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GEOMETRIC PROPERTIES OF THE SECOND-ORDER CESÀRO SPACES

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Abstract. We prove that, for any $p \in (1, \infty)$, the second-order Cesàro sequence space $\mathrm{Ces}^2(p)$ has the (β)-property and the k-NUC property for $k \geq 2$. In addition, we show that $\mathrm{Ces}^2(p)$ has the Kadec–Klee, rotundity, and uniform convexity properties. For any positive integer k, we also investigate the uniform Opial and (L) properties of the sequence space. We also establish that $\mathrm{Ces}^2(p)$ is reflexive and has the fixed-point property. Finally, we calculate the packing constant (C) of the space.

1. Introduction and preliminaries

Let ω denote the space of all real-valued sequences. Any vector subspace of ω is called a *sequence space*. We write l_∞, c, and c_0 for the spaces of all bounded, convergent, and null sequences, respectively. Let bs, cs, l_1, and l_p $(1 < p < \infty)$ denote the spaces of all bounded, convergent, absolutely convergent, and p-absolutely convergent series, respectively. Throughout the paper, we assume that (p_k) is a bounded sequence of strictly positive real numbers with $\sup\{p_k\} = H$, and we set $M = \max\{1, H\}$. The linear space $l(p)$ was defined by Maddox [10] (see also Simons [15] and Nakano [11]) as

$$l(p) = \Big\{x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty\Big\} \quad (0 < p_k \leq H < \infty).$$

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It is a complete paranormed space via the paranorm

$$g(x) = \Big(\sum_k |x_k|^{p_k}\Big)^{\frac{1}{M}}.$$

For simplicity of the notation, in what follows, a summation without limits always runs from 1 to ∞.

Next we define the second-order Cesàro sequence space by

$$\mathrm{Ces}^2(p) = \Big\{x \in \omega : \sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p < \infty\Big\},$$

for $1 \le p < \infty$, and for $p = \infty$ by

$$\mathrm{Ces}^2(\infty) = \Big\{x \in \omega : \sup_n\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big) < \infty\Big\}.$$

These spaces generalize the Cesàro sequence spaces, as shown in the following theorem.

Theorem 1.1. *The following proper inclusions hold:*

(1) $l_p \subset \mathrm{Ces}(p)$, *for* $p > 1$;
(2) $l_p \subset \mathrm{Ces}^2(p)$, *for* $p > 1$;
(3) $\mathrm{Ces}(p) \subset \mathrm{Ces}^2(p)$, *for* $p > 1$.

Proof. It is enough to prove the last inclusion, because (1) and (2) were proved in [16]. Let $x = (x_n) \in \mathrm{Ces}(p)$. Then

$$\begin{aligned}
&\Big[\sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p\Big]^{\frac{1}{p}}\\
&\quad\le \Big[\sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1)\big|x(k)\big|\Big)^p\Big]^{\frac{1}{p}}\\
&\quad= \Big[\sum_{n=1}^{\infty}\Big(\frac{1}{n+2}\sum_{k=0}^{n}\big|x(k)\big|\Big)^p\Big]^{\frac{1}{p}} < \infty.
\end{aligned}$$

On the other hand, let us consider the following sequence:

$$(x_n) = (0, 0, \ldots, 0, \underbrace{n}_{n\text{th position}}, 0, 0, \ldots).$$

Then it follows that

$$\Big[\sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p\Big]^{\frac{1}{p}} = \sum_{n=1}^{\infty}\Big(\frac{n}{(n+1)(n+2)}\Big)^p < \infty$$

for every $p > 1$, but

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=0}^{n}\big|x(k)\big|\Big) = \sum_{n=1}^{\infty} 1 = \infty.$$

□

A Banach space X is said to be *k-nearly uniformly convex* (k-NUC) if, for any $\epsilon > 0$, there exists a number $\delta > 0$ such that, for any sequence $(x_n) \subset B(X)$ with $\operatorname{sep}(x_n) \geq \epsilon$, there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that

$$\left\| \frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k} \right\| < 1 - \delta,$$

whenever $\operatorname{sep}(x_n) := \inf\{\|x_n - x_m\| : n \neq m\} > \epsilon$.

