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Recurrence of multidimensional persistent random walks. Fourier and series criteria

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The recurrence and transience of persistent random walks built from variable length Markov chains are investigated. It turns out that these stochastic processes can be seen as Lévy walks for which the persistence times depend on some internal Markov chain: they admit Markov random walk skeletons. A recurrence versus transience dichotomy is highlighted. Assuming the positive recurrence of the driving chain, a sufficient Fourier criterion for the recurrence, close to the usual Chung–Fuchs one, is given and a series criterion is derived. The key tool is the Nagaev–Guivarc'h method. Finally, we focus on particular two-dimensional persistent random walks, including directionally reinforced random walks, for which necessary and sufficient Fourier and series criteria are obtained. Inspired by (*Adv. Math.* **208** (2007) 680–698), we produce a genuine counterexample to the conjecture of (*Adv. Math.* **117** (1996) 239–252). As for the one-dimensional case studied in (*J. Theoret. Probab.* **31** (2018) 232–243), it is easier for a persistent random walk than its skeleton to be recurrent. However, such examples are much more difficult to exhibit in the higher dimensional context. These results are based on a surprisingly novel – to our knowledge – upper bound for the Lévy concentration function associated with symmetric distributions.

Keywords: concentration functions; Fourier and series recurrence criteria; Fourier perturbations; Markov operators; Markov random walks; persistent random walks; variable length Markov chain

1. Introduction

Classical random walks are usually defined from a sequence of independent and identically distributed *i.i.d.* increments $\{X_k\}_{k\geq 1}$ by $S_0 = 0$ and for every $n \geq 1$,

$$S_n := \sum_{k=1}^{n} X_k. \tag{1.1}$$

In the continuity of [12] we aim at investigating the asymptotic behaviour, and more specifically the recurrence and transience, of a multidimensional Persistent Random Walk (PRW) for which the increments are driven by a Variable Length Markov Chain (VLMC) built from some probabilized context tree (see Section 1.1 for a definition). This construction furnishes a wide class of models for the dependence of the increments which can be easily adapted to various contexts. The toy model in [12] is built from a VLMC with a double-infinite comb as context tree and the increments belongs to $\{-1, 1\} \subset \mathbb{Z}$. The characterization of the recurrent versus transient behaviour is difficult for a general probabilized context tree (see [11] for some zoology for instance). Before investigating a larger class of models, we focus in Section 1.1 on a particular context tree generalizing in $\mathbb{Z}^2$ the double-infinite comb already studied. The latter is naturally called a quadruple-infinite comb and the resulting PRW is termed the quadruple-infinite comb PRW. Originally motivated by PRWs, the aim of this paper is twofold.

First, as pointed out in Section 1.1.2, such PRWs or their time continuous linear interpolation are closely related to Markov Random Walks (MRW) and Markov Levy Walks (MLW) respectively. Those models, already studied in the litterature (see [3,19,20,26,51] and references therein) under strong moment conditions on the jumps, are natural extensions of Random Walks and Levy Walks. In this paper, a criterion, similar to the Chung–Fuchs integral test in the context of Random Walks, is proved.

Secondly, it turns out that quadruple-infinite comb PRWs are anisotropic generalization of Directionally Reinforced Random Walks (DRRWs) introduced in [38]. More precisely, DRRWs are quadruple-infinite comb PRWs for which the *i.i.d.* waiting times taking values in $\{1, 2, \ldots\}$ do not depend on some internal Markov chain and the successive directions (four possibilities) are chosen uniformly and independently among all, excepted the previous one (thus three uniform choices). Note that a DRRW is completely characterized by the distribution of the waiting times whereas the model of PRW, with more parameters, is more flexible. The authors in [48] answered partially by the negative to the conjecture in [38], Section 3, p. 247. This question is definitively closed in this paper.

1.1. The quadruple-infinite comb model

Let us start with the general construction of VLMCs built from a probabilized context tree on the alphabet $\mathcal{A} := \{\mathtt{e}, \mathtt{n}, \mathtt{w}, \mathtt{s}\}$. In the sequel, we associate with every $\ell \in \mathcal{A}$ the corresponding direction in $\mathbb{Z}^2$ in such a way that $(\overrightarrow{\mathtt{e}}, \overrightarrow{\mathtt{n}})$ stands for the canonical basis whereas $(\overrightarrow{\mathtt{w}}, \overrightarrow{\mathtt{s}})$ is the opposite one. Hence, the letters $\mathtt{e}$, $\mathtt{n}$, $\mathtt{w}$ and $\mathtt{s}$ will stand for moves to the east, north, west and south respectively.

Let $\mathcal{L} = \mathcal{A}^{-\mathbb{N}}$ be the set of left-infinite words and consider a complete tree on $\mathcal{A}$: each node has 0 or card$(\mathcal{A})$ children. The set of leaves is denoted by $\mathcal{C}$ and elements of $\mathcal{C}$ are (possibly infinite) words on $\mathcal{A}$. To each leaf $c \in \mathcal{C}$, called a context, is attached a distribution q_c on $\mathcal{A}$. Endowed with this probabilistic structure, such a tree is named a probabilized context tree. The related VLMC – here denoted by $\{U_n\}_{n\geq 0}$ – is the Markov Chain on $\mathcal{L}$ whose transitions are given by

$$\mathbb{P}(U_{n+1} = U_n\ell | U_n) = q_{\overleftarrow{\mathrm{pref}}(U_n)}(\ell), \tag{1.2}$$

where $\overleftarrow{\mathrm{pref}}(w) \in \mathcal{C}$ is defined as the shortest prefix of $w = \cdots w_{-1}w_0$, read from right to left, appearing as a leaf of the context tree. The kth increment X_k of the corresponding PRW is identified with the rightmost letter of U_k. In particular, we can write $U_n = \cdots X_{n-1}X_n$. The set of leaves of the quadruple-infinite comb encodes the memory of the VLMC and consists of words on the alphabet $\mathcal{A}$ of the form

$$\mathcal{C} := \{\ell^n \ell' : \ell \neq \ell' \in \mathcal{A}, n \geq 1\} \cup \{\ell^\infty : \ell \in \mathcal{A}\}. \tag{1.3}$$

The prefix function is then formally defined by $\overleftarrow{\mathrm{pref}}(\cdots \ell' \ell^n) = \ell^n \ell'$ and $\overleftarrow{\mathrm{pref}}(\ell^\infty) = \ell^\infty$. One can summarize the probabilistic structure as follows: for $n \geq 1$ introduce $\alpha_n : \mathcal{A} \times \mathcal{A} \to [0,1]$ and $p_n : \mathcal{A}^2 \times \mathcal{A}^2 \to [0,1]$ so that for all $\ell, \ell', \ell'' \in \mathcal{A}$ with $\ell \neq \ell'$ and $\ell' \neq \ell''$

$$q_{\ell^n \ell'}(\ell) =: 1 - \alpha_n(\ell', \ell) \quad \text{and} \quad q_{\ell^n \ell'}(\ell'') =: \alpha_n(\ell', \ell) p_n((\ell', \ell); (\ell, \ell'')). \tag{1.4}$$

For $\ell = \ell'$ or $\ell' = \ell''$, the quantities $\alpha_n(\ell')$ and $p_n((\ell', \ell); (\ell, \ell''))$ are set to zero. Also introduce $\alpha_\infty : \mathcal{A} \times \mathcal{A} \to [0,1]$ and $p_\infty : \mathcal{A}^2 \times \mathcal{A}^2 \to [0,1]$ with

$$q_{\ell^\infty}(\ell) =: 1 - \alpha_\infty(\ell, \ell) \quad \text{and} \quad q_{\ell^\infty}(\ell'') =: \alpha_\infty(\ell, \ell) p_\infty((\ell, \ell); (\ell, \ell'')), \tag{1.5}$$

for all $\ell, \ell'' \in \mathcal{A}$. These quantities are interpreted as transition probabilities – see Figure 1 – and characterize the probabilized context tree. It is worth noting that a DRRW is in particular a quadruple-infinite comb PRW for which $\alpha_n(\ell, \ell')$ depends only on $n \geq 1$ when $\ell \neq \ell'$ and such that $p_n((\ell', \ell); (\ell, \ell''))$ is constant equal to $1/3$ when $\ell \neq \ell'$ and $\ell' \neq \ell''$.

Remark 1.1. For one-dimensional PRWs, the probability for a change of direction depends only on the time spent since the last change of direction. In the higher dimensional case, this probability depends additionally on the direction of the previous run.

In the following, we refer carefully to Figure 2 below that illustrates our notations and assumptions by a realization of a linear interpolation $\{S_t\}_{t \geq 0}$ of a quadruple-infinite comb PRW.

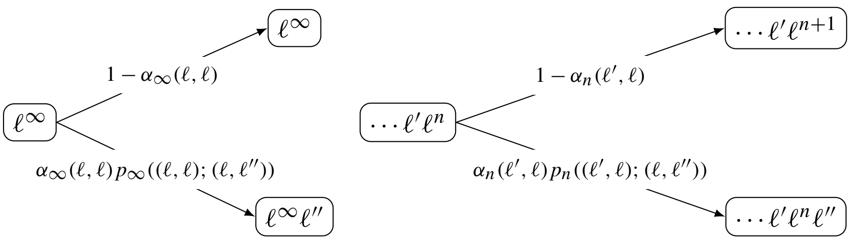

Figure 1. Transition probabilities of the quadruple-infinite comb.

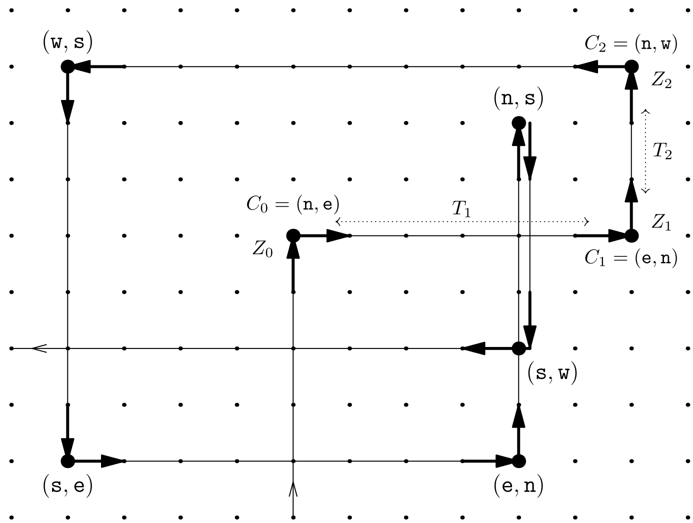

Figure 2. Piecewise interpolation of the PRW.

1.1.1. *Associated MRW*

Let P be the Markov kernel on $\mathcal{A} \times \mathcal{A}$ defined for every $\ell', \ell, \ell'' \in \mathcal{A}$ with $\ell' \neq \ell$ and $\ell'' \neq \ell$ by

$$P\big((\ell', \ell); (\ell, \ell'')\big) := \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} \big(1 - \alpha_k(\ell', \ell)\big) \right) \alpha_n(\ell', \ell) p_n\big((\ell', \ell); (\ell, \ell'')\big), \tag{1.6}$$

$$P\big((\ell, \ell); (\ell, \ell'')\big) := \sum_{n=1}^{\infty} \big(1 - \alpha_\infty(\ell, \ell)\big)^{n-1} \alpha_\infty(\ell, \ell) p_\infty\big((\ell, \ell); (\ell, \ell'')\big). \tag{1.7}$$

To get a stochastic matrix, we choose adequately the entries $P((\ell', \ell); (\ell, \ell))$ and $P((\ell, \ell); (\ell, \ell))$ when necessary and set all other entries equal to zero. For sake of simplicity, in the sequel, we make the following assumption.

Assumption 1.1. One has $(X_0, X_1) = (\mathtt{n}, \mathtt{e})$ with probability one and the state $(\mathtt{n}, \mathtt{e})$ belongs to an irreducible class $\mathcal{S} \subset \mathcal{A} \times \mathcal{A} \setminus \Delta$ of the Markov kernel P where $\Delta \subset \mathcal{A} \times \mathcal{A}$ is the diagonal subset.

