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The eternal multiplicative coalescent encoding via excursions of Lévy-type processes

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The multiplicative coalescent is a mean-field Markov process in which any pair of blocks coalesces at rate proportional to the product of their masses. In Aldous and Limic (*Electron. J. Probab.* **3** (1998) Paper no. 3) each extreme eternal version of the multiplicative coalescent was described in three different ways, one of which matched its (marginal) law to that of the ordered excursion lengths above past minima of a certain Lévy-type process.

Using a modification of the breadth-first-walk construction from Aldous (*Ann. Probab.* **25** (1997) 812–854) and Aldous and Limic (*Electron. J. Probab.* **3** (1998) Paper no. 3), and some new insight from the thesis by Uribe Bravo (Markovian bridges, Brownian excursions, and stochastic fragmentation and coalescence (2007) UNAM), this work settles an open problem (3) from Aldous (*Ann. Probab.* **25** (1997) 812–854) in the more general context of Aldous and Limic (*Electron. J. Probab.* **3** (1998) Paper no. 3). Informally speaking, each eternal version is entirely encoded by its Lévy-type process, and contrary to Aldous' original intuition, the time for the multiplicative coalescent does correspond to the linear increase in the constant part of the drift of the Lévy-type process. In the "standard multiplicative coalescent" context of Aldous (*Ann. Probab.* **25** (1997) 812–854), this result was first announced by Armendáriz in 2001, while its first published proof is due to Broutin and Marckert (*Probab. Theory Related Fields* **166** (2016) 515–552), who simultaneously account for the process of excess (or surplus) edge counts.

Keywords: entrance law; excursion; Lévy process; multiplicative coalescent; near-critical; random graph; stochastic coalescent

1. Introduction

Erdős-Rényi [32] (binomial) random graph $G(n, p)$ is one of the most studied objects of probabilistic combinatorics. In this model there are $n \geq 2$ *vertices* labeled by $\{1, 2, \dots, n\}$, and each of the $\binom{n}{2}$ edges is present with probability $p \in [0, 1]$ (and absent otherwise), independently of each other. So $G(n, p)$ is a random sub-graph of a complete graph, that looks (qualitatively, in the sense of distribution) the same when viewed from any of its vertices.

The most natural coupling of $(G(n, p), p \in [0, 1])$ is in terms of a family of $\binom{n}{2}$ independent uniform random variables, indexed by the undirected edges $\{i, j\}$; an edge $e = \{i, j\}$ is declared "open (present) in $G(n, p)$" on the event $\{U_{\{i,j\}} \leq p\}$, and "closed in (absent from) $G(n, p)$" on the complement. In this way, if $p_1 \leq p_2$, then $G(n, p_1)$ is a random subgraph of $G(n, p_2)$. A time-change $q := -\log(1 - p)$ transforms this model into a Markov chain running in continuous time. Its transitions are particularly simple: each undirected edge $\{i, j\}$ arrives as an exponential (rate 1) random variable, and stays in the graph forever after. Two different connected components of this (growing) random graph process will merge at the minimal connection time of a pair of vertices (or particles) (k, l), where k is from one, and l from the other component. The *mass* (or *size*) of any connected component (or *block*) equals the number of its particles. Using elementary properties of independent exponentials, it is simple to see that the vector of block sizes of $(G(n, 1 - e^{-q}), q \geq 0)$ is also a continuous-time Markov chain, evolving according to the *multiplicative coalescent* dynamics:

$$\begin{aligned}&\text{each pair of blocks of mass } x \text{ and } y \text{ merges at rate } xy\\&\text{into a single block of mass } x + y.\end{aligned}\tag{1.1}$$

1.1. The multiplicative coalescent in 1997

Suppose a slightly more general setting: let $(x_1, x_2, \ldots, x_m)$ be the vector of initial block masses, where x_i is now a positive integer for each i. We can represent each initial block as a collection of x_i different particles of mass 1, which have been pre-connected in some specified arbitrary way. In particular, the total number of vertices is now $n = \sum_{i=1}^m x_i$. As in $(G(n, 1 - e^{-q}), q \geq 0)$, for each edge $\{k, l\}$ let $\xi_{k,l}$ be an exponential (rate 1) arrival time of the edge connecting k and l (independent over $\{k, l\}$). Then the process of connected component masses again evolves according to (1.1).

A general *multiplicative coalescent* takes values in the space of collections of blocks, where each block has mass in $(0, \infty)$. Informally, it is a stochastic process with transitions specified by (1.1). For a given initial state with a finite number of blocks (where block masses are not necessarily integer-valued), it is easy to formalize (1.1), for example, via a similar "graph-construction", in order to define a continuous-time finite-state Markov process. Furthermore, Aldous [5] extended the state space to include l_2 configurations. More precisely, if $(l^2_\searrow, d)$ is the metric space of infinite sequences $\mathbf{x} = (x_1, x_2, \ldots)$ with $x_1 \geq x_2 \geq \cdots \geq 0$ and $\sum_i x_i^2 < \infty$, where $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_i (x_i - y_i)^2}$, then the *multiplicative coalescent* is a Feller process on $l^2_\searrow$ (see [5], Proposition 5, or Section 2.1 in [46] for an alternative argument), evolving according to description (1.1). The focus in [5] was on the existence and properties of the multiplicative coalescent, as well as on the construction of a particular eternal version $(\mathbf{X}^*(t), -\infty < t < \infty)$, called the (*Aldous'*) *standard* multiplicative coalescent. The standard version arises as a limit of the classical random graph process near the phase transition (each particle has initial mass $n^{-2/3}$ and the random graph is viewed at times $n^{1/3} + O(1)$). In particular, the marginal distribution $\mathbf{X}^*(t)$ of $\mathbf{X}^*$ was described in [5] as follows: if $(W(s), 0 \leq s < \infty)$ is standard Brownian motion and

$$W^t(s) = W(s) - \frac{1}{2}s^2 + ts, \qquad s \geq 0, \tag{1.2}$$

and B^t is its "reflection above past minima"

$$B^t(s) = W^t(s) - \min_{0 \leq s' \leq s} W^t\bigl(s'\bigr), \qquad s \geq 0, \tag{1.3}$$

then (see [5], Corollary 2) the ordered sequence of excursion (away from 0) lengths of B^t has the same distribution as $\mathbf{X}^*(t)$. Note in particular that the total mass $\sum_i X_i^*(t)$ is infinite. The entrance law $\mathbf{X}^*$ "starts from dust" and ends by forming "a giant": $\lim_{t\to-\infty} \|\mathbf{X}^*(t)\|_2 = 0$ and $\lim_{t\to\infty} \|\mathbf{X}^*(t)\|_2 = \infty$.

The author's thesis [46] was based on a related question: are there any other eternal versions of the multiplicative coalescent, and, provided that the answer is positive, what are they? Paper [6] completely described the *entrance boundary* of the multiplicative coalescent (or equivalently, the set of all of its extreme eternal laws). The *extreme eternal laws* or *versions* are conveniently characterized by the property that their corresponding tail σ-fields at time $-\infty$ are trivial. Any (other) eternal versions must be a mixture of extreme ones, see, for example, [31], Section 10. Note that the word "version" is not used here in the classical (Markov process theory) sense.

1.2. Characterizations of eternal versions in 1998

The notation to be introduced next is inherited from [6]. We write $l^3_\searrow$ for the space of infinite sequences $\mathbf{c} = (c_1, c_2, \dots)$ with $c_1 \ge c_2 \ge \cdots \ge 0$ and $\sum_i c_i^3 < \infty$. For $\mathbf{c} \in l^3_\searrow$, let $(\xi_j, j \ge 1)$ be independent with exponential (rate c_j) distributions and consider

$$V^{\mathbf{c}}(s) = \sum_j \bigl(c_j 1_{(\xi_j \le s)} - c_j^2 s\bigr), \qquad s \ge 0. \tag{1.4}$$

We may regard $V^{\mathbf{c}}$ as a Lévy-type process, where for each x only the first jump of size x is kept (cf. Section 2.5 of [6], and Bertoin [14] for background on Lévy processes). In reality, the finite number n_x of jumps of size x is kept, where n_x is the number of indices j for which $x = \mathbf{c}_j$. It is easy to see that $\sum_i c_i^3 < \infty$ is precisely the condition for (1.4) to yield a well-defined process (see also Section 2.1 of [6] or [45], Section 2).

Define the parameter space

$$\mathcal{I} := \bigl((0,\infty) \times (-\infty,\infty) \times l^3_\searrow\bigr) \cup \bigl(\{0\} \times (-\infty,\infty) \times l^3_\searrow \setminus l^2_\searrow\bigr).$$

Now modify (1.2), (1.3) by defining, for each $(\kappa, \tau, \mathbf{c}) \in \mathcal{I}$,

$$\widetilde{W}^{\kappa,\tau}(s) = \kappa^{1/2} W(s) + \tau s - \frac{1}{2}\kappa s^2, \qquad s \ge 0, \tag{1.5}$$

$$W^{\kappa,\tau,\mathbf{c}}(s) = \widetilde{W}^{\kappa,\tau}(s) + V^{\mathbf{c}}(s), \qquad s \ge 0, \tag{1.6}$$

$$B^{\kappa,\tau,\mathbf{c}}(s) = W^{\kappa,\tau,\mathbf{c}}(s) - \min_{0\le s'\le s} W^{\kappa,\tau,\mathbf{c}}\bigl(s'\bigr), \qquad s \ge 0. \tag{1.7}$$

So $B^{\kappa,\tau,\mathbf{c}}(s)$ is again the reflected process with some set of (necessarily all finite, see Theorem 1.1 below) excursions away from 0.

Denote by $\hat{\mu}(y)$ the distribution of the constant process

$$\mathbf{X}(t) = (y, 0, 0, 0, \dots), \qquad -\infty < t < \infty \tag{1.8}$$

where $y \ge 0$ is arbitrary but fixed.

Let $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots) \in l^2_{\searrow}$ be the state of a particular eternal version of the multiplicative coalescent. Then $X_j(t)$ is the mass of its jth largest block at time t. Write

$$S(t) = S_2(t) = \sum_i X_i^2(t), \quad \text{and} \quad S_3(t) = \sum_i X_i^3(t).$$

The main results of [6] are stated next.

Theorem 1.1 ([6], Theorems 2–4). (a) *For each* $(\kappa, \tau, \mathbf{c}) \in \mathcal{I}$ *there exists an eternal multiplicative coalescent* $\mathbf{X}$ *such that for each* $-\infty < t < \infty$, $\mathbf{X}(t)$ *is distributed as the ordered sequence of excursion lengths of* $B^{\kappa,t-\tau,\mathbf{c}}$.

(b) *Denote by* $\mu(\kappa, \tau, \mathbf{c})$ *the distribution of* $\mathbf{X}$ *from* (a). *The set of extreme eternal multiplicative coalescent distributions is precisely*

$$\{\mu(\kappa, \tau, \mathbf{c}) : (\kappa, \tau, \mathbf{c}) \in \mathcal{I}\} \cup \{\hat{\mu}(y) : 0 \le y < \infty\}.$$

(c) *Let* $(\kappa, \tau, \mathbf{c}) \in \mathcal{I}$. *An* (*extreme*) *eternal multiplicative coalescent* $\mathbf{X}$ *has distribution* $\mu(\kappa, \tau, \mathbf{c})$ *if and only if*

$$|t|^3 S_3(t) \to \kappa + \sum_j c_j^3 \qquad \text{a.s. as } t \to -\infty, \tag{1.9}$$

$$t + \frac{1}{S(t)} \to \tau \qquad \text{a.s. as } t \to -\infty, \tag{1.10}$$

$$|t| X_j(t) \to c_j \qquad \text{a.s. as } t \to -\infty, \forall j \ge 1. \tag{1.11}$$

In terms of the above defined parametrization, the Aldous [5] standard (eternal) multiplicative coalescent has distribution $\mu(1, 0, \mathbf{0})$. The parameters τ and κ correspond to time-centering and time/mass scaling respectively: if $\mathbf{X}$ has distribution $\mu(1, 0, \mathbf{c})$, then $\widetilde{\mathbf{X}}(t) = \kappa^{-1/3}\mathbf{X}(\kappa^{-2/3}(t - \tau))$ has distribution $\mu(\kappa, \tau, \kappa^{1/3}\mathbf{c})$. Due to (1.11), the components of $\mathbf{c}$ may be interpreted as the relative sizes of distinguished large blocks in the $t \to -\infty$ limit.