A Banach space X has property (β) if, for each $r > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $\operatorname{sep}(x_n) \geq \epsilon$, there is an index k such that

$$\left\| \frac{x + x_k}{2} \right\| \leq \delta.$$

A Banach space X is said to have the *Banach–Saks property* of type p if every weakly null sequence (x_k) has a subsequence (x_{k_l}) such that, for some $C > 0$,

$$\left\| \sum_{l=0}^{n} x_{k_l} \right\| < C(n+1)^{\frac{1}{p}}$$

for all $n \in \mathbb{N}$.

A point $x_0 \in S(X)$ is called

(1) an *extreme point* if, for every $x, y \in S(X)$, the equality $2x_0 = x + y$ implies $x = y$;
(2) a *locally uniformly rotund point* (*LUR-point*) if, for any sequence (x_n) in $B(X)$ such that $\|x_n + x\| \to 2$ as $n \to \infty$, there holds $\|x_n - x\| \to 0$ as $n \to \infty$.

A Banach space X is said to have the *rotundity property* if every point of $S(X)$ is an extreme point. A Banach space X is said to have the *Opial property* if every sequence (x_n) weakly convergent to x_0 satisfies

$$\liminf_{n\to\infty} \|x_n - x_0\| \leq \liminf_{n\to\infty} \|x_n - x\|,$$

for every $x \in X$.

A Banach space X is said to have the *uniform Opial property* if, for every $\epsilon > 0$, there exists $\tau > 0$ such that, for each weakly null sequence $(x_n) \subset S(X)$ and $x \in X$ with $\|x\| \geq \epsilon$, we have

$$1 + \tau \leq \liminf_{n\to\infty} \|x_n + x\|$$

(see [13]).

For a sequence $(x_n) \subset X$, the following notion was defined in [5],

$$A((x_n)) = \liminf_{n\to\infty} \big\{ \|x_i + x_j\| : i, j \geq n, i \neq j \big\},$$

which is related to the packing constant (see [9]) and to the Banach–Saks property as follows:

$$C(X) = \sup\big\{ A((x_n)) : (x_n) \text{ is a weakly null sequence in } S(X) \big\}.$$

For each $\epsilon > 0$, define $\Delta(\epsilon)$ to be

$$\inf\big\{ 1 - \inf\big[\|x\| : x \in A\big] : A \text{ is a closed convex subset of } B(X) \text{ with } \beta(A) \geq \epsilon \big\},$$

where

$$\beta(A) = \inf\{\epsilon > 0\colon A \text{ can be covered by finitely many balls of diameter} \le \epsilon\}.$$

The function Δ is called the *modulus of noncompact convexity* (see [7]). A Banach space X is said to have the *property* (L) if $\lim_{\epsilon\to 1^-}\Delta(\epsilon) = 1$. It was proved in [13] that the property (L) is a useful tool in the fixed-point theory and that a Banach space X has the property (L) if and only if it is reflexive and has the uniform Opial property.

Gurariĭ's modulus of convexity (see [8]) is defined by

$$\beta_X(\epsilon) = \inf\big\{1 - \inf_{0\le\alpha\le 1}\big\|\alpha x + (1-\alpha)y\big\|; x, y \in S(X), \|x-y\| = \epsilon\big\},$$

where $0 \le \epsilon \le 2$. Let X be a real vector space. A functional $\sigma : X \to [0,\infty)$ is called a *modular* if

(1) $\sigma(x) = 0$ if and only if $x = \theta$,
(2) $\sigma(\alpha x) = \sigma(x)$ for all scalars α with $|\alpha| = 1$,
(3) $\sigma(\alpha x + \beta y) \le \sigma(x) + \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

The modular σ is called *convex* if it satisfies the following:

(4) $\sigma(\alpha x + \beta y) \le \alpha\sigma(x) + \beta\sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

A modular σ is called

(5) *right continuous* if $\lim_{\alpha\to 1^+}\sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$,
(6) *left continuous* if $\lim_{\alpha\to 1^-}\sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$,
(7) *continuous* if it is both right and left continuous,

where $X_\sigma = \{x \in X : \lim_{\alpha\to 0^+}\sigma(\alpha x) = 0\}$. We define the operator σ_p on $\mathrm{Ces}^2(p)$ by

$$\sigma_p(x) = \sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p.$$

If $p \ge 1$, by convexity of the function $t \to |t|^p$, we conclude that σ_p is a convex modular in $\mathrm{Ces}^2(p)$.