Obviously, regarding the study of the asymptotic behaviour of the PRW, there is no loss of generality assuming such conditions. Note that, under this assumption, for every $c \in \mathcal{S}$,

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} \big(1 - \alpha_k(c)\big) \right) \alpha_n(c) = 1. \tag{1.8}$$

Roughly speaking, this assumption disallows a too strong reinforcement, that is a too fast decreasing rate for the transition probabilities $\alpha_n(c)$ of changing directions. As a matter of facts, the transition probabilities between two changes of letters – named breaking or moving times – are encoded by the Markov kernel P. In fact, let $\{B_n\}_{n\geq 0}$ be the almost surely finite breaking times defined inductively by $B_0 = 0$ and $B_{n+1} = \inf\{k > B_n : X_k \neq X_{k+1}\}$. These breaking times correspond to those where the PRW changes of direction. It turns out that the so called internal (configuration or driven) chain $\{C_n\}_{n\geq 0}$ defined by

$$C_n := (X_{B_n}, X_{B_{n+1}}), \tag{1.9}$$

is an irreducible Markov chain on $\mathcal{S}$ starting from $(\mathtt{n}, \mathtt{e})$ whose Markov kernel – still denoted by P abusing notation – is the restriction of P to $\mathcal{S} \times \mathcal{S}$. The waiting times $T_{n+1} := B_{n+1} - B_n$ are not independent contrary to the one-dimensional case. However, the distribution of T_{n+1} depends only on (C_n, C_{n+1}). The skeleton random walk – the PRW observed at the breaking times – $\{Z_n\}_{n\geq 0}$ on $\mathbb{Z}^2$ is then defined as

$$Z_n := S_{B_n} = \sum_{i=1}^{n} \left(\sum_{k=T_{i-1}+1}^{T_i} X_k \right), \tag{1.10}$$

where $T_0 = 0$. Obviously, Z is not a RW. Nevertheless, taking into account the additional information given by the internal Markov chain, Z is rather a MRW, also named a Markov Additive Process (MAP), semi-Markov process or hidden Markov chain (see [2,4] for instance). To be more specific, it means the process $\{(Z_n, C_n)\}_{n\geq 0}$ is Markovian on $\mathbb{Z}^2 \times \mathcal{S}$ and satisfies

$$\mathcal{L}\big((Z_{n+1} - Z_n, C_n) \mid \big\{(Z_k, C_k)\big\}_{0\leq k\leq n}\big) = \mathcal{L}\big((Z_{n+1} - Z_n, C_n) \mid C_n\big). \tag{1.11}$$

Here the latter conditional distributions do not depend on $n \geq 0$. Thereafter, we introduce for every $c, s \in \mathcal{S}$ such that $P(c, s) > 0$ the conditional jump and waiting time distributions

$$\begin{aligned} \mu_{c,s}(dx) &:= \mathbb{P}(Z_{n+1} - Z_n \in dx | C_n = c, C_{n+1} = s), \\ \mu_c(dt) &:= \sum_{s\in\mathcal{S}} P(c, s)\mu_{c,s}(dx), \\ \nu_{c,s}(dt) &= \mathbb{P}(T_{n+1} \in dt \mid C_n = c, C_{n+1} = s), \\ \nu_c(dt) &= \sum_{s\in\mathcal{S}} P(c, s)\nu_{c,s}(dt). \end{aligned} \tag{1.12}$$

Here, writing a configuration c as $(c_{\mathtt{i}}, c_{\circ})$, we get $\mu_{c,s}(n\overrightarrow{c_\circ}) = \nu_{c,s}(n)$ and $\mu_c(n\overrightarrow{c_\circ}) = \nu_c(n)$, the latter denoting the distribution of the nth term of the summand in (1.8) and the former equals to

$$\frac{p_n(c; s)\alpha_n(c) \prod_{k=1}^{n-1}(1 - \alpha_k(c))}{P(c, s)}. \tag{1.13}$$

1.1.2. *A generic MLW structure for PRWs*

Therefore, at the sight of the considerations above, a PRW can be constructed as follows:

- introduce a Markov chain $\{C_n\}_{n\geq 0}$ on $\mathcal{S}$ with transition kernel P;
- consider independent sequences of *i.i.d.* random variables $\{(\tau_n(c,s), \overrightarrow{\tau_n}(c,s))\}_{n\geq 1}$ – themselves independent of $\{C_n\}_{n\geq 0}$ – taking values in $\{1,2,\dots\}\times\mathbb{Z}^2$ for every $c,s\in\mathcal{S}$ and whose respective distributions are the push-forward images of $\mu_{c,s}$ by $x\longmapsto(\|x\|,x)$;
- the piecewise linear interpolation of the initial discrete-time PRW is then given by

$$S_t := \sum_{n=1}^{N(t)} \overrightarrow{\tau_n}(C_{n-1}, C_n) + (t - B_{N(t)})\overrightarrow{\tau_{N(t)+1}}(C_{N(t)}, C_{N(t)+1}), \tag{1.14}$$

  with

$$N(t) := \max\{n \geq 0 : B_n \leq t\} \quad \text{and} \quad B_n := \tau_1(C_0, C_1) + \cdots + \tau_n(C_{n-1}, C_n); \tag{1.15}$$

- and the skeleton MRW is obtained setting

$$Z_n := S_{B_n} = \sum_{k=1}^{n} \overrightarrow{\tau_k}(C_{k-1}, C_k). \tag{1.16}$$

Following the terminology of [35], the continuous-time persistent process $\{S_t\}_{t\geq 0}$ is virtually a Lévy Walk (LW), except that the waiting times, as well as the jumps, are no longer *i.i.d.* nor even independent. Original LWs are basically Continuous Time Random Walks (CTRWs) for which the waiting times and the sizes of jumps are coupled and usually proportional. In our context, the continuous interpolation (1.14) of the skeleton MRW is called a Markov Lévy Walk (MLW) to fit the nomenclature of the skeleton MRW and the LW structure. As a matter of facts, a LW is nothing but a MLW for which $\mu_{c,s}$ no longer depends on $c,s\in\mathcal{S}$. As well explained in [35] and also in [5,39,40,49], these kind of stochastic processes model a wide panoply of phenomena involving anomalous diffusions. These processes are termed anomalous since their standard deviations grow at a polynomial rate of order $\alpha\neq 1/2$ with the time compare to the standard square root order. In addition, it extends the notion of diffusion in the sense that the marginal densities solve a fractional Fokker–Planck equation. For instance, CTRWs are anomalous diffusions.

Obviously, there are many other possible choices for the internal chain, all of them leading to different cutting of the trajectories of the original PRW. For instance, remarking that a PRW is an additive functional of the underlying VLMC, one may choose for internal chain the VLMC itself. The jump distributions are then deterministic. With this choice, the geometry of the ambient space and the symmetries are forgotten. Somehow, the valuable information is entirely encoded in the internal chain, that is, the VLMC. On the other side, there is the choice intuitively made here for which the internal chain takes values in the finite set $\mathcal{S}$. Thus, the internal chain is much simpler whereas the jumps are no longer (conditionally) deterministic and encode a whole portion of the trajectory of the original PRW. In view of the assumptions of Theorem 2.1, it is needed to find a trade-off between the complexity of the internal chain and how faithful the additive component of the MRW describes the trajectory of the PRW.

Actually, the choice intuitively made so far for the double or the quadruple-infinite comb PRW has a natural interpretation in the terms of the context tree geometry. This involves the notion of Longest Internal Suffix (LIS) introduced in [10] and recalled here. Keeping the notation introduced in Section 1.1, consider θ the left-shift operator defined for every (possibly infinite) words $\omega := \omega_1\omega_2\cdots$ on the alphabet $\mathcal{A}$ by $\theta(\omega) = \omega_2\omega_3\cdots$ and set for every context $c \in \mathcal{C}$,

$$\ell\text{-}\operatorname{lis}(c) := \theta^{\tau(c)-1}(c) \in \mathcal{C}, \quad \text{with } \tau(c) := \inf\{n \geq 1 : \theta^n(c) \notin \mathcal{C}\}. \tag{1.17}$$

The longest internal suffix of c is defined by $\theta^{\tau(c)}(c)$. It is abbreviated as the LIS of c and it is the longest suffix of c which is an internal node of the context tree. On the contrary, the ℓ-LIS of c defined above is the shortest suffix of c which is always a context. Remark that applying the shift operator to the ℓ-LIS gives the LIS, so that the letter ℓ refers to the letter before the LIS.

We denote by $\mathcal{G} \subset \mathcal{C}$ the set of ℓ-LISs. The latter is a good candidate for the state space of the internal Markov chain (called *internal state space* in the sequel). For this choice, we retrieve for instance $\{\mathtt{e}, \mathtt{n}, \mathtt{w}, \mathtt{s}\}^2$ for the quadruple-infinite comb model. Also, we retrieve $\{\mathtt{ud}, \mathtt{du}\}$ for the double-infinite comb model. The latter is defined analogously with the binary alphabet $\{\mathtt{u}, \mathtt{d}\}$ – standing for up and down moves respectively – and leaves $\{\mathtt{u}^n\mathtt{d}, \mathtt{d}^n\mathtt{u} : n \geq 1\} \cup \{\mathtt{u}^\infty, \mathtt{d}^\infty\}$. This comb model is used in [12] to study one-dimensional PRW.

To go further, define inductively the sequence of breaking times as follows

$$B_0 = 0 \quad \text{and} \quad B_{n+1} := \inf\{k > B_n : \ell\text{-}\operatorname{lis}(\overleftarrow{\operatorname{pref}} U_k) \neq \ell\text{-}\operatorname{lis}(\overleftarrow{\operatorname{pref}} U_{B_n})\}, \tag{1.18}$$

and set as previously $T_{n+1} := B_{n+1} - B_n$ and $T_0 = 0$ but also

$$C_n := \operatorname{lis}(\overleftarrow{\operatorname{pref}} U_{B_n}) \quad \text{and} \quad Z_n := S_{B_n} = \sum_{i=1}^{n} \left(\sum_{k=T_{i-1}+1}^{T_i} X_k \right). \tag{1.19}$$

Then assuming $\overleftarrow{\operatorname{pref}} U_0 \in \mathcal{G}$, it turns out that $\{(Z_n, C_n)\}_{n\geq 0}$ is a MRW skeleton of the PRW and the latter can be recovered adding the information given by the conditional excursions

$$\mathbf{e}_{g,h}(d\xi) := \mathbb{P}\left(n \longmapsto \sum_{k=T_{i-1}+1}^{n\wedge T_i} X_k \in d\xi \,\middle|\, C_0 = g, C_1 = h \right). \tag{1.20}$$

Remark 1.2. There is no reason for a context tree to admit a finite set of LISs. That is why, in the sequel, internal Markov chains evolving in a possibly infinite countable state space are considered. It is worth noting these considerations are not artificial: consider for instance a one-dimensional PRW with increments in $\{-1, 1\}$ whose memory is encoded through the length of last rise together with the length of the last descent.

1.2. Overview of the article

Foremost, note that in Section 2 is considered a general MLW on $\mathbb{R}^d$, $d \geq 1$. Such processes are easily defined adapting slightly the construction in Section 1.1.2.

In Section 2.1, it is first proved that the MLW, as well as its embedded skeleton MRW, are either recurrent or transient supposing the internal Markov chain is recurrent (Proposition 2.1). If in addition the internal Markov chain is assumed positive recurrent, then it is shown that Z is recurrent if and only some series is infinite as for classical RWs (Proposition 2.2). This characterization consists in extending a result of [2] to multidimensional MRWs.

In Theorem 2.1 of Section 2.2 are stated Fourier and Series criteria characterizing the type (recurrent or transient) of the skeleton MRW. Eventhough, the proof of this result basically follows the ideas of the Nagaev–Guivarc'h perturbation method, it is worth noting that no moment conditions are assumed so that virtually all kinds of jump distributions can be considered. Also, let us point out that when the VLMC (U_n) no longer admits an invariant probability measure, the random time between two change of directions is no longer integrable (see the proof of this fact for the double-infinite comb VLMC in [9]).

Some probabilistic and operator Assumptions 2.1 and 2.2 together with some Sector Assumption 2.3 are obviously required and mostly relevant in the case of an infinite internal state space. As already noted, the analytic criterion (2.11) is nothing but the natural extension of the classical Chung–Fuchs integral test for MRWs: in place of the characteristic function appears the principal eigenvalue of some Fourier perturbation associated with the internal Markov operator. This principal eigenvalue admits the asymptotic expansion (2.7). Note that because of the different nature of Fourier analysis in the lattice and non lattice cases, this section only deals with MRW taking values in $\mathbb{Z}^d$. However, those results are not specific to PRWs and might be usefull in other context involving MRWs.

Unfortunately, in view of [48], Theorem 4, p. 684, Theorem 2.1 only gives a sufficient criterion for the recurrence of a MLW. The result in [48] also answers nearly by the negative to the conjecture about two-dimensional DRRWs in [38], Section 3, p. 247. Informally, it is asked whether a DRRW is recurrent simultaneously with the RW defined as the DRRW observed at the successive times of returns in its initial direction. As already pointed out by the authors, the given example in [48] do no fit well to the usual framework of DRRW since their waiting times can be equal to zero with positive probability. This may appear anecdotal, however, their ingenious and technical construction involves in a crucial way unimodality arguments that can not be applied for true DRRWs. Nevertheless, it still provides a counter-example for our general MLWs and related MRW skeletons.

In Section 2.3, returning to the quadruple-infinite comb model, a complete characterization of the recurrence of PRWs is stated in Proposition 2.1. For this specific model, the margins of the resulting skeleton are independent symmetric one-dimensional RWs. The admissible probabilistic structure is detailed in Assumptions 2.4 and includes DRRWs.

Our results are based on the fundamental Lemma 2.1 and Theorem 2.2 involving an appropriate Borel–Cantelli lemma. Let us stress that, to our knowledge, the result in Lemma 2.1 is surprisingly not mentioned anywhere. At the end of this section, the conjecture [38], Section 3, p. 247, for DRRWs, and actually for a wider class of two-dimensional PRWs, is definitively answered by the negative. We follow the constructive probabilistic approach presented in [48] which crucially relies on an unimodality assumption for the jump distribution. This assumption is dropped in the present paper by using the important Lemma 2.1.

Finally, the two remaining sections are devoted to the proofs of Proposition 2.3 and Theorem 2.1 of Section 3, the fundamental Lemma 2.1 and its consequences in Theorem 2.2, of Corollary 2.1 and Theorem 2.3 of Section 4.