1.3. The main results

The rest of this work will mostly ignore the constant eternal multiplicative coalescents. For a given $(\kappa, \tau, \mathbf{c}) \in \mathcal{I}$ we can clearly write $W^{\kappa,t-\tau,\mathbf{c}}(s) = W^{\kappa,-\tau,\mathbf{c}}(s) + ts$, $s \ge 0$. The Lévy-type process $W^{\kappa,-\tau,\mathbf{c}}$ is particularly important for this work. As we are about to see, $W^{\kappa,-\tau,\mathbf{c}}$ corresponds to the Lévy-type process from the abstract as soon as $\mathbf{X}$ has law $\mu(\kappa, \tau, \mathbf{c})$.

As noted in [6] and in [5] beforehand, at the time there was no appealing intuitive explanation of why excursions of a stochastic process would be relevant in describing the marginal laws in Theorem 1.1(a). One purpose of this work is to offer a convincing explanation (see Proposition 5 below and also in Section 4, then Lemma 9 in Section 5, and Lemma 11 in Section 6). Furthermore, open problem (3) of [5] asks about the existence of a two parameter (non-negative) process $(B^t(s), s \ge 0, t \in \mathbb{R})$ such that the excursion (away from 0) lengths of $(B^{\cdot}(s), s \ge 0)$

evolve as $X^*(\cdot)$. The statement of this problem continues by offering an intuitive explanation for why $R^{1,0,\mathbf{0}} := \{\text{reflected}(W^{1,0,\mathbf{0}}(s) + ts), s \geq 0, t \geq 0\}$ should not be the answer to this problem. Aldous' argument is more than superficially convincing, but the striking reality is that, on the contrary, the simplest guess $R^{1,0,\mathbf{0}}$ is the correct answer. Armendáriz [11] obtained but never published this result, and Broutin and Marckert [27] recently derived it, via a different approach from either [11] or the one presented here, while considering in addition the excess-edge data in agreement with [5] (thus improving on the Armendáriz claim).

Popular belief judges the breadth-first-walk construction, on which [5,6] reside, as "inadequate" and the main reason for the just described "confusion" in the statement of [5], open problem (3). One of the main points of this work is to show the contrary. Indeed, a modification of the original (Aldous') breadth-first-walk from [5,6], combined with a rigorous formulation (see Proposition 3) of Uribe's [63] graphical interpretation of the Armendáriz' representation [11], yields the following claim of independent interest, here stated for readers' benefit in the simplest (purely) homogeneous setting.

Claim (Proposition 5, special case). *Suppose that $x_1 = x_2 = \cdots = x_n = 1$, for some $n \in \mathbb{N}$, and define for $q > 0$*

$$Z^q(s) := \sum_{i=1}^{n} 1_{(\xi_i \leq q \cdot s)} - s, \qquad s \geq 0,$$

where ξ_i, $i = 1, \dots, n$ is a family of i.i.d. exponential (rate 1) random variables. For each $q > 0$, let "blocks at time q" be the finite collection of excursions (above past minima) of Z^q, and for each block let its mass be the corresponding excursion length. Set $X(0)$ to be the configuration of n blocks of mass 1, and for $q > 0$ let $X(q)$ be the configuration of masses of blocks at time q ($X(q)$ is a vector with components listed in non-increasing order, and infinitely many 0s may be appended to make it an element of $l^2_{\searrow}$). Then $(X(q), q \geq 0)$ evolves according to (1.1).

To the best of our knowledge, even the "static" statement that matches the law of $X(q)$ to the law of the component sizes of the continuous-time homogeneous Erdős-Rényi random graph for each fixed time q separately, was not previously recorded (even though the analysis of [63], on pages 111–112 is implicitly equivalent). To have a glimpse at the power of this approach, the reader is invited to fix $c > 1$, and consider the asymptotic behavior (as $n \to \infty$) of a related process $Z^{(1/n,\dots,1/n),cn}(s) := \sum_{i=1}^{n} \frac{1}{n} 1_{(n\xi_i \leq cns)} - s$, $s \geq 0$, (see Section 2 or Proposition 5 for notation) in order to determine (in a few lines only) the asymptotic size of the giant component in the supercritical regime. In addition, the simultaneous breadth-first walks framework allows for a particularly elegant treatment of surplus edges, carried out in [48].

The analysis similar to that of [6] (to be done in Sections 5 and 6) now yields the following theorem.

Theorem 1.2. *Fix a Lévy-type process $W^{\kappa,-\tau,\mathbf{c}}$, and for any $t \in (-\infty, \infty)$ define*

$$W^{\kappa,t-\tau,\mathbf{c}}(s) := W^{\kappa,-\tau,\mathbf{c}}(s) + ts, \qquad s \geq 0.$$

Let $B^{\kappa,t-\tau,\mathbf{c}}$ be defined as in (1.7). For each t, let $\mathbf{X}(t) = \mathbf{X}^{\kappa,\tau,\mathbf{c}}(t)$ be the infinite vector of ordered excursion lengths of $B^{\kappa,t-\tau,\mathbf{c}}$ away from 0. Then $(\mathbf{X}(t), t \in (-\infty, \infty))$ is a càdlàg realization of $\mu(\kappa, \tau, \mathbf{c})$.

1.4. Further comments on the literature and related work

For almost two decades the only stochastic merging process widely studied by probabilists was the (Kingman) coalescent [43,44]. Starting with Aldous [5,9], and Pitman [55], Sagitov [58], and Donnelly and Kurtz [29], the main-stream probability research on coalescents was much diversified.

The Kingman coalescent and, more generally, the mass-less (exchangeable) coalescents of [29,55,58] mostly appear in connection to the mathematical population genetics, as universal (robust) scaling limits of genealogical trees (see, for example, [19,53,59,60], or a survey [13]).

The standard multiplicative coalescent is the universal scaling limit of numerous stochastic (typically combinatorial or graph-theoretic) homogeneous (or symmetric) merging-like models [1,5,10,20,21,23,57,61]. The "non-standard" eternal extreme laws from [6] are also scaling limits of inhomogeneous random graphs and related processes, under appropriate assumptions [6,22, 24,26].

The two nice graphical constructions for coalescents with masses were discovered early on: by Aldous in [5] for the multiplicative case, and almost simultaneously by Aldous and Pitman [7] for the additive case (here any pair of blocks of mass x and y merges at rate $x + y$), via cutting the continuum random tree [4] (see also Remark 2.1(c) in the next section). The analogue of [6] in the additive coalescent case is again due to Aldous and Pitman [8]. No nice graphical construction for another (merging rate) coalescent with masses seems to have been found since. For studies of stochastic coalescents with general kernel, see Evans and Pitman [34] and Fournier [35,36]. Interest for probabilistic study of related Smoluchowski's equations (with general merging kernels) was also sparked by [9], see, for example, Norris [54], Jeon [41], then Fournier and Laurençot [37,38] and Bertoin [18] for more recent, and Merle and Normand [51,52] for even more recent developments. All of the above mentioned models are *mean-field*. See, for example, [12,39,49] for studies of (mass-less) coalescent models in the presence of spatial structure.

As already mentioned, Broutin and Marckert [27] obtain Theorem 1.2 in the standard multiplicative coalescent case, via *Prim's algorithm* construction invented for the purpose of their study, and notably different from the approach presented here. Before them Bhamidi et al. [20, 21] proved f.d.d. convergence for models similar to Erdős-Rényi random graph. For the standard additive coalescent, analogous results were obtained rather early by Bertoin [15,16] and Chassaing and Louchard [28], and are rederived in [27], again via an appropriate Prim's algorithm representation.

In parallel to and independently from the research presented here, both Martin and Ráth [50] and Uribe Bravo [62] have been studying closely related models and questions. Their approaches seem to be quite different from the one taken here, with some notable similarities. James Martin and Balázs Ráth [50] introduce a coalescence-fragmentation model called the *multiplicative coalescent with linear deletion (MCLD)*. Here in addition to (and independently of) the multiplicative coalescence, each component is permanently removed from the system at a rate proportional to its mass (this proportionality parameter is denoted by λ). In the absence of deletion (i.e., when $\lambda = 0$), their "tilt (and shift) operator" representation of the MCLD leads to an alternative proof of Theorem 1.2, sketched in detail in [50], Section 6.1 (see [50], Corollary 6.6). Further comments on links and similarities to [50] will be made along the way, most frequently in Section 3. Gerónimo Uribe [62] relies on a generalization of the construction from [27], explains its links to Armendáriz' representation, and works towards another derivation of Theorem 1.2.

The arguments presented in the sequel are partially relying on direct applications of a non-trivial result from [6], Section 2.6 (depending on [6], Section 2.5) in Section 6 (more precisely, Corollary 10). In comparison, (a) [27] also rely on the convergence results of [5] in the standard multiplicative coalescent setting, as well as additional estimates proved in [1], and (b) the analysis done in [50], Sections 4 and 5 seems to be a formal analogue of that in [6], Sections 2.5–2.6.

The present approach to Theorem 1.2 is of independent interest even in the standard multiplicative coalescent setting (where Section 5 would simplify further, since $\mathbf{c} = \mathbf{0}$, and already Lemma 8 from [5] would be sufficient for making conclusions in Section 6). In addition, it may prove useful for continued analysis of the multiplicative coalescents, as well as various other processes in the multiplicative coalescent "domain of attraction".

The reader is referred to Bertoin [17] and Pitman [56] for further pointers to stochastic coalescence literature, and to Bollobas [25] and Durrett [30] for the random graph theory and literature.

The rest of the paper is organized as follows: Section 2 introduces the simultaneous breadth-first walks and explains their link to the (marginal) law of the multiplicative coalescent, and the original breadth-first walks of [5,6]. Section 3 recalls Uribe's diagrams and includes Proposition 3, that connects the diagrams to the multiplicative coalescent. In Section 4, the simultaneous BFWs and Uribe's diagrams are linked, and as a result an important conclusion is made in Proposition 5 (the generalized version of the claim preceding Theorem 1.2). All the processes considered in Sections 2–4 have finite initial states. Section 5 serves to pass to the limit where the initial configuration is in $l^2_\searrow$. The similarities to and differences from [6] are discussed along the way, and in the accompanying paper [45]. Theorem 1.2 is proved in Section 6. Supplementary material [45] is described in the paragraph preceding the bibliography.

2. Simultaneous breadth-first walks

This section revisits the Aldous [5] breadth-first walk construction of the multiplicative coalescent started from a finite vector $\mathbf{x}$ (see, for example, [45], Section 3), with two important differences (or modifications), to be described along the way.

Recall that "breadth-first" refers here to the order in which the vertices of a given connected graph (or one of its spanning trees) are explored. Such exploration process starts at the root, visits all of its children (these vertices become the 1st generation), then all the children of all the vertices from the 1st generation (these vertices become the 2nd generation), then all the children of the 2nd generation, and keeps going until all the vertices (of all the generations) are visited, or until forever (if the tree is infinite).

