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Equivalence of some subcritical properties in continuum percolation

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We consider the Boolean model on $\mathbb{R}^d$. We prove some equivalences between subcritical percolation properties. Let us introduce some notations to state one of these equivalences. Let C denote the connected component of the origin in the Boolean model. Let $|C|$ denotes its volume. Let ℓ denote the maximal length of a chain of random balls from the origin. Under optimal integrability conditions on the radii, we prove that $\mathbb{E}(|C|)$ is finite if and only if there exists $A, B > 0$ such that $\mathbb{P}(\ell \geq n) \leq Ae^{-Bn}$ for all $n \geq 1$.

Keywords: Boolean model; continuum percolation; critical point

1. Introduction

The Boolean model. The Boolean model is defined as follows. At each point of a homogeneous Poisson point process on the Euclidean space $\mathbb{R}^d$, we center a ball of random radius. We assume that the radii of the balls are independent, identically distributed and independent of the point process. The Boolean model is the union of the balls. There are three parameters:

- An integer $d \geq 2$. This is the dimension of the ambient space $\mathbb{R}^d$.
- A real number $\lambda > 0$. The intensity measure of the Poisson point process of centers is $\lambda |\cdot|$ where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^d$.
- A probability measure ν on $(0, +\infty)$. This is the common distribution of the radii. We will also consider a random variable R whose distribution is ν.

We will denote the Boolean model by $\Sigma(\lambda, \nu, d)$ or Σ. We will also say that Σ is the Boolean model driven by the measure $\lambda\nu$.

More precisely, the Boolean model is defined as follows. Let ξ be a Poisson point process on $\mathbb{R}^d \times (0, +\infty)$ with intensity measure $\lambda|\cdot| \otimes \nu = |\cdot| \otimes \lambda\nu$. Set

$$\Sigma(\lambda, \nu, d) = \bigcup_{(c,r)\in\xi} B(c, r),$$

where $B(c, r)$ denotes the open Euclidean ball of $\mathbb{R}^d$ with center c and radius r. We refer to the book by Meester and Roy [14] for background on the Boolean model, and to the book by Schneider and Weil [18] and the book by Last and Penrose [13] for background on Poisson processes. We also denote by $S(c, r)$ the Euclidean sphere of $\mathbb{R}^d$ with center c and radius r. We write $S(r)$ when $c = 0$.

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Percolation in the Boolean model. If A and B are two subsets of $\mathbb{R}^d$, we set

$$\{A \overset{\Sigma}{\longleftrightarrow} B\} = \{\text{There exists a path in } \Sigma \text{ from } A \text{ to } B\}$$

and

$$\{0 \overset{\Sigma}{\longleftrightarrow} \infty\} = \{\text{The connected component of } \Sigma \text{ that contains the origin is unbounded}\}.$$

Let us consider the set

$$\Lambda = \left\{\lambda > 0 : \mathbb{P}(0 \overset{\Sigma}{\longleftrightarrow} \infty) = 0\right\},$$

and the associated critical threshold

$$\lambda_c = \sup \Lambda.$$

When $\lambda \in \Lambda$, all the connected components of Σ are almost surely bounded. We say that Σ does not percolate. When $\lambda \notin \Lambda$, with probability one, one of the connected components of Σ is unbounded. We say that Σ percolates.

Assume in this paragraph that $\mathbb{E}(R^d)$ is infinite. Then, for any positive λ, with probability one, $\Sigma = \mathbb{R}^d$. This can be shown easily by computing, for any $r > 0$, the probability that $B(0, r)$ is covered by one random ball of Σ. See, for example, Theorem 16.4 in [13]. In this case, the model is therefore trivial from the percolation point of view: $\Lambda = \varnothing$ and $\lambda_c = 0$. As a consequence, in what follows, we will always assume that $\mathbb{E}(R^d)$ is finite.

When $\mathbb{E}(R^d) < \infty$, the critical parameter λ_c is not degenerated, *i.e.*, $0 < \lambda_c < +\infty$ (see [9] and the remark below Theorem 3.3 in [14]), thus this model of continuum percolation exhibits a phase transition.

Background on percolation in $\mathbb{Z}^d$. Percolation in the Boolean model can be seen as a continuous analog of percolation in $\mathbb{Z}^d$, a model that was first defined and studied by Broadbent and Hammersley in the 50s, and has been widely studied in the last 60 years. Let us define it briefly. We consider the graph with vertices $\mathbb{Z}^d$ and we put an edge between two vertices if they are at Euclidean distance 1. We declare each edge to be open with probability p, for some parameter $p \in [0, 1]$, independently of each other. We consider the random graph G with vertices $\mathbb{Z}^d$ and with edges the set of the open edges. We are interested in connectivity properties of G. As previously, if A and B are two subsets of $\mathbb{R}^d$, we set

$$\{A \overset{G}{\longleftrightarrow} B\} = \{\text{There exists a path in } G \text{ from } A \text{ to } B\}$$

and

$$\{0 \overset{G}{\longleftrightarrow} \infty\} = \{\text{The connected component of } G \text{ that contains the origin is unbounded}\}.$$

We refer to the book by Grimmett [11] for background on percolation in $\mathbb{Z}^d$. This model exhibits a phase transition: the critical parameter

$$p_c = \sup\left\{p > 0 : \mathbb{P}(0 \overset{G}{\longleftrightarrow} \infty) = 0\right\}$$

is not degenerated ($p_c \in (0,1)$). When $p < p_c$, all the connected components of G are bounded (we say that G does not percolate), whereas when $p > p_c$ there exists a.s. a unique connected component of G that is not bounded (we say that G percolates). Obviously, λ_c is the analog of p_c in the continuous setting.

Other definitions of critical parameters are studied in the context of percolation in $\mathbb{Z}^d$. Let $\mathcal{C}$ be the connected component of the origin 0 in G, and $|\mathcal{C}|$ be the number of vertices in $\mathcal{C}$. Let $\Lambda(n) = [-n,n]^d$ and let $\partial\Lambda(n)$ be its boundary. We define

$$\widehat{p_c} = \sup\left\{p > 0 : \mathbb{P}\left(\Lambda(n) \overset{G}{\longleftrightarrow} \partial\Lambda(2n)\right) \to 0 \text{ as } n \to 0\right\}.$$

The definition of $\widehat{p_c}$ involves box-crossing probabilities. When $p < \widehat{p_c}$, it is possible to initialize renormalization arguments. We also define

$$\widetilde{p_c} = \sup\left\{p > 0 : \mathbb{E}\left(|\mathcal{C}|\right) < \infty\right\}.$$

It is well known that $p_c = \widetilde{p_c} = \widehat{p_c}$. This is the consequence of the following fundamental property:

$$\text{for all } p < p_c, \text{ there exist constants } A, B > 0 \text{ such that } \mathbb{P}\left(0 \overset{G}{\longleftrightarrow} \partial\Lambda(n)\right) \le A e^{-Bn}. \tag{1}$$

These results are known as a *sharp threshold* property, and have first been proved independently by Menshikov [15] and by Aizenman–Barsky [3]. Both proofs can be found in Grimmett's book [11], see Theorem 5.4. Very recently, Duminil-Copin–Tassion [7,8] and Duminil-Copin–Raoufi–Tassion [6] gave two new proofs of the sharp threshold property. Their methods apply to a wide class of models, as we will see in a few lines.

