}}}. \tag{3.16}$$

Therefore, (3.15) and (3.16) give

$$\|g\chi_B\|_{L^{\phi^*_s}} \le C\|g\chi_B\|_{L^{r'}}|B|^{-1/r'}\|\chi_B\|_{L^{\phi^*_s}}.$$

Since $1 < r' < \alpha'$, Jensen's inequality yields

$$\|\chi_B\|^{-1}_{L^{\phi^*_s}}\|g\chi_B\|_{L^{\phi^*_s}} \le C|B|^{-1/r'}\|g\chi_B\|_{L^{r'}} \le C|B|^{-1/\alpha'}\|g\chi_B\|_{L^{\alpha'}}. \qquad \square$$

The subsequent lemma is inspired by [12, Proposition 5.8] and [30, Section 3.1]. In fact, [12, Proposition 5.8] is also used to establish the atomic decomposition of the Hardy–Morrey spaces with variable exponents.

Lemma 3.11. *Let $\phi \in \mathbb{H}$. Let $u \in \mathbb{S}_\phi$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ be a sequence of scalars. If $s > s_\phi((v^u_\phi)^{-1})'$, then, for any $\{b_k\}_{k\in\mathbb{N}} \subset L^{\phi_s}$ with* $\operatorname{supp} b_k \subseteq B_k \in \mathbb{B}$ *and*

$$\|b_k\|_{L^{\phi_s}} \le A_k\|\chi_{B_k}\|_{L^{\phi_s}}, \tag{3.17}$$

where $A_k > 0$, $\forall k \in \mathbb{N}$, we have

$$\Big\|\sum_{k\in\mathbb{N}}\lambda_k b_k\Big\|_{L^{\phi_u}} \le C\Big\|\sum_{k\in\mathbb{N}}A_k|\lambda_k|\chi_{B_k}\Big\|_{L^{\phi_u}} \tag{3.18}$$

for some $C > 0$ independent of $\{A_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, and $\{\lambda_k\}_{k\in\mathbb{N}}$.

Proof. According to the definition of $\mathbb{S}_\phi$, for any $u \in \mathbb{S}_\phi$, we have $v^u_\phi < 1$. Therefore, $((v^u_\phi)^{-1})'$ is well defined and $((v^u_\phi)^{-1})' > 1$. Since $s > s_\phi((v^u_\phi)^{-1})'$, we can select an α so that $(\frac{1}{v^u_\phi})' < \alpha < \frac{s}{s_\phi}$. As $s > s_\phi((v^u_\phi)^{-1})' > s_\phi$, (2.9) asserts that $\phi_s \in \mathbb{M}$.

Since $u \in \mathbb{S}_\phi$ implies $u > s_\phi$, Theorem 2.15 ensures that $\phi_u \in \mathbb{M}$. Thus Lemma 2.11 guarantees that, for any bounded measurable function g with $\|g\|_{L^{\phi^*_u}} \le 1$,

$$\Big|\int_{\mathbb{R}^n} b_k(x)g(x)\,dx\Big| \le 2\|b_k\|_{L^{\phi_s}}\|\chi_{B_k}g\|_{L^{\phi^*_s}} \le CA_k|B|\|\chi_B\|^{-1}_{L^{\phi^*_s}}\|\chi_{B_k}g\|_{L^{\phi^*_s}}.$$

Therefore, Lemma 3.10 yields

$$\Big|\int_{\mathbb{R}^n} b_k(x)g(x)\,dx\Big| \le CA_k|B_k|^{\frac{1}{\alpha}}\Big(\int_{B_k}|g(x)|^{\alpha'}\,dx\Big)^{\frac{1}{\alpha'}},$$

where α' is the conjugate of α. Consequently,

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} b_k(x)g(x)\,dx\Big| &\leq A_k|B_k|\Big(\frac{1}{|B_k|}\int_{B_k}\big|g(x)\big|^{\alpha'}\,dx\Big)^{\frac{1}{\alpha'}}\\ &\leq CA_k|B_k|\inf_{x\in B_k} M\big(|g|^{\alpha'}\big)(x)^{\frac{1}{\alpha'}} \leq CA_k\int_{B_k} M\big(|g|^{\alpha'}\big)(x)^{\frac{1}{\alpha'}}\,dx.\end{aligned}$$

Therefore, the Hölder inequality gives

$$\begin{aligned}\int_{\mathbb{R}^n}\Big|\Big(\sum_{k\in\mathbb{N}}\lambda_k b_k(x)\Big)g(x)\Big|\,dx &\leq C\sum_{k\in\mathbb{N}} A_k|\lambda_k|\int_{B_k} M\big(|g|^{\alpha'}\big)(x)^{\frac{1}{\alpha'}}\,dx\\ &\leq C\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{N}} A_k|\lambda_k|\chi_{B_k}(x)\Big)M\big(|g|^{\alpha'}\big)(x)^{\frac{1}{\alpha'}}\,dx\\ &\leq C\Big\|\sum_{k\in\mathbb{N}} A_k|\lambda_k|\chi_{B_k}\Big\|_{L^{\phi_u}}\big\|M\big(|g|^{\alpha'}\big)^{\frac{1}{\alpha'}}\big\|_{L^{\phi_u^*}}\\ &\leq C\Big\|\sum_{k\in\mathbb{N}} A_k|\lambda_k|\chi_{B_k}\Big\|_{L^{\phi_u}}\big\|M\big(|g|^{\alpha'}\big)\big\|_{(L^{\phi_u^*})^{1/\alpha'}}^{1/\alpha'}.\end{aligned}$$