The modular σ_p is said to satisfy the δ_2-condition (see [3]) if, for every $\epsilon > 0$, there exists a constant $M > 0$ and $m > 0$ such that

$$\sigma_p(2t) \le M\sigma_p(t) + \epsilon \tag{1.1}$$

for all $t \in X_{\sigma_p}$ with $\sigma_p(t) \le m$.

2. Results

We start this section with the following lemma, whose proof is similar to that of [3, Lemma 2.1].

Lemma 2.1 ([3, Lemma 2.1])**.** *If σ_p satisfies the δ_2-condition, then, for any $A > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\big|\sigma_p(t+w) - \sigma_p(t)\big| < \epsilon \tag{2.1}$$

whenever $t, w \in X_{\sigma_p}$ with $\sigma_p(t) \le A$ and $\sigma_p(w) \le \delta$.

Theorem 2.2 ([3, Lemma 2.1]). *Suppose that σ_p satisfies the δ_2-condition.*

(1) *For any $x \in X_{\sigma_p}$, $\|x\| = 1$ if and only if $\sigma_p(x) = 1$.*
(2) *For any sequence $(x_n) \in X_{\sigma_p}$, $\|x_n\| \to 0$ if and only if $\sigma_p(x_n) \to 0$.*

Theorem 2.3. *If σ_p satisfies the δ_2-condition, then, for any $\epsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $\sigma_p(x) \leq 1-\epsilon$ implies $\|x\| \leq 1-\delta$.*

Proof. The proof of the theorem follows directly from the above two facts (see [3]). □

Theorem 2.4. *For any $x \in \mathrm{Ces}^2(p)$ and $\epsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $\sigma_p(x) \leq 1-\epsilon$ implies $\|x\| \leq 1-\delta$.*

Proof. The proof of the theorem follows directly from Theorem 2.3. □

Proposition 2.5. *If $p \geq 1$, then the modular σ_p is continuous on $\mathrm{Ces}^2(p)$, and it also satisfies the following conditions:*

(1) *if $0 < \alpha \leq 1$, then $\alpha^M \sigma_p(\frac{x}{\alpha}) \leq \sigma_p(x)$ and $\sigma_p(\alpha x) \leq \alpha \sigma_p(x)$;*
(2) *if $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha^M \sigma_p(\frac{x}{\alpha})$;*
(3) *if $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha \sigma_p(\frac{x}{\alpha})$.*

Proof. It is similar to the proof of [12, Proposition 2.1]. □

Now we will define the following two norms (the first one is known as the *Luxemburg norm* and the second one as the *Amemiya norm*) in $\mathrm{Ces}^2(p)$:

$$\|x\|_L = \inf\left\{\alpha > 0 : \sigma_p\left(\frac{x}{\alpha}\right) \leq 1\right\} \tag{2.2}$$

and

$$\|x\|_A = \inf_{\alpha > 0} \frac{1}{\alpha}\left\{1 + \sigma_p(\alpha \cdot x)\right\}. \tag{2.3}$$

Proposition 2.6. *Let $x \in \mathrm{Ces}^2(p)$. Then the following relations between σ_p and $\|\cdot\|_L$ are satisfied:*

(1) *if $\|x\|_L < 1$, then $\sigma_p(x) \leq \|x\|_L$;*
(2) *if $\|x\|_L > 1$, then $\sigma_p(x) \geq \|x\|_L$;*
(3) *$\|x\|_L = 1$ if and only if $\sigma_p(x) = 1$;*
(4) *$\|x\|_L < 1$ if and only if $\sigma_p(x) < 1$;*
(5) *$\|x\|_L > 1$ if and only if $\sigma_p(x) > 1$.*

Proof. It is similar to the proof of [1, Proposition 3.10]. □

Theorem 2.7. *The space $\mathrm{Ces}^2(p)$ is a Banach space under the Luxemburg and the Amemiya norm.*

Proof. We will prove that $\mathrm{Ces}^2(p)$ is a Banach space under the Luxemburg norm. In what follows we need to show that every Cauchy sequence in $\mathrm{Ces}^2(p)$ is convergent according to the Luxemburg norm. Let $\{x_k^n\}$ be any Cauchy sequence in $\mathrm{Ces}^2(p)$ and $\epsilon \in (0,1)$. Thus there exists a positive integer n_0 such that, for any