In fact the property $p < p_c$ is equivalent to the exponential decay of several probabilities. Let us denote by $|r|$ the number of edges in a path r, and define

$$\begin{aligned}\mathcal{L} &= \sup\left\{|r| : r \text{ starts at 0 and is an open s.a. path}\right\}\\ &= \sup\left\{|r| : r \text{ is a s.a. path starting at 0 and } r \subset \mathcal{C}\right\}.\end{aligned}$$

Let $\mathcal{D}$ be the diameter of $\mathcal{C}$, *i.e.*, $\mathcal{D} = \sup\{\|x-y\| : x, y \in \mathcal{C}\}$ where $\|\cdot\|$ denotes the Euclidean norm. Then it is known that the following properties are equivalent:

(i) $p < p_c$;
(ii) there exist constants $A, B > 0$ such that $\mathbb{P}(0 \overset{G}{\longleftrightarrow} \partial\Lambda(n)) \le Ae^{-Bn}$;
(iii) there exist constants $A, B > 0$ such that $\mathbb{P}(|\mathcal{C}| \ge n) \le Ae^{-Bn}$;
(iv) there exist constants $A, B > 0$ such that $\mathbb{P}(\mathcal{L} \ge n) \le Ae^{-Bn}$;
(v) there exist constants $A, B > 0$ such that $\mathbb{P}(\mathcal{D} \ge n) \le Ae^{-Bn}$;
(vi) $\mathbb{E}[|\mathcal{C}|] < \infty$.

The implication (i) $\Rightarrow$ (ii) was stated in (1). The proof of (i) $\Rightarrow$ (iii) is given in Grimmett's book [11] (see Theorem 6.75). Since $\mathcal{L} \le |\mathcal{C}|$ and $\mathcal{D} \le |\mathcal{C}|$, the implications (iii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v) are trivial, thus (i) implies the exponential decays described in (ii), (iii), (iv) and (v). Conversely, properties (ii), (iii), (iv) and (v) imply (vi) easily. Finally, Kesten proved in [12] (see Corollary 5.1) that (vi) implies $p < \widetilde{p_c}$, thus (vi) $\Rightarrow$ (i).

It worth noticing that the proof of (1), *that is*, (i) $\Rightarrow$ (ii), is not easy, but it is much easier to prove (vi) $\Rightarrow$ (ii), *that is*, the following property:

$$\text{if } \mathbb{E}\big(|\mathcal{C}|\big) < \infty, \text{ there exist constants } A, B > 0 \text{ such that } \mathbb{P}\big(0 \overset{G}{\longleftrightarrow} \partial\Lambda(n)\big) \leq Ae^{-Bn}. \tag{2}$$

This result has been proved by Hammersley in 1957. A modern version of his proof, relying mainly on BK inequality, is given by Grimmett in [11] (see Theorem 6.1).

Subcritical properties in continuum percolation. As in the discrete setting, several subcritical properties are of interest in continuum percolation. We denote by C the connected component of Σ that contains the origin and by $|C|$ its volume. By analogy with the discrete setting, we can define the critical thresholds

$$\widehat{\lambda}_c = \sup\big\{\lambda > 0 : \mathbb{P}\big(S(r) \overset{\Sigma}{\longleftrightarrow} S(2r)\big) \to 0 \text{ as } r \to \infty\big\},$$
$$\widetilde{\lambda}_c = \sup\big\{\lambda > 0 : \mathbb{E}\big(|C|\big) < \infty\big\}.$$

From

$$\mathbb{P}(0 \overset{\Sigma}{\longleftrightarrow} \infty) = \lim_{r\to\infty} \mathbb{P}\big(0 \overset{\Sigma}{\longleftrightarrow} S(2r)\big)$$

we get

$$\widehat{\Lambda} := \big\{\lambda > 0 : \mathbb{P}\big(S(r) \overset{\Sigma}{\longleftrightarrow} S(2r)\big) \to 0 \text{ as } r \to \infty\big\} \subset \Lambda$$

and thus

$$\widehat{\lambda}_c \leq \lambda_c.$$

The following result is implicit in [9], and a proof is given in the Appendix of [10] (Theorem 11): if $\mathbb{E}(R^d)$ is finite, then $\widehat{\Lambda}$ is open and non-empty. In particular, $\widehat{\lambda}_c$ and λ_c both belong to $(0, +\infty)$.

If the radii are bounded, then $\lambda_c = \widehat{\lambda}_c = \widetilde{\lambda}_c$: the phase transition is sharp as in the discrete setting, and it is linked to the following property, which is the analog of (1):

$$\text{if } \lambda < \lambda_c, \text{ there exist constants } A, B > 0 \text{ such that } \mathbb{P}\big(0 \overset{\Sigma}{\longleftrightarrow} S(r)\big) \leq Ae^{-Br}. \tag{3}$$

The first proof of the equality $\lambda_c = \widehat{\lambda}_c$ relied on the analogous result in the discrete setting. We refer to [14] for the proof (see Theorem 3.5) and references. Ziesche gives in [20] a short proof of the equality $\lambda_c = \widehat{\lambda}_c$ for bounded radii, using the technics developed by Duminil-Copin–Tassion in [7,8].

If the radii are unbounded, the exponential decay (3) cannot be true in general. Indeed,

$$\begin{aligned}\mathbb{P}\big(0 \overset{\Sigma}{\longleftrightarrow} S(r)\big) &\geq \mathbb{P}\big(\exists (z,s) \in \xi : 0 \in B(z,s) \text{ and } B(z,s) \cap S(r) \neq \varnothing\big) \\ &\geq \mathbb{P}\big(\exists (z,s) \in \xi : s \geq \|z\| + r\big) = 1 - e^{-\kappa(r)} \underset{r\to\infty}{\sim} \kappa(r),\end{aligned}$$

where

$$\kappa(r) = \mathbb{E}\big(\mathrm{card}\{(z,s) \in \xi : s \geq \|z\| + r\}\big) = \lambda \mathbb{E}\big(\big|B(R-r)\big|\mathbb{1}_{R\geq r}\big) \geq \lambda v_d r^d \mathbb{P}(R \geq 2r).$$

Here v_d denotes the volume of the unit ball in $\mathbb{R}^d$. We deduce from this lower bound that if the tail distribution of R does not decay exponentially fast, then (3) cannot hold.

However, this does not mean that the critical parameters λ_c, $\widehat{\lambda}_c$ and $\widetilde{\lambda}_c$ are not equal. Some results have been obtained recently in this direction. Ahlbergh, Tassion and Teixeira gave in [1] a very complete picture of percolation in the two dimensional Boolean model. In particular, they established that $\lambda_c = \widehat{\lambda}_c$ for the two dimensional Boolean model under a minimal integrability assumption ($\mathbb{E}(R^2) < \infty$), and a byproduct of their work implies that $\widetilde{\lambda}_c = \lambda_c$ if $\mathbb{E}(R^{4+\varepsilon}) < \infty$ for some positive ε. See Theorems 1.1 and 1.3 in [1]. We also refer to [2] by Ahlbergh, Tassion and Teixeira and [17] by Penrose for further results about percolation in the complement of the Boolean model. Even more recently, Duminil-Copin, Raoufi and Tassion developed new methods to prove sharp threshold properties in a wide class of models via decision trees, see for instance [5,6] for applications to other models. Concerning the Boolean model, they proved in [4] the following result: if $\mathbb{E}(R^{5d-3}) < \infty$, then $\lambda_c = \widehat{\lambda}_c$. They also proved that (3) holds as soon as $\mathbb{P}(R \geq r)$ decays exponentially fast with r, and they describe the decay of $\mathbb{P}(0 \stackrel{\Sigma}{\longleftrightarrow} S(r))$ when the tail of the distribution of R is heavy but nice. They obtain corresponding results about percolation in the complement of the Boolean model.