2. Recurrence and transience criteria

In this section, we consider a Markov chain $\{C_n\}_{n\geq 0}$ on a discrete and countable state space $\mathcal{S}$ whose Markov kernel is denoted by P. Also, we denote by $\pi(dc)$ a corresponding invariant measure, normalized to be a probability measure when possible. Let $\{\tau_n(c,s), \overrightarrow{\tau_n}(c,s)\}_{n\geq 0}$ be a sequence of *i.i.d.* random variables taking values in $[0,\infty)\times\mathbb{R}^d$ whose common distribution, depending only on (c,s), is denoted by $m_{c,s}(dt,dx)$. Moreover, let us introduce the first and second marginal distributions of $m_{c,s}$ denoted respectively, by $\nu_{c,s}(dt)$ and $\mu_{c,s}(dx)$ and set

$$\mu_c(dx) := \sum_{s\in\mathcal{S}} P(c,s)\mu_{c,s}(dx) \quad \text{and} \quad \nu_c(dt) := \sum_{s\in\mathcal{S}} P(c,s)\nu_{c,s}(dt). \tag{2.1}$$

Thereafter, one can construct as in (1.14)–(1.16) above a MLW denoted by $\{S_t\}_{t\geq 0}$ evolving in $\mathbb{R}^d$ whose skeleton $\{Z_n\}_{n\geq 0}$ is a MRW coupled with C as an internal Markov process. In order to ensure the continuity of S, it is assumed that, for every $c,s\in\mathcal{S}$ with $P(c,s)>0$,

$$m_{c,s}\big(\{0\}\times\big(\mathbb{R}^d\setminus\{0\}\big)\big)=0. \tag{2.2}$$

In the sequel $\mathbb{P}_c$ (*resp.* $\mathbb{P}_\nu$) denotes the probability distribution on the path space conditionally on $C_0=c$ (*resp.* C_0 is distributed as ν) and $S_0=Z_0=0$.

2.1. Dichotomy results

The following proposition states a zero-one law for MLWs and MRWs leading to the standard dichotomy between recurrence *versus* transience.

Proposition 2.1 (Zero-one law and dichotomy recurrence/transience). *Assume that the internal Markov chain C is irreducible and recurrent. Then, for any $c\in\mathcal{S}$ and any Borel subset $A\subset\mathbb{R}^d$, it holds*

$$\mathbb{P}_c\bigg(\bigcap_{t\geq 0}\bigcup_{u\geq t}\{S_u\in A\}\bigg)\in\{0,1\} \quad \textit{and} \quad \mathbb{P}_c\bigg(\bigcap_{n\geq 0}\bigcup_{k\geq n}\{Z_k\in A\}\bigg)\in\{0,1\}. \tag{2.3}$$

In particular, a MLW (resp. *MRW*) *is either recurrent or transient in the sense that either for any $c\in\mathcal{S}$,*

$$\mathbb{P}_c\Big(\lim_{t\to\infty}\|S_t\|=\infty\Big)=1 \quad \Big(\textit{resp.}\ \mathbb{P}_c\Big(\lim_{n\to\infty}\|Z_n\|=\infty\Big)=1\Big), \tag{T}$$

or for any $c\in\mathcal{S}$ there exists $r>0$ (a priori *depending on c*) *such that*

$$\mathbb{P}_c\Big(\liminf_{t\to\infty}\|S_t\|<r\Big)=1 \quad \Big(\textit{resp.}\ \mathbb{P}_c\Big(\liminf_{n\to\infty}\|Z_n\|<r\Big)=1\Big). \tag{R-1}$$

Remark 2.1. Considering the original set of questions about PRWs built from VLMCs, an analogous zero-one law holds substituting $\{S_t\}_{t\geq 0}$ with the discrete-time process $\{S_n\}_{n\geq 0}$.

Proof. Let $c \in \mathcal{S}$ be arbitrarily chosen and introduce the successive visit times $\{\sigma_n\}_{n\geq 0}$ of c. Define

$$\mathbb{X}_n := \left\{\overrightarrow{\tau_{\sigma_n+k}}(C_{\sigma_n+k-1}, C_{\sigma_n+k})\right\}_{1\leq k\leq \sigma_{n+1}-\sigma_n}, \tag{2.4}$$

for all $n \geq 0$. Since the excursions between two visits of c are *i.i.d.* under $\mathbb{P}_c$, so it is for $\{\mathbb{X}_n\}_{n\geq 0}$. Therefore, the zero-one law (2.3) follows from the Hewitt–Savage zero-one law [29], Theorem 3.15, p. 53, noting that the asymptotic events belong to the exchangeable σ-field of $\{\mathbb{X}_n\}_{n\geq 0}$.

Specifying the zero-one law (2.3) to the events considered in (T) and (R-1) so that they occur with probability zero or one, it only remains to prove these probabilities do not depend on the initial configuration c. To this end, suppose that S (*resp.* Z) goes to infinity for one configuration c. Then, the irreducibility of C and the translation invariance property (1.11) of Markov additive processes imply S goes to infinity with positive probability, and in turn with probability one, for any internal state. □

Assuming in addition C is π-positive recurrent, one can improve (R-1) for MRWs. To this end, introduce the recurrent set $\mathcal{R}$ and the set of possible states $\mathcal{P}$ – following the terminology and notations of [2] – defined by

$$\mathcal{R} := \left\{x \in \mathbb{R}^d : \forall \varepsilon > 0, \mathbb{P}_\pi\big(Z_n \in B(x,\varepsilon) \text{ i.o.}\big) = 1\right\} \quad \text{and,}$$
$$\mathcal{P} := \left\{x \in \mathbb{R}^d : \forall \varepsilon > 0, \exists n \geq 0, \mathbb{P}_\pi\big(Z_n \in B(x,\varepsilon)\big) > 0\right\},$$

where $B(x,\varepsilon) \subset \mathbb{R}^d$ stands for the open ball of radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^d$. Note that $\mathcal{R}$ and $\mathcal{P}$ are both closed subsets. Now let $\Gamma \subset \mathbb{R}^d$ be the smallest closed subgroup containing the support of the distribution mixture

$$\mu_\pi(dx) := \sum_{c\in\mathcal{S}} \pi(c)\mu_c(dx), \tag{2.5}$$

where $\mu_c(dx)$ is given in (2.1).

Proposition 2.2 (Recurrence features for MRWs and series criterion). *Assume that the internal Markov chain is irreducible and π-positive recurrent. Then one has $\mathcal{P} = \mathcal{R} = \Gamma$ when the MRW is recurrent. Furthermore, the alternative* (R-1) *is equivalent for the MRW to each of the following statements.*

1. *For some* (*or equivalently any*) $\varepsilon > 0$ *and some* (*or any*) *initial distribution* ν,

$$\mathbb{P}_\nu\Big(\liminf_{n\to\infty} \|Z_n\| < \varepsilon\Big) = 1. \tag{R-2}$$

2. *For some* (*or any*) $\varepsilon > 0$ *and some* (*or any*) *initial state* $c \in \mathcal{S}$,

$$\sum_{n=0}^{\infty} \mathbb{P}_c\big(Z_n \in B(0,\varepsilon)\big) = \infty. \tag{R-3}$$

Remark 2.2. Regarding the recurrence set associated with S the question seems to be more intricate since it depends strongly on the geometry of each conditional jump $\mu_{c,s}(dx)$. For instance, one can be easily convinced that it is possible for two recurrent PRWs to have both recurrent skeletons in $\mathbb{Z}^2$ but distinct recurrent set given respectively by $\mathbb{R}^2$ and $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$.

Proof. First, we deduce from the partition exhibited in [6] for stationary random walks and from the zero-one law in Proposition 2.1 that (R-1) is equivalent to (R-2) when $\nu = \pi$, and thus for any (or some) arbitrary ν since π is fully supported. Besides, one can easily see that the relevant Propositions in [2], pp. 127–130, can be adapted to a multidimensional framework. The first Proposition for instance which is originally taken from [7], p. 56, can be more generally obtained for multidimensional MRW using [6] together with the dichotomy Proposition 2.1. It follows that the recurrent alternative is equivalent to (R-3) and $\mathcal{P} = \mathcal{R} = \Gamma$. $\square$

2.2. A general sufficient Fourier criterion

In this section, an integral test using Fourier analysis is given for general MRWs. A similar criterion might be proved for MLWs, however, Fourier analysis being substantially different in the lattice and non lattice case we restrict ourself to MRWs taking values in $\mathbb{Z}^d$.

Definition 2.1 (Aperiodic MRW). *A MRW is called to be periodic if for some $c \in \mathcal{S}$, $x \in \mathbb{Z}^d$ and proper subgroup $\Gamma \subsetneq \mathbb{Z}^d$ it holds $\mu_c(x + \Gamma) = 1$. On the contrary, it is called aperiodic.*

In the sequel, we make the following probabilistic assumptions.

Assumption 2.1 (Probabilistic assumptions).

(P1) The Markov chain C is irreducible, aperiodic (classical sense) and π-positive recurrent.
(P2) The Markov random walk Z is aperiodic in $\mathbb{Z}^d$.

We introduce for every $t \in \mathbb{T}^d$ – the d-dimensional torus $\mathbb{R}^d/2\pi\mathbb{Z}^d$ – the operator on $\mathbb{L}^1(\pi)$ defined for every $f \in \mathbb{L}^1(\pi)$ and $c \in \mathcal{S}$ by

$$P_t f(c) := \mathbb{E}_c\big[e^{itZ_1} f(C_1)\big]. \tag{2.6}$$

For $t = 0$, we simply write P for P_0. There exists an extensive literature regarding such Fourier perturbations and we refer to [3,19,20,24,26,51], for instance. We recall that the peripheral spectrum – the set of spectral values with maximal modulus – is well defined for bounded operators. Moreover, we say that a Markov operator has a spectral gap when its spectrum outside a centered ball of radius $1 - \rho$ is finite for some $0 < \rho < 1$. Below, $\mathbb{1}$ stands for constant function on $\mathcal{S}$ equal to 1.

Assumption 2.2 (Operator assumptions). There exists a Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ such that:

(O1) the constant function $\mathbb{1} \in \mathcal{B}$ and the canonical injection $\mathcal{B} \longrightarrow \mathbb{L}^1(\pi)$ is continuous;

(O2) the operators P_t acts continuously on $\mathcal{B}$ for every $t \in \mathbb{T}^d$;
   (a) the restricted Markov kernel $P : \mathcal{B} \longrightarrow \mathcal{B}$ admits a spectral gap;
   (b) the map $t \longmapsto P_t$ is continuous for the subordinated operator norm induced by $\|\cdot\|_{\mathcal{B}}$;
   (c) the peripheral spectrum of $P_t : \mathcal{B} \longrightarrow \mathcal{B}$ only consists of eigenvalues when it is outside the open unit disk.

For instance, those assumptions are satisfied for $\mathcal{B} = \mathbb{L}^p(\pi)$ with $p \in [1, \infty]$ if all the P_t are quasi-compact. We allude to [18,22,23,25,31–33,41,45] for more general consideration about these properties but also – when there exist Lyapunov functions – for the interesting situation of weighted-supremum spaces corresponding to geometric ergodicity. Before stating the last assumptions, let us draw some important consequences.

Proposition 2.3 (The eigenvalue of maximal modulus). *Under Assumptions 2.1 and 2.2, for any sufficiently small neighbourhood $\mathcal{V} \subset \mathbb{T}^d$ of the origin and any $t \in \mathcal{V}$ one has:*

1. *the spectrum of P_t admits a unique element of maximal modulus $\lambda(t)$;*
2. *$\lambda(t)$ is an eigenvalue of algebraic multiplicity one;*
3. *$|\lambda(t)| \leq 1$ and $|\lambda(t)| = 1$ if and only if $t = 0$;*
4. *the map $t \longmapsto \lambda(t)$ is continuous and admits the asymptotic expansion*

$$\lambda(t) \underset{t \to 0}{=} \widehat{\mu}_\pi(t) + \mathcal{O}\big(\|P_t - P\|^2\big), \tag{2.7}$$

   where $\widehat{\mu}_\pi$ is the characteristic function of the mixture distribution (2.5).

As it shall appear in the proof, it is possible to continue the asymptotic expansion given in (2.7). In addition to Assumptions 2.1 and 2.2, to avoid some tangential convergence making the recurrence criteria more intricate to expose, we need the following technical hypothesis.

Assumption 2.3 (Sector condition). For any sufficiently small neighbourhood $\mathcal{V} \subset \mathbb{T}^d$ of the origin, there exists $K > 0$ such that for all $t \in \mathcal{V}$,

$$\big|\Im\big(\lambda(t)\big)\big| \leq K \Re\big(1 - \lambda(t)\big). \tag{2.8}$$

Roughly speaking, the sector condition forbids a too strong drift term. In the classical context of Random Walks, if the jumps have a second order moment this condition is equivalent for their characteristic function to the null drift condition. On the contrary, if their tails satisfy some general regularly varying conditions then (see [43] for more details) the sector condition is always satisfied. Finally, most of the operators encountered are sectorial and one can consult [34], Chapter 2, and [36] for a rigorous definition and elementary properties.

As an example, when the underlying Banach space is $\mathbb{L}^2(\pi)$, this assumption can be stated in a more handy way in terms of the associated sectorial forms (we refer to [30] and particularly its Chapters Five and Six for more details). Introduce the sesquilinear form

$$\mathcal{E}_t[f, g] = \sum_{c \in \mathcal{S}} (f - P_t f)(c)\overline{g(c)}\pi(c). \tag{2.9}$$

Note that for $t=0$, it is nothing but the usual Dirichlet form associated with the driving chain. Then one can consider the real and imaginary part of the latter form respectively, given by

$$\mathfrak{R}_t(f,g) := \frac{\mathcal{E}_t[f,g] + \overline{\mathcal{E}_t[g,f]}}{2} \quad \text{and} \quad \mathfrak{I}_t(f,g) := \frac{\mathcal{E}_t[f,g] - \overline{\mathcal{E}_t[g,f]}}{2i}. \tag{2.10}$$

It turns out that $\mathfrak{R}_t$ is symmetric and positive (definite when $t \neq 0$) but also that condition (2.8) is equivalent to the usual sector condition $|\mathfrak{I}_t| \leq C\mathfrak{R}_t$. Typically, this inequality trivially holds when the internal chain C is π-reversible and the jumps satisfy the symmetry $\mu_{c,s}(dx) = \mu_{s,c}(-dx)$ for every $c, s \in \mathcal{S}$. In that case, the imaginary part vanishes and the spectrum is real.