$n,m \geq n_0$, we get $\|x^{(n)} - x^{(m)}\|_L < \epsilon$. From Proposition 2.6 we obtain

$$\sigma_p(x^{(n)} - x^{(m)}) \leq \|x^{(n)} - x^{(m)}\|_L < \epsilon, \tag{2.4}$$

for all $n,m \geq n_0$. This implies that

$$\sum_{k=1}^{\infty}\Big(\frac{1}{(k+1)(k+2)}\sum_{i=0}^{k}(k+1-i)|x_i^{(n)} - x_i^{(m)}|\Big)^p < \epsilon. \tag{2.5}$$

For each fixed k and for all $n,m \geq n_0$,

$$\frac{1}{(k+1)(k+2)}\sum_{i=0}^{k}(k+1-i)|x_i^{(n)} - x_i^{(m)}| < \epsilon.$$

Hence $(y_k^{(n)})_k = (\frac{1}{(k+1)(k+2)}\sum_{i=0}^{k}(k+1-i)|x_i^{(n)}|)_k$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is a complete normed space, there exists

$$(y_k)_k = \Big(\frac{1}{(k+1)(k+2)}\sum_{i=0}^{k}(k+1-i)|x_i|\Big)_k$$

in $\mathbb{R}$ such that $(y_k^{(n)}) \to y_k$ as $n \to \infty$. Therefore, as $n \to \infty$ by relation (2.4), we have

$$\sum_{k=1}^{\infty}\Big(\frac{1}{(k+1)(k+2)}\sum_{i=0}^{k}(k+1-i)|x_i - x_i^{(m)}|\Big)^p < \epsilon,$$

for all $m \geq n_0$. In the sequel, we will show that (y_k) is a sequence from $\mathrm{Ces}^2(p)$. From Proposition 2.5 and relation (2.5) we have

$$\lim_{n\to\infty}\sigma_p(x^{(n)} - x^{(m)}) = \sigma_p(x - x^{(m)}) \leq \|x - x^{(m)}\|_L < \epsilon,$$

for all $m \geq n_0$. This implies that $(x^{(n)}) \to x$ as $m \to \infty$. We therefore have $x = x^{(n)} - (x^{(n)} - x) \in \mathrm{Ces}^2(p)$. And this proves that $\mathrm{Ces}^2(p)$ is a complete normed space under the Luxemburg norm. $\square$

In what follows, we will show some results related to the Luxemburg norm, and due to this reason we will denote it by $\|\cdot\|$.

Theorem 2.8. *The space* $\mathrm{Ces}^2(p)$ *is rotund if and only if* $p > 1$.

Proof. Let $\mathrm{Ces}^2(p)$ be rotund, and choose $p = 1$. Consider the following two sequences given by

$$x = \Big(0,0,\ldots,0,\underbrace{\frac{(n+1)(n+2)}{2^n}}_{n\text{th term}},0,0,\ldots\Big)$$

and

$$y = \Big(0,0,\ldots,0,\underbrace{\frac{2(n+1)(n+2)}{3^n}}_{n\text{th term}},0,0,\ldots\Big).$$

Then obviously $x \neq y$ and

$$\sigma_p(x) = \sigma_p(y) = \sigma_p\Big(\frac{x+y}{2}\Big) = 1.$$

Then it follows from Proposition 2.6(3) that $x, y, \frac{x+y}{2} \in S[\mathrm{Ces}^2(p)]$, which leads to the conclusion that the sequence space $\mathrm{Ces}^2(p)$ is not rotund. Hence $p > 1$.

Conversely, let $x \in S[\mathrm{Ces}^2(p)]$, where $1 < p < \infty$ and $y, z \in S[\mathrm{Ces}^2(p)]$ such that $x = \frac{y+z}{2}$. By convexity of σ_p and property (3) from Proposition 2.6, we have

$$1 = \sigma_p(x) \le \frac{\sigma_p(y) + \sigma_p(z)}{2} \le \frac{1}{2} + \frac{1}{2} = 1,$$

which gives that $\sigma_p(y) = \sigma_p(z) = 1$ and

$$\sigma_p(x) = \frac{\sigma_p(y) + \sigma_p(z)}{2}. \tag{2.6}$$

From the last relation we obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p \\
&\quad= \frac{1}{2}\Big\{\sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|y(k)\big|\Big)^p \\
&\qquad+ \sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|z(k)\big|\Big)^p\Big\}.
\end{aligned}$$