Refer to $\mathbf{x} = (x_1, x_2, x_3, \dots) \in l^2_\searrow$ as *finite*, if for some $i \in \mathbb{N}$ we have $x_i = 0$. Let the *length* of $\mathbf{x}$ be the number $\mathrm{len}(\mathbf{x})$ of non-zero coordinates of $\mathbf{x}$. Fix a finite initial configuration $\mathbf{x} \in l^2_\searrow$. For each $i \le \mathrm{len}(\mathbf{x})$ let ξ_i have exponential (rate x_i) distribution, independently over i.

As before, $\mathbf{X}$ is used to denote a stochastic process evolving according to the multiplicative coalescent dynamics. In this section, the initial configuration $\mathbf{X}(0) = \mathbf{x}$ is discrete (finite), so $\mathbf{X}$ is the law of the continuous-time random graph connected component masses. Given ξ, simultaneously for all $q > 0$, we next construct the modified (with respect to [5,6]) breadth-first walk, coupled with $\mathbf{X}(q)$ started from $\mathbf{X}(0) = \mathbf{x}$ at time 0 (see Propositions 1 and 5). This simultaneity in q is a *new feature* with respect to [5,6]. To the best of our knowledge, the powerful (full law) coupling of Proposition 5 (see also the simplified Claim in the introduction) was previously unknown. The family of processes defined in (2.1) below will be henceforth called the *simultaneous breadth-first walks*.

Fix $q > 0$, and consider the sequence $(\xi_i/q)_{i \le \mathrm{len}(\mathbf{x})}$. Let us introduce the abbreviation $\xi_i^q := \xi_i/q$. The order statistics of $(\xi_i^q)_{i \le \mathrm{len}(\mathbf{x})}$ are $(\xi_{(i)}^q)_{i \le \mathrm{len}(\mathbf{x})}$. Define

$$Z^{\mathbf{x},q}(s) := \sum_{i=1}^{\mathrm{len}(\mathbf{x})} x_i 1_{(\xi_i^q \le s)} - s = \sum_{i=1}^{\mathrm{len}(\mathbf{x})} x_{(i)} 1_{(\xi_{(i)}^q \le s)} - s, \qquad s \ge 0, q > 0. \tag{2.1}$$

In words, $Z^{\mathbf{x},q}$ has a unit negative drift and successive positive jumps, which occur precisely at times $(\xi_{(i)}^q)_{i \le \mathrm{len}(\mathbf{x})}$, and where the magnitude of the ith successive jump is denoted by $x_{(i)}$. Figure 1 shows graphs of $Z^{\mathbf{x},q}$ and of $Z^{\mathbf{x},4q/3}$ for the same realization as that depicted on Figure 2. The three ticks on the x-axis of each graph correspond to ξ_1^q, ξ_5^q and ξ_6^q (resp. to $\xi_1^{4q/3}$, $\xi_5^{4q/3}$ and $\xi_6^{4q/3}$). The meaning of the intervals indicated in gray or blue below each graph will become clear shortly (see also Figure 2).

Here is the first important observation. For each q, the multiplicative coalescent started from $\mathbf{x}$ and evaluated at time q can be constructed in parallel to $Z^{\mathbf{x},q}$ via a breadth-first walk coupling, similar to the one from [5,6]. The interval $F_1^q := [0, \xi_{(1)}^q]$ is the first "load-free" period. Set $J_0 := \{1, 2, \dots, \mathrm{len}(\mathbf{x})\}$. At the time of the first jump of $Z^{\mathbf{x},q}$ we note

$$\pi_1 := i \quad \text{if and only if} \quad \xi_i = \xi_{(1)}, \quad \text{and} \quad J_1 := J_0 \setminus \{\pi_1\},$$

so that π_1 is the index of the first size-biased pick from $(\mathbf{x}_i)_{i=1}^{\mathrm{len}(\mathbf{x})}$ using ξs (or equally, ξ^qs).

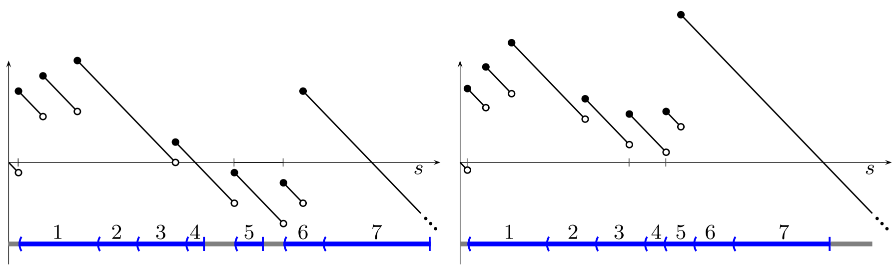

Figure 1. Graphs of joint realizations of $Z^{\mathbf{x},q}$ and $Z^{\mathbf{x},4q/3}$.

Furthermore, let us define for $l \le \mathrm{len}(\mathbf{x})$

$$\pi_l := i \quad \text{if and only if} \quad \xi_i = \xi_{(l)}, \qquad l \in \{1, \dots, \mathrm{len}(\mathbf{x})\}, \quad \text{and} \quad J_l := J_{l-1} \setminus \{\pi_l\}. \tag{2.2}$$

In this way, $(x_{\pi_1}, x_{\pi_2}, \dots, x_{\pi_{\mathrm{len}(\mathbf{x})}})$ is the size-biased random ordering of the initial non-trivial block masses, and in particular x_{π_i} equals $x_{(i)}$ from (2.1). As already noted, the random permutation π does not depend on q.

Let $\mathcal{F}_s^q := \sigma\{\{\{\xi_i^q > u\} : i \in J_0\}, u \le s\}$. Then $\mathcal{F}^q = \{\mathcal{F}_s^q, s \ge 0\}$ is the filtration generated by the arrivals of ξ^qs. Due to elementary properties of independent exponentials, it is clear that the above defined process $Z^{\mathbf{x},q}$ is a continuous-time Markov chain with respect to $\mathcal{F}^q$. Indeed, given $\mathcal{F}_s^q$, the (residual) clocks $\xi_i^q \vee s - s$ are again mutually independent, and moreover on the event $\{\xi_i^q > s\}$ we clearly have $P(\xi_i^q - s > u | \mathcal{F}_s^q) = e^{-x_i q u} = P(\xi_i^q > u)$. Furthermore, $\xi_{(1)}^q$ is a finite stopping time with respect to $\mathcal{F}^q$ and

$$P\big(\xi_i^q - \xi_{(1)}^q > u | \mathcal{F}_{\xi_{(1)}}^q\big) 1_{(i \in J_1)} = e^{-x_i q u} 1_{(i \in J_1)} = P\big(\xi_i^q > u\big) 1_{(i \in J_1)}. \tag{2.3}$$

Let $I_0 = \varnothing$ and $I_1 := (\xi_{(1)}^q, \xi_{(1)}^q + x_{\pi_1}]$. Note that the length of the interval I_1 is the same (positive) quantity x_{π_1} for all $q > 0$. During the time interval I_1 the dynamics "listens for the children of π_1". More precisely, if for some j we have $\xi_j^q \in I_1$, or equivalently, if $\xi_j^q - \xi_{(1)}^q \le x_{\pi_1}$, we can interpret this as

edge $j \leftrightarrow \pi_1$ appears before time q in the multiplicative coalescent.

Indeed, as argued above, $P(\xi_j^q - \xi_{(1)}^q > x_{\pi_1} | \mathcal{F}_{\xi_{(1)}^q}^q) = e^{-q x_j x_{\pi_1}}$, and this is precisely the multiplicative coalescent probability of the jth and the π_1st block not merging before time q.

For any two reals $a < b$ and an interval $[c, d]$ where $0 \le c < d$, define the concatenation

$$(a, b] \oplus [c, d] := (a + c, b + d].$$

Recall that $I_1 = (\xi_{(1)}^q, \xi_{(1)}^q + x_{\pi_1}]$, and define N_1 to be the number of ξ^qs that rung during I_1 (this is the size of the 1st generation in the exploration process). For any $l \ge 2$ define recursively: if I_{l-1} is defined

$$I_l^q \equiv I_l := \begin{cases} I_{l-1} \oplus [0, x_{\pi_l}], & \text{provided } \dfrac{\xi_{(l)}}{q} \in I_{l-1}, \\ \text{undefined}, & \text{otherwise}, \end{cases} \tag{2.4}$$

and if I_l is defined in (2.4), let

$$N_l^q \equiv N_l := \text{the number of } \xi^q\text{s that rung during } I_l, \tag{2.5}$$

and otherwise let N_l be (temporarily) undefined. Since ξ^qs decrease in q, the intervals $I_\cdot^q$ defined in this (coupling) construction do vary over q (their endpoints decrease in q), but all of their lengths are constant in q. In fact, if defined, I_l equals $(\xi_{(1)}^q, \xi_{(1)}^q + \sum_{m=1}^l x_{\pi_m}]$. We henceforth abuse the notation and mostly omit the superscript q when referring to Is or Ns.

During each $I_l \setminus I_{l-1}$ the coupling dynamics "listens for the children of π_l", among all the ξ^qs which have not been heard before (i.e., they did not ring during I_{l-1}). If I_l is defined in (2.4), the set of children of π_l in the above breadth-first order is precisely $J_{N_{l-1}} \setminus J_{N_l}$ (Js were defined in (2.2)), which will be empty if and only if $N_l = N_{l-1}$. The same memoryless property of exponential random variables as used above (e.g. in (2.3)) ensures that

$$\begin{aligned} &P\big(\xi_k^q \in I_l \setminus I_{l-1} | \mathcal{F}_{\xi_{(1)}^q + x_{\pi_1} + \cdots + x_{\pi_{l-1}}}\big) 1_{(k \in J_{N_{l-1}})} \\ &\quad = P(k \in J_{N_{l-1}} \setminus J_{N_l} | \mathcal{F}_{\xi_{(1)}^q + x_{\pi_1} + \cdots + x_{\pi_{l-1}}}) 1_{(k \in J_{N_{l-1}})} = e^{-q x_k x_{\pi_l}} 1_{(k \in J_{N_{l-1}})} \qquad \text{a.s.} \end{aligned} \tag{2.6}$$

Due to independence of ξs, the residual clocks have again the (conditional) multi-dimensional product law. So for each l, the set of children of π_l equals in law to the set of blocks which are connected by an edge to the π_lth block in the multiplicative coalescent at time q, given that they did not get connected by an edge (before time q) to any of the previously recorded blocks $\pi_1, \ldots, \pi_{l-1}$.

The above procedure may (and typically will) stop at some $l_1 \le \text{len}(\mathbf{x})$, due to $\xi_{(l_1)}^q$ not arriving in I_{l_1-1}. This will happen if and only if the whole connected component of the π_1st initial block (in the multiplicative coalescent, observed at time q) was explored during I_{l_1-1}, and the π_{l_1-1}st initial block was its last visited "descendant", while the rest of the graph was not yet "seen" during $F_1^q \cup I_{l_1-1}$. Indeed, if $a_1 = \xi_{(1)}^q$ and $b_1 = \xi_{(1)}^q + x_{\pi_1} + \cdots + x_{\pi_{l_1-1}}$, it is straight-forward to see that

$$Z^{\mathbf{x},q}(s) > Z^{\mathbf{x},q}(a_1) = Z^{\mathbf{x},q}(b_1), \qquad \forall s \in (a_1, b_1). \tag{2.7}$$

In words, the interval $Cl(I_{l_1-1}) = [a_1, b_1]$ is an *excursion* of $Z^{q,\mathbf{x}}$ *above past minima* of length $b_1 - a_1 = x_{\pi_1} + \cdots + x_{\pi_{l_1-1}}$, which is the total mass of the first (explored) spanning tree in the breadth-first walk. Due to (2.3), (2.6) and the related observations made above, this (random) tree matches the spanning tree of the connected component of π_1 in the coupled multiplicative coalescent, observed at time q. This (first) spanning tree is rooted at π_1 (cf. Figures 2 and 3) for all $q > 0$. It will be clear from construction, that the roots of subsequently explored spanning trees can (and inevitably do) change at some $q > 0$.