Our approach. We aim at understanding the properties of C in the subcritical regime, with as few moment conditions on R as possible. For that purpose, we introduce new quantities describing C. We denote by D the diameter of C. We denote by $\#C$ the number of random balls contained in C. In other words, $\#C$ is the following cardinality:

$$\#C = \mathrm{card}\big(\{(c,r) \in \xi : c \in C\}\big).$$

A chain of length $n \geq 1$ is a sequence $((c_1,r_1),\dots,(c_n,r_n))$ of distinct points of ξ such that

$$\forall i \in \{2,\dots,n\}, \quad B(c_{i-1},r_{i-1}) \cap B(c_i,r_i) \neq \varnothing.$$

We say that the chain starts in $A \subset \mathbb{R}^d$ if $B(c_1,r_1)$ touches A. We say that the chain stops in $A \subset \mathbb{R}^d$ if $B(c_n,r_n)$ touches A. We denote by ℓ the largest length of a chain starting in $B(0,1)$. More precisely,

$$\ell = \sup\big\{n \geq 0 : \exists x_1,\dots,x_n \in \xi \text{ s.t. } (x_1,\dots,x_n) \text{ is a chain starting in } B(0,1)\big\}. \tag{4}$$

Obviously, D is the analog of $\mathcal{D}$ in the discrete setting, and both $|C|$ and $\#C$ can bee seen as analogs of $|\mathcal{C}|$ in the discrete setting. The variable ℓ is one possible analog of $\mathcal{L}$. The study of the quantity ℓ is less standard, but in our context it plays a crucial role, since we will prove that the tail distribution of ℓ is the one that decays exponentially fast in the subcritical regime with minimal integrability hypotheses on R. Note that, in contrast to what happens for similar quantities in the discrete setting, the decay of the tail distributions of $|C|$, $\#C$ or D is not always exponential in the subcritical regime. See Corollary 3 below.

We investigate in this paper the connection between different percolation properties, such as the behavior of $\mathbb{P}(S(r) \overset{\Sigma}{\longleftrightarrow} S(2r))$ as r goes to ∞, the integrability properties of $|C|$, $\#C$ and D, and the tail of the distribution of ℓ.

Main results. We state first the following result.

Theorem 1. *Let $s > 0$. The following statements are equivalent*:

- *For small enough λ, $\mathbb{E}(|C|^{s/d})$ is finite.*
- *For small enough λ, $\mathbb{E}(\#C^{s/d})$ is finite.*
- *For small enough λ, $\mathbb{E}(D^s)$ is finite.*
- *$\mathbb{E}(R^{d+s})$ is finite.*

Moreover, if $\mathbb{E}(R^{d+s})$ is finite, then $\mathbb{E}(|C|^{s/d})$, $\mathbb{E}(\#C^{s/d})$ and $\mathbb{E}(D^s)$ are finite as soon as $\lambda \in \widehat{\Lambda}$.

In particular, if $\mathbb{E}(R^{2d}) = \infty$ then $\mathbb{E}(|C|) = \mathbb{E}(\#C) = \mathbb{E}(D^d) = \infty$ for any λ. Since we are interested in the finiteness of those expectations, we will naturally suppose that $\mathbb{E}(R^{2d}) < \infty$ in the following theorem, which is the main result of this article.

Theorem 2. *Assume that $\mathbb{E}(R^{2d})$ is finite. The following statements are equivalent*:

1. $\mathbb{P}(S(r) \overset{\Sigma}{\longleftrightarrow} S(2r)) \to 0$ *as $r \to \infty$.*
2. *There exists $A, B > 0$ such that, for all $n \geq 1$,*

$$\mathbb{P}(\ell \geq n) \leq A \exp(-Bn). \tag{5}$$

3. *$\mathbb{E}(D^d)$ is finite.*
4. *$\mathbb{E}(|C|)$ is finite.*
5. *$\mathbb{E}(\#C)$ is finite.*

The main contribution of our work is the proof of $4 \Rightarrow 2$, *that is*, the fact that $\mathbb{E}(|C|) < \infty$ implies the exponential decay of ℓ, and this will be the core of the paper (see Section 3.2). Note that (5) does not imply that the decay of the tail of $\#C$ is exponential. One can for example prove the following result, which is a simple consequence of Theorems 1 and 2.

Corollary 3. *Let $s > d$. Assume $\mathbb{E}(R^{2d}) < \infty$ and $\mathbb{E}(R^{d+s}) = \infty$. Let $\lambda \in \widehat{\Lambda} = (0, \widehat{\lambda}_c)$. Then there exists A, B such that*

$$\mathbb{P}(\ell \geq n) \leq A \exp(-Bn).$$

However,

$$\mathbb{E}\big(D^s\big) = \mathbb{E}\big(|C|^{s/d}\big) = \mathbb{E}\big(\#C^{s/d}\big) = \infty.$$

Combining Theorems 1 and 2 one also gets the following corollary.

Corollary 4. *Let $s \geq d$. Assume that $\mathbb{E}(R^{d+s})$ is finite. The following statements are equivalent*:

1. $\mathbb{P}(S(r) \stackrel{\Sigma}{\longleftrightarrow} S(2r)) \to 0$ *as* $r \to \infty$.
2. *There exists* $A, B > 0$ *such that, for all* $n \geq 1$, $\mathbb{P}(\ell \geq n) \leq A\exp(-Bn)$.
3. $\mathbb{E}(D^s)$ *is finite.*
4. $\mathbb{E}(|C|^{s/d})$ *is finite.*
5. $\mathbb{E}(\#C^{s/d})$ *is finite.*

The above results also yield equalities between some percolation thresholds. Such equalities were already proven in the case where R is bounded. We refer to Sections 3.4 and 3.5 of [14] and references therein for such results.

The proof of Theorem 1 is given in Section 2. The proof of Theorem 2 is given in Section 3. Corollaries 3 and 4 are straightforward consequences of Theorems 1 and 2, thus no additional proof is needed.

2. Proof of Theorem 1

The proof of Theorem 1 is divided into two parts. In Section 2.1, we prove the following result.

Lemma 5. *Let* $s > 0$. *If* $\mathbb{E}[R^{d+s}] < \infty$ *and* $\lambda \in \widehat{\Lambda}$, *then* $\mathbb{E}(|C|^{s/d})$, $\mathbb{E}(\#C^{s/d})$ *and* $\mathbb{E}(D^s)$ *are finite.*

Section 2.2 is devoted to the proof of the following result.

Lemma 6. *Let* $s > 0$. *If* $\mathbb{E}[R^{d+s}] = \infty$ *then for every* $\lambda > 0$, $\mathbb{E}(|C|^{s/d})$, $\mathbb{E}(\#C^{s/d})$ *and* $\mathbb{E}(D^s)$ *are infinite.*

Theorem 1 is a straightforward consequence of Lemmas 5 and 6.

2.1. Proof of Lemma 5

We first establish the following result.

Theorem 7. *Let* $s > 0$. *Assume* $\mathbb{E}(R^{d+s}) < \infty$. *Let* $\lambda \in \widehat{\Lambda}$. *Then*

$$\int_0^\infty \alpha^{s-1}\mathbb{P}\big(S(\alpha) \stackrel{\Sigma}{\longleftrightarrow} S(2\alpha)\big)\, d\alpha < \infty$$

and

$$\mathbb{E}\big(D^s\big) < \infty.$$

The result is implicit in [9]. We choose to give a detailed proof using intermediate results in Appendix A in [10] which themselves rely on results in [9].

Let us recall some notation from [9] or [10]. Let $\alpha > 0$.

- $\Sigma(B(0,\alpha))$ is the union of random balls of the Boolean model with centers in $B(0,\alpha)$.
- $G(0,\alpha)$ is the event "there exists a path from $S(\alpha)$ to $S(8\alpha)$ in $\Sigma(B(0,10\alpha))$".
- $\Pi(\alpha) = P(G(0,\alpha))$.

Set

$$\varepsilon(\alpha) = \int_{[\alpha,+\infty)} r^d \nu(dr). \tag{6}$$

Note that, when $\mathbb{E}(R^{d+s})$ is finite,

$$\int_0^\infty \alpha^{s-1}\varepsilon(\alpha)\,d\alpha < \infty. \tag{7}$$

The following proposition is stated in the same way in [10] as Proposition 12 in Appendix A.