As $\alpha' < 1/v_\phi^u$, the definition of v_ϕ^u guarantees that there exists a $\alpha' < t < 1/v_\phi^u$ such that M is bounded on $(L^{\phi_u^*})^{1/t}$. By using Jensen's inequality, M is also bounded on $(L^{\phi_u^*})^{1/\alpha'}$. For any bounded measurable function g with $\|g\|_{L^{\phi_u^*}}\leq 1$, we obtain

$$\int_{\mathbb{R}^n}\Big|\Big(\sum_{k\in\mathbb{N}}\lambda_k b_k(x)\Big)g(x)\Big|\,dx \leq C\Big\|\sum_{k\in\mathbb{N}} A_k|\lambda_k|\chi_{B_k}\Big\|_{L^{\phi_u}}\|g\|_{L^{\phi_u^*}}.$$

For any $g\in L^{\phi_u^*}$, $g_m\uparrow g$ where $g_m = g\chi_{\{x\in\mathbb{R}^n:|g(x)|<m\}}$, $m\in\mathbb{N}$. Thus Fatou's lemma assures that $\|g_m\|_{L^{\phi_u^*}}\uparrow\|g\|_{L^{\phi_u^*}}$. Finally, Corollary 2.5 yields (3.18). $\square$

We are ready to present the proof of Theorem 3.9. It is close to the ideas in [27, Section 4].

Proof of Theorem 3.9. Write

$$\begin{aligned}&\Big\|\mathcal{M}\Big(\sum_{j\in\mathbb{N}}|\lambda_j|a_{B_j}\Big)\Big\|_{L^\phi}\\ &\quad\leq\Big\|\sum_{j\in\mathbb{N}}|\lambda_j|\chi_{3B_j}\mathcal{M}(a_{B_j})\Big\|_{L^\phi}+\Big\|\sum_{j\in\mathbb{N}}|\lambda_j|\chi_{\mathbb{R}^n\setminus 3B_j}\mathcal{M}(a_{B_j})\Big\|_{L^\phi} = I + II.\end{aligned}$$

We first deal with I. As for any $\psi\in\mathcal{S}$, ψ has a radial majorant that is nonincreasing, bounded, and integrable. According to [32, Chapter II, (16)], we have

$$\sup_{t>0}\big|\psi_t * a_{B_j}(x)\big| \leq M(a_{B_j})(x)\int_{\mathbb{R}^n}\big|\psi(x)\big|\,dx \leq C\mathfrak{N}_N(\psi)M(a_{B_j})(x), \quad \forall x\in 3B_j$$

for some $N, C > 0$ independent of $j\in\mathbb{N}$, $\psi\in\mathcal{S}$, $x\in\mathbb{R}^n$, and $t>0$.

By taking supreme over those $\psi\in\mathcal{S}$ with $\mathfrak{N}_N(\psi)\leq 1$, we obtain

$$\mathcal{M}a_{B_j}(x)\leq CM(a_{B_j})(x), \quad \forall x\in 3B_j. \tag{3.19}$$

As $0<\theta\le 1$, the θ-inequality gives

$$\begin{aligned}I &\le C\Big\|\sum_{j\in\mathbb{N}}|\lambda_j|M(a_{B_j})\Big\|_{L^\phi} \le C\Big\|\Big(\sum_{j\in\mathbb{N}}\big(|\lambda_j|M(a_{B_j})\big)^\theta\Big)^{1/\theta}\Big\|_{L^\phi}\\ &= C\Big\|\sum_{j\in\mathbb{N}}\big(|\lambda_j|M(a_{B_j})\big)^\theta\Big\|^{1/\theta}_{L^{\phi_{1/\theta}}}.\end{aligned}$$

Since $q>s_\phi$, (2.9) asserts that $\phi_q\in\mathbb{M}$. Consequently,

$$\big\|\big(M(a_{B_j})\big)^\theta\big\|_{L^{\phi_{q/\theta}}} = \big\|M(a_{B_j})\big\|^\theta_{L^{\phi_q}} \le C\|a_{B_j}\|^\theta_{L^{\phi_q}} \le C\|\chi_{B_j}\|^{\theta(\frac{1}{q}-1)}_{L^\phi}.$$

As $\frac{1}{\theta}\in\mathbb{S}_\phi$ and $q/\theta > s_\phi((v_\phi^{1/\theta})^{-1})'$, we apply Lemma 3.11 with $u=1/\theta$, $b_j=(M(a_{B_j}))^\theta$, $s=q/\theta$, and $A_j=\|\chi_{B_j}\|^{-\theta}_{L^\phi}$ to obtain

$$I \le C\Big\|\sum_{j\in\mathbb{N}}\big(|\lambda_j|M(a_{B_j})\big)^\theta\Big\|^{1/\theta}_{L^{\phi_{1/\theta}}} \le C\Big\|\sum_{j\in\mathbb{N}}\Big(\frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^\phi}}\Big)^\theta\chi_{3B_j}\Big\|^{1/\theta}_{L^{\phi_{1/\theta}}}.$$