The following theorem deal with a Fourier-like criterion for MRWs. Compared to (R-3), the series criterion given here is slightly more general since it holds for an arbitrary initial distribution ν dominated by π, *that is*, there exists $c > 0$ such that $\nu \leq c\pi$.

Theorem 2.1 (Fourier and series criterion). *Under Assumptions 2.1, 2.2 and 2.3 the MRW is recurrent or transient accordingly as*

$$\lim_{r\uparrow 1} \int_{\mathcal{V}} \mathfrak{R}\left(\frac{1}{1-r\lambda(t)}\right) dt = \infty \quad \textit{or} \quad \lim_{r\uparrow 1} \int_{\mathcal{V}} \mathfrak{R}\left(\frac{1}{1-r\lambda(t)}\right) dt < \infty, \tag{2.11}$$

for some (or any) neighbourhood $\mathcal{V}$ of the origin for which $\lambda(t)$ is well-defined. Besides, the integral above is infinite or finite accordingly as

$$\sum_{n=0}^{\infty} \mathbb{P}_\nu(Z_n = 0) = \infty \quad \textit{or} \quad \sum_{n=0}^{\infty} \mathbb{P}_\nu(Z_n = 0) < \infty, \tag{2.12}$$

for some (or any) initial distribution ν dominated by π.

Recall that the first term in the expansion (2.7) is $\widehat{\mu}_\pi(t)$ so that the integral criterion (2.11) can be interpreted as a perturbation of the classical one obtained for a random walk with μ_π as jump distribution. Also, one can note using the sector condition that this criterion can be rewritten in terms of

$$\int_{\mathcal{V}} \frac{1}{\mathfrak{R}(1-\lambda(t))} dt = \infty \quad \text{or} \quad \int_{\mathcal{V}} \frac{1}{\mathfrak{R}(1-\lambda(t))} dt < \infty. \tag{2.13}$$

Besides, it could be interesting to compare (2.11) with the conjecture stated in [2], p. 126, namely: *such MRW is recurrent if, and only if*

$$\sum_{c\in\mathcal{S}} \pi(c) \int_{(-\varepsilon,\varepsilon)} \mathfrak{R}\left(\frac{1}{1-\widehat{\mu}_c(t)}\right) dt = \infty. \tag{2.14}$$

Here $\widehat{\mu}_c(t)$ denote the characteristic function of $\mu_c(dx)$ defined in (2.1).

Finally, the series criterion (2.12) is obvious for any initial distribution ν when the internal state space is finite or when the internal operator satisfies some Doeblin's condition. It is also possible to suppose the MRW satisfying some scaling limit as in [27] to get the same series criterion.

Remark 2.3. Theorem 2.1 may applies in many contexts: PRWs built from VLMCs obviously but also, for instance, additive functional of Markov chains and non-homogeneous random walks. We can refer to [46] and [28] respectively. To a lesser extent, it could apply to random walks in random environment subject to generalizing it to uncountable internal state spaces (the environments viewed from the particles) and non-lattice jumps.

An illustration of Theorem 2.1

To see what imply the Assumptions of Theorem 2.1, we give a concrete and simple application in the field of non-homogeneous random walks, similar to those investigated in [28], pp. 6–9.

First, assume that the internal Markov chain C is given by the house-of-cards process on $\mathbb{N}$ whose transitions $P(n,0) = p_n$ and $P(n,n+1) = 1 - p_n$ with $p_n \in (0,1)$ and $\epsilon := \inf_{n\geq 0} p_n > 0$. One can easily check that C is irreducible, π-positive recurrent and aperiodic with

$$\pi(n) = \frac{\prod_{k=0}^{n-1}(1-p_k)}{\sum_{n\geq 0}\prod_{k=0}^{n-1}(1-p_k)}.$$

Besides, since $\epsilon > 0$, P satisfy the classical Doeblin's condition. Therefore, it is well-known – see, for instance, [50], Chapter 2, – that $|||P^n - \pi(\cdot)\mathbb{1}\|_\infty \leq (1-\epsilon)^n$. In other words, the Markov kernel $P : \mathbb{L}^\infty(\pi) \longrightarrow \mathbb{L}^\infty(\pi)$ has a spectral gap. Note that the hypothesis (O1) is trivially satisfied here.

At this stage, we need to specify our assumptions on the conditional jumps in (2.1) associated with Z. To this end, introduce $\mu_{n,n+1}$, $\mu_{n,0}$ and $\mu_n = (1-p_n)\mu_{n,n+1} + p_n\mu_{n,0}$ the associated conditional distributions and assume that Z is aperiodic and

$$\lim_{t\to 0}\sup_{n\in\mathbb{N}}\left|\widehat{\mu}_n(t) - 1\right| = 0.$$

It is not difficult to see that $t \longmapsto P_t$ is then continuous as bounded operator on $\mathbb{L}^\infty(\pi)$. Also, we need to illustrate Assumption (O2(c)). As a matter of facts, this condition is in many cases a direct product of the other hypotheses.

Regarding the Assumption (O2(c)), it can be deduced from previous assumptions — in many cases, such a deduction is possible. In facts, for this example, let us consider, for every $k \geq 0$, the linear contraction $T_k : \mathbb{L}^\infty(\pi) \longrightarrow \mathbb{L}^\infty(\pi)$ defined by $T_k f(n) = f(n)\mathbb{1}_{\{n\leq k\}}$ and remark that

$$\gamma_k := \sup_{\|f\|_\infty\leq 1} \pi\big(|f - T_k f|\big) \xrightarrow[k\to\infty]{} 0.$$

Besides,

$$\left\|P_t^n f\right\|_\infty \leq \left\|P^n|f|\right\|_\infty \leq (1-\epsilon)^n\|f\|_\infty + \pi\big(|f|\big).$$

It turns out for n and k large enough that

$$\left\|P_t^n - P_t^n T_k\right\|_\infty \leq 2(1-\epsilon)^n + \gamma_k < 1.$$

Since $P_t^n T_k$ is a finite-rank operator, it is compact, and since P_t is a contraction on $\mathbb{L}^\infty$ the Lemma of [14], Chap. VIII.8.2, p. 709, applies so that (O2(c)) is satisfied. The reader could consult [31] and [21] for other tools in this spirit.

In order to verify the sector Assumption 2.3 and to apply Theorem 2.1, suppose for instance that for every $n \in \mathbb{N}$ the distributions $\mu_{n,0}$ and $\mu_{n,n+1}$ are supported on $\mathbb{Z}$ and respectively given for every $k \in \mathbb{Z}$ by

$$\mu_{n,0}(k) = \frac{c_\alpha}{1 + |k - m_n|^{1+\alpha}} \quad \text{and} \quad \mu_{n,n+1}(k) = \frac{C_\beta}{1 + |k - M_n|^{1+\beta}}. \tag{2.15}$$

We assume furthermore that $0 < \alpha, \beta < 2$ with $\alpha, \beta \neq 1$ and $(p_n m_n)_{n\in\mathbb{N}}$, $((1 - p_n)M_n)_{n\in\mathbb{N}}$ are two bounded sequences. Note that c_α and C_β are two positive normalizing constants.

The aperiodicity condition is then trivially satisfied and we can deduce from [1] that

$$\|P_t - P\| \leq \sup_{n\in\mathbb{N}} \bigl|\widehat{\mu}_n(t) - 1\bigr| = \mathcal{O}\bigl(|t|^{\alpha\wedge\beta\wedge 1}\bigr),$$

where $\wedge$ stand the minimum operators. Furthermore, one can prove from [1] that

$$1 - \widehat{\mu}_\pi(t) = \gamma_\alpha |t|^\alpha + \gamma_\beta |t|^\beta - i\left[\sum_{n\in\mathbb{N}} \pi_n\bigl(p_n m_n + (1 - p_n)M_n\bigr)\right] t + o\bigl(|t|^{\alpha\wedge\beta\wedge 1}\bigr),$$

where γ_α and γ_β are two positive constants. Therefore, one can see that the sector condition holds according to

$$\sum_{n\in\mathbb{N}} \pi_n\bigl(p_n m_n + (1 - p_n)M_n\bigr) = 0 \quad \text{or} \quad \alpha \wedge \beta < 1.$$

In the two latter situation Theorem 2.1 applies. In the first one, the recurrence of Z is equivalent to $\alpha \wedge \beta > 1$ whereas it is always transient in the second one.

2.3. Necessary and sufficient criteria for the quadruple-infinite comb

We first need to extend an oscillation criterion used in [48] for unimodal symmetric distributions.

Theorem 2.2 (Series and Fourier criterion). *Let $\{H_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ be two independent RWs on $\mathbb{Z}$ starting from the origin, the second one being symmetric. Then*

$$\begin{aligned} &\mathbb{P}(H_{n+1} = 0, V_n V_{n+1} \leq 0, i.o.) = 1 \\ \iff \quad &\sum_{n=0}^{\infty} \mathbb{P}(H_{n+1} = 0)\mathbb{P}(0 \leq V_n \leq V_{n+1} - V_n) = \infty. \end{aligned} \tag{2.16}$$

Furthermore, if the two-dimensional random walk $\{(H_n, V_n)\}_{n\geq 0}$ is transient, this criterion is equivalent to the Fourier integral criterion

$$\lim_{r\uparrow 1} \int_{\mathcal{V}} \int_{\mathcal{V}} \Re\left(\frac{\Phi_V(r,s)}{1 - \varphi_H(t)\varphi_V(s)}\right) ds\, dt = \infty, \tag{2.17}$$

where $\mathcal{V} \subset \mathbb{T}^2$ is any sufficiently small neighbourhood of the origin, φ_H and φ_V are the characteristic functions of the jumps associated with H and V and Φ_V is the trigonometric series

$$\Phi_V(r,s) = \sum_{n=0}^{\infty} r^n \mathcal{T}_V(n)\cos(ns). \tag{2.18}$$

Here $\mathcal{T}_V(n)$ denotes the two-sided tail distribution of the symmetric jumps of V.

This result for symmetric and unimodal distribution involves in [48] a Borel–Cantelli lemma which crucially relies on the unimodality assumption. Actually, instead of unimodality, applying Lemma 2.1 below on Lévy concentration functions, only symmetry is required. Recall that the Lévy concentration function of a real random variable X is defined for all $\lambda \geq 0$ by

$$Q(X,\lambda) = \sup_{x\in\mathbb{R}} \mathbb{P}(x \leq X \leq x+\lambda). \tag{2.19}$$

The following fundamental Lemma means – roughly speaking – that the supremum is reached near the origin for the symmetric distributions.

Lemma 2.1 (Lévy concentration function). *Let $\{M_n\}_{n\geq 0}$ be a symmetric random walk. Then there exist two positive universal constants L and C such that for all $p > 0$ for which the characteristic function of the jumps is non-negative on $[-p, p]$ and all $n \geq 1$ and $\lambda \geq L/p$,*

$$\mathbb{P}(0 \leq M_n \leq \lambda) \leq Q(M_n,\lambda) \leq C\mathbb{P}(0 \leq M_n \leq \lambda). \tag{2.20}$$

Remark 2.4. One can obtain similar bounds replacing the condition $0 \leq M_n \leq \lambda$ in (2.20) by the symmetric one $|M_n| \leq \lambda/2$. Besides, one can note that $Q(M_n,\lambda) = \mathbb{P}(|M_n| \leq \lambda/2)$ when the jump distribution is unimodal and non-atomic.

At this stage, Theorem 2.2 provides a necessary and sufficient criteria for a wide class of PRWs, built from a quadruple infinite comb as in Section 1.1, under the following assumptions. We refer also to the beginning of Section 2.

Assumption 2.4 (Generalized DRRWs). Let $\{S_n\}_{n\geq 0}$ be a quadruple-infinite comb PRW starting from the origin, the initial time being a vertical-to-horizontal move and such that

(H1) the distribution associated with the persistence time $\tau_n((\ell,\ell'),(\ell',\ell''))$ depend only on the current horizontal $\mathtt{h} = \{\mathtt{e},\mathtt{w}\}$ or vertical $\mathtt{v} = \{\mathtt{n},\mathtt{s}\}$ direction ℓ' of the walker. They are denoted by $\nu_{\mathtt{h}}(dt)$ or $\nu_{\mathtt{v}}(dt)$ respectively;

(H2) the probabilities to change from the current direction into an orthogonal one only depend on the final direction (among east, north, west or south) and are constant with respect to the absolute directions (horizontal or vertical). Those are denoted by

$$p_{\mathtt{e}} = p_{\mathtt{w}} = \frac{1-p_{\mathtt{v}}}{2} \quad \text{and} \quad p_{\mathtt{n}} = p_{\mathtt{s}} = \frac{1-p_{\mathtt{h}}}{2}, \tag{2.21}$$

in such way that p_{h} and p_{v} stand respectively for the probabilities to stay in the current horizontal and vertical direction at each breaking time.

Remark 2.5. In [48], Theorem 2, p. 682, it is proved that DRRWs are transient in $\mathbb{Z}^d$ for $d \geq 3$. Therefore, the higher dimensional cases modeled by Assumption 2.4 seem to be irrelevant (it still would require a proof) and are not investigated in this paper.

This framework includes two types of PRWs which are of particular interest when the waiting time distributions are equal, that is $\nu_{\mathrm{h}}(dt) = \nu_{\mathrm{v}}(dt)$:

- Original DRRWs if $p_{\mathrm{h}} = p_{\mathrm{v}} = 1/3$.
- DRRWs without U-turns (non-backtracking DRRWs) if $p_{\mathrm{h}} = p_{\mathrm{v}} = 0$.