Proposition 8. *There exists a constant $K = K(d)$ such that, for any $\alpha > 0$,*

$$\Pi(\alpha) \le P\big(S(\alpha) \overset{\Sigma}{\longleftrightarrow} S(2\alpha)\big) \le K\Pi(\alpha/10) + \lambda K\varepsilon(\alpha/10), \tag{8}$$

$$\Pi(10\alpha) \le K\Pi(\alpha)^2 + \lambda K\varepsilon(\alpha), \tag{9}$$

$$\Pi(\alpha) \le \lambda K\alpha^d. \tag{10}$$

Proof of Theorem 7. This is a consequence of Proposition 8 above and Lemma 3.7 in [9]. Showing how to apply Lemma 3.7 would not be much shorter than adapting the proof in our context. Therefore, we choose to give a full proof. Let $s > 0$. Assume $\mathbb{E}(R^{d+s}) < \infty$ and let $\lambda \in \widehat{\Lambda}$. By (8), $\Pi(\alpha)$ tends to 0 as α tends to ∞ Therefore, we can fix α_0 large enough such that, for all $\alpha \ge \alpha_0/10$,

$$10^s K\Pi(\alpha) \le \frac{1}{2}.$$

Then, for any $\alpha \ge \alpha_0$, using (9) and the definition of α_0,

$$\begin{aligned}
\int_{\alpha_0}^{\alpha} r^{s-1}\Pi(r)\,dr &\le \int_{\alpha_0}^{\alpha} r^{s-1}K\Pi(r/10)^2\,dr + \int_{\alpha_0}^{\alpha} r^{s-1}\lambda K\varepsilon(r/10)\,dr \\
&\le 10^s \int_{\alpha_0/10}^{\alpha/10} r^{s-1}K\Pi(r)^2\,dr + \int_{\alpha_0}^{\infty} r^{s-1}\lambda K\varepsilon(r/10)\,dr \\
&\le \frac{1}{2}\int_{\alpha_0/10}^{\alpha/10} r^{s-1}\Pi(r)\,dr + \int_{\alpha_0}^{\infty} r^{s-1}\lambda K\varepsilon(r/10)\,dr.
\end{aligned}$$

Therefore, for any large enough α,

$$\int_{\alpha_0}^{\alpha} r^{s-1}\Pi(r)\,dr \le \frac{1}{2}\int_{\alpha_0/10}^{\alpha_0} r^{s-1}\Pi(r)\,dr + \frac{1}{2}\int_{\alpha_0}^{\alpha} r^{s-1}\Pi(r)\,dr + \int_{\alpha_0}^{\infty} r^{s-1}\lambda K\varepsilon(r/10)\,dr.$$

Then, rearranging and using (7),

$$\int_{\alpha_0}^{\alpha} r^{s-1}\Pi(r)\,dr \le \int_{\alpha_0/10}^{\alpha_0} r^{s-1}\Pi(r)\,dr + 2\int_{\alpha_0}^{\infty} r^{s-1}\lambda K\varepsilon(r/10)\,dr < \infty.$$

Therefore,

$$\int_{\alpha_0}^{\infty} r^{s-1}\Pi(r)\,dr < \infty$$

and thus

$$\int_{0}^{\infty} r^{s-1}\Pi(r)\,dr < \infty.$$

By (7) and (8), this yields the first required result. The other result then follows from the fact that, for any $\alpha > 0$,

$$\{D \ge 4\alpha\} \subset \bigl\{S(\alpha) \overset{\Sigma}{\longleftrightarrow} S(2\alpha)\bigr\}. \qquad \square$$

Proof of Lemma 5. We suppose that $\mathbb{E}(R^{d+s}) < \infty$ and $\lambda \in \widehat{\Lambda}$. By Theorem 7, we know that $\mathbb{E}(D^s) < \infty$. Since $C \subset B(0, D)$, this implies $\mathbb{E}(|C|^{s/d}) < \infty$. It remains to prove that $\mathbb{E}(\#C^{s/d}) < \infty$.

Let $\kappa > 0$ be such that

$$\lambda v_d \kappa^d = \frac{1}{2},$$

where v_d denotes the volume of the unit ball in $\mathbb{R}^d$. For every $u > 0$, we have

$$\begin{aligned}\mathbb{P}(\#C \ge u) &\le \mathbb{P}\bigl(C \subset B\bigl(0, \kappa u^{1/d}\bigr) \text{ and } \#C \ge u\bigr) + \mathbb{P}\bigl(C \not\subset B\bigl(0, \kappa u^{1/d}\bigr)\bigr)\\ &\le \mathbb{P}\bigl(\#\bigl\{(c,r) \in \xi : c \in B\bigl(0, \kappa u^{1/d}\bigr)\bigr\} \ge u\bigr) + \mathbb{P}\bigl(D \ge \kappa u^{1/d}\bigr). \end{aligned} \tag{11}$$

Since $\#\{(c,r) \in \xi : c \in B(0, \kappa u^{1/d})\}$ is a Poisson random variable with parameter $u/2$, we obtain

$$\mathbb{P}\bigl(\#\bigl\{(c,r) \in \xi : c \in B\bigl(0, \kappa u^{1/d}\bigr)\bigr\} \ge u\bigr) \le \exp\biggl(u\biggl(\frac{1}{2} - \ln(2)\biggr)\biggr)$$

and thus

$$\int_0^{\infty} du\, \frac{s}{d} u^{\frac{s}{d}-1} \mathbb{P}\bigl(\#\bigl\{(c,r) \in \xi : c \in B\bigl(0, \kappa u^{1/d}\bigr)\bigr\} \ge u\bigr) < \infty.$$

Since $\mathbb{E}(D^s) < \infty$, we have

$$\int_0^{\infty} du\, \frac{s}{d} u^{\frac{s}{d}-1} \mathbb{P}\bigl(D \ge \kappa u^{1/d}\bigr) < \infty.$$

We conclude by (11) that $\mathbb{E}(\#C^{s/d})$ is finite. $\square$

2.2. Proof of Lemma 6

Set

$$A = \sup\{r > 0 : \exists c \in B(0, r/2) \text{ s.t. } (c, r) \in \xi\},$$

with the convention $A = 0$ if the set is empty. Note that $B(0, A/2)$ is covered by Σ. We first state the following preliminary result, which is essentially implicit in [9].

Lemma 9. *Let $\lambda > 0$ and $s > 0$. Assume that $\mathbb{E}(R^{d+s})$ is infinite. Then $\mathbb{E}(A^s)$ is infinite.*

Proof of Lemma 9. For any $a > 0$,

$$\mathbb{P}(A > a) = 1 - \exp\left(-\lambda v_d 2^{-d} \int_{(a,+\infty)} r^d \nu(dr)\right).$$

If $\mathbb{E}(R^d)$ is infinite, then $\mathbb{P}(A > a) = 1$ for all $a > 0$ and thus $A = +\infty$ almost surely and therefore $\mathbb{E}(A^s) = \infty$.

Assume henceforth that $\mathbb{E}(R^d)$ is finite. Then

$$\mathbb{P}(A > a) \underset{a\to\infty}{\sim} \lambda v_d 2^{-d} \int_{(a,+\infty)} r^d \nu(dr).$$

Therefore, for some constant $\gamma > 0$, for all $a > 0$,

$$\mathbb{P}(A > a) \geq \gamma \int_{(a,+\infty)} r^d \nu(dr)$$

and then

$$\begin{aligned}
\mathbb{E}\big(A^s\big) &= \int_{(0,+\infty)} da\, s a^{s-1} \mathbb{P}(A > a) \\
&\geq \gamma \int_{(0,+\infty)} da\, s a^{s-1} \int_{(a,+\infty)} \nu(dr) r^d \\
&= \gamma \int_{(0,+\infty)} \nu(dr) r^d \int_{(0,r)} da\, s a^{s-1} \\
&= \gamma \int_{(0,+\infty)} \nu(dr) r^{d+s} \\
&= \gamma \mathbb{E}\big(R^{s+d}\big)
\end{aligned}$$

which is infinite by assumption. □

Proof of Lemma 6. Let $\lambda > 0$ and $s > 0$. We suppose that $\mathbb{E}(R^{d+s})$ is infinite. By Lemma 9, we obtain that $\mathbb{E}(A^s) = \infty$. Since $B(0, A/2)$ is covered by Σ, we know that $D \geq A$ and $|C| \geq v_d A^d / 2^d$, thus $\mathbb{E}(D^s) = \infty$ and $\mathbb{E}(|C|^{s/d}) = \infty$. It remains to prove that $\mathbb{E}(\#C^{s/d}) = \infty$.