The next interval of time $F_2^q := (\xi_{(1)}^q + x_{\pi_1} + \cdots + x_{\pi_{l_1-1}}, \xi_{(l_1)}^q]$ is again "load-free" for the breadth-first walk. Repeating the above exploration procedure starting from $\xi_{(l_1)}^q$ amounts to defining $I_{l_1}^q \equiv I_{l_1} := (\xi_{(l_1)}^q, \xi_{(l_1)}^q + x_{\pi_{l_1}}]$ and listening for the children of π_{l_1}st block during I_{l_1}, and then running the recursion (2.4), (2.5) for $l \ge l_1 + 1$ until it stops, which occurs when all the vertices (blocks) of the second connected component are explored. This exploratory coupling construction continues until all the initial blocks of positive mass are accounted for, or equivalently until $\xi_{(\text{len}(\mathbf{x}))}^q$. Clearly no ξ can ring during $I_{\text{len}(\mathbf{x})} \setminus I_{\text{len}(\mathbf{x})-1}$ (which is open on the left), and $Z^{\mathbf{x},q}$ continues its evolution as a deterministic process (line of slope -1) starting from the left endpoint of $I_{\text{len}(\mathbf{x})}$.

Figures 1 and 2 illustrate the just described coupling. In the current notation $\mathbf{x} = (1.1, 0.8, 0.5, 0.4, 0.4, 0.3, 0.2, 0, 0, \ldots)$ so that $\text{len}(\mathbf{x}) = 7$. The three "load-free" intervals F_i^q, $i = 1, 2, 3$ are indicated in gray. The interval I_i^q or $I_i^q \setminus I_{i-1}^q$ (the latter corresponds to non-leading blocks) is indicated in blue with marker i on top. The excursions of $Z^{\mathbf{x},q}$ above past minima are the (closed) disjoint unions of blue intervals. Each excursion of $Z^{\mathbf{x},q}$ above past minima corresponds

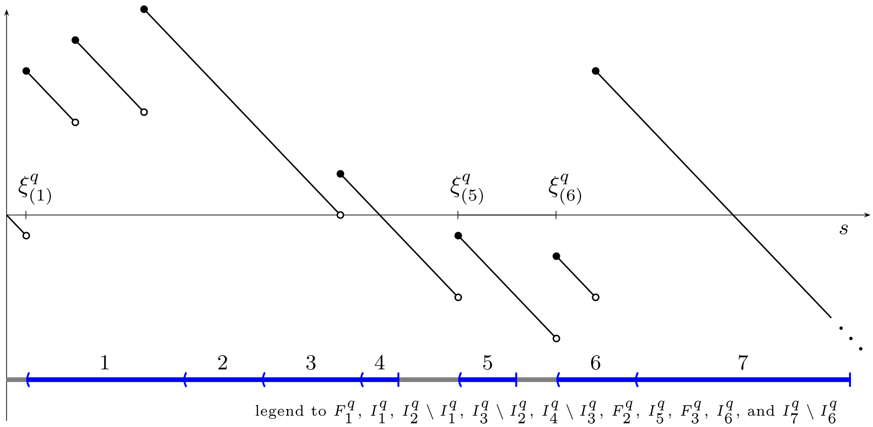

Figure 2. A magnified view of the left graph in Figure 1.

uniquely to a connected component in the coupled multiplicative coalescent evaluated at time q. It is clear from (2.4), (2.5) that the order of blocks visited within any given connected component is breadth-first. Note as well that the connected components are explored in the size-biased order. Indeed, the fact that the initial block of the next component to be explored is picked in a size-biased way, with respect to block masses, induces *size-biasing* of connected components (again with respect to mass) in the multiplicative coalescent at time q.

The above reasoning can be also summarized as follows:

Proposition 1. *Let* $\mathbf{x}$ *be finite, and* $q > 0$. *Let the breadth-first walk* $Z^{\mathbf{x},q}$ *encode* $\mathbf{X}(q) \in l^2_\searrow$ *as follows: for each excursion (above past minima) of* $Z^{\mathbf{x},q}$ *record its length, and let* $\mathbf{X}(q)$ *be the vector of thus obtained decreasingly ordered excursion lengths, appended with infinitely many* 0*s. Then* $\mathbf{X}(q)$ *has the marginal law of the multiplicative coalescent started from* $\mathbf{X}(0) = \mathbf{x}$ *and observed at time* q.

Since $(Z^{\mathbf{x},q})_{q>0}$ exist on one and the same probability space, the process

$$\mathbf{X} := \big(\mathbf{X}(q), q > 0\big) \quad \text{and} \quad \mathbf{X}(0) = \mathbf{x}$$

is well defined, and we refer to it temporarily as the *multiplicative coalescent marginals coupled to* $Z^{\mathbf{x},\cdot}$.

In the original breadth-first construction (coupling) in [5,6], the leading block of each component did not correspond to a jump of the walk. It was chosen instead via an auxiliary source of randomness. In comparison, each non-leading block was uniquely matched to a jump of the breadth-first walk, and for a non-leading block of mass m, this jump had size m and was exponentially distributed with rate m. As in the simultaneous construction, all the exponential jumps were mutually independent. The reader is also referred to [45], Section 3 for further details.

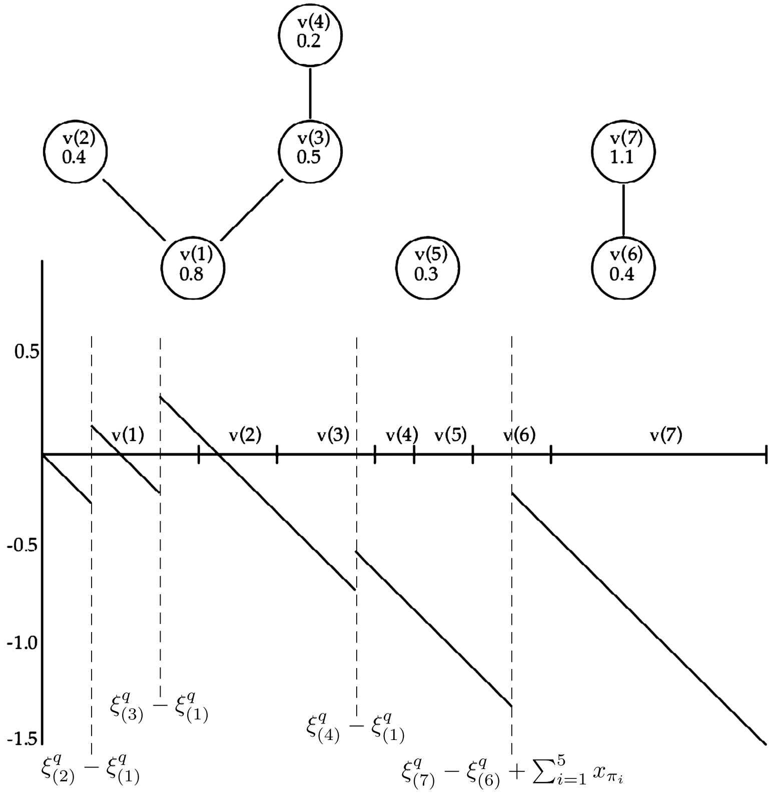

Figure 3. Comparison of the original and the simultaneous breadth-first walk paths.

Figure 3 (without the vertical dashed lines and their labels) is a duplicate of [6], Figure 1. Here we assume that it shows the graph of the original breadth-first walk (at multiplicative coalescent-time q), corresponding to the same realization as the one used in Figure 2. In particular $\mathbf{x} = (1.1, 0.8, 0.5, 0.4, 0.4, 0.3, 0.2, 0, 0, \ldots)$ and $x_{\pi_i} \equiv v(i)$. Note that the segment marked by i in Figure 2 has exactly the same length as the segment marked by $v(i)$ in Figure 3. One can read off from Figure 3 that π belongs to $\{(2,4,3,7,6,5,1), (2,5,3,7,6,4,1)\}$. However, $\xi_{(1)} = \xi_2$, $\xi_{(5)} = \xi_6$ and $\xi_{(6)} = \xi_5$ (or $\xi_{(6)} = \xi_4$, depending on π) are not observed. In comparison, in the simultaneous breadth-first walk construction, the $\xi^q_{(1)}$ (here it equals ξ_2), $\xi_{(5)}$ (here it equals ξ_6) and $\xi_{(6)}$ all influence (see (2.1) or Figure 2) the walk. The additional vertical dashed lines (not existing in [6], Figure 1) and their labels illustrate the link between the two breadth-first walk constructions (see also Figure 2 and the explanations provided below it). In particular, the first jump of the original breadth-first walk happens at time (distributed as) $\xi^q_{(2)} - \xi^q_{(1)}$, the second one happens at time (distributed as) $\xi^q_{(3)} - \xi^q_{(1)}$, and this continues until the first component is exhausted. The next jump happens at the time the next non-leading block is encountered.

Moreover, due to elementary properties of residual exponentials, the following is true.

Lemma 2. *Fix a finite initial configuration* $\mathbf{x}$ *and* (*multiplicative coalescent*) *time* $q > 0$, *recall* $Z^{\mathbf{x},q}$ *from* (2.1), *and the load-free intervals* F_i^q, $i \geq 1$. *If* F_i^q, $i \geq 1$ *are all cut from the abscissa, and the jumps which happen at the end points of* $F_\cdot^q$ *are all ignored* (*deleted*), *then* (*the graph of*) $Z^{\mathbf{x},q}$ *transformed in this way has the law of* (*the graph of*) *the breadth-first walk from* [5,6], *corresponding to* q.

Most of the argument is included in the above made observations and explanations. Figures 2 and 3 illustrate the claim. Time s for $Z^{\mathbf{x},q}$ (on Figure 2) corresponds to time $s - \xi_{(L+1)}^q + \sum_{i=1}^K x_{\pi_i}$ in the original breadth-first walk (on Figure 3), where L is the number of connected components completely explored via $Z^{\mathbf{x},q}$ before time s, and $(x_{\pi_i})_{i=1}^K$ is the total mass of these L connected components (on $L = 0$ this mass is naturally 0). The consecutive load-free intervals and their final jumps, which are cut out by the transformation, serve as the auxiliary source of randomness used for choosing leading blocks in the exploration of [5,6]. The details are left to the reader.

Remark 2.1. (a) At the moment it may seem that the main (potential) gain of the just described modified breadth-first construction is in "compactifying" the input data (compare with [6], Section 2.3 or [27], Section 6.1 for alternatives). It will become apparent in the sequel (see Proposition 5 and Section 5) that this construction is quite natural, in that stronger convergence results can be obtained from it with less effort.

(b) As $q \searrow 0$, the ξ^q diverge to ∞, but more importantly they diverge from each other, so $\mathbf{X}(q) \to \mathbf{x} = \mathbf{X}(0)$ almost surely. It is not difficult to see that for any $q \geq 0$, $\mathbf{X}$ is also almost surely right-continuous at q (see Section 4).