Non-backtracking DRRWs are natural generalizations of the symmetric 1-dimensional PRWs investigated in [12] and was the original motivation of this work.

Let us introduce a symmetric Rademacher random variable ε, two geometric random variables G_{h} and G_{v} with parameters $1 - p_{\mathrm{h}}$ and $1 - p_{\mathrm{v}}$, and two sequences of *i.i.d.* random variables $\{\tau_k^{\mathrm{h}}\}_{k\geq 1}$ and $\{\tau_k^{\mathrm{v}}\}_{k\geq 1}$ distributed as $\nu_{\mathrm{h}}(dt)$ and $\nu_{\mathrm{v}}(dt)$. We assume that all of these are independent of each other. Then we can consider two independent random walks $\{H_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ whose respective jumps are distributed as

$$\varepsilon \sum_{k=1}^{G_{\mathrm{h}}} (-1)^{k-1} \tau_k^{\mathrm{h}} \quad \text{and} \quad \varepsilon \sum_{k=1}^{G_{\mathrm{v}}} (-1)^{k-1} \tau_k^{\mathrm{v}}, \tag{2.22}$$

and state a necessary and sufficient criterion for the recurrence of these specific PRWs.

Corollary 2.1 (Necessary and sufficient criteria). *Under Assumption 2.4 the origin is recurrent for $\{S_n\}_{n\geq 0}$ if and only if*

$$\begin{gathered}\sum_{n=0}^{\infty} \mathbb{P}(H_{n+1} = 0)\mathbb{P}(0 \leq V_n \leq V_{n+1} - V_n) \quad \textit{or} \\ \sum_{n=0}^{\infty} \mathbb{P}(V_{n+1} = 0)\mathbb{P}(0 \leq H_n \leq H_{n+1} - H_n) = \infty.\end{gathered} \tag{2.23}$$

Thereafter, we answer by the negative to the conjecture in [38] (by presenting a constructive method to build recurrent PRWs with transient MRW skeletons).

Theorem 2.3 (Definitive invalidation of the conjecture). *There exist waiting time distributions on the positive integers* $\{1, 2, \ldots\}$ *such that the associated DRRWs and non-backtracking DRRWs in* $\mathbb{Z}^2$ *are recurrent whereas their MRW skeletons are transient.*

We can deduce from the Fourier criterion (2.17) that such distributions are necessarily non-integrable. Recall that in the case of non integrable persistent times, there is no invariant probability measure for the associated VLMC. Inspired by [47], one could ask for a proof relying on

Fourier analysis. Such an approach has seemed to us tedious. That is why our preferences go to a more concrete probabilistic proof in the spirit of [17,48].

3. Proofs of Section 2.2

Let us start with Proposition 2.3 which lay the groundwork for Theorem 2.1.

Proof of Proposition 2.3. First, we get from Assumptions (P1) and (O1) that in the complex Banach space $\mathcal{B}$ we have $\ker(P-I) = \mathrm{span}(\mathbb{1})$. Together with the spectral gap condition (O2(a)) and since any isolated element of the spectrum is an eigenvalue, the spectral radius of P is necessarily equal to 1. Besides, the eigenvalue 1 is necessarily of algebraic multiplicity one. Otherwise, the operator $P-I$ would induce a surjective map from $\ker(P-I)^2 \supsetneq \mathrm{span}(\mathbb{1})$ to $\mathrm{span}(\mathbb{1})$ and thus there would exist $f \in \mathcal{B}$ such that $Pf = f + \mathbb{1}$, in contradiction with the Perron-Frobenius theorem. Furthermore, another application of this theorem shows that if λ is an eigenvalue of P on the unit circle then $\lambda = 1$. This ends the proof of the first and second point of the proposition in the case when $t = 0$.

Furthermore, the two properties shown above are somehow open in the sense that they still hold for small continuous perturbations of P. More precisely, in virtue of the continuity hypothesis (O2(b)), we can apply [30], Theorem 3.16, p. 212. It follows there exist $\delta, \epsilon > 0$ and a positively-oriented curve Γ enclosing 1 such that, for any bounded linear perturbation H smaller than δ for the operator norm, there exists a unique element $\lambda(H) \in \sigma(P+H)$ – the spectrum of $P+H$ – in the interior of Γ. The latter is an element of maximal modulus but also an eigenvalue of algebraic multiplicity one and the distance $d(\sigma(P+H), \Gamma)$ is lower-bounded by ϵ. The proof of the two first points then follows by setting $H = P_t - P$.

To prove the third point, let us denote by $\rho(T)$ the spectral radius of a bounded linear operator T on a Banach space. Another application of [30], Theorem 3.16, implies that $T \longmapsto \rho(T)$ is upper semi-continuous so it is for $t \longmapsto \rho(P_t)$. Consequently, the function $t \longmapsto \rho(P_t)$ reaches its maximum M on any compact set $K \subset \mathbb{T}^d$ at some point $t^* \in K$. Let λ be a peripheral spectral value of P_{t^*} so that $|\lambda| = M$. Assume that $M \geq 1$. It comes from Assumption (O2(c)) that $P_{t^*}h = \lambda h$ for some eigenvector $h \in \mathcal{B}$. Since the modulus of a characteristic function is lower than one, the triangle inequality implies that for every $c \in \mathcal{S}$,

$$M\big|h(c)\big| \leq \sum_{s\in\mathcal{S}} \big|\widehat{\mu}_{c,s}\big(t^*\big)\big|\big|h(s)\big|P(c,s) \leq P|h|(c), \tag{3.1}$$

where $\widehat{\mu}_{c,s}$ stands for the characteristic function of $\mu_{c,s}$ given in (2.1). Since $P^n|h|$ converge pointwise towards $\pi(|h|)\mathbb{1}$ then necessarily $M = 1$. Supposing now $M = 1$ and $t^* \neq 0$ we obtain, again from the latter convergence, that $|h| \leq \pi(|h|)\mathbb{1}$ and then $|h| = \pi(|h|)\mathbb{1}$ since π has full support and $\pi(\pi(h)\mathbb{1} - h) = 0$. This implies that $|\widehat{\mu}_{c,s}(t^*)| = 1$ as soon as $P(c,s) \neq 0$. However, this property is equivalent for the MRW to be periodic, which is excluded. As a consequence, for any neighbourhood $\mathcal{V} \subset \mathbb{T}^d$ of the origin,

$$\sup_{t\in\mathbb{T}^d\setminus\mathcal{V}} \rho(P_t) < 1. \tag{3.2}$$

Since $|\lambda(t)| = \rho(P_t)$ on $\mathcal{V}$ the proof of the third point is completed.

It remains to prove the last and fourth point. Again, we use the perturbation theory exposed in [30] but taking care about the (possibly) infinite dimensional context using the Cauchy holomorphic functional calculus. With the notations above, since the resolvent $R_H(\xi) := (P + H - \xi)^{-1}$ is holomorphic outside the (compact) spectrum of $P + H$, we can consider for all H with $\|H\| \leq \delta$ the following so called Dunford integral

$$Q_H := -\frac{1}{2\pi i} \int_\Gamma R_H(\xi)\, d\xi. \tag{3.3}$$

Let H_0 be any perturbation with $\|H_0\| \leq \delta$. Recall that $d(\sigma(P + H_0), \Gamma) \geq \epsilon$ and thus

$$\sup_{\xi \in \Gamma} \big\| R_{H_0}(\xi) \big\| \leq \frac{1}{\epsilon}.$$

As a consequence, for any H such that $\|H\| < \epsilon$, and any $\xi \in \Gamma$, one can write

$$R_{H+H_0}(\xi) = R_{H_0}(\xi) + \sum_{p \geq 1} \big(-H R_{H_0}(\xi)\big)^p. \tag{3.4}$$

Recall that $\|\cdot\|$ is a subordinated norm. It follows that $Q_{H+H_0} - Q_{H_0} = \mathcal{O}(H)$ and hence $H \longmapsto Q_H$ is continuous on a neighbourhood of the origin.

Moreover, writing the Laurent series expansion of $R_H(\xi)$ around $\lambda(H)$, it is well known that Q_H is the continuous projector on the generalized eigenspace associated with $\lambda(H)$. Since the latter is of multiplicity one, this space is one-dimensional so that, denoting by Tr the linear trace defined on the finite rank operator ideal,

$$\lambda(H) - 1 = \operatorname{Tr}\big(Q_H(P + H)\big) - 1 = \operatorname{Tr}\big((P + H - 1)Q_H\big). \tag{3.5}$$

Lemma 3.1. *The trace operator is continuous on the space E of rank-one bounded linear operators endowed with the induced subordinated norm and $|\operatorname{Tr}(T)| \leq \|T\|$ for all $T \in E$.*

Proof. Let (T_n) be a sequence of continuous rank-one operators converging to T. There exist continuous linear forms (φ_n), φ and vectors (v_n), v such that $\operatorname{Tr}(T_n) = \varphi_n(v_n)$ and $\operatorname{Tr}(T) = \varphi(v)$, where T_n and T are respectively represented as $\varphi_n \otimes v_n$ and $\varphi \otimes v$. We can assume without loss of generality that (v_n) and v are of norm 1. Thereafter, if we consider e such that $\varphi(e) = 1$ we get that $\lim_{n\to\infty} |\varphi_n(e)| = 1$ and $\varphi_n(e)v_n \longrightarrow v$. Thus replacing v_n by $(\varphi_n(e))v_n$ and φ_n by $(\varphi_n(e))^{-1}\varphi_n$ we can assume that $v_n \longrightarrow v$. Then, necessarily $\varphi_n \longrightarrow \varphi$ for the subordinated norm. In particular, we deduce the convergence $\operatorname{Tr}(T_n) \longrightarrow \operatorname{Tr}(T)$. Since the upper-bound is obvious this ends the proof of the Lemma. □

It turns out from the continuity of the perturbed eigenprojector, the representation (3.5) and Lemma 3.1 above that $H \longmapsto \lambda(H)$ is continuous on a neighbourhood of the origin. As a consequence, we get the continuity of $\lambda(t)$ on a neighbourhood of the origin writing $\lambda(t) = \lambda(P_t - P)$. Finally, we shall prove the asymptotic expansion (2.7). To this end, we continue the work initiated for (3.5), (3.4) and (3.3). First note $(P + H - 1)R_H(\xi) = 1 + (\xi - 1)R_H(\xi)$ but also the identity $Q_H(P+H-1)Q_H = (P+H-1)Q_H$. Then, one can write

$$\lambda(H)-1 = \mathrm{Tr}\bigg(\frac{1}{2\pi i}\int_\Gamma (1-\xi)Q_H R_H(\xi)\,d\xi\bigg).$$

Set $R := R_0$ and $Q := Q_0$. Using (3.4) with $H_0 = 0$ and since one can exchange the integral and the series, the problem reduces to the trace of the normally convergent series

$$\sum_{p\geq 0}\frac{1}{2\pi i}\int_\Gamma (1-\xi)Q_H R(\xi)\big(-HR(\xi)\big)^p\,d\xi. \tag{3.6}$$

Following [30], Chap. I.5.3, the Laurent series expansion of the resolvent is given by

$$R(\xi) = -\frac{Q}{\xi-1}+\sum_{n\geq 0}(\xi-1)^n T^{n+1}, \tag{3.7}$$

where $T := \sum_{n=0}^{\infty} P^n(I-Q)$. Therefore, to evaluate each integrals in (3.6) it only remains to identify the Cauchy principal singularity (the residue) of the integrand. For $p=0$, the integral in (3.6) vanishes whereas for $p=1$ it is equal to $Q_H QHQ$, its trace is given by $\pi Q_H H\mathbb{1}$. For the following terms, the pth principal singularity is a finite sum of multiplicative terms involving Q, T, HQ, HT and most importantly Q_H as common left-factor. A thorough study would leads to their explicit expression. However, for our purpose, it is only needed to note the pth principal singularity is a rank-one operator whose range do not depend on p. As a consequence, we can apply the upper-bound in Lemma 3.1 to the sum over $p\geq 2$ and we obtain

$$\mathrm{Tr}\bigg(\sum_{p\geq 2}\frac{1}{2\pi i}\int_\Gamma (1-\xi)Q_H R(\xi)\big(-HR(\xi)\big)^p\,d\xi\bigg) = \mathcal{O}\big(\|H\|^2\big).$$

Since $\pi Q = \pi$, we deduce $\lambda(H)-1 = \pi H\mathbb{1}+\pi(Q_H-Q)H\mathbb{1}+\mathcal{O}(\|H\|^2) = \pi H\mathbb{1}+\mathcal{O}(\|H\|^2)$. To conclude, the asymptotic expansion (2.7) is proved by setting $H := P_t - P$. □

Proof of Theorem 2.1. First, the Markov additive property implies for every $n\geq 1$ and every π-integrable or non-negative function f on $\mathcal{S}$ the identity

$$P_t^n f(c) = \mathbb{E}_c\big[e^{it.Z_n} f(C_n)\big]. \tag{3.8}$$

We show that

$$\sum_{n\geq 0}\mathbb{P}_\nu(Z_n=0) = \lim_{r\uparrow 1}\sum_{n\geq 0} r^n\mathbb{P}_\nu(Z_n=0) = \lim_{r\uparrow 1}\int_{\mathbb{T}^d}\Re\big(\nu(1-rP_t)^{-1}\mathbb{1}\big)\,dt. \tag{3.9}$$