Since $x = \frac{y+z}{2}$, we get

$$\begin{aligned}
&\sum_{k=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|y(k)+z(k)\big|\Big)^p \\
&\quad= \frac{1}{2}\Big(\sum_{k=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|y(k)\big|\Big)^p \\
&\qquad+ \sum_{k=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|z(k)\big|\Big)^p\Big).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|y(k)+z(k)\big|\Big)^p \\
&\quad= \frac{1}{2}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|y(k)\big|\Big)^p \\
&\qquad+ \frac{1}{2}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p.
\end{aligned}$$

From the last relation we get that $y_i = z_i$ for all $i \in \mathbb{N}$, whence $z = y$. It means that the sequence space $\mathrm{Ces}^2(p)$ is rotund. $\square$

In what follows we will give two facts without proof, because their proofs follow directly from Propositions 2.5 and 2.6.

Theorem 2.9. *Let $x \in \mathrm{Ces}^2(p)$. Then the following statements hold:*

(i) *if $0 < \alpha < 1$ and $\|x\| > \alpha$, then $\sigma_p(x) > \alpha^M$;*
(ii) *if $\alpha \geq 1$ and $\|x\| < \alpha$, then $\sigma_p(x) < \alpha^M$.*

Theorem 2.10. *Let (x_n) be a sequence in $\mathrm{Ces}^2(p)$. Then the following statements hold:*

(i) $\lim_{n\to\infty} \|x_n\| = 1$ *implies* $\lim_{n\to\infty} \sigma_p(x_n) = 1$.
(ii) $\lim_{n\to\infty} \sigma_p(x_n) = 0$ *implies* $\lim_{n\to\infty} \|x_n\| = 0$.

Theorem 2.11. *Let $x \in \mathrm{Ces}^2(p)$, and let $(x^{(n)}) \subset \mathrm{Ces}^2(p)$. If $\sigma_p(x^{(n)}) \to \sigma_p(x)$ and $x_k^{(n)} \to x_k$ as $n \to \infty$ for all $k \in \mathbb{N}$, then $\|x^{(n)} - x\| \to 0$ as $n \to \infty$.*

Proof. The proof of the theorem is similar to Theorem 2.9 in [12]. □

Theorem 2.12. *The Banach space $\mathrm{Ces}^2(p)$ has the (β)-property.*

Proof. Let us suppose for the contrary that $\mathrm{Ces}^2(p)$ does not have the (β)-property. Then there exists $\epsilon > 0$ such that, for any $\delta \in (0, \frac{\epsilon}{1+2^{1+p}})$, there is a sequence $(x_n) \subset S(\mathrm{Ces}^2 p)$ with $\mathrm{sep}(x_n) > \epsilon^{\frac{1}{p}}$ and an element $x_0 \in S(\mathrm{Ces}^2(p))$ such that

$$\left\| \frac{x_n + x_0}{2} \right\|_{\mathrm{Ces}^2(p)}^p > 1 - \delta,$$

for every $n \in \mathbb{N}$. Let us consider δ as a fixed value from $(0, \frac{\epsilon}{1+2^{1+p}})$. We claim that

$$\lim_{j\to\infty} \sup_k \sum_{n=j+1}^{\infty} \Big(\frac{1}{(n+1)(n+2)} \sum_{i=1}^{n} (n+1-i)\big|x(i)\big| \Big)^p \leq \frac{2^{p+1}\delta}{2^p - 1}. \tag{2.7}$$

Otherwise, we can assume that there exists a sequence (j_k) such that $j_k \to \infty$ as $k \to \infty$ and

$$\sum_{n=j_k+1}^{\infty} \Big(\frac{1}{(n+1)(n+2)} \sum_{i=1}^{n} (n+1-i)\big|x(i)\big| \Big)^p > \frac{2^{p+1}\delta}{2^p - 1}, \tag{2.8}$$

for every $k \in \mathbb{N}$. Let $\delta_1 > 0$ be a real number corresponding to $\epsilon = \delta$ and $A = 1$ in Lemma 2.1. Then there exists n_1 such that

$$\|x_0 \cdot \chi_{\{n_1, n_1+1, \ldots\}}\|_{\mathrm{Ces}^2(p)}^p = \sum_{n=n_1+1}^{\infty} \Big(\frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} (i+1-k)\big|x_0(i)\big| \Big)^p < \delta_1.$$