(c) A little thought is needed to realize that as q increases, the excursion families of $Z^{\mathbf{x},\cdot}$ are "nested": with probability 1, if $q_1 < q_2$ and two blocks k, l are merged in $\mathbf{X}(q_1)$, they are also merged in $\mathbf{X}(q_2)$ (an example of this is depicted in Figure 1). This fact is encouraging, but cannot ensure on its own that the multiplicative coalescent marginals coupled to $Z^{\mathbf{x},\cdot}$ is in fact a multiplicative coalescent process. Moreover, while the nesting is encouraging, the following observation will likely increase the level of reader's skepticism about $\mathbf{X}$ having the multiplicative coalescent law: if $e_1^{q_1}$, $e_2^{q_1}$ and $e_3^{q_1}$ are three different excursions of $Z^{\mathbf{x},q_1}$ explored in the increasing order of their indices, and if the initial blocks k, l, m are contained in the connected components matched to $e_1^{q_1}$, $e_2^{q_1}$, $e_3^{q_1}$, respectively, then it is impossible that k and m are merged in $X(q_2)$ without l being merged with k (and therefore with m) in $X(q_2)$. If there is a simultaneous (for all q) scaling limit of $(Z^{\mathbf{x},q}, \mathbf{X}(q))$ (under well chosen hypotheses), the just mentioned property persists in the limit. This observation is perhaps the strongest intuitive argument pointing against the claim of Proposition 5 and Theorem 1.2. On the other hand, analogous representations of the standard additive coalescent are well-known (cf. [15,16,28]). One may be less surprised there, due to the "cutting the CRT" dual (from [7]), and the well-known connection between the exploration process of continuum trees and forests on the one hand, and Brownian excursions on the other (cf. [4,17,56]). As Nicolas Broutin (personal communication) points out, any (binary) fragmentation can be formally represented as a "stick-breaking" process, in which the two broken pieces of any split block remain nearest neighbors (in some arbitrary but fixed way). The reversed "coalescent" will then have the above counterintuitive property by definition. However, one is particularly fortunate if both of these processes (time-reversals of each other) are Markov, and if in addition the "sticks" are the excursions of a (generalized) random walk or a related process.

(d) Let us denote by $\mathcal{C}$ the operation on paths (i.e., the cutting and pasting transformation) from Lemma 2. In [6], the non-trivial multiplicative coalescent extreme entrance laws were obtained by taking limits of Aldous' breadth-first walks (see Section 5 below), and the limits of their excursions (nearly) above past minima. It was shown that these excursion lengths, considered as an $l^2_\searrow$-valued random object, converge in law to the excursion lengths above past minima of the limiting "walk" (a member of the family defined in (1.6)). Lemma 2 makes this latter (somewhat technical) step redundant in the present setting. More precisely, if one can show that under the same hypotheses as those in [6], $(Z^{t+n^{1/3},\mathbf{x}^{(n)}})_n$ (see Section 5 for precise definitions) converges to the same $W^{\kappa,t-\tau,\mathbf{c}}$ as their $(\mathcal{C}(Z^{t+n^{1/3},\mathbf{x}^{(n)}}))_n$, the conclusion about the ordered excursion lengths is immediate. Indeed, the sequence of excursion above past minima of $Z^{t+n^{1/3},\mathbf{x}^{(n)}}$ almost surely matches the sequence of excursions (nearly) above past minima of $\mathcal{C}(Z^{t+n^{1/3},\mathbf{x}^{(n)}})$, for which Propositions 7 and 9 of [6] (including the results in [6], Section 2.6) apply verbatim.

3. Uribe's diagram

We start by recalling the insight given in Chapter 4 of Uribe [63], in the notation analogous to that of Section 2. In particular, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is a finite-dimensional vector with $n \geq 2$, ξ_i is an exponential (rate x_i) random variable, and ξs are mutually independent. Denote by π the size-biased random reordering of $\mathbf{x}$, which is determined by ξs, so that $\xi_{\pi_i} \equiv \xi_{(i)}$, $\forall i$ (almost surely).

Define n different half-lines: for $s \geq 0$

$$
\begin{aligned}
L'_1: &\quad s \mapsto \xi_{(1)} - 0 \cdot s, \\
L'_2: &\quad s \mapsto \xi_{(2)} - x_{\pi_1} s, \\
L'_3: &\quad s \mapsto \xi_{(3)} - (x_{\pi_1} + x_{\pi_2}) s, \\
&\quad \cdots\cdots\cdots \\
&\quad \cdots\cdots\cdots \\
L'_n: &\quad s \mapsto \xi_{(n)} - (x_{\pi_1} + x_{\pi_2} + \cdots + x_{\pi_{n-1}}) s.
\end{aligned}
$$

Consider two integers k, j such that $1 \leq j < k \leq n$. Since L'_k starts (a.s.) at a strictly larger value than L'_j, and it has (absolute) slope strictly greater than L'_j, it is clear that L'_k and L'_j intersect at some $s_{k,j} > 0$. For each $k = 2, \dots, n$ define

$$
s_k := \min_{j<k} s_{k,j}, \qquad \ell_k := \{1 \leq j < k : s_k = s_{k,j}\}.
$$

There are (almost surely) no ties among s_k or $s_{k,j}$ for different indices k, j (see also Remark 3.1 below). Uribe's diagram consists of line segments (see Figures 4 and 5)

$$
\begin{aligned}
L_1: &\quad s \mapsto \xi_{(1)} - 0 \cdot s, \qquad s \in [0, s_2 \vee \cdots \vee s_n], \\
L_2: &\quad s \mapsto \xi_{(2)} - x_{\pi_1} s, \qquad s \in [0, s_2],
\end{aligned}
$$

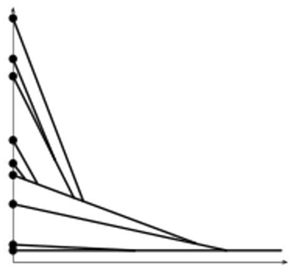

Figure 4. An illustration of Uribe's diagram.

$$
\begin{aligned}
L_3: &\quad s \mapsto \xi_{(3)} - (x_{\pi_1} + x_{\pi_2})s, &\quad s \in [0, s_3],\\
&\quad \cdots\cdots\cdots \\
&\quad \cdots\cdots\cdots \\
L_n: &\quad s \mapsto \xi_{(n)} - (x_{\pi_1} + x_{\pi_2} + \cdots + x_{\pi_{n-1}})s, &\quad s \in [0, s_n].
\end{aligned}
$$

The image of a realization of L in Figure 4 is inspired by [63], Chapter 4, Figure 1. It is interesting to note that its reflection in the x axis is a redrawing of [50], Figure 1.3. A more detailed figure, which will correspond in terms of the values of $\mathbf{x}$, q and ξs to the images in Figures 2 and 3 is provided in Section 4.

According to [63], Chapter 4, Section 2, *Armendáriz' representation of the multiplicative coalescent* [11] is this graphical summary (introduced in [63] for enhanced understanding) joint with an informal description of the following kind:

- the state space is $\mathbb{R}_+^n$, its elements are interpreted as lists of block masses, not necessarily ordered,
- at time 0 match the block of mass x_{π_i} to L_i',
- whenever two of the lines intersect, merge the corresponding blocks, form the new vector of block masses accordingly, continue drawing the lowest indexed line (now matched to the new block), as well as any line that has not participated in the intersection,
- keep going as long as there is more than one line remaining.

Another construction, advocated as "essentially Armendáriz' representation" (but better for certain applications) is described at the beginning of [63], Chapter 4, Section 3. This latter construction strongly resembles the "tilt" representation of Martin and Ráth (see [50], Definition 2.7 and Theorem 2.8).

The rest of this section is only one possible rigorous formulation of the aforementioned picture. Here Proposition 3 is stated and proved (in [45]) in a particularly convenient partition-valued framework. Corollary 4 features a process that should be the analogue of the Armendáriz representation according to [63]. It serves here as a step in obtaining Theorem 1.2, through the equivalence obtained in Section 4, suggesting its potential relevance elsewhere. An alternative approach is the particle representation of Martin and Ráth [50], Section 3.2, developed in the

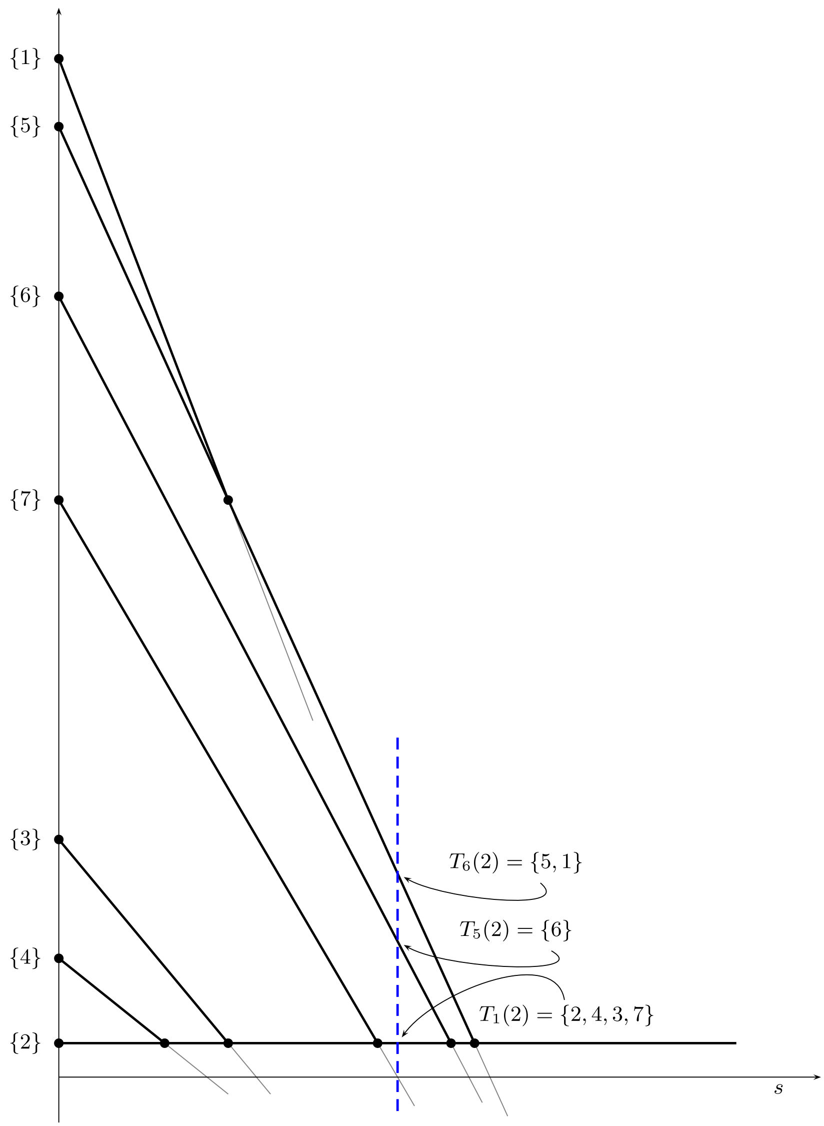

Figure 5. Uribe's diagram of the realization for which the simultaneous and the original breadth-first walks, corresponding to the coalescent time $q = 2$, are depicted in Figures 2 and 3.

more general setting of multiplicative coalescent with linear deletion. Interestingly, the analysis of [50] is again based on an analogue of (a reflection of) Uribe's diagram.

Uribe's diagram could be interpreted as a "genealogical tree". More precisely, let us match each point on the diagram $\bigcup_{i=1}^{n} \bigcup_{s\in[0,s_i]} L_i(s)$ to a subset of $\{1, 2, \ldots, n\}$ in the following way.

For each i, set $T_i(0) := \{\pi_i\}$. Each T_i is piece-wise constant, and jumps according to the following algorithm:

$$T_i(s) := T_i(s-) \cup \bigcup_{j=i+1:s_j=s,\ell_j=i}^{n} T_j(s-), \qquad i \in \{1, 2, \dots, n\},$$

and $T_i(s) := \emptyset$, $\forall s \geq s_i$. In words, for each i, the contents of T_i are moved (without replacement) to T_{ℓ_i} at time s_i, and there is no other copying, cutting or pasting done. Define $T(s) := \{T_1(s), \dots, T_n(s)\}$, $s \geq 0$, where the empty sets are ignored. Note that in this way each $s \geq 0$ is mapped to a partition $T(s)$ of $\{1, 2, \dots, n\}$. The reader is referred to an analogous *look-down* construction of [29], which has since been extensively used in the setting of massless (usually called exchangeable) coalescents.