To this end, write the resolvent $(1-rP_t)^{-1}$ on $\mathcal{B}$ as the classical series expansion of the bounded operators $r^n P_t^n$ when $0<r<1$. Remark that $t\longmapsto \nu(r^n P_t^n)\mathbb{1}$ is continuous by assumptions and bounded by r^n from (3.8). Then, the equalities of (3.9) follow from the monotone convergence theorem for the first one and the dominated convergence theorem for the second one. To go further, let $0 < \epsilon < 1$ be given, so that in (3.2) one has $\rho(P_t) \leq 1 - \epsilon$ for every $t \notin \mathcal{V}$. Again, the series expansion of the resolvent yield the existence of $C > 0$ such that $\|(1 - rP_t)^{-1}\|_{\mathcal{B}} \leq C\epsilon^{-1}$ for every $0 < r < 1$ and $t \notin \mathcal{V}$. Since ν is assumed to be a continuous linear form on $\mathbb{L}^1(\pi)$, it is also continuous on $\mathcal{B}$ from (O1). Denoting by N_1, N_2 the $\mathcal{B}$-norm of the continuous linear forms ν and $\mathbb{1}$, it follows

$$\left|\lim_{r\uparrow 1} \int_{\mathbb{T}^d \setminus \mathcal{V}} \Re\big(\nu(1 - rP_t)^{-1}\mathbb{1}\big)\, dt\right| \leq C\epsilon^{-1} N_1 N_2 < \infty. \tag{3.10}$$

Therefore, the r-limit on the right-hand side of (3.9) is finite or infinite accordingly as the same r-limit, but integrating on any (or some) neighbourhood of the origin, is finite or infinite. Let us write $P_t = \lambda(t)Q_t + E_t$ where $Q_t := Q_{P_t - P}$ is the one-dimensional projector on the eigenspace associated with $\lambda(t)$ defined by (3.3). Note that E_t can be seen as the restriction of P_t to the stable subspace $\ker(Q_t) = \Im(1 - Q_t)$ and Q_t the restriction of P_t to the one-dimensional supplementary subspace $\ker(1 - Q_t) = \Im(Q_t)$. Besides, another use of [30], Theorem 6.17, p. 178, with the spectral gap condition (O2(b)) gives $0 < \epsilon < 1$ such that $\rho(E_t) \leq 1 - \epsilon$ for every $t \in \mathcal{V}$. In addition, the operators Q_t and E_t commute so that $P_t^n = \lambda(t)^n Q_t + E_t^n$ for every $n \geq 1$. Thus, it follows

$$\Re\big(\nu(1 - rP_t)^{-1}\mathbb{1}\big) = \Re\left(\frac{\nu Q_t \mathbb{1}}{1 - r\lambda(t)}\right) + \Re\big(\nu(1 - rE_t)^{-1}\mathbb{1}\big).$$

As for (3.10), similar arguments imply that the second term in the right-hand side of the latter equality is bounded by some positive constant, uniformly with respect to $0 < r < 1$ and $t \in \mathcal{V}$, in such way that the finiteness or not of the r-limit depend only on the first term. Moreover, this latter integrand can be rewritten up to the multiplicative term $|1 - r\lambda(t)|^{-2}$ as

$$\Re(\nu Q_t \mathbb{1})\Re\big(1 - r\lambda(t)\big) - \Im(\nu Q_t \mathbb{1})\Im\big(r\lambda(t)\big).$$

Note that $|\Im(r\lambda(t))| \leq K\Re(1 - r\lambda(t))$ for every $0 < r < 1$ and $t \in \mathcal{V}$ by the sector Assumption 2.3. Also, remark that $\nu Q_t \mathbb{1}$ converges toward 1 as t goes to 0. Then, we deduce easily that

$$\sum_{n\geq 0} \mathbb{P}_\nu(Z_n = 0) = \infty \quad \Longleftrightarrow \quad \lim_{r\uparrow 1} \int_{\mathcal{V}} \Re\left(\frac{1}{1 - r\lambda(t)}\right) dt = \infty,$$

for some (or any) neighbourhood of the origin for which $\lambda(t)$ is well-defined. It is worth noting that ν disappears of the integral condition. Therefore, the necessary and sufficient recurrence criterion follows from Proposition 2.2. This ends the proof of the theorem. □

4. Proofs of Section 2.3

We begin with the proof of Lemma 2.1. Thereafter, we will be able to prove Theorem 2.2 applying a generalized Borel–Cantelli argument and then obtain Corollary 2.1. The proof of Theorem 2.3 requires these three results and is given at the end of this section.

Proof of Lemma 2.1. We follow and make more precise the results of [15,16] connecting the behaviour of the Lévy concentration function with the integral near the origin of the characteristic function. Let $\varphi(t)$ be the characteristic function of the jumps associated with $\{M_n\}_{n\geq 0}$. Since the latter is symmetric, one can find $p>0$ such that $\varphi(t)\geq 0$ on $[-p,p]$. Let us introduce now

$$h(t) := \big(1-|t|\big)^+ \quad \text{and} \quad H(x) := \int e^{ixt} h(t)\,dt = \left(\frac{\sin(x/2)}{x/2}\right)^2.$$

A direct consequence of the Fourier-duality implies, for any $\lambda>0$ and $n\in\{1,2,\dots,\}$, the following crucial identity

$$\frac{\lambda}{2\pi}\int_{-2\pi/\lambda}^{2\pi/\lambda} \varphi(t)^n h(\lambda t/2\pi) e^{-it\xi}\,dt = \int H\big(2\pi(x-\xi)/\lambda\big)\mathbb{P}_{M_n}(dx). \tag{4.1}$$

Setting $\xi=0$ in the equality above, it follows, for any $\lambda>2\pi/p$ and $n,N\in\{1,2,\dots\}$, that

$$\frac{\lambda}{4\pi}\int_{-\pi/\lambda}^{\pi/\lambda}\varphi(t)^n\,dt \leq 2\mathbb{P}(0\leq M_n\leq N\lambda) + \left(2\sum_{k\geq N}\frac{1}{\pi^2k^2}\right)Q(M_n,\lambda). \tag{4.2}$$

To conclude, we need the following well known result. Because its proof does not appear clearly in the literature, a brief proof of this fact is given below.

Lemma 4.1. *There exist universal constants $0<m\leq M$ such that for any $\lambda>2\pi/p$ and $n\geq 1$,*

$$m\lambda\int_{-\pi/\lambda}^{\pi/\lambda}\varphi(t)^n\,dt \leq Q(M_n,\lambda)\leq M\lambda\int_{-\pi/\lambda}^{\pi/\lambda}\varphi(t)^n\,dt. \tag{4.3}$$

Proof of Lemma 4.1. First, the upper bound is nothing but the classical concentration inequality stated in [16], p. 292. Indeed – since $\varphi(t)^n$ is positive on $[-p,p]$ – we can remove the absolute value around it in [16]. To get the lower bound, one can adapt the proof of that in [16], p. 292.

Indeed, the latter allows us to write

$$m\lambda\int_{-\pi/\lambda}^{\pi/\lambda}\big|\varphi(t)\big|^{2n}\,dt \leq Q(M_n,\lambda). \tag{4.4}$$

Again, the absolute value can be removed but the characteristic function is replaced by its square. As a matter of facts, the proof of (4.4) in [16], pp. 292–293, is based on the fundamental relation (4.1). It is applied to the symmetric random variable M_n^s with characteristic function $|\varphi(t)|^{2n}$ and some standard concentrations inequality for $V:=M_n^s+U$ are used, where U is an independent random variable having $h(\lambda t/2\pi)$ for characteristic function (a difference of two *i.i.d.* uniform random variable). More precisely, it is mainly used that $Q(M_n,\lambda)\geq Q(V,\lambda)$ and $|\varphi(t)|^{2n}\geq 0$ on $[-2\pi/\lambda, 2\pi/\lambda]$. Since all the latter properties remains valid without symmetrization, we deduce the lower bound in (4.3) and this ends the proof of the theorem. □

Thus let us choose N such that the series in (4.2) is lower than $1/(8\pi M)$. Then the right-hand side of (4.3), the inequality (4.2) and classical results, stated for instance at the beginning of

[16], imply the inequality of Lemma 2.1 with $\lambda_0 := 2\pi N/p$ and $C := 16\pi M(N+1)$, where M is given by Lemma 4.1. $\square$

Proof of Theorem 2.2. First, it comes from the symmetry of $\{V_n\}_{n\geq 0}$ and its increments, denoted by $\{Z_n\}_{n\geq 0}$, that it is only needed to focus on the events $E_n := \{H_{n+1} = 0, 0 \leq V_n \leq -Z_{n+1}\}$. Besides, denoting by F_V the cumulative distribution function of the jumps of V, conditioning successively with respect to the filtrations generated by $\{H_k : 1 \leq k \leq n+1\}$ and $\{(V_k, Z_k) : 1 \leq k \leq n\}$, and finally applying the usual conditional Borel–Cantelli lemma, we get that

$$\{E_n, \text{for infinitely many } n\} = \left\{\sum_{n\geq 0} \mathbb{1}_{\{H_{n+1}=0\}} F_V(-V_n) \mathbb{1}_{\{V_n \geq 0\}} = \infty\right\} \quad \text{a.s.}$$

Consequently, one can replace the $-Z_n$ in the definition of the E_n by an identically distributed sequence $\{Z_n^{\perp}\}_{n\geq 1}$ independent of $\{V_n\}_{n\geq 1}$ since the resulting events – say $E_n^{\perp}$ – lead exactly to the same criterion. Furthermore, the limit superior of these events belong to the exchangeable σ-algebra associated with an *i.i.d.* sequence of random variables in such way that the Hewitt–Savage zero-one law applies and we only need to prove that $E_n^{\perp}$ occur for infinitely many n with positive probability. To this end, using conditional arguments and Lemma 2.1, for any $n > k \geq 1$,

$$\mathbb{P}\big(E_n^{\perp} \cap E_k^{\perp}\big) \leq \mathbb{P}\big(E_k^{\perp}\big)\mathbb{E}\big[\mathbb{1}_{\{H_{n-k}=0\}}\mathbb{P}_{V_k}\big(0 \leq V_{n-k} \leq Z_{n+1}^{\perp}\big)\big] \leq C\mathbb{P}\big(E_k^{\perp}\big)\mathbb{P}\big(E_{n-k-1}^{\perp}\big).$$

Thereafter, we can conclude with a classical step – see [44], p. 726, for instance. In fact, the inequality above implies the sequence $\sum_{k=1}^{n} \mathbb{1}_{E_k^{\perp}} / \sum_{k=1}^{n} \mathbb{P}(E_k^{\perp})$ is bounded in $\mathbb{L}^2$ and thus equi-integrable. Then we can apply the generalized Fatou-lemma so that

$$\mathbb{E}\left[\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} \mathbb{1}_{E_k^{\perp}}}{\sum_{k=1}^{n} \mathbb{P}(E_k^{\perp})}\right] \geq 1.$$

Therefore, if the sequence of partial sums at the denominator is unbounded, then the series at the numerator is divergent with positive probability. This ends the proof of the series criterion since the reciprocal implication is a straightforward consequence of the standard Borel–Cantelli lemma. Regarding the Fourier-like criterion (2.17), remark that

$$\mathbb{P}(H_{n+1} = 0) = \int_{\mathbb{T}^d} \Re\big(\varphi_H^{n+1}(t)\big)\,dt \quad \text{and}$$

$$\mathbb{P}(0 \leq V_n \leq Z_{n+1}) = \sum_{k=0}^{\infty} \mathcal{T}_V(k) \int_{\mathbb{T}^d} \cos(ks)\varphi_V^n(s)\,ds.$$

Then, multiplying by the geometric terms u^n and r^k respectively and using standard inversion theorems, it follows the series in the criterion is infinite if and only if

$$\lim_{r,u\uparrow 1} \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \Re\left(\frac{\varphi_H(t)}{1 - u\varphi_H(t)\varphi_V(s)}\right)\Phi_V(r,s)\,ds\,dt = \infty.$$

When (H, V) is transient, it follows from the Ornstein–Chung–Fuchs criterion that the u-limit can be removed since $1/(1 - u\varphi_H(t)\varphi_V(s))$ is uniformly integrable on $\mathbb{T}^d \times \mathbb{T}^d$ for $0 < u < 1$. It is then not difficult to drop $\varphi_H(t)$ and integrate around the origin to get the integral criterion. $\square$

Proof of Corollary 2.1. First remark that the original generalized DRRW pass through the origin during either a horizontal or vertical move. Besides, it turns out that $\{(H_n, V_n)\}_{n\geq 0}$ and $\{(H_{n+1}, V_n)\}_{n\geq 0}$ are respectively the skeleton random walks associated with the horizontal-to-vertical and the vertical-to-horizontal changes of direction. Due to Theorem 2.2, the recurrence of the origin follows from the divergence of one of the series in (2.23), each of them corresponding to a walker passing through the origin infinitely often during vertical or horizontal moves, respectively. Thus, it remains to prove that the convergence of both series in (2.23) leads to the transience of the origin. We only consider the first series since the other one can be treated analogously. With the settings in (2.22), we observe that it suffices to show

$$\sum_{n=0}^{\infty} \mathbb{P}(H_{n+1} = 0)\mathbb{P}\Big(0 \leq V_n \leq \max_{1\leq l\leq G_{\mathrm{v}}} \varepsilon A_l\Big) < \infty, \quad \text{where } A_l := \sum_{k=1}^{l} (-1)^{k-1}\tau_k^{\mathrm{v}}. \tag{4.5}$$

Here, all the involved random variables are independent. In particular, applying the standard Borel–Cantelli lemma, we deduce from (4.5) that the origin is not recurrent – for a walker passing through the origin during vertical moves. Since a similar argument holds for horizontal movements, it suffices to check that the divergence of the first series in (2.23) implies (4.5). Finally, this is obvious when $p_{\mathrm{v}} = 0$, otherwise it is a consequence of the following lemma.