Since $\chi_{3B}\le CM(\chi_B)$, $\forall B\in\mathbb{B}$, for some $C>0$, Theorem 2.15 yields

$$I \le C\Big\|\Big(\sum_{j\in\mathbb{N}}\Big(\frac{|\lambda_j|^{\theta/2}}{\|\chi_{B_j}\|^{\theta/2}_{L^\phi}}M\chi_{B_j}\Big)^2\Big)^{\frac{1}{2}}\Big\|^{2/\theta}_{L^{\phi_{2/\theta}}} \le C\Big\|\sum_{j\in\mathbb{N}}\frac{|\lambda_j|^\theta}{\|\chi_{B_j}\|^\theta_{L^\phi}}\chi_{B_j}\Big\|^{1/\theta}_{L^{\phi_{1/\theta}}}. \qquad (3.20)$$

Next, we consider II. Let $d_s=[ns_\phi-n]$. For $x\in\mathbb{R}^n\backslash 3B_j$, we use the vanishing moment condition satisfied by a_j to obtain

$$\big|(a_{B_j}*\psi_t)(x)\big| \le \int_{\mathbb{R}^n}\Big|a_{B_j}(y)\Big(\psi_t(x-y)-\sum_{|\gamma|\le d_\omega}\frac{(y-x_{B_j})^\gamma}{\gamma!}\partial^\gamma\psi_t(x-x_{B_j})\Big)\Big|\,dy.$$

By using the reminder terms of the Taylor expansion of ψ_t, we have

$$\big|(a_{B_j}*\psi_t)(x)\big| \le \int_{\mathbb{R}^n}\big|a_{B_j}(y)\big|\sum_{|\gamma|=d_s+1}\Big|\frac{(y-x_{B_j})^\gamma}{\gamma!}\partial^\gamma\psi_t\big(x-y+h(y-x_{B_j})\big)\Big|\,dy$$

for some $0\le h\le 1$. Since $y\in B_j$, we have $|(y-x_{B_j})^\gamma|\le |B_j|^{\frac{d_s+1}{n}}$ for any $|\gamma|=d_s+1$. Moreover, for any $y\in B_j$,

$$\big|x-y+h(y-x_{B_j})\big| \ge |x-x_{B_j}|-(1-h)|y-x_{B_j}| \ge \frac{1}{2}|x-x_{B_j}|.$$

We obtain

$$\big|(a_{B_j}*\psi_t)(x)\big| \le C\mathfrak{N}_N(\psi)t^{-(d_s+n+1)}|B_j|^{\frac{d_s+1}{n}}\big(1+t^{-1}|x-x_{B_j}|\big)^{-L}\int_{3B_j}\big|a_{B_j}(y)\big|\,dy$$

for some sufficient large $L>n+d_s+1$ and some $C>0$ independent of $t>0$ and ψ.

As $\phi_q\in\mathbb{M}$, we have $\|\chi_{3B_j}\|_{L^{\phi_q}}\|\chi_{3B_j}\|_{L^{\phi_q^*}}\le C|B_j|$. The Hölder inequality and the definition of a_{B_j} yield

$$\int_{3B_j}\big|a_{B_j}(y)\big|\,dy \le 2\|a_{B_j}\|_{L^{\phi_q}}\|\chi_{3B_j}\|_{L^{\phi_q^*}} \le C\frac{|B_j|}{\|\chi_{B_j}\|_{L^\phi}} \qquad (3.21)$$

for some $C > 0$; that is,

$$\big|(a_{B_j} * \psi_t)(y)\big| \le C\mathfrak{N}_N(\psi)t^{-(d_s+n+1)}\frac{|B_j|^{\frac{n+d_s+1}{n}}}{\|\chi_{Q_j}\|_{L^\phi}}\big(1+t^{-1}|y-x_{B_j}|\big)^{-L}.$$

As $L > n + d_s + 1$, by taking supreme over $t > 0$, $|x - y| \le t$ and $\psi \in \mathcal{S}$ with $\mathfrak{N}_N(\psi) \le 1$ on both sides of the above inequality, we obtain

$$\mathcal{M}a_{B_j}(x) \le C\frac{|B_j|^{\frac{n+d_s+1}{n}}}{\|\chi_{B_j}\|_{L^\phi}}\frac{1}{|x-x_{B_j}|^{n+d_s+1}}, \quad \forall x \in \mathbb{R}^n\backslash 3B_j.$$