Clearly the partitions along each path of T are nested: if $s_1 < s_2$ and l and k are in the same equivalence class of $T(s_1)$, they are (almost surely) in the same equivalence class of $T(s_2)$. So T can also be regarded as a random coalescent on the space of partitions of $\{1, 2, \dots, n\}$. Its initial state is the trivial partition $\theta_0 := \{\{\pi_1\}, \{\pi_2\}, \dots, \{\pi_n\}\} = \{\{1\}, \{2\}, \dots, \{n\}\}$. Denote by $\mathcal{G}_t := \sigma\{T(s), s \leq t\}$, $t \geq 0$, so that $\mathcal{G} := (\mathcal{G}_t)_{t \geq 0}$ is the filtration generated by T. The process T will be referred to in the sequel as *Uribe's coalescent* process.

It is evident that Uribe's diagram L is a deterministic function of $\mathbf{x}$ and ξs, and when needed we shall underline this fact by writing $L(\xi_1, \xi_2, \dots, \xi_n; \mathbf{x})$.

Remark 3.1. The ξs are independent and continuous, and therefore (with probability 1) no two pairs of lines in L' can meet simultaneously. Therefore T (viewed as path-valued) takes value in the space of step functions, such that successive values on a typical path are nested (sub)partitions of θ_0, each having exactly one fewer equivalence class than the prior one. In particular, $\mathcal{G}_t$ is generated by events of the following type: for $k \geq 1$

$$\{T(0) = \theta_0, T(t_1) = \theta_1, T(t_2) = \theta_2, \dots, T(t_k) = \theta_k\}, \qquad 0 < t_1 < \dots < t_k \leq t,$$

where θ_{j+1} is either equal to θ_j or to a "coarsening" of θ_j obtained by merging two different equivalent classes in θ_j, $0 \leq j \leq k-1$.

Let us now account for the masses: for any i and $s \geq 0$ define $M_i(s) := \sum_{l \in T_i(s)} x_l$, with the convention that a sum over an empty set equals 0. In this way, to each non-trivial equivalence class of $T(s)$ a positive mass is uniquely assigned, and the sum of the masses $\sum_i M_i(s)$ is the identity $\sum_{i=1}^n x_i$, almost surely.

Suppose for a moment that $n = 2$. For Uribe's diagram, there are two possibilities: either π is the identity, or π is the transposition. In either case, the two initial equivalence classes $\{1\}$ and $\{2\}$ merge at random time s_2 which we denote by S. Note that the event $\{S > s\}$ is (almost surely) identical to the union of the following two disjoint events $\{\xi_2 > \xi_1 + s x_1\}$ and $\{\xi_1 > \xi_2 + s x_2\}$. Thus

$$P(S > s) = \int_0^\infty x_1 e^{-x_1 u} e^{-x_2(u + s x_1)}\, du + \int_0^\infty x_2 e^{-x_2 u} e^{-x_1(u + s x_2)}\, du, \tag{3.1}$$

and the reader can easily verify that the RHS equals $e^{-x_1 x_2 s}$. So if $n = 2$, the coalescent time is distributed equally in the random graph (component masses) and in Uribe's coalescent.

Even with this hint in mind, the next result will likely seem at least counterintuitive if not striking to an non-expert reader.

Proposition 3. *Uribe's coalescent process T has the law of the partition-valued process generated by the connected components of the continuous-time random graph. More precisely, it is a continuous-time Markov chain, such that any two equivalence classes in T merge, independently of all the other merger events, at the rate equal to the product of their masses* (*where the mass of an equivalence class is the sum of xs over its elements*).

The argument given in the supplement [45], Section 4 is based on a sequence of elementary observations, and its outline is comparable to that of [11], Lemma 15. Some care is however needed in correctly setting up the conditioning (otherwise the statement of the proposition would seem obvious from the start). In particular, there seems to be no way of a priori knowing that T with respect to $\mathcal{G}$ has the Markov property. The argument exhibits the transition rates in the process of checking for Markovianity.

Recall that, for $i = 1, 2, \dots, n$, $M_i(s)$ is the mass of the equivalence class $T_i(s)$, provided that $T_i(s) \neq \varnothing$, and it is defined to be 0 otherwise. Now let $\mathbf{Y}(s)$ be a $l^2_{\searrow}$-valued random variable, formed by listing the components of $(M_1(s), M_2(s), \dots, M_n(s))$ in decreasing order, and appending infinitely many zeros (to obtain a vector in $l^2_{\searrow}$). It is clear that $\mathbf{Y} = (\mathbf{Y}(s), s \geq 0)$ is adapted to $\mathcal{G}$. Moreover, Proposition 3 can be restated as the following.

Corollary 4. *The process* $\mathbf{Y}$ *is a multiplicative coalescent started from the decreasing ordering of* $(x_1, x_2, \dots, x_n, 0, \dots)$.

In the case where all the n initial masses are equal, a subset of the just derived identities was already known to Gumbel [40]. It is well known (see, for example, the discussion in [63], Chapter 4, or [25], Chapter 7, or [30], Chapter 2, Section 8) that the connectivity time of the (classical) Erdős-Rényi random graph, is of the order $(\log n + G + o(1))/n$, where G has Gumbel's law $P(G \leq g) = e^{-e^{-g}}$, $g \in \mathbb{R}$.

4. Breadth-first walks meet Uribe's diagram

In this section, we will compare the simultaneous breadth-first random walks of Section 2 with Uribe's diagram of Section 3. More precisely, a coupling of these random objects will be realized on one and the same probability space, so that the multiplicative coalescent marginals $\mathbf{X}$ coupled to $Z^{\mathbf{x},\cdot}$ (see Section 2) can be matched to $\mathbf{Y}$ derived from Uribe's diagram (see Section 3).

As an immediate corollary we obtain the following.

Proposition 5. *Let* $\mathbf{x}$ *be finite. Then the multiplicative coalescent marginals* $\mathbf{X}$ *coupled to* $Z^{\mathbf{x},\cdot}$ *has the law of a multiplicative coalescent started from* $\mathbf{x}$.

Let $\mathbf{x}$ be finite, set $n := \mathrm{len}(\mathbf{x})$, and recall the construction from Section 2. Use the same $(\xi_1, \dots, \xi_n)$ to form the corresponding Uribe diagram $L(\xi_1, \xi_2, \dots, \xi_n; \mathbf{x})$. The notation is slightly abused here since in Section 2 (resp. 3) vectors have infinite (resp. finite) length, but the correspondence between the two is clear (appending infinitely many zeros to the finite vector will give the infinite one).

Assume for a moment that $\mathbf{x} = (1.1, 0.8, 0.5, 0.4, 0.4, 0.3, 0.2)$, and let us pretend that the realization from Figures 2 and 3 corresponds to the time parameter q equal to 2. Let us assume in addition that $\pi = \tau := (2, 4, 3, 7, 6, 5, 1)$. This means that $(\xi_{(1)}, \xi_{(2)}, \dots, \xi_{(7)}) = (\xi_2, \xi_4, \xi_3, \xi_7, \xi_6, \xi_5, \xi_1) = (0.2, 0.7, 1.4, 3.4, 4.6, 5.6, 6)$.

The corresponding Uribe's diagram $L(\xi, \mathbf{x})$ is shown in Figure 5. For any time s, the partition $T(s)$ can be read from the graph as it could be read from a genealogical tree. Each of the "active" lines represents a different equivalence class. The blue vertical dashed line marks time $q = 2$. In $T(2)$ there are three equivalence classes, matched to L_1, L_5 and L_6, as shown in the figure. This partition is, of course, the same as the one given in terms of trees depicted on Figure 3 (recall that the same realization is being illustrated in Figures 2, 3 and 5).

The coupling stated at the beginning of the section is realized in the most natural way. Recall that both $(Z^{\mathbf{x},q}, q > 0)$ and $L(\xi; \mathbf{x})$ (and therefore T and $\mathbf{Y}$) are functions of ξs and $\mathbf{x}$. It is important here to let the finite family of independent exponential random variables $\xi_\cdot$, used in the construction of $Z^{\mathbf{x},\cdot}$ and $L(\xi; \mathbf{x})$, be the same, almost surely.

As already noted (see Remark 2.1(c)), the partition structure induced by the evolution of $Z^{\mathbf{x},\cdot}$ gets coarser as q increases. In addition, only pairs of neighboring blocks or families of blocks, with respect to the random order established by π, can coalesce either in $\mathbf{X}$ or in T (that is, in $\mathbf{Y}$). Note that, for each multiplicative coalescent time q, the relation of being connected by a path of edges $\leftrightarrow$ that occurred before time q is an equivalence relation on the initial set of blocks. Hence it suffices to show that, almost surely, for each $q > 0$ and $i \in \{1, 2, \dots, n-1\}$ it is true that $\pi_i \leftrightarrow \pi_j$ with respect to $Z^{\mathbf{x},q}$ (see Section 2) if and only if $\pi_i \sim \pi_j$ with respect to T (see Section 3). At time $q = 0$ the just made claim is clearly correct, since there are no edges $\leftrightarrow$ in $\mathbf{X}$, and the partition of T is trivial. Suppose that random time $Q_1 > 0$ is such that $T(Q_1-) = \theta_0$ and $T(Q_1)$ contains $\{\pi_i, \pi_{i+1}\}$. This means that the lines L'_i and L'_{i+1} intersect at time Q_1, and no other pair of lines intersects before time Q_1. Or equivalently, $\xi_{(i+1)} - \xi_{(i)} = Q_1 x_{\pi_i}$ and $\xi_{(j+1)} - \xi_{(j)} > Q_1 x_{\pi_j}$ for $j \neq i$. Or equivalently, $\xi^{Q_1}_{(i+1)} - \xi^{Q_1}_{(i)} = x_{\pi_i}$, and $\xi^{Q_1}_{(j+1)} - \xi^{Q_1}_{(j)} > x_{\pi_j}$ for $j \neq i$. A quick check of the construction in Section 2 suffices to see that, on the above event, the edge $\pi_i \leftrightarrow \pi_{i+1}$ arrives at time Q_1 in $\mathbf{X}$, and no edge arrives to $\mathbf{X}$ before time Q_1. At time Q_1, the line L'_{i+1} stops being active in Uribe's diagram. The new neighbors of $\{\pi_i, \pi_{i+1}\}$ are π_{i+2} and π_{i-1}. All the active lines above L'_i account for the new mass $x_{\pi_i} + x_{\pi_{i+1}}$ of $\{\pi_i, \pi_{i+1}\}$, since this quantity is built into their slope (together with masses corresponding to any other active lines underneath them). Similarly, $Z^{\mathbf{x},q}$ for $q > Q_1$ does not need to observe $\xi^q_{(i+1)}$ any longer, it suffices to attribute the cumulative "listening length" $x_{\pi_i} + x_{\pi_{i+1}}$ to the breadth-first walk time $\xi^q_{(i)}$ at which the leading particle of the component $\{\pi_i, \pi_{i+1}\}$ is seen by the walk. Due to these two observations, one can continue the comparison of the coalescence of the remaining blocks driven by $(T(q), q \geq Q_1)$ to that driven by $(Z^{\mathbf{x},q}, q \geq Q_1)$, and conclude by induction that in both processes the sequence of pairs of blocks that coalesce, and their respective times of coalescence, are identical, almost surely.

As already noted, Proposition 5 is a direct consequence. Note that in the sense of the just produced coupling, the simultaneous breadth-first walks of Section 2 are equivalent to Uribe's diagram. In comparison, the original breadth-first walk of [5] coupling is "static" (it works for one q at a time), and it seems difficult to turn it into a "dynamic" version due to a certain (small but present) loss of information (see Lemma 2).

5. Scaling limits for simultaneous breadth-first walks

This section imitates the approach of Section 2.4 in [6]. It is interesting to note that these scaling limits are simpler to derive here than they were for the original multiplicative coalescent encoding walks in [6].