Lemma 4.2. *The following estimate holds for all $n \geq 0$*

$$\mathbb{P}\Big(0 \leq V_n \leq \max_{1\leq l\leq G_{\mathrm{v}}} \varepsilon A_l\Big) \leq 2(1 + p_{\mathrm{v}})\mathbb{P}(0 \leq V_n \leq V_{n+1} - V_n).$$

Proof. First, note that for fixed j and any $x \geq 0$,

$$\mathbb{P}\Big(\max_{1\leq l\leq j} -A_l \geq x\Big) \leq \mathbb{P}\Big(\max_{1\leq l\leq j} A_l \geq x\Big).$$

Thus, by conditioning with respect to ε,

$$\mathbb{P}\Big(\max_{1\leq l\leq j} \varepsilon A_l \geq x\Big) \leq \mathbb{P}\Big(\max_{1\leq l\leq j} A_l \geq x\Big).$$

Besides, we remark that $A_{2i+1} \geq A_{2i+2}$ for every $i \geq 0$. Hence, the local maxima of A_l are reached for odd indices. Since A_{2i+1} can be rewritten as $A_{2i+1} = \tau_1^{\mathrm{v}} + \sum_{k=1}^{i}(\tau_{2k+1}^{\mathrm{v}} - \tau_{2k}^{\mathrm{v}})$, a direct application of [42], Chapter 3, Theorem 10, p. 50, for the symmetric increments $\tau_{2k+1}^{\mathrm{v}} - \tau_{2k}^{\mathrm{v}}$ yields

$$\mathbb{P}\Big(\max_{1\leq l\leq 2i+2} A_l \geq x\Big) = \mathbb{P}\Big(\max_{1\leq l\leq 2i+1} A_l \geq x\Big) \leq 2\mathbb{P}(A_{2i+1} \geq x),$$

for every $x \in \mathbb{R}$. The result then follows by collecting bounds and conditioning with respect to the random variables V_n and G_{V}. □

This ends the proof of Corollary 2.1. □

Proof of Theorem 2.3. We shall construct inductively appropriate jump distributions as in the concise and elegant paper [17] in which a probabilistic proof of a result of [47] is given. This result states that a recurrent symmetric random walk on the line may have jump with arbitrary large tails. We also follow the clever path borrowed by the authors in [48]. We stress that we do not require unimodality assumptions making the construction more general but also handier and easier to state.

Step 1. We define inductively $\{(l_k, y_k, p_{k+1})\}_{k\geq 1}$ where l_k and y_k are non-negative integers satisfying $y_{k+1} \geq y_k + l_k$. The couple l_k and y_k, $k \geq 1$, represent some spatial parameters of the distribution explained below in Step 2. whereas the p_k's are non-negative numbers whose sum is less than 1. Each p_k can be thought as conditional probabilities. The following quantities will be fixed throughout all the procedure: $(v_k)_{k\geq 2}$ and $(u_k)_{k\geq 2}$ are two sequences of positive numbers such that for some fixed $\delta > 0$ and $c > 0$ one has for every $k \geq 2$,

$$v_k \underset{k\to\infty}{=} \mathrm{o}(u_k) \quad \text{and} \quad \frac{1}{v_k} < \frac{c}{k^{2+\delta}}. \tag{4.6}$$

Thereafter, we proceed as follows: choose $y_1, l_1 \geq 1$ and, given some fixed $r \in (0, 1)$, choose $0 < p_2 < 1-r$. Knowing the first $k-1$ terms of the sequence $\{(l_k, y_k, p_{k+1})\}_{k\geq 1}$, we may choose (l_k, y_k, p_{k+1}) respecting the following constraints for some positive constants α and β:

1. first choose (l_k, y_k) such that $y_k \geq y_{k-1} + l_{k-1}$ and $l_k \geq 2$ for all k sufficiently large with

   - $$\frac{1}{l_k^2 p_k^2} \ln\left(\frac{1}{r p_k}\right) \leq \frac{1}{v_k}; \tag{4.7}$$
   - $$\sum_{i=1}^{k-1} p_i (y_i + l_i)^2 \leq \alpha p_k (y_k + l_k)^2; \tag{4.8}$$
   - $$\frac{p_k^2 l_k^2 y_k}{(y_k + l_k)^2} \geq u_k; \tag{4.9}$$
   - $$\frac{y_k^2}{p_k (y_k + l_k)^2} \leq \beta; \tag{4.10}$$

2. in second step, choose p_{k+1} such that

   - $$\frac{1}{v_k} \leq \frac{1}{l_k^2 p_k^2} \ln\left(\frac{1}{p_{k+1}}\right) \leq \frac{c}{k^{(2+\delta)}}. \tag{4.11}$$

Actually, it is even possible to choose $y_k = l_1 + \cdots + l_{k-1}$. Note that, together with (4.7), condition (4.11) implies $p_{k+1} \leq r p_k$. It follows that $p_2 + p_3 + \cdots$ is lower than $p_2/(1-r) < 1$

so that one can choose $0 \le p_0 \le 1 - p_2/(1-r)$ arbitrary and find $0 \le p_1 \le 1$ such that

$$q_k := 1 - (p_0 + \cdots + p_k) \xrightarrow[k\to\infty]{} 0. \tag{4.12}$$

Step 2. We shall associate with $\{(l_k, y_k, p_{k+1})\}_{k\ge1}$ a sequence of defective laws $\{\mu_k\}_{k\ge1}$. Those will have $\{1-q_k\}_{k\ge1}$ – all less than 1 – defined above for respective total masses on $\mathbb{Z}$, that is why they are called defective. Then we shall construct coupled and defective random walks $\{W_n^k\}_{n\ge0}$, $k \ge 1$, with respective jumps $\{\mu_k\}_{k\ge1}$, such that the limiting random walk $\{H_n\}_{n\ge0}$ – see below for the precise meaning – obtained by these approximations satisfies

$$\sum_{n=1}^{\infty} \mathbb{P}(H_n = 0)^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \mathbb{P}(H_n = 0)\mathbb{P}(0 \le H_n \le H_{n+1} - H_n) = \infty. \tag{4.13}$$

To this end, adjoin a cemetery $\Delta \notin \mathbb{Z}$. We set $y_0 = 0$, $l_0 = 1$ and $\mathbb{I}_0 = \{0\}$ and for every $k \ge 1$,

$$\mathbb{I}_k := \mathbb{I}_k^+ \sqcup \mathbb{I}_k^- \quad \text{with } \mathbb{I}_k^+ := [y_k, y_k + l_k) \cap \mathbb{Z} \quad \text{and} \quad \mathbb{I}_k^- := -\mathbb{I}_k^+. \tag{4.14}$$

Then consider for every $k \ge 0$ the distribution μ_k on $\mathbb{I}_0 \sqcup \cdots \sqcup \mathbb{I}_k \sqcup \{\Delta\}$, symmetric on $\mathbb{Z}$ and uniform on each $\mathbb{I}_0, \dots, \mathbb{I}_k$ with respective masses $p_0, \dots, p_k$ and $\mu_k(\Delta) = q_k$ as in Figure 3. Note that such sequences converge in distribution to some symmetric probability measure μ on $\mathbb{Z}$. In addition, one can choose $p_0 = 0$ avoiding possibly trivial jumps. For every $k \ge 0$, introduce an *i.i.d.* sequence of random variables $(X_j^k)_{j\ge1}$ distributed as μ_k with the following coupling properties along $k \ge 1$:

$$X_j^k = X_j^{k-1} \quad \text{on } \{X_j^{k-1} \ne \Delta\} \quad \text{and} \quad \mathbb{P}\big(X_j^k \in dx \,|\, X_j^{k-1} = \Delta\big) = \mathbb{P}\big(\xi^k U^k + \big(1-\xi^k\big)\Delta \in dx\big),$$

where ξ^k is distributed as $\mathcal{B}(p_k/q_{k-1})$, U^k is uniform on $\mathbb{I}_k$ and ξ^k and U^k are independent. With these sequences of jumps, associate the so called defective random walks $\{W_n^k\}_{n\ge0}$ – they fall into the cemetery as soon as one of their jumps does – starting from the origin. It follows from the coupling properties that $W_n^r = W_n^k$ for every $r \ge k$ as soon as $W_n^k \ne \Delta$ in such way that we

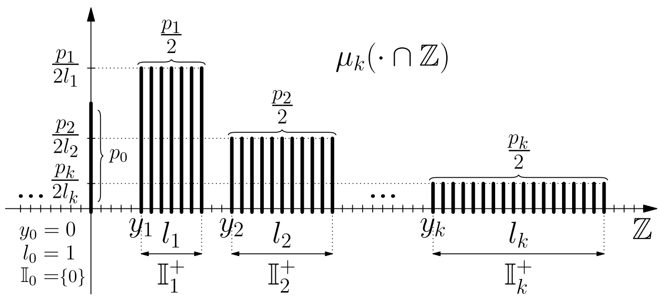

Figure 3. The kth symmetric distribution.

can consider the non-defective almost-sure limit random walk given for every $n \geq 0$ by

$$H_n := \lim_{k\to\infty} W_n^k.$$

Note that the latter has for jump distribution the limit μ of the μ_k and we denote in an obvious meaning by $\{X_n\}_{n\geq 1}$ the corresponding *i.i.d.* jumps. In the sequel, we say for two non-negative sequences u and v that $u_k \preceq v_k$ if there exists $c > 0$ such that $u_k \leq c v_k$ for all k sufficiently large and $u_k \asymp v_k$ whenever $u_k \preceq v_k$ and $v_k \preceq u_k$.

Lemma 4.3. *The following estimates hold*

$$\sqrt{\sum_{n=1}^{\infty} \mathbb{P}\big(W_n^k = 0\big)^2} \preceq \sum_{i=1}^{k} \frac{1}{l_i\, p_i} \sqrt{\ln\Big(\frac{1}{p_{i+1}}\Big)} \tag{4.15}$$

and

$$\sum_{n=1}^{\infty} \mathbb{P}\big(W_n^k = 0\big)\mathbb{P}\big(0 \leq W_n^k \leq X_{n+1}^k\big) \succeq \frac{y_k}{(y_k + l_k)^2} \ln\Big(\frac{1}{p_{k+1}}\Big). \tag{4.16}$$

Proof. We begin with the inequality (4.15). First, we shall prove that

$$\mathbb{P}\big(W_n^k = 0\big) \leq \mathbb{P}\big(W_n^{k-1} = 0\big) + \frac{1}{l_k} \frac{\omega(p_k/(1-q_k))}{\sqrt{n(p_k/(1-q_k))}} (1-q_k)^n, \tag{4.17}$$

where $\omega : (0,1) \longrightarrow (0,\infty)$ is defined by

$$\omega(p) := \min_{u\in(0,1)} \left(\sqrt{\frac{2}{u}} + \sqrt{\frac{1}{2ep}} \frac{1}{1-u}\right). \tag{4.18}$$

To this end, first note that $\mathbb{P}(W_n^k = 0) = \mathbb{P}(W_n^{k-1} = 0) + \mathbb{P}(W_n^k = 0, W_n^{k-1} = \Delta)$. Let $\mathbb{Q}_n^k$ be the conditional probability given $\{X_1^k, \ldots, X_n^k \neq \Delta\}$ – an event of $\mathbb{P}$-measure equal to $(1-q_k)^n$. In order to lighten the notations, we set $\mathbb{Q} := \mathbb{Q}_n^k$ in the following. We can write

$$\mathbb{P}\big(W_n^k = 0, W_n^{k-1} = \Delta\big) = (1-q_k)^n \sum_{m=1}^{n} \mathbb{Q}\big(W_n^k = 0 | Z_n^k = m\big)\mathbb{Q}\big(Z_n^k = m\big), \tag{4.19}$$

where $Z_n^k = \operatorname{card}\{1 \leq i \leq n : X_i^k \in \mathbb{I}_k\}$. Given one of the $\binom{n}{m}$ partitions $J \sqcup I$ of $[1,n] \cap \mathbb{Z}$ with $\operatorname{card}(J) = m$, we can define

$$F_J := \left(\bigcap_{j\in J} \{X_j^k \in \mathbb{I}_k\}\right) \cap \left(\bigcap_{i\in I} \{X_i^k \notin \mathbb{I}_k\}\right) \subset \{Z_n^k = m\}.$$

Besides, all these events form a partition of $\{Z_n^k = m\}$ itself. Under $\mathbb{Q}(\star | F_J)$ the random variables X_j^k are independent and, in addition, for every $j \in J$, X_j^k can be written as $\theta_j Y_j$ where $\{Y_j\}_{j\in J}$

and $\{\theta_j\}_{j\in J}$ are two independent families of *i.i.d.* random variables uniformly distributed on $\mathbb{I}_k^+$ and $\{\pm 1\}$, respectively. Also, observe that under $\mathbb{Q}$ the random variable Z_n^k is a binomial of parameters n and $p_k/(1-q_k)$. Finally, lemma 4.3 follows from the two technical lemmas below.

Lemma 4.4. *Let $\{Y_j\}_{j\geq 1}$ be a sequence of independent random variables distributed uniformly on integers intervals of length $l \geq 2$. Then for every $m \geq 1$ one has*

$$\sup_{x\in\mathbb{Z}} \mathbb{P}(Y_1 + \cdots + Y_m = x) \leq \frac{1}{l}\sqrt{\frac{2}{m}}. \tag{4.20}$$

Lemma 4.5. *Let Z have a binomial distribution with parameters n and p. Then*

$$\mathbb{E}\left[\frac{\mathbb{1}_{\{Z\geq 1\}}}{\sqrt{Z}}\right] \leq \frac{\omega(p)}{\sqrt{2np}}, \tag{4.21}$$

where $\omega : (0, 1) \longrightarrow (0, \infty)$ is defined in (4.18).