Let $r_0 > 0$ be such that $\mathbb{P}(R \le r_0) > 0$. Set

$$A_{>r_0} = A \quad \text{if } A > r_0 \quad \text{and} \quad A_{>r_0} = 0 \quad \text{otherwise.}$$

In other words,

$$A_{>r_0} = \sup\{r > r_0 : \exists c \in B(0, r/2) \text{ s.t. } (c, r) \in \xi\}$$

with the convention $A_{>r_0} = 0$ if the set is empty. Note that $B(0, A_{>r_0}/2)$ is covered by Σ and that $A_{>r_0}$ is measurable with respect to $\xi_{>r_0} = \xi \cap \mathbb{R}^d \times (r_0, +\infty)$. Set

$$N = \operatorname{card}\bigl(\{(c, r) \in \xi : c \in B(0, A_{>r_0}/2) \text{ and } r \le r_0\}\bigr).$$

Conditionally on $\xi_{>r_0}$, N is a Poisson random variable with parameter $\alpha A^d_{>r_0}$ where

$$\alpha = \lambda \mathbb{P}(R \le r_0) v_d / 2^d > 0.$$

But as $B(0, A_{>r_0}/2)$ is contained in Σ, any random ball centered in $B(0, A_{>r_0}/2)$ is contained in C. Therefore

$$\#C \ge N$$

and thus

$$\mathbb{E}\bigl(\#C^{s/d}\bigr) \ge \mathbb{E}\bigl(N^{s/d}\bigr) = \mathbb{E}\bigl(\mathbb{E}\bigl(N^{s/d} | \xi_{>r_0}\bigr)\bigr).$$

Let μ_0 be such that, for any $\mu \ge \mu_0$, if $X(\mu)$ is a Poisson random variable with parameter μ, then $\mathbb{P}(X(\mu) \ge \mu/2) \ge 1/2$. Then,

$$\begin{aligned}\mathbb{E}\bigl(\#C^{s/d}\bigr) &\ge \mathbb{E}\bigl(\mathbb{E}\bigl(N^{s/d} | \xi_{>r_0}\bigr) \mathbb{1}_{\alpha A^d_{>r_0} \ge \mu_0}\bigr) \\ &\ge \mathbb{E}\left(\frac{1}{2}\left(\frac{\alpha A^d_{>r_0}}{2}\right)^{s/d} \mathbb{1}_{\alpha A^d_{>r_0} \ge \mu_0}\right) = \beta \mathbb{E}\bigl(A^s_{>r_0} \mathbb{1}_{\alpha A^d_{>r_0} \ge \mu_0}\bigr),\end{aligned}$$

where

$$\beta = \frac{\alpha^{s/d}}{2^{1+s/d}}.$$

To end the proof, it is therefore sufficient to check that $\mathbb{E}(A^s_{>r_0})$ is infinite. But this is a consequence of the infiniteness of $\mathbb{E}(A^s)$ obtained by Lemma 9. $\square$

3. Proof of Theorem 2

3.1. Preliminary results

For any $r > 0$, we denote by $C(r)$ the connected component of $\Sigma \cup B(0, r)$ which contains $B(0, r)$. If A and B are two subsets of $\mathbb{R}^d$, we denote by $A + B$ the Minkowski sum of A and B defined by

$$A + B = \{a + b, a \in A, b \in B\}.$$

Lemma 10. *Let $\lambda > 0$. Assume that $\mathbb{E}(|C|)$ is finite. Then*

$$\mathbb{E}\big(\big|C(1) + B(0,1)\big|\big) < \infty.$$

Proof. This is a consequence of FKG inequality, see, for instance, Theorem 2.2 in [14]. Set

$$p = \mathbb{P}\big(B(0,1) \subset \Sigma\big)$$

and note that p is positive. For any $x \in \mathbb{R}^d$, using FKG inequality in the third step and stationarity and definition of p in the fourth step, we get

$$\begin{aligned}
\mathbb{P}(x \in C) &= \mathbb{P}\big(\{0\} \xleftrightarrow{\Sigma} \{x\}\big) \\
&\geq \mathbb{P}\big(\big\{B(0,1) \xleftrightarrow{\Sigma} B(x,1)\big\} \cap \big\{B(0,1) \subset \Sigma\big\} \cap \big\{B(x,1) \subset \Sigma\big\}\big) \\
&\geq \mathbb{P}\big(B(0,1) \xleftrightarrow{\Sigma} B(x,1)\big)\mathbb{P}\big(B(0,1) \subset \Sigma\big)\mathbb{P}\big(B(x,1) \subset \Sigma\big) \\
&= p^2\mathbb{P}\big(B(0,1) \xleftrightarrow{\Sigma} B(x,1)\big) \\
&= p^2\mathbb{P}\big(x \in C(1) + B(0,1)\big).
\end{aligned}$$

Therefore,

$$\mathbb{E}\big(\big|C(1) + B(0,1)\big|\big) = \int_{\mathbb{R}^d} dx\, \mathbb{P}\big(x \in C(1) + B(0,1)\big) \leq p^{-2} \int_{\mathbb{R}^d} dx\, \mathbb{P}(x \in C) = p^{-2}\mathbb{E}\big(|C|\big).$$

As $\mathbb{E}(|C|)$ is finite, the lemma is proven. □

Lemma 11. *Let $K = K(d)$ be such that, for any $r > 0$, the ball $B(0,r)$ can be covered by $K(1+r)^d$ balls of radius 1. Let $\lambda > 0$. Let $r, s > 0$. Then*

$$\begin{aligned}
\int_{\mathbb{R}^d} dx\, \mathbb{P}\big(B(0,r) \xleftrightarrow{\Sigma} B(x,s)\big) &= \mathbb{E}\big(\big|C(r) + B(0,s)\big|\big) \\
&\leq K^2(1+r)^d(1+s)^d\mathbb{E}\big(\big|C(1) + B(0,1)\big|\big).
\end{aligned}$$

Proof. Write

$$\mathbb{E}\big(\big|C(r) + B(0,s)\big|\big) = \int_{\mathbb{R}^d} dx\, \mathbb{P}\big(x \in C(r) + B(0,s)\big) = \int_{\mathbb{R}^d} dx\, \mathbb{P}\big(B(0,r) \xleftrightarrow{\Sigma} B(x,s)\big).$$

Cover $B(0,r)$ with at most $K(1+r)^d$ balls of radius 1. Cover $B(x,s)$ with at most $K(1+s)^d$ balls of radius 1. If $\{\mathbb{P}(B(0,r) \xleftrightarrow{\Sigma} B(x,s)\}$ holds, then there exists a path in Σ from one of the balls that cover $B(0,r)$ to one of the balls that cover $B(x,s)$. Using union bound, stationarity and a change of variable, we thus get

$$\begin{aligned}\mathbb{E}\big(\big|C(r)+B(0,s)\big|\big) &\leq K^2(1+r)^2(1+s)^2\int_{\mathbb{R}^d} dx\,\mathbb{P}\big(B(0,1)\overset{\Sigma}{\longleftrightarrow} B(x,1)\big)\\ &= K^2(1+r)^d(1+s)^d\mathbb{E}\big(\big|C(1)+B(0,1)\big|\big).\end{aligned}$$

The lemma is proven. □

3.2. Proof of 4 $\Longrightarrow$ 2

This is the main part of the proof. For any $n\geq 2$, any $r,s>0$ and any $x,y\in\mathbb{R}^d$ we consider the event

$$\begin{aligned}&L_n(x,r,y,s)\\ &\quad=\big\{\text{There exists a chain of length } n-1 \text{ starting in } B(x,r) \text{ and stopping in } B(y,s)\big\}\end{aligned}$$

and we set

$$a_n(r,s)=\int_{\mathbb{R}^d} dy\,\mathbb{P}\big(L_n(0,r,y,s)\big). \tag{12}$$

Recall that R is a random variable with distribution ν. Let S_i, $i\geq 0$ be independent copies of $R+1$.