By using (3.6), we find that

$$\mathcal{M}a_{B_j}(x) \le C\frac{(M\chi_{B_j}(x))^{\frac{n+d_s+1}{n}}}{\|\chi_{B_j}\|_{L^\phi}}, \quad \forall x \in \mathbb{R}^n\backslash 3B_j \tag{3.22}$$

for some $C > 0$ independent of the atoms $\{a_{B_j}\}$. Write $\gamma = \frac{n+d_s+1}{n}$. Consequently,

$$II \le C\Big\|\Big(\sum_{j\in\mathbb{N}}\frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^\phi}}(M\chi_{B_j})^\gamma\Big)^{1/\gamma}\Big\|^\gamma_{L^{\phi_\gamma}}.$$

Since $\gamma = \frac{n+d_s+1}{n} \ge \frac{n+[ns_\phi-n]+1}{n} > s_\phi$, the Fefferman–Stein vector-valued maximal inequality asserts that

$$II \le C\Big\|\Big(\sum_{j\in\mathbb{N}}\frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^\phi}}\chi_{B_j}\Big)^{1/\gamma}\Big\|^\gamma_{L^{\phi_\gamma}} = C\Big\|\sum_{j\in\mathbb{N}}\frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^\phi}}\chi_{B_j}\Big\|_{L^\phi}$$

for some $C > 0$. Then the θ-inequality gives

$$II \le C\Big\|\sum_{j\in\mathbb{N}}\frac{|\lambda_j|^\theta}{\|\chi_{B_j}\|^\theta_{L^\phi}}\chi_{Q_j}\Big\|^{1/\theta}_{L^{\phi_{1/\theta}}}. \tag{3.23}$$

In conclusion, (3.20) and (3.23) yield (3.13). □

4. Molecular characterization

The molecular decomposition for the classical Hardy spaces H^p was introduced by Coifman, Taibleson, and Weiss in [2] and [33]. In this section, we extend the molecular decomposition to the Hardy–Musielak–Orlicz spaces. We begin with the definition of molecules for H^ϕ (see [34, Chapter XIV, Section 6.6]).

Definition 4.1. Let $\phi \in \mathbb{H}$, and let $p > s_\phi$. For any $N, M \in \mathbb{N}$, a family of measurable functions $\{M_B\}_{B\in\mathbb{B}}$ is called a (ϕ, p, N, M)-*molecular family* if

$$\big\|M_B(\cdot)\big\|_{L^{\phi_p}} \le C\|\chi_B\|^{\frac{1}{p}-1}_{L^\phi}, \tag{4.1}$$

$$\big\|M_B(\cdot)|\cdot - x_B|^M\big\|_{L^{\phi_p}} \le Cr_B^M\|\chi_B\|^{\frac{1}{p}-1}_{L^\phi}, \tag{4.2}$$

$$\int_{\mathbb{R}^n} x^\gamma M_B(x)\,dx = 0, \quad \forall \gamma \in \mathbb{N}^n \text{ with } 0 \le |\gamma| \le N \tag{4.3}$$

for some $C > 0$. We call M_B a *molecule centered at* B.

It is easy to see that a (ϕ, p, N)-atomic family is a (ϕ, p, N, M)-molecular family. Therefore, an atomic decomposition is also a molecular decomposition.

Consequently, similar to the molecular decomposition of the classical Hardy spaces, the only nontrivial part for the molecular decomposition of H^ϕ is the reconstruction theorem.

Theorem 4.2. *Let $\phi \in \mathbb{H}$. Suppose that*

(1) $0 < \theta \leq 1$ *satisfies* $\frac{1}{\theta} \in \mathbb{S}_\phi$,
(2) $q > \max(s_\phi, \theta s_\phi((v_\phi^{1/\theta})^{-1})')$, *and*
(3) $M > \max(n - \frac{n\theta d_\phi}{q}, [ns_\phi] - nd_\phi)$.

Then, for any $(\phi, q, [ns_\phi - n], M)$-molecular family, $\{M_B\}_{B\in\mathbb{B}}$, and sequence of scalars $\{\lambda_B\}_{B\in\mathbb{B}}$ satisfying

$$\Big\|\sum_{B\in\mathbb{B}}\Big(\frac{|\lambda_B|}{\|\chi_B\|_{L^\phi}}\Big)^\theta \chi_B\Big\|_{L^{\phi_{1/\theta}}}^{\frac{1}{\theta}} < \infty, \tag{4.4}$$

the series $f = \sum_{B\in\mathbb{B}} \lambda_B M_B$ converges in $\mathcal{S}'$ and $f \in H^\phi$ with

$$\|f\|_{H^\phi} \leq C\Big\|\sum_{B\in\mathbb{B}}\Big(\frac{|\lambda_B|}{\|\chi_B\|_{L^\phi}}\Big)^\theta \chi_B\Big\|_{L^{\phi_{1/\theta}}}^{\frac{1}{\theta}} \tag{4.5}$$

for some $C > 0$ independent of f.

Proof. We follow the ideas given in the proof of [6, Theorem 7.16]. We first show that, for any molecule M_B, we can rewrite it as a series of atoms. Let $\{M_B\}_{B\in\mathbb{B}}$ be a (ϕ, p, N, M)-molecular family.