Proof of Lemmas 4.4 and 4.5. The proof of Lemma 4.4 in [48], pp. 697–698, contains some misunderstandings. To overcome these difficulties, we refer to [37]. This probabilistic estimate relies on combinatorics considerations, the so called polynomial coefficients. To go further, we allude for instance to [13], Section 1.16, pp. 77–78.

For Lemma 4.5, the inequality follows from

$$\mathbb{E}\left[\frac{\mathbb{1}_{\{Z\geq 1\}}}{\sqrt{Z}}\right] \leq \frac{1}{\sqrt{npu}}\left[1 + \sqrt{npu}e^{-2n(1-u)^2p^2}\right],$$

obtained with a truncation argument along $\{Z \geq npu\}$ for any $u \in (0, 1)$ and the Hoeffding's inequality. Since $x\exp(-x^2) \leq 1/\sqrt{2e}$ and $u \in (0, 1)$ is arbitrary the result follows. □

Thereafter, after conditioning with respect to the θ_j, applying Lemma 4.4 it comes for any J,

$$\mathbb{Q}\big(W_n^k = 0 | F_J\big) \leq \frac{1}{l_k}\sqrt{\frac{2}{m}}, \quad \text{and thus} \quad \mathbb{Q}\big(W_n^k = 0 | Z_n^k = m\big) \leq \frac{1}{l_k}\sqrt{\frac{2}{m}}.$$

Lemma 4.5 and (4.19) then imply (4.17). Since the p_k and the q_k go to zero (recall $p_{k+1} \leq rp_k$) and simple estimations lead to $\omega(p) \sim (2ep)^{-1/2}$ as p goes to 0, we get for all k large enough,

$$\mathbb{P}\big(W_n^k = 0\big) \leq \mathbb{P}\big(W_n^{k-1} = 0\big) + \frac{1}{l_k p_k}\frac{(1-q_k)^n}{\sqrt{n}}. \tag{4.22}$$

Therefore, the upper bound (4.15) follows by induction from the Minkowski inequality, the series expansion of $\ln(1+x)$ near the origin and the inequality $q_k \geq p_{k+1}$.

It remains to prove the lower bound (4.16). In the sequel, we denote by $(H_n^k)_{n\geq 1}$ the random walk on $\mathbb{Z}$ whose common distribution of the *i.i.d.* jumps, denoted by $\{Y_n^k\}_{n\geq 0}$, is given by the conditional law $\mu_k(\star|\mathbb{Z})$. The corresponding characteristic function is denoted by φ_k. It is worth noting that, under the probability measure $\mathbb{Q}$, W_n^k is distributed as H_n^k and X_1^k is distributed as Y_1^k. If φ stands for the characteristic function of μ, then

$$\|\varphi - \varphi_k\|_\infty \le 2q_k \xrightarrow[k\to\infty]{} 0. \tag{4.23}$$

Since φ and the φ_k are continuous and non-negative with $\varphi(0) = 1$, there exists $p > 0$ such that $\varphi_k(t) \ge 0$ for k large enough and every $t \in [-p, p]$. Hence, Lemma 2.1 provides positive universal constants C, L such that for all k sufficiently large and every $\lambda \ge L/p$ and $n \ge 1$,

$$\mathbb{P}\big(0 \le H_n^k \le \lambda\big) \ge \frac{Q(H_n^k, \lambda)}{C}.$$

We will apply [8], Theorem 1.1, but before we shall give a lower and an upper estimate of the variance of X_1^k given $X_1^k \in \mathbb{Z}$, denoted below by σ_k^2. Using (4.8) and the fact that $1/2 \le 1 - q_k \le 1$ for all k sufficiently large, it is not difficult to see that for such k,

$$\frac{1}{3} p_k (y_k + l_k - 1)^2 \le \sigma_k^2 \le 2(\alpha + 1) p_k (y_k + l_k)^2, \tag{4.24}$$

where α is given by (4.8). Therefore, looking at the event $\{X_n^k \ge y_k\}$, [8], Theorem 1.1, and (4.10) imply there exists a positive constant γ such that for every $n \ge 1$ and k sufficiently large

$$\mathbb{P}\big(0 \le H_n^k \le Y_{n+1}^k\big) \ge \frac{\gamma p_k y_k}{\sqrt{n p_k (y_k + l_k)^2}}. \tag{4.25}$$

Since φ is the characteristic function of an aperiodic symmetric distribution, we deduce from (4.23) that there exist $\delta \in (0, 1)$ and a neighbourhood $\mathcal{V}$ of the origin for which $|\varphi_k(t)| \le \delta$ for every $t \in \mathbb{T}^1 \setminus \mathcal{V}$ and k large enough. Once again Lemma 4.1 and the lower bound in [8] together with (4.10) guarantees the existence of a positive constant θ such that for all k large enough,

$$\forall n \ge 1, \quad \mathbb{P}\big(H_n^k = 0\big) \ge \frac{\theta}{\sqrt{n p_k (y_k + l_k)^2}} - \theta \delta^n. \tag{4.26}$$

We remark that the two latter lower bounds hold for the defective random walks W_n^k by adding the multiplicative term $(1 - q_k)^{n+1}$. Finally, one deduce (4.16) noting that $q_k \le p_{k+1}/(1 - r)$ from remarks above (4.12). □

Therefore, letting $k \longrightarrow \infty$ in (4.15) and (4.16) and using conditions (4.6) and (4.11), we obtain (4.13) by the monotone convergence theorem.

From now, one choose such a distribution μ with $p_0 = 0$ and consider its positive part ν normalized to be a probability. It is nothing but the distribution of $|X|$ when X is distributed as μ. Then one can see with the help of Corollary 2.1 that the latter is the waiting times distribution announced in Theorem 2.3 but only for some non-backtracking DRRW. Indeed, let $\{V_n\}_{n\ge 0}$ be an independent copy of H chosen above and set for every $n \ge 0$,

$$Z_{2n} := (H_n, V_n) \quad \text{and} \quad Z_{2n+1} := (H_{n+1}, V_n). \tag{4.27}$$

It turns out that Z is the skeleton of a non-backtracking DRRW, having ν for waiting time distribution, starting from the origin, and moving initially toward the east or the west with probability $1/2$ respectively. The transience of the MRW skeleton follows from the convergence of the first series in (4.13) whereas the recurrence of the non-backtracking DRRW is due to the divergence of the second one. For original DRRW, we still have to work.

Step 3. Indeed, consider now a DRRW. The skeleton – say $\widehat{Z}$ – in the conjecture of [38] refer to the times when the walker come back its original direction. This skeleton can not be described from a couple of independent symmetric random walks as previously. However, looking at the times when the walker moves orthogonally gives an another skeleton – say $\widetilde{Z}$ – naturally associated with DRRW and always embedded into our "true" skeleton Z. As for (4.27) this skeleton can be viewed as the "true" skeleton of a non-backtracking DRRW. Here, the waiting times are distributed as

$$\varepsilon \sum_{k=1}^{G} (-1)^{k-1} \tau_k, \tag{4.28}$$

where G is a geometric random variable with parameter $2/3$, ε is a symmetric Rademacher random variable and $\{\tau_k\}_{k\geq 1}$ is a sequence of *i.i.d.* random variables distributed as ν defined at the end of the latter step – all of them being independent of each other. Introducing the random walk $\{\widetilde{H}_n\}_{n\geq 0}$ whose jumps are distributed as (4.28) and an independent copy $\widetilde{V}$ one can write

$$\widetilde{Z}_{2n} := (\widetilde{H}_n, \widetilde{V}_n) \quad \text{and} \quad \widetilde{Z}_{2n+1} := (\widetilde{H}_{n+1}, \widetilde{V}_n).$$

We refer to see Figure 4 where it is represented the skeletons Z, $\widetilde{Z}$ and $\widehat{Z}$ of some DRRW. We need the following result whose proof is postponed to the end.

Lemma 4.6. *The random walk $\{\widetilde{H}_n\}_{n\geq 0}$ satisfies the same estimates as $\{H_n\}_{n\geq 0}$ in (4.13). In particular, the associated MRW $\widetilde{Z}$ given in (4) is transient whereas the corresponding non-backtracking DRRW is recurrent.*

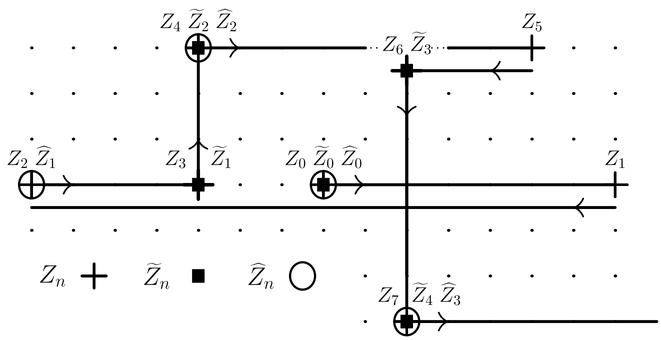

Figure 4. The three distinct skeleton of a DRRW.

Assume now Lemma 4.6. It is clear that the recurrence of the original DRRW is a consequence of the recurrence of the non-backtracking one. Furthermore, we can show that the transience of $\widetilde{Z}$ implies also the transience of Z and thus that of $\widehat{Z}$. To this end, by contraposition, if $\widehat{Z}$ is recurrent introduce $\{\sigma_n\}_{n\geq 0}$ the successive return times of Z to 0 and consider

$$A_{n+1} := \{Z_{\sigma_n+1} \perp Z_{\sigma_n-1}\}, \quad n \geq 0, \tag{4.29}$$

that is the event "Z breaks orthogonally at time σ_n". Since $A_n \in \mathcal{F}_{\sigma_n}$ and $\mathbb{P}(A_{n+1}|\mathcal{F}_{\sigma_n}) = 2/3$, with $\mathcal{F}_{\sigma_n}$ the natural σ-field up to the time σ_n, applying the conditional Borel–Cantelli lemma gives that A_n holds for infinitely many n and we deduce easily that $\widetilde{Z}$ is recurrent. This completes the proof of Theorem 2.3 for the original DRRW excepted Lemma 4.6. □

Proof of Lemma 4.6. To begin with, the claims of the lemma concerning recurrence or transience are direct consequences of Corollary 2.1 and (4.13). To show the latter estimates, look carefully at the proof of Lemma 4.3 above, especially when the defective jumps $\{X_n^k\}_{n\geq 1}$ are replaced by *i.i.d.* random variables distributed as

$$\widetilde{X}_n^k = \varepsilon_n \sum_{i=1}^{G_n} (-1)^{i-1} \tau_{n,i}^k, \tag{4.30}$$

where $\{\tau_{n,i}^k\}_{n,i\geq 1}$ is an *i.i.d.* array of random variables having $\nu_k(dt) := \mu_k(dt|\mathbb{N} \cup \{\Delta\})$ for distribution, $\{G_n\}_{n\geq 1}$ is a sequence of *i.i.d.* geometric random variables with parameter 2/3 and $\{\varepsilon_n\}_{n\geq 1}$ is a sequence of *i.i.d.* symmetric Rademacher random variables – all of them being independent of each other. We denote by $\{\widetilde{W}_n^k\}_{n\geq 0}$ the coupled defective random walks associated with the jumps $\{\widetilde{X}_j^k\}_{j\geq 0}$, $k \geq 0$. Conditioning with respect to $G_1, \dots, G_n$, the upper-bound (4.22) still holds since

$$\frac{(1-q_k)^{G_1+\cdots+G_n}}{\sqrt{G_1+\cdots+G_n}} \leq \frac{(1-q_k)^n}{\sqrt{n}} \quad a.s.$$

It turns out that the limit random walk $\{\widetilde{H}_n\}_{n\geq 0}$ satisfies the left-hand side of (4.13). It remains to show the lower bound. Let us remark that the variance of the jump distribution in (4.30) still satisfies the former lower and upper bounds in (4.24) with possibly different universal constants. The lower bound is straightforward since $G_n \geq 1$ for all $n \geq 1$ whereas for the upper bound we distinguish between even G_n and odd G_n and use basic conditional arguments. Furthermore, similar arguments apply to the characteristic functions since

$$\Phi_k(t) = \sum_{i=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{2i} |\phi_k(t)|^{2i} \varphi_k(t) + \sum_{i=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{2i-1} |\phi_k(t)|^{2i},$$

where φ_k, ϕ_k and Φ_k are respectively the characteristic functions of $\mu_k(dx|\mathbb{Z})$, $\nu_k(dt|\mathbb{N})$ and the random variable (4.30), given it does not fall into the cemetery point. It follows that the lower bound (4.25) is true for this new distribution (we suppose $G_{n+1} = 1$ into the jump $n+1$) but also the lower bound (4.26). Obviously, the constant are possibly different. To conclude, it suffices to take the conditional expectation with respect to $G_1, \dots, G_{n+1}$ in the multiplicative additional terms and observe by independence and the Jensen inequality that

$$\mathbb{E}\big[(1-q_k)^{G_1+\cdots+G_{n+1}}\big] \geq (1-q_k)^{3(n+1)/2}.$$

Since the additional factor $3/2$ does not change the nature of the series, we deduce that the second series of (4.13) is also infinite for $\{\widetilde{H}_n\}_{n\geq 0}$, ending the proof of the lemma. □

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