Lemma 12. *For any $\lambda>0$, any $p\geq 2$ and any $k\geq 1$,*

$$\mathbb{P}(\ell\geq kp)\leq\big(\lambda\big[\mathbb{E}\big(a_p(S_1,S_2)^2\big)\big]^{1/2}\big)^k.$$

S. et $c_0=0$ and $r_0=1$. Let $\lambda>0$, $p\geq 2$ and $k\geq 1$. We will use BK inequality, see the main theorem in [19] (and the remark (iii) above it concerning the choice of the definition of disjoint occurrence of increasing events). Before stating the inequality, let us recall informally some notations. The BK inequality apply directly to increasing events living on bounded region, thus define

$$L_p^n(x,r,y,s)=\left\{\begin{array}{c}\text{There exists a chain } \big((x_1,r_1),\dots,(x_{p-1},r_{p-1})\big) \text{ of length } p-1\\ \text{starting in } B(x,r) \text{ and stopping in } B(y,s)\\ \text{s.t. for all } i\in\{1,\dots,p-1\}, x_i\in[-n,n]^d\end{array}\right\}.$$

If $((c_i,r_i))_{1\leq i\leq k}$ are points of $\mathbb{R}^d\times(0,+\infty)$, we say that the increasing events $L_p(c_{i-1},r_{i-1},c_i,r_i)$ (respectively $L_p^n(c_{i-1},r_{i-1},c_i,r_i)$), $i\in\{1,\dots,k\}$, occur disjointly if there exists k chains, each of length $p-1$, using in total $k(p-1)$ distinct random balls such that, for all $i\in\{1,\dots,k\}$, the ith chain starts in $B(c_{i-1},r_{i-1})$ and stops in $B(x_i,r_i)$ (respectively, and all the centers of the balls of these chains belong to $[-n,n]^d$). We denote these events by

$$L_p(c_0,r_0,c_1,r_1)\circ\cdots\circ L_p(c_{k-1},r_{k-1},c_k,r_k)\quad\text{and}$$

$$L_p^n(c_0, r_0, c_1, r_1) \circ \cdots \circ L_p^n(c_{k-1}, r_{k-1}, c_k, r_k)$$

or simply by

$$\underset{i}{\circ} L_p(c_{i-1}, r_{i-1}, c_i, r_i) \quad \text{and} \quad \underset{i}{\circ} L_p^n(c_{i-1}, r_{i-1}, c_i, r_i).$$

By BK inequality, for all $n \in \mathbb{N}$ we have

$$\mathbb{P}\big(L_p^n(c_0, r_0, c_1, r_1) \circ \cdots \circ L_p^n(c_{k-1}, r_{k-1}, c_k, r_k)\big) \le \prod_{i=1}^{k} \mathbb{P}\big(L_p^n(c_{i-1}, r_{i-1}, c_i, r_i)\big).$$

Taking the limit as n goes to infinity, we obtain

$$\mathbb{P}\big(L_p(c_0, r_0, c_1, r_1) \circ \cdots \circ L_p(c_{k-1}, r_{k-1}, c_k, r_k)\big) \le \prod_{i=1}^{k} \mathbb{P}\big(L_p(c_{i-1}, r_{i-1}, c_i, r_i)\big). \tag{13}$$

If $\ell \ge kp$, then there exists a chain of kp distinct balls starting in $B(0, 1)$. Taking one ball every pth balls in this chain, we get a sequence $(c_1, r_1), \dots, (c_k, r_k)$ of distinct points of ξ such that, with a slight abuse of notation, the event

$$\underset{i}{\circ} L_p(c_{i-1}, r_{i-1}, c_i, r_i)$$

holds for $\xi \setminus \{(c_1, r_1), \dots, (c_k, r_k)\}$. Therefore, again with a slight abuse of notation,

$$\mathbb{P}(\ell \ge kp) \le \mathbb{E}\Bigg(\sum_{(c_1,r_1),\dots,(c_k,r_k)\in\xi \text{ distinct}} \mathbb{1}_{\underset{i}{\circ} L_p(c_{i-1},r_{i-1},c_i,r_i)}\big(\xi \setminus \big\{(c_1, r_1), \dots, (c_k, r_k)\big\}\big)\Bigg).$$

From Slivnyak's theorem, see Proposition 4.1.1 in [16], we get

$$\mathbb{P}(\ell \ge kp) \le \lambda^k \int_{(\mathbb{R}^d)^k} dc_1 \cdots dc_k \int_{(0,+\infty)^k} \nu(dr_1) \cdots \nu(dr_k) \mathbb{P}\big(\underset{i}{\circ} L_p(c_{i-1}, r_{i-1}, c_i, r_i)\big).$$

By (13), this yields

$$\mathbb{P}(\ell \ge kp) \le \lambda^k \int_{(\mathbb{R}^d)^k} dc_1 \cdots dc_k \int_{(0,+\infty)^k} \nu(dr_1) \cdots \nu(dr_k) \prod_{i=1}^{k} \mathbb{P}\big(L_p(c_{i-1}, r_{i-1}, c_i, r_i)\big).$$

Using stationarity and (12), we then get

$$\mathbb{P}(\ell \ge kp) \le \lambda^k \int_{(0,+\infty)^k} \nu(dr_1) \cdots \nu(dr_k) \prod_{i=1}^{k} a_p(r_{i-1}, r_i).$$

Distinguishing according to parity, we get

$$\mathbb{P}(\ell \ge kp) \le \lambda^k \int_{(0,+\infty)^k} \nu(dr_1) \cdots \nu(dr_k) \prod_{1\le i\le k, i \text{ odd}} a_p(r_{i-1}, r_i) \prod_{1\le i\le k, i \text{ even}} a_p(r_{i-1}, r_i).$$

Then, by Cauchy–Schwarz inequality,

$$\mathbb{P}\big(L_{kp}(1)\big) \le \lambda^k A_p(k) B_p(k),$$

where

$$A_p(k) = \left[\int_{(0,+\infty)^k} \nu(dr_1)\cdots\nu(dr_k) \prod_{1\le i\le k, i \text{ odd}} a_p(r_{i-1}, r_i)^2\right]^{1/2}$$

and

$$B_p(k) = \left[\int_{(0,+\infty)^k} \nu(dr_1)\cdots\nu(dr_k) \prod_{1\le i\le k, i \text{ even}} a_p(r_{i-1}, r_i)^2\right]^{1/2}.$$

Let us get rid of the (easy but annoying) special case $r_0 = 1$ as follows. Recall that R is a random variable with distribution ν and that $S_i, i \ge 0$ are independent copies of $R+1$. As $a_p(r,s)$ is non-decreasing in r and s, we have

$$A_p(k) \le \left[\mathbb{E}\left(\prod_{1\le i\le k, i \text{ odd}} a_p(S_{i-1}, S_i)^2\right)\right]^{1/2}$$

and

$$B_p(k) \le \left[\mathbb{E}\left(\prod_{1\le i\le k, i \text{ even}} a_p(S_{i-1}, S_i)^2\right)\right]^{1/2}.$$

As $(S_i)_{i\ge 0}$ is an i.i.d. sequence, we get

$$A_p(k)B_p(k) \le \left[\mathbb{E}\big(a_p(S_1, S_2)^2\big)\right]^{k/2}$$

and then

$$P(\ell \ge kp) \le \left(\lambda\left[\mathbb{E}\big(a_p(S_1, S_2)^2\big)\right]^{1/2}\right)^k.$$

The lemma is proven. □

Recall that ℓ is defined by (4).

Lemma 13. *Let $\lambda > 0$. Assume that $\mathbb{E}(R^d)$ and $\mathbb{E}(|C|)$ are finite. Then ℓ is finite with probability one.*

Remark. We could remove the assumption $\mathbb{E}(R^d)$ finite, as it is a consequence of the finiteness of $\mathbb{E}(|C|)$.