For any fixed $B \in \mathbb{B}$, write $B_k = B(x_B, 2^k r_B)$ and $E_k = B_k \backslash B_{k-1}$, $k \in \mathbb{N} \cup \{0\}$. Write $M_{B,k}(x) = \chi_{E_k}(x) M_B(x)$, and let P_B^k be the unique polynomial of degree $[ns_\phi - n]$ such that $\int_{E_k}(M_{B,k}(x) - P_B^k(x))x^\alpha\, dx = 0$, $\forall |\alpha| \leq [ns_\phi - n]$.

Define $P_{B,k}(x) = \chi_{E_k}(x)P_B^k(x)$. Therefore, we have $M_B = \sum_{k=0}^\infty (M_{B,k} - P_{B,k}) + \sum_{k=0}^\infty P_{B,k}$. Write $f = \sum_{B\in\mathbb{B}} \lambda_B M_B = \sum_{B\in\mathbb{B}}\sum_{k=0}^\infty \lambda_B(M_{B,k} - P_{B,k}) + \sum_{B\in\mathbb{B}}\sum_{k=0}^\infty \lambda_B P_{B,k} = F + G$.

We first consider F. The function $M_{B,k} - P_{B,k}$ is supported in B_k and satisfies the vanishing moment conditions up to order $[ns_\phi - n]$.

It remains to deal with the size condition. We find that

$$\begin{aligned}\|M_{B,k} - P_{B,k}\|_{L^{\phi_p}} &\leq C\|M_{B,k}\|_{L^{\phi_p}} = C\big\|M_{B,k}|\cdot - x_B|^M|\cdot - x_B|^{-M}\big\|_{L^{\phi_q}}\\ &\leq C2^{-kM}r_B^{-M}\big\|M_{B,k}|\cdot - x_B|^M\big\|_{L^{\phi_q}} \leq C2^{-kM}\|\chi_B\|_{L^\phi}^{\frac{1}{q}-1},\end{aligned}$$

where we obtain the last inequality by using (4.2); that is,

$$\|M_{B,k} - P_{B,k}\|_{L^{\phi_p}} \leq C2^{-kM}\Big(\frac{\|\chi_B\|_{L^\phi}}{\|\chi_{B_k}\|_{L^\phi}}\Big)^{\frac{1}{q}-1}\|\chi_{B_k}\|_{L^\phi}^{\frac{1}{q}-1} = C\mu_{B,k}\|\chi_{B_k}\|_{L^\phi}^{\frac{1}{q}-1}, \tag{4.6}$$

where $\mu_{B,k} = 2^{-kM}(\frac{\|\chi_B\|_{L^\phi}}{\|\chi_{B_k}\|_{L^\phi}})^{\frac{1}{q}-1}$.

Therefore, $\{A_{B,k}\}$ is a $(\phi, q, [ns_\phi - n])$-atomic family where $A_{B,k} = \frac{M_{B,k} - P_{B,k}}{\mu_{B,k}}$. Moreover, $F = \sum_{B\in\mathbb{B}} \sum_{k=0}^{\infty} \lambda_B \sum_{k=0}^{\infty}(M_{B,k} - P_{B,k}) = \sum_{B\in\mathbb{B}} \sum_{k=0}^{\infty} \lambda_B \mu_{B,k} A_{B,k}$.

Fix a $\kappa > 1$ and $d < d_\phi$ such that $M > \kappa n - \frac{dn\theta}{q}$. In view of (2.12), we have $\chi_{B_k} \le c2^{kn\kappa}(M\chi_B)^\kappa$. Therefore,

$$\begin{aligned}&\sum_{B\in\mathbb{B}} \sum_{k=0}^{\infty}\Big(\frac{\lambda_B\mu_{B,k}}{\|\chi_{B_k}\|_{L^\phi}}\Big)^\theta \chi_{B_k}\\ &\qquad \le C\sum_{B\in\mathbb{B}}\Big(\frac{\lambda_B}{\|\chi_B\|_{L^\phi}}\Big)^\theta (M\chi_B)^\kappa \sum_{k=0}^{\infty} 2^{-kM\theta}2^{kn\kappa}\Big(\frac{\|\chi_B\|_{L^\phi}}{\|\chi_{B_k}\|_{L^\phi}}\Big)^{\theta/q}.\end{aligned}$$

Proposition 2.17 assures that, for that given d, there exists a constant $C > 0$ such that

$$\sum_{k=0}^{\infty} 2^{kn-kM\theta}\Big(\frac{\|\chi_B\|_{L^\phi}}{\|\chi_{B_k}\|_{L^\phi}}\Big)^{\theta/q} \le C\sum_{k=1}^{\infty} 2^{k(\kappa n - M - \frac{dn\theta}{q})} < \infty$$

because $M > \kappa n - \frac{dn\theta}{q}$.