Take k large enough such that $j_k > n_1$. Then from Lemma 2.1, the convexity of the function $|\cdot|^p$, and relation (2.8), we have

$$\begin{aligned}
1-\delta &< \sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\Big|\frac{x_k(i)+x_0(i)}{2}\Big|\Big)^p \\
&= \sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\Big|\frac{x_k(i)+x_0(i)}{2}\Big|\Big)^p \\
&\quad + \sum_{n=n_1+1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\Big|\frac{x_k(i)+x_0(i)}{2}\Big|\Big)^p \\
&\le \frac{1}{2}\sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|x_0(i)\big|\Big)^p \\
&\quad + \frac{1}{2}\sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|x_k(i)\big|\Big)^p \\
&\quad + \sum_{n=n_1+1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\Big|\frac{x_k(i)}{2}\Big|\Big)^p + \delta \\
&\le \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|x_k(i)\big|\Big)^p \\
&\quad + \frac{1}{2^p}\sum_{n=n_1+1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|x_k(i)\big|\Big)^p + \delta \\
&\le \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|x_k(i)\big|\Big)^p \\
&\quad - \frac{2^p-1}{2^p}\sum_{n=n_1+1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|x_k(i)\big|\Big)^p + \delta \\
&< 1-2\delta+\delta = 1-\delta.
\end{aligned}$$

Hence, relation (2.8) is valid. Now, from the inequality

$$\begin{aligned}
&\Big(\frac{1}{(n_1+1)(n_2+2)}\sum_{i=0}^{n}(i+1-k)\big|x_k(i)\big|\Big)^p \\
&\qquad \le \sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|x_k(i)\big|\Big)^p,
\end{aligned}$$

it follows that

$$\big|x_k(i)\big| \le \frac{(n_1+1)(n_1+2)}{n_1+1-i},$$

for every $k \in \mathbb{N}$ and $i = 1, 2, \ldots, n_1$. It means that there exists a subsequence (y_n) of (x_n) and numerical sequence (a_n) such that $\lim_{k\to\infty} y_k(i) = a_k$, for $i =$

$1, 2, \ldots, n_1$. Therefore

$$\sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|y_k(i)-y_m(i)\big|\Big)^p < \delta$$

for sufficiently large n and m. Consequently,

$$\begin{aligned}
\|y_k - y_m\|^p_{\mathrm{Ces}^2(p)} &= \sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|y_k(i)-y_m(i)\big|\Big)^p \\
&= \sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|y_k(i)-y_m(i)\big|\Big)^p \\
&\quad + \sum_{n=n_1+1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|y_k(i)-y_m(i)\big|\Big)^p \\
&\leq \sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|y_k(i)-y_m(i)\big|\Big)^p \\
&\quad + 2^p\sum_{n=n_1+1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|y_k(i)\big|\Big)^p \\
&\quad + 2^p\sum_{n=n_1+1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}(i+1-k)\big|y_m(i)\big|\Big)^p \\
&\leq \delta + 2^{p+1}\delta \\
&< \epsilon_0,
\end{aligned}$$

whence $\mathrm{sep}(x_n) \leq \mathrm{sep}(y_n) < (\epsilon_0)^{\frac{1}{p}}$. This contradiction shows that $\mathrm{Ces}^2(p)$ has the property (β). $\square$

Corollary 2.13. *The space* $\mathrm{Ces}^2(p)$ *has the Kadec–Klee property.*

Corollary 2.14. *The space* $\mathrm{Ces}^2(p)$ *has the k-NUC property for every* $k \geq 2$.

Corollary 2.15. *The spaces* $\mathrm{Ces}^2(p)$ *and* $(\mathrm{Ces}^2(p))^*$ *have the Banach–Saks property.*

The proof of Corollary 2.15 follows from [4, Theorem 1].