Proof. Let $\lambda > 0$ be such that ℓ is infinite with positive probability. We aim at proving that $\mathbb{E}(|C|)$ is infinite. For any $\eta > 0$, set

$$\Sigma_{\le\eta} = \bigcup_{(c,r)\in\xi : r\le\eta} B(c,r).$$

If η is small enough, then $\Sigma_{\leq\eta}$ does not percolate. Indeed, $\Sigma_{\leq\eta}$ is a Boolean model driven by the measure

$$\lambda\nu\big(\cdot\cap(0,\eta]\big).$$

We can therefore couple $\Sigma_{\leq\eta}$ with the Boolean model $\Sigma^+_{\leq\eta}$ driven by the measure

$$\lambda\nu\big((0,\eta]\big)\delta_\eta$$

in such a way that

$$\Sigma_{\leq\eta}\subset\Sigma^+_{\leq\eta}.$$

But

$$\eta^{-1}\Sigma^+_{\leq\eta}$$

is a Boolean model driven by

$$\lambda\nu\big((0,\eta]\big)\eta^d\delta_1.$$

Therefore, as soon as

$$\lambda\nu\big((0,\eta]\big)\eta^d<\lambda_c(\delta_1,d),$$

$\eta^{-1}\Sigma^+_{\leq\eta}$ does not percolate, thus $\Sigma^+_{\leq\eta}$ does not percolate and then $\Sigma_{\leq\eta}$ does not percolate. In the remaining of the proof, we fix $\eta>0$ such that $\Sigma_{\leq\eta}$ does not percolate.

Let $C_{>\eta}$ denote the union of all the random balls of C whose radius is greater than η. We will repeatedly use the following property, that holds since $\mathbb{E}(R^d)$ is finite: almost surely, whatever the bounded region B of $\mathbb{R}^d$ we consider, the number of random balls of Σ that touches B is finite. If ℓ is infinite, then for any n there exists a chain $((c_1^n,r_1^n),\dots,(c_n^n,r_n^n))$ of balls starting in $B(0,1)$. For all n the ball $B(c_1^n,r_1^n)$ touches $B(0,1)$, but by the previously stated property we know that a.s. there exists only a finite number of balls of Σ that touches $B(0,1)$, thus an infinite number of those balls $B(c_1^n,r_1^n)$ are equal to the same ball that we will denote by B_1. In other words, there exists an infinite number of chains of arbitrarily large length starting in B_1. But by the same property, we know that a.s. there exists only a finite number of balls of Σ that touches $B(0,1)\cup B_1$, thus an infinite number of those chains (with arbitrarily large length) use a common first ball of Σ that we denote by B_2. By induction, we construct an infinite chain of distinct balls $(B_i=B(x_i,s_i),i\geq1)$ starting in $B(0,1)$. Suppose that only a finite number of the radii s_i are bigger than η. Then there exists an infinite chain of balls of radii smaller than or equal to η. By the previously stated property, we know that a.s. this infinite number of balls cannot stay in any bounded region, thus $\Sigma_{\leq\eta}$ has to percolate, which is absurd by our choice of η. We conclude that an infinite number of these balls $(B_i=B(x_i,s_i),i\geq1)$ satisfy $s_i>\eta$. A.s. these balls cannot stay in any bounded region, thus up to extraction we obtain a sequence $(c_i,r_i)_{i\geq0}$ of points of ξ such that, for all $i\geq1$,

$$B(c_i,r_i)\subset C,\qquad \|c_i\|\geq\|c_{i-1}\|+1,\quad r_i\geq\eta.$$

Therefore, $|C|=\infty$ almost surely on the event $\{\ell=\infty\}$. As this event occurs with positive probability, we get $\mathbb{E}(|C|)=\infty$. □

Lemma 14. *Let $\lambda > 0$. Assume that $\mathbb{E}(R^{2d})$ and $\mathbb{E}(|C|)$ are finite. Then there exists $p \geq 2$ such that*

$$\lambda\big[\mathbb{E}\big(a_p(S_1, S_2)^2\big)\big]^{1/2} < 1.$$

Proof. As $\mathbb{E}(|C|)$ is finite, ℓ is almost surely finite by Lemma 13. For any $r > 0$, $B(0, r)$ can be covered by a finite number of balls of radius 1. Therefore, by stationarity, the maximal number $\ell(r)$ of balls in a chain starting in $B(0, r)$ is almost surely finite. As a consequence, for all $r > 0$,

$$\mathbb{P}\big(\ell(r) \geq p\big) \to 0 \quad \text{as } p \to \infty.$$

Thus, for all $r, s > 0$ and $y \in \mathbb{R}^d$,

$$\mathbb{P}\big(L_p(0, r, y, s)\big) \to 0 \quad \text{as } p \to \infty,$$

as $L_p(0, r, y, s) \subset \{\ell(r) \geq p - 1\}$. Moreover, for all $r, s > 0$, $y \in \mathbb{R}^d$ and $p \geq 2$,

$$\mathbb{P}\big(L_p(0, r, y, s)\big) \leq \mathbb{P}\big(B(0, r) \overset{\Sigma}{\longleftrightarrow} B(y, s)\big). \tag{14}$$

But, by Lemma 11,

$$\int_{\mathbb{R}^d} dy\, \mathbb{P}\big(B(0, r) \overset{\Sigma}{\longleftrightarrow} B(y, s)\big) \leq K^2(1 + r)^d(1 + s)^d\mathbb{E}\big(\big|C(1) + B(0, 1)\big|\big). \tag{15}$$

Moreover, as $\mathbb{E}(|C|)$ is finite, we get

$$\mathbb{E}\big(\big|C(1) + B(0, 1)\big|\big) < \infty \tag{16}$$

by Lemma 10. Therefore, by dominated convergence theorem, for any $r, s > 0$,

$$\int_{\mathbb{R}^d} dy\, \mathbb{P}\big(L_p(0, r, y, s)\big) \to 0 \quad \text{as } p \to \infty$$

that is, using Definition (12),

$$a_p(r, s) \to 0 \quad \text{as } p \to \infty.$$

By (14) and (15),

$$a_p(S_1, S_2) \leq K^2(1 + S_1)^d(1 + S_2)^d\mathbb{E}\big(\big|C(1) + B(0, 1)\big|\big).$$

Using (16) and the finiteness of $\mathbb{E}(R^{2d})$ we get that the square of the right-hand side of the above inequality is integrable. Using dominated convergence theorem again, we then get

$$\mathbb{E}\big(a_p^2(S_1, S_2)\big) \to 0 \quad \text{as } p \to \infty.$$

The lemma is proven. $\square$

Proof of $4 \implies 2$**.** By Lemma 14, there exits $p \geq 2$ such that

$$\kappa := \lambda\big[\mathbb{E}\big(a_p(S_1, S_2)^2\big)\big]^{1/2} < 1.$$

By Lemma 12, for any $k \geq 1$,

$$\mathbb{P}(\ell \geq kp) \leq \kappa^k.$$

Therefore there exists $A, B > 0$ such that, for any $n \geq 1$, $\mathbb{P}(\ell \geq n) \leq A\exp(-Bn)$. □

3.3. Proof of the others implications

Proof of $2 \implies 1$**.** Let $K = K(d)$ be such that, for any $r \geq 1$, the sphere $S(r)$ can be covered by Kr^{d-1} balls of radius 1. Let $r \geq 1$. Let us first prove

$$\mathbb{P}\big(S(r) \xleftrightarrow{\Sigma} S(2r)\big) \leq \mathbb{P}\big(H(r)\big) + Kr^{d-1}\mathbb{P}(\ell \geq \sqrt{r}/2), \tag{17}$$

where

$$H(r) = \big\{\exists (c, s) \in \xi : s \geq \sqrt{r} \text{ and } B(c, s) \cap B(0, 2r) \neq \varnothing\big\}.$$

Indeed, if $\{S(r) \xleftrightarrow{\Sigma} S(2r)\}$ holds, then there exists a chain of random balls from a point of $S(r)$ to a point of $S(2r)$. If moreover $H(r)$ does not hold, then the number of balls of this chain is at least $\sqrt{r}/2$. The sphere $S(r)$ can be covered by Kr^{d-1} balls of radius 1 and the starting point of the chain is in one of these balls. Using the union bound and stationarity, we get (17).