Thus,

$$\begin{aligned}\Big\|\sum_{B\in\mathbb{B}} \sum_{k=0}^{\infty}\Big(\frac{\lambda_B\mu_{B,k}}{\|\chi_{B_k}\|_{L^\phi}}\Big)^\theta \chi_{B_k}\Big\|_{L^{\phi_{1/\theta}}} &\le C\Big\|\sum_{B\in\mathbb{B}}\Big(\frac{\lambda_B}{\|\chi_B\|_{L^\phi}}\Big)^\theta (M\chi_B)^\kappa\Big\|_{L^{\phi_{1/\theta}}}\\ &\le C\Big\|\Big(\sum_{B\in\mathbb{B}}\Big(\frac{\lambda_B}{\|\chi_B\|_{L^\phi}}\Big)^\theta (M\chi_B)^\kappa\Big)^{\frac{1}{\kappa}}\Big\|^\kappa_{L^{\phi_{\kappa/\theta}}}.\end{aligned}$$

As $\kappa > 1$ and $\frac{\kappa}{\theta} > \frac{1}{\theta} > s_\phi$, Theorem 2.15 guarantees that

$$\Big\|\sum_{B\in\mathbb{B}} \sum_{k=0}^{\infty}\Big(\frac{\lambda_B\mu_{B,k}}{\|\chi_{B_k}\|_{L^\phi}}\Big)^\theta \chi_{B_k}\Big\|_{L^{\phi_{1/\theta}}} \le C\Big\|\sum_{B\in\mathbb{B}}\Big(\frac{\lambda_B}{\|\chi_B\|_{L^\phi}}\Big)^\theta \chi_B\Big\|_{L^{\phi_{1/\theta}}}.$$

Consequently, Theorem 3.9 yields that $A \in H^\phi$ and

$$\|F\|_{H^\phi} \le C\Big\|\sum_{B\in\mathbb{B}}\Big(\frac{\lambda_B}{\|\chi_B\|_{L^\phi}}\Big)^\theta \chi_B\Big\|_{L^{\phi_{1/\theta}}}. \tag{4.7}$$

Next, we deal with B. We use the ideas given in [6, pp. 332–334].

For any $k \in \mathbb{N}\cup\{0\}$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \le [ns_\phi - n]$, let $\phi^k_{\alpha,B}$ be the unique function on E_k such that $\frac{1}{|E_k|}\int_{E_k} \phi^k_{\alpha,B}(x)(x - x_B)^\gamma\, dx = \delta_{\alpha,\gamma}$, $\forall |\gamma| \le [ns_\phi - n]$, where $\delta_{\alpha,\gamma} = 1$ when $\alpha = \gamma$ and $\delta_{\alpha,\gamma} = 0$ otherwise.

Therefore, $P_{B,k}(x) = \sum_{|\alpha|\le[ns_\phi - n]} m^k_{\alpha,B}\phi^k_{\alpha,B}(x)$, where $m^k_{\alpha,B} = \frac{1}{|E_k|}\int M_{B,k}(x) \times (x - x_B)^\alpha\, dx$.

According to [6, pp. 332–334], we have $\sum_{k=0}^{\infty} P_{B,k}(x) = \sum_{k=0}^{\infty} N^k_{\alpha,B}\psi^k_{\alpha,B}(x)$, where $N^k_{\alpha,B} = \sum_{j=k+1}^{\infty} m^j_{\alpha,B}|E_j|$ and $\psi^k_{\alpha,B} = |E_{k+1}|^{-1}\phi^{k+1}_{\alpha,B} - |E_k|^{-1}\phi^k_{\alpha,B}$. We also have $|\psi^k_{\alpha,B}| \le C(2^k r_B)^{-|\alpha|-n}$ for some $C > 0$ independent of $B \in \mathbb{B}$ and $\alpha \in \mathbb{N}^n$.

Thus, the Hölder inequality assures that

$$\left|N^k_{\alpha,B}\psi^k_{\alpha,B}(x)\right| \leq C\sum_{j=k+1}^{\infty}\|M_{B,j}\|_{L^{\phi_q}}\|\chi_{E_j}\|_{L^{\phi_q^*}}(2^j r_B)^{|\alpha|}(2^k r_B)^{-|\alpha|-n}.$$

The definition of molecular family and the fact that $E_j \subset B_j$ guarantees that

$$\begin{aligned}&\left|N^k_{\alpha,B}\psi^k_{\alpha,B}(x)\right|\\ &\quad\leq C\sum_{j=k+1}^{\infty}\left\|M_{B,j}|\cdot-x_B|^M\big(l(B)+|\cdot-x_B|\big)^{-M}\right\|_{L^{\phi_q}}\|\chi_{B_j}\|_{L^{\phi_q^*}}2^{(j-k)|\alpha|}|B_k|^{-1}\\ &\quad\leq C\sum_{j=k+1}^{\infty}2^{-jM}\|\chi_B\|_{L^\phi}^{\frac{1}{q}-1}\frac{\|\chi_{B_j}\|_{L^{\phi_q^*}}}{|B_j|}2^{(j-k)|\alpha|}\frac{|B_j|}{|B_k|}.\end{aligned}$$

Since $q > s_\phi$, Theorem 2.15 ensures that $\phi_q \in \mathbb{M}$. We are allowed to apply Lemma 2.11 to obtain

$$\left|N^k_{\alpha,B}\psi^k_{\alpha,B}(x)\right| \leq C2^{-kM}\sum_{j=k+1}^{\infty}\frac{\|\chi_B\|_{L^\phi}^{\frac{1}{q}-1}}{\|\chi_{B_j}\|_{L^{\phi_q}}}2^{(j-k)(|\alpha|+n-M)}.$$