Theorem 2.16. *For any* $1 < p < \infty$, *the space* $\mathrm{Ces}^2(p)$ *has the uniform Opial property.*

Proof. Let $\epsilon > 0$, and let $x \in \mathrm{Ces}^2(p)$. Then there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{n=n_1+1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p < \Big(\frac{\epsilon_0}{4}\Big)^p,$$

for $\epsilon_0 \in (0,\epsilon)$ and $1+\frac{\epsilon^p}{2} \geq (1+\epsilon_0)^p$. On the other hand, from $\|x\|_{\mathrm{Ces}^2(p)} \geq \epsilon$, we obtain that

$$\begin{aligned}
\epsilon^p &\leq \sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p \\
&\quad+\sum_{n=n_1+1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p \\
&< \sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p+\Big(\frac{\epsilon_0}{4}\Big)^p \\
&< \sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p+\frac{\epsilon^p}{4},
\end{aligned}$$

whence

$$\sum_{n=1}^{n_1}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p \geq \frac{3\epsilon^p}{4}.$$

Let $(x_m) \subset S(\mathrm{Ces}^2(p))$ be any weakly null sequence. From $x_m(i) \to 0$, for $i = 1,2,\dots$, it follows that there exists $m_0 \in \mathbb{N}$ such that

$$\Big\|\sum_{i=1}^{n_1} x_m(i)e_i\Big\|_{\mathrm{Ces}^2(p)} < \frac{\epsilon_0}{4},$$

for every $m > m_0$. Therefore,

$$\begin{aligned}
\|x_m+x\|_{\mathrm{Ces}^2(p)} &= \Big\|\sum_{i=1}^{n_1}\big(x_m(i)+x(i)\big)e_i+\sum_{i=n_1+1}^{\infty}\big(x_m(i)+x(i)\big)e_i\Big\|_{\mathrm{Ces}^2(p)} \\
&\geq \Big\|\sum_{i=1}^{n_1}x(i)e_i+\sum_{i=n_1+1}^{\infty}x_m(i)e_i\Big\|_{\mathrm{Ces}^2(p)}-\Big\|\sum_{i=1}^{n_1}x_m(i)e_i\Big\|_{\mathrm{Ces}^2(p)} \\
&\quad-\Big\|\sum_{i=n_1+1}^{\infty}x(i)e_i\Big\|_{\mathrm{Ces}^2(p)} \\
&\geq \Big\|\sum_{i=1}^{n_1}x(i)e_i+\sum_{i=n_1+1}^{\infty}x_m(i)e_i\Big\|_{\mathrm{Ces}^2(p)}-\frac{\epsilon_0}{2}, \qquad (2.9)
\end{aligned}$$

for every $m > m_0$. Moreover,

$$\begin{aligned}
&\Big\|\sum_{i=1}^{n_1}x(i)e_i+\sum_{i=n_1+1}^{\infty}x_m(i)e_i\Big\|_{\mathrm{Ces}^2(p)}^p \\
&\quad= \big\|\big(x(1),x(2),\dots,x(n_1),x_m(n_1+1),\dots\big)\big\|_{\mathrm{Ces}^2(p)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n_1}(n+1-i)\big|x(i)\big| \\
&\quad + \frac{1}{(n+1)(n+2)}\sum_{i=n_1+1}^{\infty}(n+1-i)\big|x_m(i)\big|\Big)^p \\
&\geq \sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n_1}(n+1-i)\big|x(i)\big|\Big)^p \\
&\quad + \sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{i=n_1+1}^{\infty}(n+1-i)\big|x_m(i)\big|\Big)^p \qquad (2.10) \\
&\geq \frac{3\epsilon^p}{4} + \Big(1-\frac{\epsilon^p}{4}\Big) \\
&= 1 + \frac{\epsilon^p}{2} \\
&> (1+\epsilon_0)^p. \qquad (2.11)
\end{aligned}
$$

Now, from equations (2.9) and (2.10), we get

$$\|x_m + x\| \geq 1 + \frac{\epsilon_0}{2}.$$

This means that $\mathrm{Ces}^2(p)$ has the uniform Opial property. □

Corollary 2.17. *For $1 < p < \infty$, the space $\mathrm{Ces}^2(p)$ has the property (L) and the fixed-point property.*

Theorem 2.18. *The equality $C(\mathrm{Ces}^2(p)) = 2^{\frac{1}{p}}$ holds for any $p \geq 1$.*

The technique of the proof is similar to that of [5, Theorem 3], and so we omit it.