As

$$\mathbb{P}(\ell \geq \sqrt{r}/2) \leq A\exp(-B\sqrt{r}/2),$$

we get

$$Kr^{d-1}\mathbb{P}(\ell \geq \sqrt{r}/2) \to 0 \quad \text{as } r \to \infty. \tag{18}$$

Furthermore,

$$\begin{aligned}
\mathbb{P}\big(H(r)\big) &\leq \mathbb{E}\big(\mathrm{card}\big[\big\{(c, s) \in \xi : s \geq \sqrt{r} \text{ and } B(c, s) \cap B(0, 2r) \neq \varnothing\big\}\big]\big) \\
&\leq \lambda \int_{[\sqrt{r}, \infty)} v_d(2r + s)^d \nu(ds) \\
&\leq \lambda \int_{[\sqrt{r}, \infty)} v_d\big(2s^2 + s\big)^d \nu(ds),
\end{aligned}$$

where v_d denotes the volume of the unit ball of $\mathbb{R}^d$. Therefore

$$\mathbb{P}\big(H(r)\big) \to 0 \quad \text{as } r \to \infty. \tag{19}$$

The result follows from (17), (18) and (19). □

Remark. Replacing $\sqrt{r}$ by $\alpha r/\log r$ for some big enough α in the definition of the event $H(r)$, and replacing accordingly the event $\{\ell \geq \sqrt{r}/2\}$ by $\{\ell \geq \log r/(2\alpha)\}$, we could weakened the hypothesis $\mathbb{E}(R^{2d}) < \infty$ to $\mathbb{E}(R^d(\log R)^\beta) < \infty$ for some β, but only for the implication $2 \Rightarrow 1$.

Proof of $1 \implies 3$**.** This is a consequence of Theorem 1. □

Proof of $3 \implies 4$**.** Note that C is a subset of $B(0, D)$. As a consequence, $|C| \leq v_d D^d$. The result follows. □

Proof of $4 \iff 5$**.** There exits actually simple inequalities between $\mathbb{E}(|C|)$ and $\mathbb{E}(\#C)$. Using Slivnyak's theorem, see Proposition 4.1.1 in [16], we get

$$\begin{aligned}\mathbb{E}(\#C) &= \mathbb{E}\left(\sum_{c\in\chi} 1_{c\in C}\right) \\ &= \lambda \int_{\mathbb{R}^d} dc \int_{(0,+\infty)} \nu(dr)\, \mathbb{P}\big(c \in C\big(\xi \cup (c,r)\big)\big),\end{aligned}$$

where $C(\xi \cup (c,r))$ denotes the connected component containing the origin in the Boolean model with the extra ball $B(x,r)$. Thus,

$$\begin{aligned}\mathbb{E}(\#C) &= \lambda \int_{\mathbb{R}^d} dc \int_{(0,+\infty)} \nu(dr)\, \mathbb{P}\big(B(c,r) \text{ touches } 0 \text{ or } B(c,r) \text{ touches } C\big) \\ &= \lambda \int_{\mathbb{R}^d} dc \int_{(0,+\infty)} \nu(dr)\, \mathbb{P}\big(c \in B(0,r) \cup \big(C + B(0,r)\big)\big) \\ &= \lambda \mathbb{E}\big(\big|B(0,R) \cup \big(C + B(0,R)\big)\big|\big),\end{aligned}$$

where R is independent of ξ and where the distribution of R is ν. Thus,

$$\lambda \mathbb{E}\big(|C|\big) \leq \mathbb{E}(\#C). \tag{20}$$

Moreover, using Lemma 11,

$$\begin{aligned}\mathbb{E}(\#C) &\leq \lambda v_d \mathbb{E}\big(R^d\big) + \lambda \mathbb{E}\big(\big|C(1) + B(0,R)\big|\big) \\ &\leq \lambda v_d \mathbb{E}\big(R^d\big) + \lambda K^2 2^d \mathbb{E}\big([1+R]^d\big) \mathbb{E}\big(\big|C(1) + B(0,1)\big|\big).\end{aligned} \tag{21}$$

But, by Lemma 10, $\mathbb{E}(|C(1) + B(0,1)|)$ is finite. As $\mathbb{E}(R^d)$ is also finite, the equivalence follows from (20) and (21). □

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References

[1] Ahlberg, D., Tassion, V. and Teixeira, A. (2018). Existence of an unbounded vacant set for subcritical continuum percolation. *Electron. Commun. Probab.* **23** Paper No. 63, 8. MR3863919

[2] Ahlberg, D., Tassion, V. and Teixeira, A. (2018). Sharpness of the phase transition for continuum percolation in $\mathbb{R}^2$. *Probab. Theory Related Fields* **172** 525–581. MR3851838

[3] Aizenman, M. and Barsky, D.J. (1987). Sharpness of the phase transition in percolation models. *Comm. Math. Phys.* **108** 489–526. MR0874906

[4] Duminil-Copin, H., Raoufi, A. and Tassion, V. (2018). Subcritical phase of d-dimensional Poisson–Boolean percolation and its vacant set. Preprint. Available at arXiv:1805.00695.

[5] Duminil-Copin, H., Raoufi, A. and Tassion, V. (2019). Exponential decay of connection probabilities for subcritical Voronoi percolation in $\mathbb{R}^d$. *Probab. Theory Related Fields* **173** 479–490. MR3916112

[6] Duminil-Copin, H., Raoufi, A. and Tassion, V. (2019). Sharp phase transition for the random-cluster and Potts models via decision trees. *Ann. of Math.* (2) **189** 75–99. MR3898174

[7] Duminil-Copin, H. and Tassion, V. (2016). A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. *Comm. Math. Phys.* **343** 725–745. MR3477351

[8] Duminil-Copin, H. and Tassion, V. (2016). A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$. *Enseign. Math.* **62** 199–206. MR3605816

[9] Gouéré, J.-B. (2008). Subcritical regimes in the Poisson Boolean model of continuum percolation. *Ann. Probab.* **36** 1209–1220. MR2435847

[10] Gouéré, J.-B. and Théret, M. (2017). Positivity of the time constant in a continuous model of first passage percolation. *Electron. J. Probab.* **22** Paper No. 49, 21. MR3661663

[11] Grimmett, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **321**. Berlin: Springer. MR1707339

[12] Kesten, H. (1982). *Percolation Theory for Mathematicians*. *Progress in Probability and Statistics* **2**. Boston, MA: Birkhäuser. MR0692943

[13] Last, G. and Penrose, M. (2018). *Lectures on the Poisson Process*. *Institute of Mathematical Statistics Textbooks* **7**. Cambridge: Cambridge Univ. Press. MR3791470

[14] Meester, R. and Roy, R. (1996). *Continuum Percolation*. *Cambridge Tracts in Mathematics* **119**. Cambridge: Cambridge Univ. Press. MR1409145

[15] Men'shikov, M.V. (1986). Coincidence of critical points in percolation problems. *Dokl. Akad. Nauk SSSR* **288** 1308–1311. MR0852458

[16] Møller, J. (1994). *Lectures on Random Voronoĭ Tessellations*. *Lecture Notes in Statistics* **87**. New York: Springer. MR1295245

[17] Penrose, M.D. (2018). Non-triviality of the vacancy phase transition for the Boolean model. *Electron. Commun. Probab.* **23** Paper No. 49, 8. MR3841410

[18] Schneider, R. and Weil, W. (2008). *Stochastic and Integral Geometry*. *Probability and Its Applications* (*New York*). Berlin: Springer. MR2455326

[19] van den Berg, J. (1996). A note on disjoint-occurrence inequalities for marked Poisson point processes. *J. Appl. Probab.* **33** 420–426. MR1385351

[20] Ziesche, S. (2018). Sharpness of the phase transition and lower bounds for the critical intensity in continuum percolation on $\mathbb{R}^d$. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 866–878. MR3795069

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