As $\operatorname{supp}\psi^k_{\alpha,B} \subseteq B_k$, by applying the norm $\|\cdot\|_{L^{\phi_q}}$ on both sides of the above inequality, we find that

$$\|N^k_{\alpha,B}\psi^k_{\alpha,B}\|_{L^{\phi_q}} \leq C2^{-kM}\|\chi_B\|_{L^\phi}^{\frac{1}{q}-1}\sum_{j=k+1}^{\infty}\frac{\|\chi_{B_k}\|_{L^{\phi_q}}}{\|\chi_{B_j}\|_{L^{\phi_q}}}2^{(j-k)(|\alpha|+n-M)}.$$

Let d be selected so that $d < d_\phi$ and $M > [ns_\phi] - nd$. Proposition 2.17 assures that

$$\sum_{j=k+1}^{\infty}\frac{\|\chi_{B_k}\|_{L^{\phi_q}}}{\|\chi_{B_j}\|_{L^{\phi_q}}}2^{(j-k)(|\alpha|+n-M)} \leq \sum_{j=k+1}^{\infty}2^{(j-k)(|\alpha|+n-M-nd)} < \infty$$

because $M > [ns_\phi - n] + n - nd$.

Therefore, we have $\|N^k_{\alpha,B}\psi^k_{\alpha,B}\|_{L^{\phi_q}} \leq C2^{-kM}\|\chi_B\|_{L^\phi}^{\frac{1}{q}-1}$, and hence the rest of the arguments follow from the corresponding arguments from $M_{B,k} - P_{B,k}$ after (4.6). Thus, for simplicity, we omit the details and conclude that

$$\|G\|_{H^\phi} \leq C\left\|\sum_{B\in\mathbb{B}}\left(\frac{\lambda_B}{\|\chi_B\|_{L^\phi}}\right)^\theta\chi_B\right\|_{L^{\phi_{1/\theta}}}. \tag{4.8}$$

Finally, by using (4.7) and (4.8), we establish (4.5) because $\|\cdot\|_{H^\phi}$ is a quasi-norm. □

When $\phi = \psi_p$, $0 < p \leq 1$, we have $s_{\psi_p} = d_{\psi_p} = 1/p$. Therefore, $[ns_\phi] - nd_\phi = [n/p] - n/p \leq 0$. Moreover, $n - \frac{n\theta d_\phi}{q} = n - \frac{n}{q}$. Thus, the condition imposed on M becomes $M > n - \frac{n}{q}$. This reduces to the usual condition imposed on the molecules for the classical Hardy spaces (see [2], [33]).

As special cases of the above theorem, we have the molecular decompositions of the Hardy–Orlicz spaces and the Hardy spaces with variable exponents.

Finally, we give an application of the intrinsic atomic and molecular decompositions on the boundedness of the operator on H^ϕ.

Theorem 4.3. *Let $\phi \in \mathbb{H}$. Let T be a convolution operator*

$$Tf(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} f(y)K(x-y)\,dy.$$

Suppose that there exists a $0 < \delta \le 1$ such that K satisfies

$$\int_{\epsilon<|x|<R} K(x)\,dx = 0, \quad 0 < \epsilon < R < \infty, \tag{4.9}$$

$$\big|K(x-y) - K(x)\big| \le C|y|^\delta |x|^{-n-\delta}, \quad |x| > 2|y| \tag{4.10}$$

for some $C > 0$. If

(1) $1 < s_\phi < \frac{n+1}{n}$,

(2) *T is bounded on L^{ϕ_q} for some $q > \max(s_\phi, \theta s_\phi((v_\phi^{1/\theta})^{-1})')$ with $\frac{1}{\theta} \in \mathbb{S}_\phi$, and*

(3) $\delta > \max(n - \frac{n\theta d_\phi}{q}, [ns_\phi] - nd_\phi)$,

then T is bounded on H^ϕ.

Proof. Let $\{a_B\}_{B\in\mathbb{B}}$ be a $(\phi, q, [ns_\phi - n])$-atomic family. We consider the action of T on a_B. Write $m_B = Ta_B$. We are going to show that $\{m_B\}$ is a constant multiple of a $(\phi, q, [ns_\phi - n], M)$-molecular family with $\delta > M > \max(n - \frac{n\theta d_\phi}{q}, [ns_\phi] - nd_\phi)$.

Since $1 < s_\phi < \frac{n+1}{n}$, we have $[ns_\phi - n] = 0$. In view of (4.9), we find that $\int m_B(x)\,dx = \int Ta_B(x)\,dx = 0$. Therefore, $\{m_B\}_{B\in\mathbb{B}}$ fulfills (4.3). As T is bounded on L^{ϕ_q}, we have $\|m_B\|_{L^{\phi_q}} = \|Ta_B\|_{L^{\phi_q}} \le C\|a_B\|_{L^{\phi_q}} \le C\|\chi_B\|_{L^\phi}^{\frac{1}{q}-1}$ for some $C > 0$ independent of $B \in \mathbb{B}$. Hence $\{m_B\}_{B\in\mathbb{B}}$ satisfies (4.1).