Theorem 2.19. *The Gurarii̇̆ modulus of convexity for the sequence space $\mathrm{Ces}^2(p)$ ($1 \leq p < \infty$) is*

$$\beta_{\mathrm{Ces}^2(p)} \leq 1 - \Big(1 - \Big(\frac{\epsilon}{2}\Big)^p\Big)^{\frac{1}{p}},$$

for every $\epsilon > 0$.

Proof. We follow some techniques given in [14]. Let $x \in \mathrm{Ces}^2(p)$. Then

$$\sum_{n=1}^{\infty}\Big(\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(n+1-k)\big|x(k)\big|\Big)^p < \infty.$$

If we denote by A the matrix which represents the sequence space defined by the above relation, then it can be expressed in the following form:

$$A = (a_{nk}) = \begin{cases} \frac{(n+1-k)}{(n+1)(n+2)} & \text{for } 0 \leq k \leq n; n,k \in \{0,1,2,3,4,\ldots\}, \\ 0 & \text{for } k > n. \end{cases}$$

Let $\epsilon > 0$. From the definition of matrix A, it follows that there exists the inverse matrix B. We define the following two sequences:

$$x = (x_n) = \left(B\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}, B\left(\frac{\epsilon}{2}\right), 0, \ldots\right),$$
$$y = (y_n) = \left(B\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}, B\left(-\frac{\epsilon}{2}\right), 0, \ldots\right).$$

The norms of the above sequences are

$$\|x\|^p_{\mathrm{Ces}^2(p)} = \big\|A(x)\big\|^p_{l_p} = \left|\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}\right|^p + \left|\frac{\epsilon}{2}\right|^p = 1,$$
$$\|y\|^p_{\mathrm{Ces}^2(p)} = \big\|A(y)\big\|^p_{l_p} = \left|\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}\right|^p + \left|-\frac{\epsilon}{2}\right|^p = 1,$$

and

$$\begin{aligned}\|x - y\|_{\mathrm{Ces}^2(p)} &= \big\|A(x-y)\big\|_{l_p} \\ &= \left(\left|\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}} - \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}\right|^p + \left|\frac{\epsilon}{2} - \left(-\frac{\epsilon}{2}\right)\right|^p\right)^{\frac{1}{p}} = \epsilon.\end{aligned}$$

Now we will estimate the infimum of the expression

$$\inf_{0 \le \alpha \le 1} \big\|\alpha \cdot x + (1-\alpha) \cdot y\big\|_{\mathrm{Ces}^2(p)},$$

for every $x, y \in S(\mathrm{Ces}^2(p))$. We have

$$\begin{aligned}&\inf_{0 \le \alpha \le 1} \big\|\alpha \cdot x + (1-\alpha) \cdot y\big\|_{\mathrm{Ces}^2(p)} \\ &= \inf_{0 \le \alpha \le 1} \big\|\alpha \cdot A(x) + (1-\alpha) \cdot A(y)\big\|_{l_p} \\ &= \inf_{0 \le \alpha \le 1} \left\{\left|\alpha\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}} + (1-\alpha)\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}\right|^p \right. \\ &\quad \left. + \left|\alpha\left(\frac{\epsilon}{2}\right) + (1-\alpha)\left(-\frac{\epsilon}{2}\right)\right|^p\right\}^{\frac{1}{p}} \\ &= \inf_{0 \le \alpha \le 1} \left\{1 - \left(\frac{\epsilon}{2}\right)^p + (2\alpha - 1)\left(\frac{\epsilon}{2}\right)^p\right\}^{\frac{1}{p}} \\ &= \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}.\end{aligned}$$

Hence, for every $p \ge 1$, we get the estimate

$$\beta_{\mathrm{Ces}^2(p)} \le 1 - \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}. \qquad \square$$

Corollary 2.20.

(1) *If* $\epsilon = 2$, *then* $\beta_{\mathrm{Ces}^2(p)} \le 1$ *and* $\mathrm{Ces}^2(p)$ *is strictly convex.*
(2) *If* $0 < \epsilon < 2$, *then* $0 < \beta_{\mathrm{Ces}^2(p)} < 1$ *and* $\mathrm{Ces}^2(p)$ *is uniformly convex.*
(3) *Under conditions from (2),* $\mathrm{Ces}^2(p)$ *is reflexive.*

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