Given $\mathbf{x} \in l^2_\searrow$ let

$$\sigma_r(\mathbf{x}) := \sum_i x_i^r, \qquad r = 1, 2, 3.$$

For each $n \geq 1$, let $\mathbf{x}^{(n)}$ be a finite vector (in the sense of Section 2). Let $((Z^{\mathbf{x}^{(n)},q}(s), s \geq 0), q \geq 0)$ and $(\mathbf{X}^{(n)}(q), q \geq 0)$ be the simultaneous breadth-first walks, and the multiplicative coalescent coupled to $Z^{\mathbf{x}^{(n)},\cdot}$ (see Section 2 and Proposition 5), respectively.

Suppose that for some $\kappa \in [0, \infty)$ and $\mathbf{c} \in l^3_\searrow$, the following hypotheses are true:

$$\frac{\sigma_3(\mathbf{x}^{(n)})}{(\sigma_2(\mathbf{x}^{(n)}))^3} \to \kappa + \sum_j c_j^3, \tag{5.1}$$

$$\frac{x_j^{(n)}}{\sigma_2(\mathbf{x}^{(n)})} \to c_j, \qquad j \geq 1, \tag{5.2}$$

$$\sigma_2\bigl(\mathbf{x}^{(n)}\bigr) \to 0, \tag{5.3}$$

as $n \to \infty$. It is easy to convince oneself (or see Lemma 8 of [6] or [45], Section 6) that for any $(\kappa, 0, \mathbf{c}) \in \mathcal{I}$ there exists a finite vector valued sequence $(\mathbf{x}^{(n)})_{n\geq 1}$ satisfying (5.1)–(5.3).

As in [6], we furthermore pick an integer valued sequence $(m(n))_{n\geq 1}$, which increases to infinity sufficiently slowly so that

$$\left|\sum_{i=1}^{m(n)} \frac{(x_i^{(n)})^2}{(\sigma_2(\mathbf{x}^{(n)}))^2} - \sum_{i=1}^{m(n)} c_i^2\right| \to 0, \qquad \left|\sum_{i=1}^{m(n)} \left(\frac{x_i^{(n)}}{\sigma_2(\mathbf{x}^{(n)})} - c_i\right)^3\right| \to 0, \tag{5.4}$$

$$\text{and} \quad \sigma_2\bigl(\mathbf{x}^{(n)}\bigr) \sum_{i=1}^{m(n)} c_i^2 \to 0. \tag{5.5}$$

Fix $t \in \mathbb{R}$ and let $q_n := \frac{1}{\sigma_2(\mathbf{x}^{(n)})} + t$. Recall (2.1), and define

$$Z_n := Z^{\mathbf{x}^{(n)}, q_n},$$
$$R_n(s) := \sum_{i=1}^{m(n)} \left(x_i^{(n)} 1_{(\xi_i^{q_n} \le s)} - \frac{(x_i^{(n)})^2}{\sigma_2(\mathbf{x}^{(n)})} s \right), \quad \text{and} \quad Y_n(s) := Z_n(s) - R_n(s), \qquad s \ge 0.$$

It is implicit in the notation that $\xi_i^{q_n} := \xi_i^{(n)}/q_n$, where $\xi_i^{(n)}$ has exponential (rate $x_i^{(n)}$) distribution, and where $(\xi_i^{(n)})_i$ are independent over i, for each n.

Define $\bar{Z}_n$, $\bar{R}_n$, $\bar{Y}_n$ to be respectively, Z_n, R_n, Y_n multiplied by $\frac{1}{\sigma_2(\mathbf{x}^{(n)})}$, so that $\bar{Z}_n \equiv \bar{Y}_n + \bar{R}_n$. It should not be surprising that both the shift in the multiplicative coalescent time and the spatial scaling applied to the walks are the same as in [6]. It is clear that, for each n, R_n and Y_n are independent (the former depends only on the first $m(n)$ terms of the sequence $(\xi_i^{(n)})_i$, and the latter only on the other terms).

Recall the definitions (1.5)–(1.7). The following result is a direct analogue of [6], Proposition 9.

Proposition 6. *If $(\kappa, 0, \mathbf{c}) \in \mathcal{I}$, and provided (5.1)–(5.3) are satisfied as $n \to \infty$, then*

$$(\bar{Y}_n, \bar{R}_n) \xrightarrow{d} (\widetilde{W}^{\kappa,t}, V^{\mathbf{c}}), \qquad \text{as } n \to \infty,$$

where $\widetilde{W}^{\kappa,t}$ and $V^{\mathbf{c}}$ are independent, and therefore $\bar{Z}_n \xrightarrow{d} W^{\kappa,t,\mathbf{c}}$.

The rest of this section is devoted to the proof of the above proposition, and some of its consequences. As already mentioned, the argument is a simplification of that given in Section 2.4 of [6], for the main reason that the current Y_n has a simpler explicit form. From now on assume that $t \in \mathbb{R}$ is the one fixed in Proposition 6 via the definition of q_n.

Note that the independence of Y_n and R_n clearly implies that of $\bar{Y}_n$ and $\bar{R}_n$. So provided that each of the sequences converges in law, the joint convergence in law to the product limit law is a trivial consequence. Furthermore, the convergence of $\bar{R}_n$ can be verified in a standard way (for each k, the kth largest jump of $\bar{R}_n$ converges to the kth largest jump of $V^{\mathbf{c}}$, and the second-moment MG estimates are used to bound the tails), as was already done in [6] (see the supplement [45], Section 7).

Lemma 7. *We have*

$$\bar{R}_n \xrightarrow{d} V^{\mathbf{c}}(s), \qquad \text{as } n \to \infty. \tag{5.6}$$

It remains to study the convergence of $(\bar{Y}_n)_n$. This sequence of processes differs from the equally named sequence in [6]. Write σ_r^n for $\sigma_r(\mathbf{x}^{(n)})$, $r = 1, 2, 3$ in the sequel. An important observation is that

$$\bar{Y}_n(s) := \sum_{i=m(n)+1}^{\mathrm{len}(\mathbf{x}^{(n)})} \frac{x_i^{(n)}}{\sigma_2^n} 1_{(\xi_i^{q_n} \le s)} - \frac{s}{\sigma_2^n} + \sum_{i=1}^{m(n)} \frac{(x_i^{(n)})^2}{(\sigma_2^n)^2} s. \tag{5.7}$$

The infinitesimal drift and variance calculations are now straightforward. Let $\mathcal{F}_s^n := \sigma\{\bar{Y}_n(u) : u \le s\}$, so that $\mathcal{F}^n := (\mathcal{F}_s^n)_{s\ge 0}$ is the filtration generated by $\bar{Y}_n$. The proof of the following result is also provided in [45], Section 7.

Lemma 8. *For each fixed s,*

$$E\big(d\bar{Y}_n(s)|\mathcal{F}_s^n\big) \xrightarrow{p} (t-\kappa s)\,ds, \qquad \text{as } n\to\infty, \tag{5.8}$$

$$E\big(\big(d\bar{Y}_n(s)\big)^2|\mathcal{F}_s^n\big) \xrightarrow{p} \kappa\,ds, \qquad \text{as } n\to\infty. \tag{5.9}$$

Since the largest jump of $\bar{Y}_n$ is of size $x_{m(n)+1}^{(n)}/\sigma_2^n = o_n(1)$, the classical martingale central limit theorem (cf. [33]) implies that

$$\bar{Y}_n \xrightarrow{d} \widetilde{W}^{\kappa,t}, \qquad \text{as } n\to\infty,$$

and, as already argued, this concludes the proof of the proposition.

Remark 5.1. By the Cauchy–Schwarz inequality, $(\sigma_2^n)^2 \le \sigma_1^n\sigma_3^n$, and so (5.1), (5.3) imply that $\sigma_1^n \to \infty$ as $n\to\infty$. While this fact was needed in the proof of the analogous [6], Proposition 9, here it could slip by unnoticed. If $\kappa > 0$, it is easy to see that the limit $W^{\kappa,t,\mathbf{c}}$ of $\bar{Z}_n$ has (countably) infinitely many excursions above past minima. If $\kappa = 0$ and $\mathbf{c}\in l^3_{\searrow}\setminus l^2_{\searrow}$, the same was proved in [6], Proposition 14.

Using the Skorokhod representation theorem, we may assume that the convergence stated in Proposition 6 holds in the almost sure sense. To state the next result (essential for the conclusions to be made in Section 6), redefine $q_n(t) := t + \frac{1}{\sigma_2^n}$, for $t\in\mathbb{R}$. Then let $Z_n^t := Z^{\mathbf{x}^{(n)},q_n(t)}$ and $\bar{Z}_n^t := Z_n^t/\sigma_2^n$. The (almost sure version of) Proposition 6 says that there exists a Brownian motion W and an independent jump process $V^{\mathbf{c}}$, such that

$$\bar{Z}_n^t \to W^{\kappa,t,\mathbf{c}}, \qquad \text{almost surely, as } n\to\infty,$$

where the convergence of paths is considered in the Skorokhod J_1 topology. Let

$$A_t := \big\{\omega : \bar{Z}_n^t(\cdot)(\omega) \to W^{\kappa,t,\mathbf{c}}(\cdot)(\omega) \text{ in the Skorokhod } J_1 \text{ topology}\big\}.$$

Lemma 9. *On the event A_t, for any $z\in\mathbb{R}$*

$$\big(\bar{Z}_n^z(s), s\ge 0\big) \to \big(W^{\kappa,t,\mathbf{c}}(s) + (z-t)s, s\ge 0\big) \equiv W^{\kappa,z,\mathbf{c}}, \qquad \text{as } n\to\infty,$$

in the Skorokhod J_1 topology.

Proof. Recall the explicit form (2.1) of $Z^{\cdot,\cdot}$ Observe the following identity:

$$Z_n^z\left(s\cdot\frac{q_n(t)}{q_n(z)}\right) = Z_n^t(s) + s\left(1-\frac{q_n(t)}{q_n(z)}\right), \qquad \forall s\ge 0.$$

Since clearly $\lim_{n\to\infty}\frac{q_n(t)}{q_n(z)}=1$, and moreover since

$$\frac{1}{\sigma_2^n}\left(1-\frac{q_n(t)}{q_n(z)}\right)=z-t+O_{z,t}\left(\sigma_2^n\right),$$

the convergence stated in the lemma follows omega-by-omega on A_t.

6. Conclusions

Propositions 7 and 9 from [6] are stated in [45], Section 5. Here is an immediate consequence of them and Lemma 2, as announced in Remark 2.1(d).

Corollary 10. *Assume the hypotheses of Proposition 6. Then for each fixed t, the sequence $(\mathbf{X}^{(n)}(q_n(t)))_n$ converges in law (with respect to $l^2_\searrow$-metric) to the sequence of ordered excursions of $B^{\kappa,t,\mathbf{c}}$ (away from 0).*

But in fact more is true in view of Lemma 9. From now on we take the families $(W^{\kappa,t,\mathbf{c}},t\in\mathbb{R})$ and $(B^{\kappa,t,\mathbf{c}},t\in\mathbb{R})$ to be jointly defined on a common probability space, as in Section 1.3 (and in Theorem 1.2) via a given pair $(W,V^{\mathbf{c}})$, where W is Brownian motion and $V^{\mathbf{c}}$ is an independent jump process from (1.4).