Moreover, we have

$$\big\|m_B(\cdot)\chi_{B(x_B,2r_B)}(\cdot)|\cdot - x_B|^M\big\|_{L^{\phi_q}} \le r_B^M \|m_B\|_{L^{\phi_q}} \le C r_B^M \|\chi_B\|_{L^\phi}^{\frac{1}{q}-1}. \tag{4.11}$$

Whenever $y \in B$ and $x \in \mathbb{R}^n \backslash B(x_B, 2r_B)$, by using the vanishing moment condition satisfied by $\{a_B\}$ and the Hölder inequality, we find that

$$\begin{aligned}
\big|Ta_B(x)\big| &= \Big|\int_{3B} \big(K(x-y) - K(x-x_B)\big)a_B(y)\,dy\Big| \\
&\le C\int_{3B} \frac{|y-x_B|^\delta}{|x-x_B|^{n+\delta}}\big|a_B(y)\big|\,dy \\
&\le C r_B^\delta \frac{\|\chi_B\|_{L^{\phi_q^*}}\|a_B\|_{L^{\phi_q}}}{|x-x_B|^{n+\delta}} \\
&\le C r_B^{\delta+n} \frac{\|\chi_B\|_{L^{\phi_q^*}}\|\chi_B\|_{L^\phi}^{\frac{1}{q}-1}}{|B||x-x_B|^{n+\delta}}.
\end{aligned}$$

As $q > s_\phi$, Theorem 2.15 shows that $\phi_q \in \mathbb{M}$. Lemma 2.11 yields that

$$\left|Ta_B(x)\right| \le Cr_B^{\delta+n}\frac{\|\chi_B\|_{L^\phi}^{\frac{1}{q}-1}}{\|\chi_B\|_{L^{\phi_q}}}\frac{1}{|x-x_B|^{n+\delta}} \le Cr_B^{\delta+n}\frac{1}{|x-x_B|^{n+\delta}}\|\chi_B\|_{L^\phi}^{-1}.$$

Consequently,

$$\begin{aligned}&\left\|m_B\chi_{\mathbb{R}^n\setminus B(x_B,2r_B)}|\cdot - x_B|^M\right\|_{L^{\phi_q}}\\ &\quad\le Cr_B^{\delta+n}\|\chi_B\|_{L^\phi}^{-1}\left\|(1-\chi_B)|\cdot - x_B|^{M-n-\delta}\right\|_{L^{\phi_q}}.\end{aligned} \tag{4.12}$$

To deal with $\|(1-\chi_B)|\cdot - x_B|^{M-n-\delta}\|_{L^{\phi_q}}$, we first obtain an estimate of $\|\chi_{B(x_B,2^jr_B)}\|_{L^{\phi_q}}$ in terms of $\|\chi_B\|_{L^{\phi_q}}$.

Write $B_j = B(x_B, 2^jr_B)$. In view of (2.12), the fact that $\phi_q \in \mathbb{M}$ yields $\|\chi_{B_j}\|_{L^{\phi_q}} \le C2^{jn}\|M\chi_B\|_{L^{\phi_q}} \le C2^{jn}\|\chi_B\|_{L^{\phi_q}}$. Since $M < \delta$, we have

$$\begin{aligned}\left\|(1-\chi_B)|\cdot - x_B|^{M-n-\delta}\right\|_{L^{\phi_q}} &\le C\sum_{j=0}^\infty 2^{j(M-n-\delta)}r_B^{M-n-\delta}\|\chi_{B_j}\|_{L^{\phi_q}}\\ &\le \sum_{j=0}^\infty 2^{j(M-\delta)}r_B^{M-n-\delta}\|\chi_B\|_{L^{\phi_q}} \le Cr_B^{M-n-\delta}\|\chi_B\|_{L^{\phi_q}}.\end{aligned}$$

By applying this estimate on (4.12), we obtain

$$\left\|m_B\chi_{\mathbb{R}^n\setminus B(x_B,2r_B)}|\cdot - x_B|^M\right\|_{L^{\phi_q}} \le Cr_B^M\|\chi_B\|_{L^\phi}^{\frac{1}{q}-1}$$

for some $C > 0$ independent of $B \in \mathbb{B}$.

The above inequality and (4.11) guarantee that $\{m_B\}$ fulfills (4.2); that is, $\{m_B\}$ is a constant multiple of a $(\phi, q, [ns_\phi - n], M)$-molecular family.

Therefore, Theorems 3.3 and 4.2 yield our result on the boundedness of T on H^ϕ. □

Notice that, in [12] and [27], the boundedness of the singular integral operators T on the Hardy spaces with variable exponents and the Hardy–Morrey spaces with variable exponents relies on the boundedness of T on Lebesgue spaces. In particular, Theorem 4.2 gives us another result on the boundedness of T on Hardy spaces with variable exponents with the assumption on the boundedness of T on the corresponding Lebesgue spaces with variable exponents only.

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