Let us denote by A the event A_t of full probability from Lemma 9. For each $t\in\mathbb{R}$, define $\Xi^{(n)}(t)$ to be the point process on $[0,\infty)\times(0,\infty)$ such that (x,y) is in $\Xi^{(n)}(t)$ if and only if there is an excursion above past minima of $\bar{Z}_n^t$ (see (2.7) and Figure 2), starting from x and ending at $x+y$. Similarly, let $\Xi^{(\infty)}(t)$ be the point process on $[0,\infty)\times(0,\infty)$ such that (x,y) is in $\Xi^{(\infty)}(t)$ if and only if there is an excursion away from 0 of $B^{\kappa,t,\mathbf{c}}$, starting from x and ending at $x+y$. One can then apply deterministic result stated as [5], Lemma 7 to conclude the following: on the event A of full probability, for each $t\in\mathbb{R}$, one has

$$\lim_{n\to\infty}\Xi^{(n)}(t)=\Xi^{(\infty)}(t), \tag{6.1}$$

in the sense of vague convergence of counting measures on $[0,\infty)\times(0,\infty)$ (see, e.g., [42]). As in [5,6], write π for the "project onto the y-axis" defined on $\mathbb{R}^2$, and "ord" for the "decreasing ordering" map defined on infinite-length vectors, respectively. For a fixed (think large) $K<\infty$, define in addition π_K to be the "project the strip $[0,K]\times(0,\infty)$ onto the y-axis" analogue of π. Then one can recognize $\mathrm{ord}(\pi(\Xi^{(n)}(t)))$ as $\mathbf{X}^{(n)}(q_n(t))$, and $\mathbf{X}^{(\infty)}(t):=\mathrm{ord}(\pi(\Xi^{(\infty)}(t)))$ as the infinite vector of excursion lengths of $B^{\kappa,t,\mathbf{c}}$. Similarly $\pi_K(\Xi^{(\infty)}(t))$ (resp. $\pi_K(\Xi^{(n)}(t))$) is the collection of all the excursions of $B^{\kappa,t,\mathbf{c}}$ (resp. $\bar{Z}_n^t$), which start before time K.

We already know that the law of $\mathbf{X}^{(\infty)}(t)$ is that of the marginal of $\mu(\kappa,0,\mathbf{c})$ at time t (or equivalently, the marginal of $\mu(\kappa,t,\mathbf{c})$ at time 0). The vague convergence (6.1) now easily implies that there exists a (random) order of $\pi_K(\Xi^{(n)}(t))$, here temporarily denoted by $\mathrm{ord}_{\Xi^{(n)},\Xi^{(\infty)}}$, since it is induced by the similarity of the Ξs, such that

$$\left\|\mathrm{ord}_{\Xi^{(n)},\Xi^{(\infty)}}\left(\pi_K\left(\Xi^{(n)}(t)\right)\right)-\mathrm{ord}\left(\pi_K\left(\Xi^{(\infty)}(t)\right)\right)\right\|_2\to 0,\qquad \text{on the event } A. \tag{6.2}$$

In words, if considering only the starts before time K, it is possible to order the excursions of $\bar{Z}_n^t$ so that the corresponding infinite vector (obtained by appending an infinite sequence of 0s to the elements of $\pi_K(\Xi^{(n)}(t))$) matches the infinite vector $\mathrm{ord}(\pi_K(\Xi^{(\infty)}(t)))$ in l_1-norm up to an $o_n(1)$ error term. Moreover, convergence in l_1-norm implies convergence in l_2-norm.

Take $z > t$, another real number, and let $\varepsilon > 0$ be fixed but arbitrarily small. From now on, u will denote either t or z. In order to upgrade (6.2) to the convergence of $\mathbf{X}^{(n)}(q_n(u))$ to $\mathbf{X}^{(\infty)}(u)$ with respect to distance $d(\cdot,\cdot)$, one can make the following observations. For $\mathbf{x} \in l^2_{\searrow}$, let $f_m(\mathbf{x}) := (x_1, \dots, x_m, 0, 0, \dots)$ be the "projection" onto the first m components. Take some arbitrarily large integer k, and choose $m_k \in \mathbb{N}$ such that

$$P\big(d\big(\mathbf{X}^{(\infty)}(u), f_{m_k}\big(\mathbf{X}^{(\infty)}(u)\big)\big) > \varepsilon\big) < \frac{1}{2^k}. \tag{6.3}$$

Since $\mathbf{X}^{(n)}(q_n(t)) \xrightarrow{d} \mathbf{X}^{(\infty)}(t)$ with respect to d, for this k and possibly larger but still finite m_k we can have in addition

$$\limsup_n P\big(d\big(\mathbf{X}^{(n)}\big(q_n(u)\big), f_{m_k}\big(\mathbf{X}^{(n)}\big(q_n(u)\big)\big)\big) > \varepsilon\big) < \frac{1}{2^k}. \tag{6.4}$$

In words, with an overwhelming probability, all the (random) infinite vectors under consideration are well-approximated (in the l_2-norm) by their first m_k components.

Since $\pi(\Xi^{(\infty)}(\cdot))$ is l_2-valued, and since its elements are listed in size-biased order, one can easily deduce that for the above m_k, there exists some large time $K_k := K(m_k) < \infty$, such that

$$P\big(f_{m_k}\big(\mathrm{ord}\big(\pi_K\big(\Xi^{(\infty)}(u)\big)\big)\big) \neq f_{m_k}\big(\mathbf{X}^{(\infty)}(u)\big)\big) < \frac{1}{2^k}. \tag{6.5}$$

In words, K is sufficiently large so that with high probability the largest m_k elements of $\pi(\Xi^{(\infty)}(u))$, all correspond to excursions that started before time K. Again due to $\mathbf{X}^{(n)}(q_n(u)) \xrightarrow{d} \mathbf{X}^{(\infty)}(u)$, the analogous

$$\limsup_n P\big(f_{m_k}\big(\mathrm{ord}\big(\pi_K\big(\Xi^{(n)}\big(q_n(u)\big)\big)\big)\big) \neq f_{m_k}\big(\mathbf{X}^{(n)}\big(q_n(u)\big)\big)\big) < \frac{1}{2^k} \tag{6.6}$$

is implied for some (possibly larger but) finite $K = K(m_k)$.

Apply the triangle inequality to bound $d(\mathbf{X}^{(n)}(q_n(u)), \mathbf{X}^{(\infty)}(u))$ by the sum of the following terms: $d(\mathbf{X}^{(n)}(q_n(u)), f_{m_k}(\mathbf{X}^{(n)}(q_n(u))))$, $d(f_{m_k}(\mathbf{X}^{(n)}(q_n(u))), f_{m_k}(\mathbf{X}^{(\infty)}(u)))$, and $d(f_{m_k}(\mathbf{X}^{(\infty)}(u)), \mathbf{X}^{(\infty)}(u))$. The initial and the final term are controlled by (6.3)–(6.4), while the middle term is controlled by (6.5)–(6.6) and (6.2), where one makes use of the elementary inequality: for $\mathbf{x}, \mathbf{y} \in l^2$,

$$d\big(\mathrm{ord}(\mathbf{x}), \mathrm{ord}(\mathbf{y})\big) \leq \sum_i (x_i - y_{b(i)})^2,$$

regardless of the choice of bijection $b : \mathbb{N} \to \mathbb{N}$.

Remark 6.1. It is clear (for example from (5.8)–(5.9), think about redefining $q_n(t)$ as $\frac{1}{\sigma_2(\mathbf{x}^{(n)})} + t - \tau$), that the parameter τ corresponds to the time-shift of the eternal multiplicative coalescent, and so the above conclusions automatically extend to the setting where $\tau \neq 0$.

An immediate conclusion is the following.

Lemma 11. *If* $((\bar{Z}^{x^{(n)}, q_n(t)}(s), s \geq -1/\sigma_2^n), t \in \mathbb{R}) \longrightarrow ((W^{\kappa, t-\tau, \mathbf{c}}(s), s \geq 0), t \in \mathbb{R})$, *as* $n \to \infty$, *in the sense of Lemma 9, and if* $\mathbf{X}^{(\infty)}(t) \in l^2_{\searrow}$ *is the vector of ordered excursion lengths of* $B^{\kappa, t-\tau, \mathbf{c}}$, *then*

(i) *for any* $t \in \mathbb{R}$

$$d\big(\mathbf{X}^{(n)}\big(q_n(t)\big), \mathbf{X}^{(\infty)}(t)\big) \xrightarrow{p} 0, \qquad \text{as } n \to \infty,$$

(ii) *for any finite sequence of times* $t_1 < t_2 < \cdots < t_m$, *one can find a subsequence* $(n_j)_j$ *such that almost surely*

$$d\big(\mathbf{X}^{(n_j)}\big(q_{n_j}(t_k)\big), \mathbf{X}^{(\infty)}(t_k)\big) \to 0, \qquad \text{for all } k = 1, \dots, m, \text{as } j \to \infty.$$

Recalling that $(\mathbf{X}^{(n)}(q_n(s)), s \geq -1/\sigma_2^n)$ has the law of the multiplicative coalescent (see Proposition 5), and applying the Feller property together with Lemma 11(i), where one should identify $\mathbf{X}^{(\infty)}$ with $\mathbf{X}$, will complete the proof of the claim about the distribution of $\mathbf{X}$ in Theorem 1.2. It is easy to see (using arguments analogous to those given above) that the realization $\mathbf{X}^{(\infty)} \equiv \mathbf{X}$ of each eternal version from Theorem 1.2 is a càdlàg (rcll) process on an event of full probability.

Remark 6.2. Recall the COL operation of [6], Section 5. In particular, each $c_i > 0$ is interpreted as the rate of Poisson coloring (per unit mass) by marks of the ith "color", applied to the standard Aldous' multiplicative coalescent $\mathbf{X}^*$. Once all the color marks are deposited, any two blocks of $\mathbf{X}^*$ that share at least one mark of the same color are instantaneously and simultaneously merged together. The jump in $W^{\kappa, \cdot-\tau, \mathbf{c}}$ at time ξ_i of size c_i corresponds precisely to the effect of coloring by the ith color. Moreover, one could argue that if $\widetilde{W}^{\kappa, \cdot-\tau, \mathbf{0}}$ and $\widetilde{W}^{\kappa, \cdot-\tau, \mathbf{c}}$ are given in (1.6) using the same Brownian motion W, then the excursions of the corresponding $(B^{\kappa, t-\tau+\|\mathbf{c}\|_2, \mathbf{c}}, t \in \mathbb{R})$ (suppose for simplicity that $\mathbf{c} \in l^2_{\searrow}$) away from 0 are almost surely the result of the above COL operation executed on the excursions of $(B^{\kappa, t-\tau, \mathbf{0}}, t \in \mathbb{R})$. The fact that, as time increases, each color "spreads" in this coupling (almost surely) only over the "neighboring" blocks may again seem counterintuitive. The point is that COL commutes with the multiplicative coalescent dynamics, and that therefore it can be pushed to $-\infty$. The infinitesimally small dust particles of $\mathbf{X}^*(-\infty)$ are mutually interchangeable. The ith color at time $-\infty$ is represented as an additional dust particle, of mass much superior to standard dust, but still negligible (a formal statement of this is (1.11) or (5.2)). One can naturally couple the representation of the multiplicative coalescent using simultaneous breadth-first walks (or Uribe's diagram) started only from standard dust as $t \to -\infty$, with the same representation of the multiplicative coalescent started from the union of two types of dust as $t \to -\infty$. Proposition 6 and Lemma 9 do this formally (their predecessor is [6], Proposition 41). In this coupling every color gradually spreads only to neighboring blocks of those already marked by it.

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Supplementary Material

Supplement to "The eternal multiplicative coalescent encoding via excursions of Lévy-type processes" (DOI: 10.3150/18-BEJ1060SUPP; .pdf). The accompanying text consists of eight sections (including a short introductory note). The title of each section summarizes its contents. A fraction of the material presented ([45], Sections 2, 3, 5 and 6) is intended to help readers gain time (reduce the need for consulting external literature) while reading this article. The rest ([45], Sections 4 and 7) contains novel arguments or open problems ([45], Section 8), of which some are related to recent studies [2,3].

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